ON A NEW ALGORITHM FOR TIME STEP INTEGRATION OF NONLINEAR SYSTEMS

Edoardo Anderheggen and Gianni Bazzi
Swiss Federal Institute of Technology Zürich

SUMMARY

A new implicit algorithm for time step integration of finite element structural dynamic equations is presented. Convergence, stability and numerical damping properties are discussed. Due to the way nonlinear structural behavior is taken into account the algorithm is expected to compare favourably with existing ones. Some simple numerical results are presented. A related explicit algorithm is also derived and shortly discussed.

INTRODUCTION

Lately much attention is being paid to finite element structural dynamic problems where nonlinear behavior is taken into account. It is typical for this kind of problems that the internal nodal forces developed by the structure and resisting external loads and inertia forces are not linear functions of nodal displacements but have to be evaluated from the actual stress state of the deformed structure by virtual work integration or similar procedures. The displacement time history is then obtained integrating step-by-step the nonlinear dynamic equations, a very cumbersome procedure in most practical cases.

Time step algorithms can be classified as implicit, requiring the solution of a system of coupled and generally nonlinear equations at each time step or as explicit where the unknown problem parameters at the end of each step are obtained directly. A clear and concise discussion of requirements and applicability for both kinds of algorithms can be found in reference [1].

In the present paper an implicit algorithm is first derived and discussed using simple numerical tests to show some of its properties. A corresponding explicit algorithm (which has not been implemented yet) is then also derived. It should be clear, however, that the present paper has the limited scope of reporting some early results of a research project presently in progress. More extensive numerical tests and comparisons with different algorithms are needed for general conclusions.

IMPLICIT METHOD

Within a time step Δt, i.e. between t_o and $t_1 = t_o + \Delta t$ (with $\tau = t - t_o$), the following system of differential equations, a dot indicating derivation with respect to time, has to be solved:

$$\{R(\tau)\} = [M]\{\ddot{W}(\tau)\} + [C]\{\dot{W}(\tau)\} + \{F(\tau)\} - \{P(\tau)\} = 0$$

where: $\{R(\tau)\}$ = residual vector; $\{W(\tau)\}$ = vector of nodal displacement parameters; $[M]$ = mass matrix; $[C]$ = viscous damping matrix; $\{F(\tau)\}$ = vector of internal nodal forces; $\{P(\tau)\}$ = vector of prescribed external loads. The internal structural nodal forces $\{F(\tau)\}$ are evaluated as follows (see Fig. 1):

$$\{F(\tau)\} = \{F_o\} + [\tilde{K}](\{W(\tau)\} - \{W_o\}) + \{\bar{F}(\tau)\}$$

where: $\{F_o\}$ = internal forces at $t = t_o$ (subscripts 0 and 1 always refer to the beginning and the end of the time step); $[\tilde{K}]$ = an approximation of the actual secant stiffness matrix between t_o and t_1 (generally the tangent stiffness matrix evaluated at $t = t_o$ or at the beginning of some earlier time step will be used for $[\tilde{K}]$); $\{\bar{F}(\tau)\}$ = vector of corrective internal forces due to the nonlinear behavior of the structure within the time step. The distribution of these corrective forces $\{\bar{F}(\tau)\}$ is assumed to be given by:

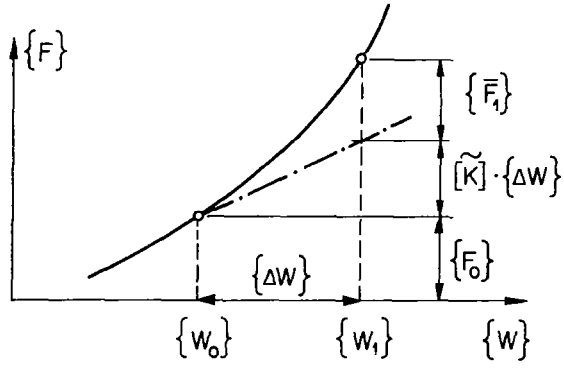


Fig. 1 : Internal Forces

$$\{\bar{F}(\tau)\} = [b_{ii}(\tau)]\{\bar{F}_1\}$$

where the $b_{ii}(\tau)$, coefficients of the diagonal interpolation matrix $[b_{ii}(\tau)]$, are time functions (with $b_{ii}(0) = 0$ and $b_{ii}(\Delta t) = 1$) to be chosen according to the expected distribution of each single corrective force $\bar{F}_i(\tau)$ as explained later. The internal corrective forces $\{\bar{F}_1\}$ at the end of the time step will have to be evaluated by stress-virtual-strain integration or by similar procedures (taking into account strain-stress history, large displacements, etc.) for all elements where nonlinear behavior is expected.

For the displacement parameters the following assumptions are used:

$$\{W(\tau)\} = N_1(\tau)\cdot\{W_o\} + N_2(\tau)\cdot\{\dot{W}_o\} + N_3(\tau)\cdot\{W_1\}$$

the N_i's being shape functions in the time dimension. In order to insure the necessary continuity of $\{W\}$ and $\{\dot{W}\}$ (but not of $\{\ddot{W}\}$) between time steps the following conditions have to be satisfied:

$$N_1(0) = 1 \qquad \dot{N}_1(0) = N_1(\Delta t) = 0$$
$$\dot{N}_2(0) = 1 \qquad N_2(0) = N_2(\Delta t) = 0$$
$$N_3(\Delta t) = 1 \qquad N_3(0) = \dot{N}_3(0) = 0$$

We also impose the following conditions:

$$\dot{N}_1(\Delta t) = -\frac{1+\rho}{\Delta t} \qquad \dot{N}_2(\Delta t) = -\rho \qquad \dot{N}_3(\Delta t) = \frac{1+\rho}{\Delta t}$$

$$\int_0^{\Delta t} N_1 \cdot d\tau = \frac{\rho}{1+\rho} \cdot \Delta t \qquad \int_0^{\Delta t} N_2 \cdot d\tau = 0 \qquad \int_0^{\Delta t} N_3 \cdot d\tau = \frac{1}{1+\rho} \cdot \Delta t$$

the parameter ρ ($0 < \rho \leq 1$) being the spectral radius in the limit $\Delta t \to \infty$ of the operator matrix for the linear case as explained later. Polynomials of 4th degree in s (with $s = \tau/\Delta t$) can be chosen for the shape functions N_i:

$$N_1 = 1 - N_3$$
$$N_2 = (-5(1+\rho)s^4 + 4(3+2\rho)s^3 - 3(3+\rho)s^2 + 2s) \cdot \frac{\Delta t}{2}$$
$$N_3 = (5(\rho^2-4\rho+7)s^4 - 8(\rho^2-5\rho+9)s^3 + 3(\rho^2-6\rho+13)s^2) \cdot \frac{1}{2+2\rho}$$

For the important special case where $\rho = 1$ (no numerical damping) the shape of the N_i functions is shown in Fig. 2.

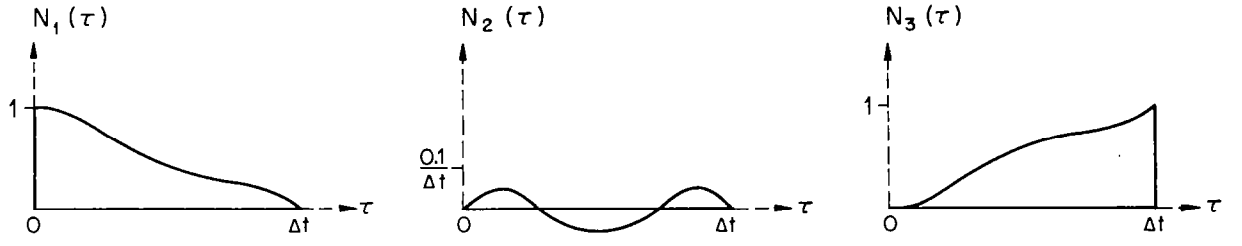


Fig. 2 : Shape Functions $N_1(\tau)$ $N_2(\tau)$ $N_3(\tau)$

In order to evaluate $\{W_1\} = \{W(t_1)\}$ the weighted integral of the residual vector $\{R(\tau)\}$ is set to zero at each time step:

$$\int_0^{\Delta t} G \cdot \{R(\tau)\} \cdot d\tau = 0 \qquad \text{with: } G = \frac{\Delta t}{1+\rho}$$

where the choice of a constant weighting function G insures that the integral of all external and internal forces vanishes at each time step. The following system of equations for $\{\Delta W\} = \{W_1\} - \{W_0\}$ is found:

$$([M] + \frac{\Delta t}{1+\rho}[C] + \frac{\Delta t^2}{(1+\rho)^2}[\tilde{K}])\{\Delta W\} = \Delta t[M]\{\dot{W}_0\} - \frac{\Delta t^2}{1+\rho}(\{F_0\} + [B_{ii}]\{\bar{F}_1\} - \{P^*\})$$

where:

$$[B_{ii}] = \frac{1}{\Delta t}\int_0^{\Delta t} [b_{ii}(\tau)] \cdot d\tau \qquad \{P^*\} = \frac{1}{\Delta t}\int_0^{\Delta t} \{P(\tau)\} \cdot d\tau$$

As the matrix multiplying the unknown vector $\{\Delta W\}$ has the same structure as the stiffness matrix $[\tilde{K}]$, the algorithm is called implicit. The corrective forces $\{\bar{F}_i\}$ can only be evaluated when $\{\Delta W\}$ is known; the system of equations is therefore nonlinear. Unless the influence of $\{\bar{F}_1\}$ is neglected, equilibrium iterations within each time step are necessary, requiring reiterate evaluations of $\{\bar{F}_1\}$. A norm for the changes of $\{\bar{F}_1\}$ (rather than $\{\Delta W\}$) was used in all numerical examples as a convergence criterion.

The displacement velocity vector $\{\dot{W}_1\}$ is found from the assumed shape functions:

$$\{\dot{W}_1\} = \frac{1+\rho}{\Delta t}\{\Delta W\} - \rho\{\dot{W}_o\}$$

The convergence and stability properties of the algorithm for proportionally damped, linear-elastic free vibrations can be discussed applying a modal transformation (see reference [2]). For a mode Y(t) associated with a frequency ω and with a damping coefficient ξ the following operator relations are obtained ($\Omega = \omega \cdot \Delta t$):

$$\begin{Bmatrix} Y_1 \\ \Delta t \cdot \dot{Y}_1 \end{Bmatrix} = \frac{1}{1 + \frac{2\xi\Omega}{1+\rho} + \frac{\Omega^2}{(1+\rho)^2}} \cdot \begin{bmatrix} 1 + \frac{2\xi\Omega}{1+\rho} - \frac{\rho\Omega^2}{(1+\rho)^2} & 1 \\ -\Omega^2 & 1 - \frac{2\rho\xi\Omega}{1+\rho} - \frac{\rho\Omega^2}{(1+\rho)^2} \end{bmatrix} \begin{Bmatrix} Y_o \\ \Delta t \cdot \dot{Y}_o \end{Bmatrix}$$

The operator matrix shows the following properties:

1. For small Δt's i.e. in the limit $\Omega \to 0$ convergence to the true solution is insured.
2. The factor ρ represents the spectral radius of the two-roots operator matrix in the limit $\Omega \to \infty$. The algorithm is therefore unconditionally stable for $\rho \leq 1$.
3. With $\rho = 1$ the operator matrix becomes identical to Newmark's with $\gamma = 1/2$ and $\beta = 1/4$ ("trapezoidal rule", see references [3] and [2]). No numerical damping, second order accuracy, minimal period elongation and no amplitude decay are obtained.
4. For $\rho < 1$ numerical damping used to "filter out" disturbing high frequencies is introduced. In fact, the choice of ρ represent a convenient and natural way of controlling numerical damping. This, however, decreases linearly with Ω (as in Newmark's method with $\gamma > 1/2$) and not quadratically as might be desirable (see references [2], [4], [5]). For $\rho < 1$ second order accuracy is therefore lost, which limits the choice of ρ to values close to unity.
5. The so-called "overshoot" effect both for displacements and velocities (see reference [6]) is avoided for all values of ρ.

All this is of course only valid in the linear case. However, the algorithm has two other properties worth mentioning as they appear to be valuable in the nonlinear case.

The first one concerns the way the $b_{ii}(\tau)$ functions describing the distribution of the internal corrective forces $\{\bar{F}(\tau)\}$ within the time step or the corresponding average values B_{ii} are chosen.

By setting $B_{ii} = 0$ the influence of the corrective force $\bar{F}_i$ is neglected, i.e. linear behavior as described by the stiffness matrix $[\tilde{K}]$ is assumed in that zone of the structure which affects $\bar{F}_i$. If all B_{ii} are set to zero, $\{\bar{F}_1\}$ is neglected and no equilibrium iterations are necessary, which may lead to rapidly diverging results unless sufficiently small time steps are used.

If nonlinear behavior is expected and the tangent stiffness matrix evaluated at the beginning of the time step is used for $[\tilde{K}]$ then $B_{ii} = 1/3$, corresponding to $b_{ii} = (\tau/\Delta t)^2$, should be used, because the time derivatives $\dot{\bar{F}}_i(0)$ of the corrective functions at $\tau = 0$ are known to vanish.

If for $[\tilde{K}]$ the tangent stiffness matrix of an earlier time step is used, then $B_{ii} = 1/2$ may be used corresponding to the assumption of a linearly distributed corrective force $\bar{F}_i$, i.e. to $b_{ii} = \tau/\Delta t$. In the commonly used undamped version of the Newmark algorithm (with $\gamma = 1/2$ and $\beta = 1/4$) implemented with the so-called "out-of-balance-load" procedure as well as in the original iterative formulation of the Newmark method (see reference [3]) the B_{ii}'s are implicitly always set to 1/2, a rather poor choice when the actual tangent stiffness matrix is used.

In some parts of the structure, where highly nonlinear behavior is expected, it might be usefull to evaluate the $\bar{F}_i$'s not only at $\tau = \Delta t$ but also at $\tau = \Delta t/2$ or even at several points between $\tau = 0$ and $\tau = \Delta t$. This allows evaluation of the B_{ii} coefficients more exactly by simple numerical integration procedures.

The second welcome property of the algorithm is due to the fact that for each time step average values $\{P^*\}$ of the external loads $\{P(t)\}$ are used instead of considering only instantaneous values of $\{P(t)\}$ as in most other well-known algorithms. If the time step Δt is large compared to the typical period of the loads and if the load values are known in intervals smaller than Δt, quite relevant improvements can be obtained by using very little-time-consuming numerical integration procedures to evaluate $\{P^*\}$.

NUMERICAL RESULTS

The single-degree-of-freedom, numerically and physically undamped ($\rho = 1$; $\xi = 0$) examples presented below are intended to show the beneficial influence of choosing the proper B_{ii} coefficients and of correctly evaluating the average external loads $\{P^*\}$.

In the first two examples the homogeneous equation of motion is given by:

$M \cdot \ddot{W} + F(W) = 0$

with a constant mass M, a prescribed starting velocity $\dot{W}(o)$ and a vanishing displacement W(o) at t = 0. In the first example a quadratic relation between F(W) and W (nonlinear elasticity) is assumed as shown in Fig. 3. In the second example the internal force F(W) is produced by five linear-elastic brittle springs

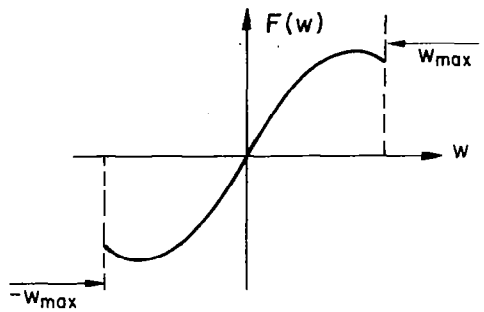


Fig. 3 : First Example
F(w)-w-Relation

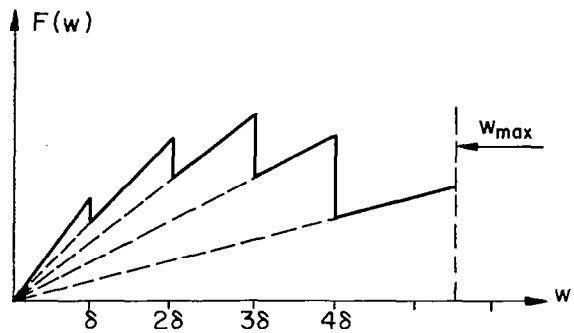
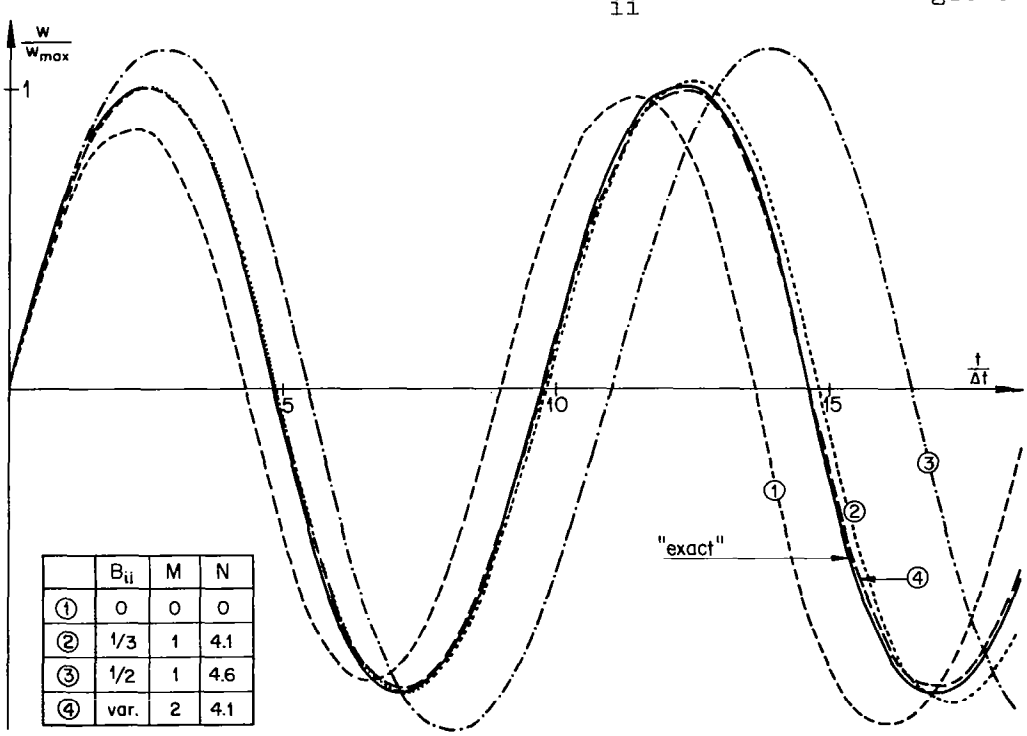


Fig. 4 : Second Example
F(w)-w-Relation

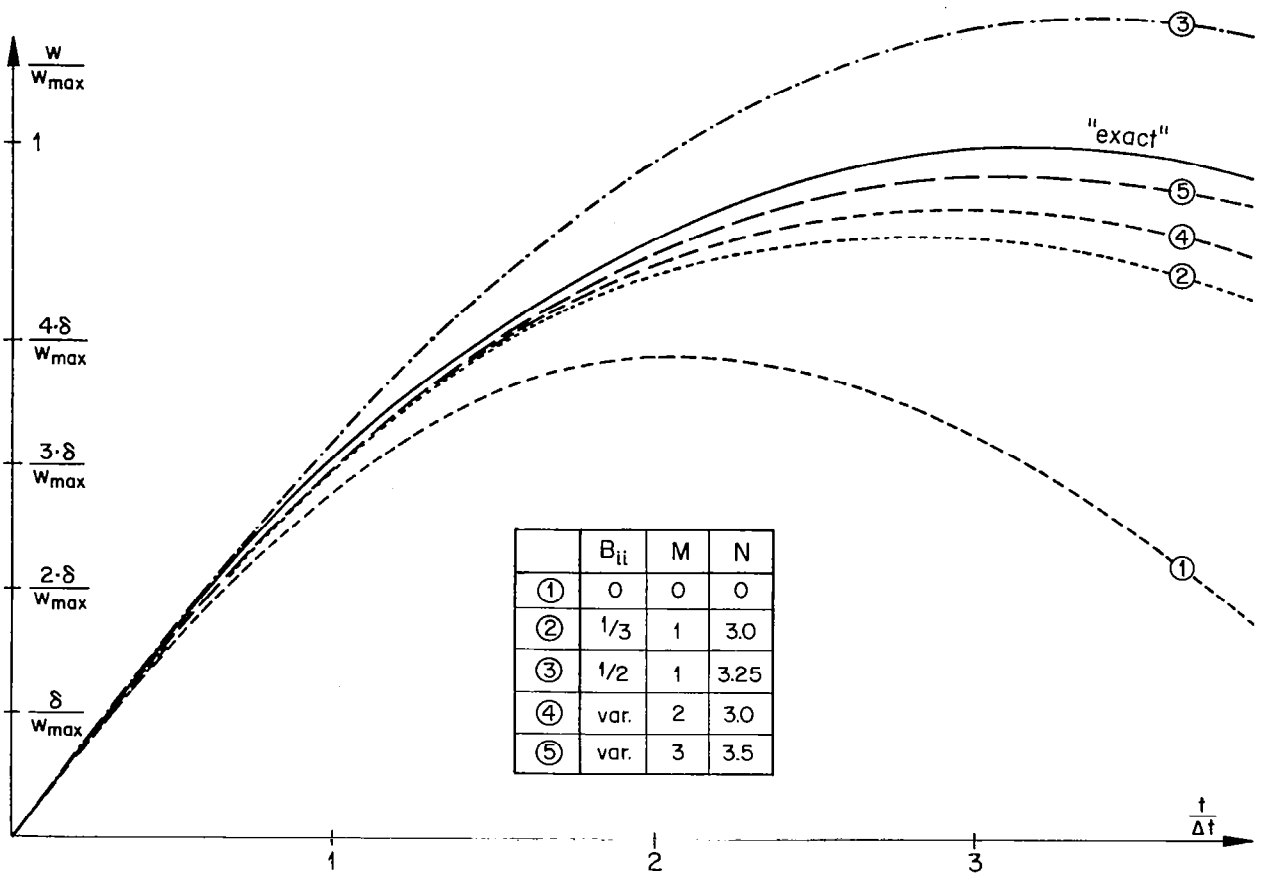
with identical stiffnesses. Four of them are assumed to break at W = δ, 2δ, 3δ and 4δ leading to the F-W-diagram shown in Fig. 4 remindful of crack propagation problems.

Different time histories for different B_{ii}'s are shown in Figs. 5 and 6.



	B_{ii}	M	N
①	0	0	0
②	1/3	1	4.1
③	1/2	1	4.6
④	var.	2	4.1

Fig. 5 : First Example. Time History for Several B_{ii} Coefficients



	B_{ii}	M	N
①	0	0	0
②	1/3	1	3.0
③	1/2	1	3.25
④	var.	2	3.0
⑤	var.	3	3.5

Fig. 6 : Second Example. Time History for Several B_{ii} Coefficients

The "exact" ones were obtained using very small time steps. For the others large time steps were used. The tangent stiffness (i.e. the derivative of F(W) with respect to W) was updated at each time step and equilibrium iterations (except for $B_{ii} = 0$) were repeated until the change of the corrective force $\overline{F}_1$ became negligible. "N" is the average number of such equilibrium iterations. "M" is the number of times the internal corrective forces $\{\overline{F}(\tau)\}$ have to be evaluated at each time step for the different B_{ii}'s: $M = N = 0$ for $B_{ii} = 0$; $M = 1$ for $B_{ii} = 1/3$ or $B_{ii} = 1/2$; for $M > 1$, B_{ii} is variable as M values of $\{\overline{F}(\tau)\}$ are used to evaluate the time integral leading to B_{ii}.

The advantage, when using the actual tangent stiffness matrix, of choosing $B_{ii} = 1/3$ instead of $B_{ii} = 1/2$ (which, as explained earlier, would correspond to the most used version of Newmark's algorithm) is evident. An even more marked improvement, as clearly demonstrated in Fig. 6, is obtained by evaluating B_{ii} more exactly ($M > 1$). This, however, requires time-consuming evaluations of $\{\overline{F}(\tau)\}$ for different τ's within each time step.

The third example (Fig. 7) shows the response of a linear undamped system with a natural period $T_s = 4$ to a sinusoidal load with a period $T_p = 1$. The time step used ($\Delta t = 0.3$) is very large compared with T_p but reasonable if compared

with T_s. The parameter "K" given in Fig. 7 represents the number of discrete values of the load function P(t) used for the evaluation of the load average P* within each time step (K = 2 if only $P(t_0)$ and $P(t_1)$ are used as in most well-known algorithms). Of course convergence to the exact solution due to the large time step used is not obtained; nevertheless the advantage of correctly evaluating P* is evident, the computational effort needed being negligible.

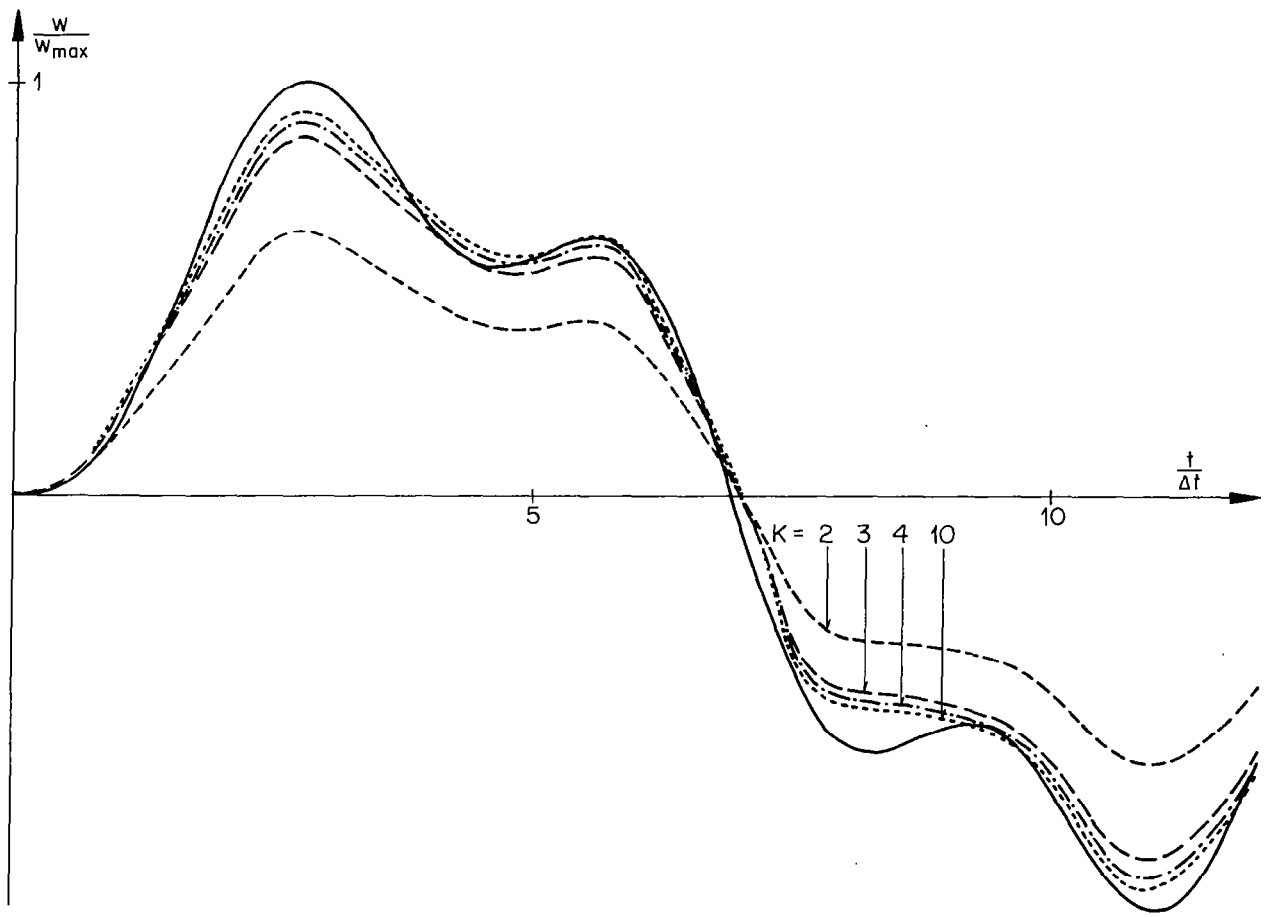


Fig. 7 : Third Example. Time History for Different Evaluations of $\int_0^{\Delta t} P(\tau)\,dt$

EXPLICIT METHOD

An explicit algorithm closely related to the implicit one discussed above has also been derived. The main difference lies in the way the integral of the internal forces $\{F(\tau)\}$ within the time step is evaluated:

$$\int_0^{\Delta t} \{F(\tau)\} \cdot dt \simeq \Delta t \cdot \{\tilde{F}\}$$

where $\{\tilde{F}\}$ is determined assuming a linear distribution of $\{F(\tau)\}$ as well as

uniform motion within the time step:

$$\{\tilde{F}\} = \{F(\{\tilde{W}\})\} \qquad \text{with:} \quad \{\tilde{W}\} = \{W_o\} + \frac{\Delta t}{2}\{\dot{W}_o\}$$

Requiring the weighted integral of the residuals to vanish while using the same shape functions $N_i(\tau)$ and the same constant weighting function G as before, the following system of equations for $\{\Delta W\}$ is obtained:

$$([M] + \frac{\Delta t}{1+\rho}[C])\{\Delta W\} = \Delta t[M]\{\dot{W}_o\} - \frac{\Delta t^2}{1+\rho}(\{\tilde{F}\} - \{P^*\})$$

Assuming the matrices [M] and [C] to be diagonal, i.e. lumping masses and viscous dampers in the joints, these equations are trivial. The algorithm is then called explicit.

If, as for the implicit method, a modal transformation is performed, the following operator equations are found:

$$\begin{Bmatrix} Y_1 \\ \Delta t \cdot \dot{Y}_1 \end{Bmatrix} = \frac{1}{1 + \frac{2\xi\Omega}{1+\rho}} \begin{bmatrix} 1 + \frac{2\xi\Omega}{1+\rho} - \frac{\Omega^2}{1+\rho} & 1 - \frac{\Omega^2}{2+2\rho} \\ -\Omega^2 & 1 - \frac{2\xi\Omega\rho}{1+\rho} - \frac{\Omega^2}{2} \end{bmatrix} \begin{Bmatrix} Y_o \\ \Delta t \cdot \dot{Y}_o \end{Bmatrix}$$

As expected the operator matrix shows that the algorithm is only conditionally stable. For $\rho = 1$ the stability condition obtained by setting the spectral radius equal to unity is:

$$\Omega_{cr} = 2$$

or, for the more stringent "bifurcation" condition (see reference [7]):

$$\Omega_{bif} = 2 \cdot \sqrt{1-\xi^2}$$

For damped systems ($\xi > 0$) these conditions are less stringent than those given in references [7] or [8], due to the fact that here a diagonal damping maxtrix [C] is assumed. For $0.8 \leq \rho < 1.0$ the changes in Ω_{cr} and Ω_{bif} are found to be minimal. However, as in the implicit method, with $\rho < 1$ numerical damping is introduced and second order accuracy is lost. The parameter ρ has not the same meaning as before but still controls numerical damping.

In fact it is questionable if a choice different from $\rho = 1$ would make much sense when using the explicit method. Because, due to stability reasons, explicit methods require very small time steps, they are mainly used for problems where all or almost all frequencies of the finite element model have to be taken into account (e.g. for shock or wave propagation problems) so that a "filtering out" of frequencies by numerical damping is not necessary and mostly not desirable.

The main reason why ρ was left as a variable in the above derivations is due to the fact that it seems possible to combine the implicit and the explicit method following the procedure suggested by Hughes et. al. in reference [8], where their damping coefficient γ is also used in both cases.

FURTHER DEVELOPMENTS

As stated in the introduction this paper is only concerned with early results of a research project presently in progress. The following developments are planned:

1. Extensive tests of the implicit algorithm for reinforced concrete structures under earthquake loads. Different element formulations are to be compared.
2. Implementation and tests of iterative methods (e.g. overrelaxation) for the solution of the equations arising at each time step, possibly combining iterative solution steps with equilibrium iterations. As good guesses for starting iteration are available from earlier time steps, iterative methods might well prove to be quite efficient.
3. Implementation and tests of the explicit algorithm and of the coupled implicit-explicit-method as suggested by Belytschko and Mullen in reference [8] or by Hughes et. al. in reference [7]. Also in this case the use of iterative methods for the solution of the implicit equations looks promising.
4. Development of efficient, state-of-the art computer programs to be applied in practical cases. This is a critical point, the applicability of step-by-step procedures being severely limited by the great amount of computations involved.

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