ADAPTIVE FINITE ELEMENT ANALYSIS BASED ON P-CONVERGENCE

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SUMMARY

The results of numerical experiments are presented in which a posteriori estimators of error in strain energy were examined on the basis of a typical problem in linear elastic fracture mechanics. Two estimators were found to give close upper and lower bounds for the strain energy error. The potential significance of this is that the same estimators may provide a suitable basis for adaptive redistribution of the degrees of freedom in finite element models.

INTRODUCTION

One of the most important problems remaining in finite element research is the development of adaptive finite element software systems, i.e., finite element computer programs which have a local error estimation capability and a capability to increase the number of degrees of freedom selectively such that the quality of approximation is nearly uniform over the entire solution domain and the error does not exceed some pre-specified tolerance.

Research concerned with the development of adaptive finite element software systems has been underway at Washington University for several years. This work has resulted in the development of an approach for improving the quality of approximation without mesh refinement. In this approach the number of degrees of freedom is increased by increasing the polynomial orders (p) or introducing non-polynomial basis functions over fixed finite element mesh divisions. This process of reducing the error of approximation through the addition of new basis functions is called "p-convergence" to distinguish it from the conventional approach (called "h-convergence") in which the size of finite elements (h) is reduced while the number and type of basis functions for each element are fixed.

The efficiency of p-convergent adaptive procedures has been established in a series of numerical experiments, reported in references 1 and 2 and it has been shown that the rate of p-convergence cannot be slower than the rate of h-convergence (I. Babuska, private communication). In fact, for the vast majority of practical problems, p-convergence is substantially faster. This and other computational advantages suggest that adaptive finite element software systems should be based on p-convergence.

The key problem is to find suitable estimators of error which would indicate when and where the number of degrees of freedom should be increased over the solution domain. Some estimators have been proposed already:

1. Babuska and Rheinboldt developed a local, asymptotic, a posteriori error estimator for h-convergent approximations (ref. 3). The estimator requires element level computations only and measures the error in strain energy associated with an element. An optimal distribution of the degrees of freedom is obtained when this error measure is the same for all elements.

2. Melosh and Marcal (ref. 4) proposed to measure the specific energy difference (SED), defined as the largest difference over the element domain between the computed strain energy density function and the same function evaluated at the origin of the elemental coordinate system. This measures the effect of modes higher than those associated with the (generalized) constant strain states on the distribution of strain energy density within finite elements. The criterion for mesh refinement is that SED be approximately the same for each finite element. In practical computations SED is approximated by the largest of the strain energy density differences evaluated at quadrature points only.

3. Peano et al. (ref. 5) proposed a criterion for p-convergent approximations. This criterion is based on the rate of change of the total potential energy with respect to higher order displacement modes, evaluated before the higher displacement modes are actually introduced. When the rate of change of the potential energy exceeds a prescribed tolerance, the stiffness terms corresponding to the higher modes are assembled and the new system of equations is solved. The hierarchic structure of the elemental stiffness matrices permits efficient use of block relaxation procedures in obtaining improved solutions.

In this paper we examine the numerical characteristics of a criterion similar to that proposed by Babuska and Rheinboldt in ref. 3, but modified for p-convergent approximations. Our study is preliminary in nature and is restricted to one specific problem.

ERROR MEASURES

We have examined two measures for the error in strain energy on the basis of a problem in two-dimensional elasticity containing a geometric singularity. This problem is typical for a large class of problems in linear elastic fracture mechanics (fig. 1). The error measures were as follows. First we define the ith component of the residual vector, which represents the unbalanced body force, as

$$r_i = G\,\hat{u}_{i,jj} + (\lambda + G)\,\hat{u}_{j,ji} + X_i \qquad (1)$$

in which

G and λ are Lame's constants;
$\hat{u}_i$ is the ith component of the displacement vector computed by the finite element method. The subscripts range over 1,2.
X_i is the ith component of the body force vector.

One of the measures, to be called the "r-estimator," is defined for the kth finite element as

$$R_k(\alpha) = \frac{1}{p_k^{\alpha}} \int_{A_k} (r_1^2 + r_2^2)\, dA \tag{2}$$

in which

- α is a constant, to be determined by numerical experiments;
- p_k is the polynomial order of the displacement approximation over the kth element;
- A_k is the area of the kth element.

The other measure, to be called the "t-estimator", is defined over interelement boundaries and external boundaries on which tractions are specified, as the square of the unbalanced tractions. Specifically, at the boundary of two elements, the vector of unbalanced tractions is:

$$t_i(s) = [\hat{\sigma}_{ij}^{(a)}(s) - \hat{\sigma}_{ij}^{(b)}(s)]\, n_j \tag{3}$$

in which

- $\hat{\sigma}_{ij}^{(a)}$ is the finite element approximation to the stress tensor for the ath element;
- n_j is the unit normal to the interelement boundary;
- s is the variable along the element boundary.

At external boundaries the unbalanced traction vector is the difference between the computed traction vector and the applied traction vector. When displacement vector components are specified, the corresponding unbalanced traction vector component is zero.

The t-estimator is defined as

$$T_k(\beta) = \frac{1}{p_k^{\beta}} \int_{\Gamma_k} (t_1^2 + t_2^2)\, ds \tag{4}$$

in which

- β is a constant, to be determined by numerical experiments;
- p_k is the polynomial order of the displacement approximation at the kth interelement or external boundary segment;
- Γ_k denotes the kth elemental boundary.

Both the r and t measures were found to give close indications of the total error in strain energy in p-convergent approximations. The details are as follows.

The sample problem represented in fig. 1 does not have a known exact solution. For this reason it was necessary first to estimate the exact value of the total strain energy U. This was possible by utilizing the asymptotic relationship given in reference 6:

$$U = U_p + \frac{c}{NDF} \tag{5}$$

in which

U_p	is the computed (total) strain energy, based on pth order polynomial approximation;
c	is a constant;
NDF	is the (net) number of degrees of freedom.

Extrapolating on the basis of eq. (5), U was found to be $0.7702\ \sigma^2\ell^2/E$ ($+ 0.0002\ \sigma^2\ell^2/E$) in computations involving two different finite element mesh divisions. The computed dimensionless strain energy values, the corresponding errors and the values of the two error estimators for parameter values $\alpha = 3$, $\beta = 2$ are given in Table I. The percent changes in the estimators as p is increased tend to bound the corresponding percent change in the strain energy error with increasing precision such that the change in the r-estimator is smaller and the change in the t-estimator is larger than that of the strain energy error. Furthermore, the values of these percentage changes are monotonically decreasing for $p > 2$. This suggests the possibility that two constants c_r and c_t could be found such that a relationship,

$$c_r \sum_k R_k(3) \leq U - U_p \leq c_t \sum_k T_k(2) \tag{6}$$

remains valid for all p-values and that the upper and lower bounds become progressively closer to the strain energy error as p is increased. This is, of course, highly speculative at the present because no theoretical justification exists, but consistent with the observations of this numerical experiment. For example, if we choose $c_r = 0.7008$ and $c_t = 0.2657$, the two estimators will give the value of the strain energy error at $p = 7$. The resulting relationship between the strain energy error and the two estimators is illustrated in fig. 2.

The question naturally arises whether the same estimators would bound the energy error at the element level as well. Clearly, this approach will be useful only if the indicators tend to zero with increasing p at about the same rate as the error in strain energy does, not only for the entire solution domain but for individual finite elements as well. The presently available information is sufficient only to indicate trends.

The problem chosen for study does not have a known exact solution; thus the energy error cannot be computed with precision. For the global solution the rate of convergence formula (eq. 5) provided a basis for extrapolation to the limit value of the computed strain energy values. Such formulas are not available for predicting convergence at the element level. For this reason an ad hoc procedure for extrapolation had to be devised. It was assumed that the strain energy converges at the element level as

$$u^{(k)} = u_p^{(k)} + \frac{\mu}{p^\nu} \tag{7}$$

where μ and ν are constants, $u^{(k)}$ is the strain energy of the kth element, and $u_p^{(k)}$ is the strain energy of the kth element computed on the basis of uniform pth order polynomial approximation over all of the finite elements. Taking two consequtive values of polynomial orders q and p, μ can be eliminated and the extrapolated value of $u^{(k)}$ is

$$u^{(k)} = \frac{p^\nu u_p^{(k)} - q^\nu u_q^{(k)}}{p^\nu - q^\nu} \tag{8}$$

where ν and $u^{(k)}$ were chosen such that for q = 5, p = 6 and q = 6, p = 7 the value of $u^{(k)}$ was constant. The resulting estimate of the strain energy for element 2 is $u^{(k)} = 0.0844\ \sigma^2 \ell^2/E$. When the p-distribution is uniform over the entire mesh, the estimators vary over the finite elements by several orders of magnitude. As could be expected, their value is the greatest for the crack tip elements (element numbers 1,3,4) and least for the elements remote from the crack tip (element numbers 5,6,8). In those elements which are not on the crack tip, the estimators apparently approach zero faster than the error in strain energy.

When the distribution of p is altered such that the element to element variation in the estimators is reduced, then the estimators tend to become closer to the energy error. Specifically, letting p = 7 for the crack tip elements and p = 3 for the remote elements, the variation in the estimators is reduced somewhat but is still 5 orders of magnitude for the r-estimators and 4 orders of magnitude for the t-estimators. In this case we have complete third order polynomial approximation in the transition elements 2 and 7 with some additional shape functions ranging in order from 4 to 7. The estimators for element 2 were computed as $c_r R_2(3) = 0.2 \times 10^{-5}$, $c_t T_2(2) = 0.2 \times 10^{-4}$, and the dimensionless strain energy error as 0.4×10^{-4}. Thus the indications are that the bounding property of the estimators may be preserved at the element level only if the p-distribution is such that the estimators do not vary by more than 2 or 3 orders of magnitude.

In all computational experiments conducted to date the r- and t-estimators were found to be consistent indicators of the source of energy error, in this case the crack tip singularity.

CONCLUDING REMARKS

A preliminary numerical investigation has indicated that readily computable bounds may exist for the error in strain energy in p-convergent finite element analysis. These bounds would provide an indication of where the degrees of freedom should be increased over the solution domain in adaptive finite element analysis to achieve uniform quality of approximation.

REFERENCES

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TABLE I
ESTIMATORS FOR THE ERROR IN STRAIN ENERGY

[Centrally cracked panel, 8-element mesh, uniform p-distribution.*]

p	$U_p^{\dagger}$	$U-U_p$	$\sum_k R_k(3)$	$\sum_k T_k(2)$
2	0.6936	0.0766	0.0864	0.4662
3	0.7305	0.0397	0.0534	0.1989
4	0.7461	0.0241	0.0321	0.1042
5	0.7538	0.0164	0.0218	0.0658
6	0.7584	0.0118	0.0161	0.0452
7	0.7613	0.0089	0.0127	0.0335

*To convert entries into dimensioned values, multiply by $\sigma^2 \ell^2/E$.
†Poisson's ratio: 0.3.

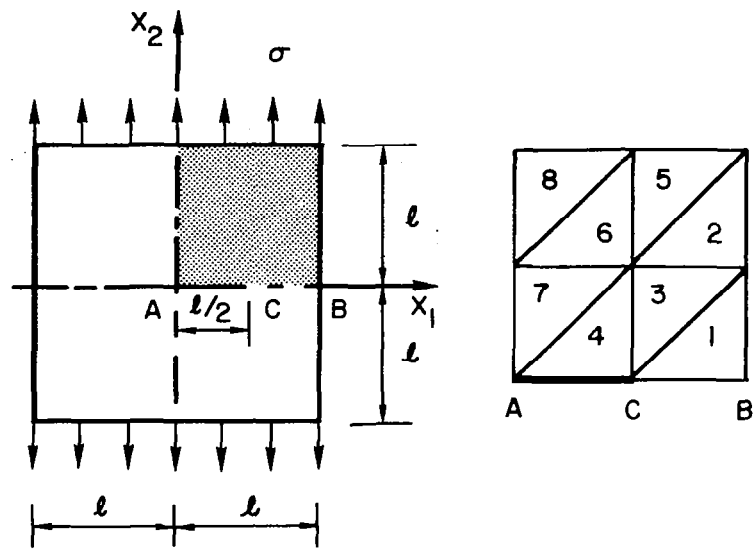

Fig. 1. Centrally cracked square panel mesh division and element numbering.

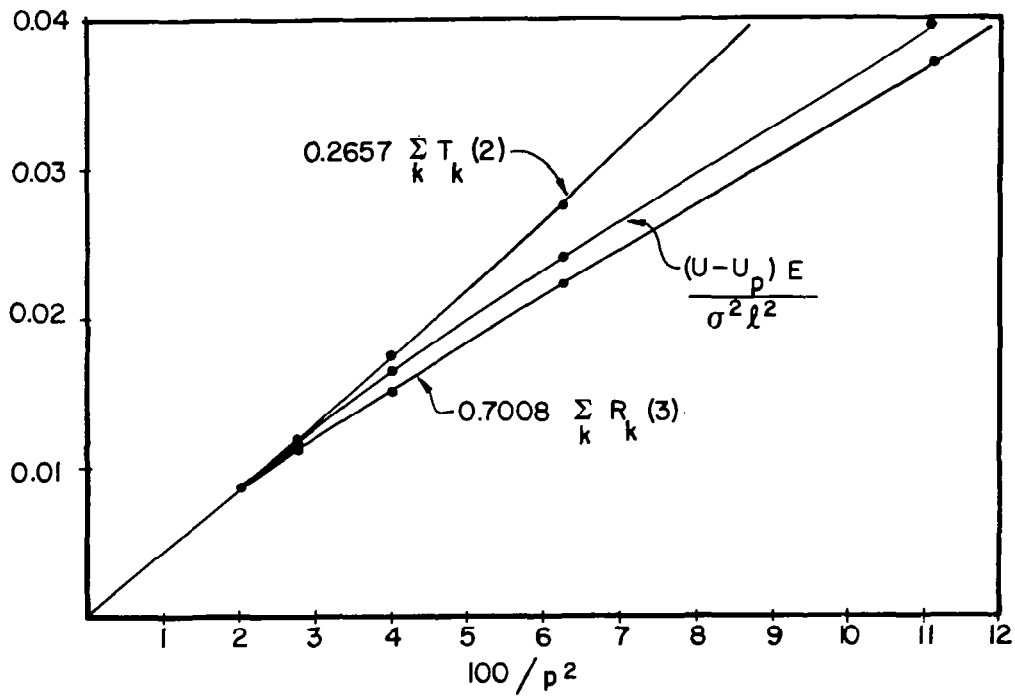

Fig. 2. Variation of the error indicators and the error in strain energy with the polynomial order (p).