THE INCORPORATION OF TRUNCATED FOURIER SERIES INTO FINITE DIFFERENCE APPROXIMATIONS OF STRUCTURAL STABILITY EQUATIONS

Steven R. Hannah
Former Graduate Student
Air Force Institute of Technology
Presently at Space and Missile Systems Organization (SAMSO)

Anthony N. Palazotto
Department of Aeronautics and Astronautics
AFIT School of Engineering

SUMMARY

A new trigonometric approach to the finite difference calculus was applied to the problem of beam buckling as represented by virtual work and equilibrium equations. The trigonometric functions were varied by adjusting a wavelength parameter in the approximating Fourier series. Values of the critical force obtained from the modified approach for beams with a variety of boundary conditions were compared to results using the conventional finite difference method. The trigonometric approach produced significantly more accurate approximations for the critical force than the conventional approach for a relatively wide range in values of the wavelength parameter; and the optimizing value of the wavelength parameter corresponded to the half-wavelength of the buckled mode shape. Thus, selection of the wavelength parameter is a simple process if the half-wavelength is known. Methods for selecting this parameter in the more general case are also presented. It was found from a modal analysis that the most accurate solutions are obtained when the approximating function closely represents the actual displacement function and matches the actual boundary conditions. It is more difficult to select a satisfactory value of the wavelength parameter for the equilibrium equation which makes the virtual work equation more attractive for practical applications. A comparison of the virtual work and the Galerkin approaches identified marked similarities between the two methods.

INTRODUCTION

Numerical analysis has played an important part in furthering the understanding of structures over the past decades. Finite element and difference are just two techniques which may be considered part of the overall numerical approach. The similarities between the two have come to light recently as they are applied to energy equations, and therefore further research into finite difference methods, with respect to structures, has been pursued. Stein and Housner (ref. 1) initiated work into a trigonometric approach to finite difference as applied to plate instability which appeared very appealing because of its convergence characteristics (ref. 2). Thus, the authors investigated this relatively new trigonometric approach and applied it to problems of beam buckling, incorporating both the virtual work and equilibrium equations by extending the method's fundamental mathematical concepts. In order to accomplish this, a trigonometric function was varied by adjusting a wavelength parameter in an approximating Fourier series. Values of the critical force obtained from the modified approach for beams with a variety of boundary conditions were, for the first time, at least to the authors' knowledge, compared to results using the conventional finite difference method. The trigonometric approach produced significantly more accurate approximations for the critical force than the conventional approach for a relatively wide range in values of the wavelength parameter; and the optimizing value of the wavelength parameter corresponded to the half-wavelength of the buckled mode shape. It was found from a modal analysis that the most accurate solutions are obtained when the approximating function closely represents the actual displacement function. It is more difficult to select a satisfactory value of the wavelength parameter for the equilibrium equations which makes the virtual work equation more attractive for practical applications. The buckled mode shape (or eigenfunction) is predicted with high accuracy regardless of the value of the wavelength parameter. A comparison of the virtual work and the Galerkin approaches identified marked similarities between the two methods.

MATHEMATICAL EQUATIONS AND NUMERICAL APPROXIMATIONS

Virtual Work Equation. The virtual work principle can be stated mathematically as

$$\delta W_e = \delta U \tag{1}$$

(W_e = external work; U = strain energy)

For a one-dimensional beam with an axial force P, this relationship reduces to the following equation (ref. 3):

$$EI \int_0^L \frac{d^2v}{dx^2} \frac{d^2\delta v}{dx^2} dx = P \int_0^L \frac{dv}{dx} \frac{d\delta v}{dx} dx = 0 \tag{2}$$

where v represents the vertical displacements during buckling. The derivatives in equation (2) can be replaced by trigonometric finite difference approximations:

$$v'_{i+\frac{1}{2}} = \frac{\pi}{2\lambda \sin(\frac{\pi h}{2\lambda})} (-v_i + v_{i+1}) \tag{3}$$

$$v_i'' = \frac{\pi^2}{4\lambda^2 \sin^2(\frac{\pi h}{2\lambda})} (v_{i-1} - 2v_i + v_{i+1}) \tag{4}$$

where h = mesh spacing and λ = buckle wavelength parameter. (Refer to the Appendix for complete derivation of equations (3) and (4)). Note that the virtual work equation has been evaluated using half-station approximations for the first derivative and full-station approximations for the second derivative. The trigonometric finite difference expressions represented by equations (3) and (4) reduce to the conventional polynomial expressions as λ approaches infinity since

$$\lim_{\lambda\to\infty} \frac{\pi}{2\lambda \sin(\frac{\pi h}{2\lambda})} = \frac{1}{h} \tag{5}$$

Integration in equation (2) is performed using the trapezoid rule. The virtual displacements in the resulting equation can be ordered to produce

$$\sum_{i=1}^{N} f_i(v,P)\,\delta v_i = 0 \tag{6}$$

For this equality to hold, the coefficients of the individual δv_i must be equal to zero which leads to an eigenvalue problem of the form

$$\sum_{i=1}^{N} g_{ij}(P) v_i = 0 \qquad j = 1,2,\cdots N \tag{7}$$

from which the critical force can be calculated.

Equilibrium Differential Equation. The elastic form of the differential equation describing the slightly bent equilibrium configuration of an initially flat beam can be stated as (see ref. 4)

$$\frac{d^4v}{dx^4} + \frac{P}{EI}\frac{d^2v}{dx^2} = 0 \tag{8}$$

The trigonometric finite difference approximation for the fourth derivative was derived incorporating a five term Fourier series with the following result:

$$v_i^{iv} = \left(\frac{\pi}{\lambda}\right)^4 \left[\frac{1}{A_1^2} (v_{i-2} - 4v_{i-1} + 6v_i - 4v_{i+1} + v_{i+2}) + 16T_5\left(1 - \frac{\sin^4\theta}{A_1^2}\right)\right] \tag{9}$$

where

$$T_5 = \frac{A_1v_{i+2} + A_2v_{i+1} + A_3v_i + A_2v_{i-1} + A_1v_{i-2}}{A_4} \tag{10}$$

$$\theta = \frac{\pi h}{\lambda} \tag{11}$$

$$A_1 = -2 \cos \theta + 2 \tag{12}$$

$$A_2 = 2 \cos(2\theta) - 2 \tag{13}$$

$$A_3 = -4 \cos(2\theta) + 4 \cos \theta \tag{14}$$

$$A_4 = 16 \sin^2\theta - 48 \cos^2\theta + 32 \cos^3\theta + 48 \cos^4\theta - 32 \cos^5(\theta) \tag{15}$$

Equation (8) can be evaluated at each of the N grid points placed along the length of the beam, resulting in a set of equations with the following form:

$$\frac{A}{h^2}(v_{i-2} - 4v_{i-1} + 6v_i - 4v_{i+1} + v_{i+2}) + \frac{P}{EI}(v_{i-1} - 2v_i + v_{i+1}) = 0 \; ; \; i = 1,2,\cdots N \tag{16}$$

where A represents the combined coefficient resulting from the evaluation of equation (9). The displacements in this set of equations can be ordered to produce an eigenvalue problem in the form of equation (7).

Virtual Work and Galerkin Approaches. To complete the discussion of finite differences, it is instructive to compare the virtual work approach, as applied in this paper, to the Galerkin approach. Both methods are based on the principle of minimum potential energy. The Galerkin approach approximates the displacement function by the following series expansion:

$$v(x) = \sum_{i=1}^{M} a_i \phi_i(x) \tag{17}$$

where a_i are undetermined coefficients and ϕ_i represent continuous functions. By substituting this approximating function in the potential energy expression and performing a sequence of variational operations, the following system of equations is obtained:

$$\sum_{i=1}^{M} f_{ij}(P)a_j = 0 \qquad i = 1,2\cdots M \tag{18}$$

where f_{ij} denotes a functional relationship in terms of the external force. The concept underlying the Galerkin approach is based on the fact that the error in the approximation of equation (17) is minimized for any value of M if the a_i are chosen such that equations (18) are simultaneously satisfied (ref. 5). The virtual work approach incorporating finite difference approximations can be described in a similar manner. The Fourier coefficients, T_i, are implicitly selected (through the computational properties of the virtual work algorithm) such that the error due to the approximation of v(x) is minimized. This hypothesis was substantiated numerically by an analysis of the Fourier series approximating function.

CONCLUSIONS AND RESULTS

This paper compares the results of using the trigonometric and conventional approaches to the finite difference calculus in order to solve the equilibrium and virtual work equations. A wide range of boundary conditions were investigated. Various values of λ were used in the trigonometric approach to determine the optimum value as well as to determine the range over which the trigonometric approach gives more accurate approximations than the conventional approach. In addition, an in-depth search was conducted to provide plausible explanations for the superiority of one method over the other and one value of λ over other values. Finally, the effect of decreasing the number of grid points and the use of full-station approximations in the virtual work equation were investigated.

The virtual work method was found to be an efficient and simple approach and provided excellent results for both the trigonometric and conventional techniques with as few as five grid points. As predicted from theory, computational data revealed that the magnitude of error in computing P_{cr} varies directly with the square of the grid size. The variable input parameter, λ, has the effect of adjusting the wavelength of the Fourier series approximating function, and the optimum value of λ corresponds to the half-wavelength of the buckled mode shape for each boundary condition. There is a range in the values of λ for which the trigonometric approach is superior. This range extends from approximately 25% below the optimum value to infinity. Thus, a large value of λ is guaranteed to provide more accurate results than the conventional approach. Of course, if λ is chosen to be too large, the error from the conventional and trigonometric techniques approaches the same value; and the benefit of using the trigonometric technique is lost. It can be shown that $\lambda=1.5L$ produces satisfactory results for all boundary conditions. An illustration of this can be observed in figures 1 and 2 which depict the error in the calculation of P_{cr} for pinned-pinned and free-guided beams. A potential explanation for the superiority of one method over the other was found from an investigation of the Fourier and Taylor series approximating functions. The λ value which yields the closest series approximation to the theoretical displacement function corresponds to the optimum value of λ. That is, the key to calculating an accurate estimate of the critical force is to supply an approximating function which very closely reproduces the buckled mode shape as well as to satisfy the geometric and force boundary condition.

The similarities between the virtual work technique as employed in this paper and the Galerkin approach have been explored. In both cases, equilibrium expressions are used to derive potential energy relationships; and the displacement functions are approximated by series expansions. In addition, there is a strong relationship between the resulting sets of equations developed by the two methods. Both methods attempt to minimize the error in approximating the displacement function. When the approximating function is altered such that this minimized error is larger, the error in the computed critical force will increase proportionally. This concept was demonstrated by the use of the full-station finite difference approximation for the first derivative. The decreased accuracy in the approximation of v(x) caused a significant increase in the error of the calculated value of the critical force.

The equilibrium approach was also found to be efficient, and excellent results were obtained using the trigonometric technique. Figures 3 and 4 depict the results of calculating the critical force for pinned-pinned and free-guided beams. However, it is more difficult to select an effective value of λ using this approach than was found to be true for the virtual work approach. There are two optimum values of λ due to the fact that five terms were used in the Fourièr series for the derivation of the fourth derivative. The first value corresponds to the half-wavelength of the buckled mode shape. A precise estimate of the buckled wavelength is required in this case, however, since there is little margin for error. The range around this optimum value for which the trigonometric approach is superior to the conventional approach is very small, and the error builds rapidly as estimates of the optimum value worsen. There is a large range around the second optimum value for which the trigonometric approach is superior. This range extends from approximately 27% below the optimum value to infinity. This provides a comfortable margin of error for selecting λ. The problem is that there is no known physical parameter from which this second optimal value can be estimated other than the realization that each term of the series is superimposed upon each other, yielding a wavelength equal to the theoretical value at this particular λ. It appears from the available data that a value which is 2.75 times the half-wavelength provides a reasonably close estimate in most cases, but specific boundary conditions vary considerably from this figure. Despite the uncertainty, it is much safer to attempt an estimate of the second optimal value of λ due to the larger error margin. An attempt to use the first optimal value is probably unwise unless the buckled mode shape is known a priori with reasonable accuracy. It can be shown that λ=3.75L provides more accurate results than the conventional approach for all boundary conditions.

In comparing the results of the virtual work and equilibrium approaches, many similarities were noticed despite the major conceptual differences in the derivation of these methods. The interpretation of the wavelength parameter, λ, is the same in both cases as already discussed. In addition, the virtual work and equilibrium methods give the same value for P_{cr} when conventional finite difference expressions are used. The two methods do not give the same result when trigonometric expressions are used due to the presence of the two additional Fourier series terms in the equilibrium equation. Several major differences were also noted in the two methods. For example, it is more difficult to predict the optimum value of λ for the equilibrium approach. Additionally, it was found that an error in the estimate of λ produces a larger error in the computed value of P_{cr} for the equilibrium method than for the virtual work method. For these reasons, the virtual work method is recommended for general use over the equilibrium method. The trigonometric approach to the finite difference calculus is recommended over the conventional approach, particularly in those cases when the shape of the displacement function is known within rather broad tolerances.

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APPENDIX

Trigonometric finite difference approximations are derived in a manner similar to the conventional expressions with the exception that the following form of the Fourier series is used rather than the Taylor series:

$$v(x) = T_1 + T_2 \sin \frac{\pi(x - x_o)}{\lambda} + T_3 \cos \frac{\pi(x - x_o)}{\lambda} \quad (19)$$

The derivative of equation (19) evaluated at the reference point, x_o, is given by

$$v'(x_o) = T_2 \frac{\pi}{\lambda} \quad (20)$$

and

$$T_2 = v'(x_o) \frac{\lambda}{\pi} \quad (21)$$

Evaluation of equation (19) at $x_o + h/2$ and $x_o - h/2$ results in

$$v_{+\frac{1}{2}} = T_1 + T_2 \sin \frac{\pi h}{2\lambda} + T_3 \cos \frac{\pi h}{2\lambda} \quad (22)$$

$$v_{-\frac{1}{2}} = T_1 - T_2 \sin \frac{\pi h}{2\lambda} + T_3 \cos \frac{\pi h}{2\lambda} \quad (23)$$

Subtract equation (23) from equation (22) to obtain

$$v_{+\frac{1}{2}} - v_{-\frac{1}{2}} = 2T_2 \sin \frac{\pi h}{2\lambda} \quad (24)$$

If equation (21) is substituted in equation (24) and the terms rearranged, the following expression is obtained

$$v'(x_o) = \frac{1}{\hat{h}} (v_{+\frac{1}{2}} - v_{-\frac{1}{2}}) \tag{25}$$

where

$$\hat{h} = \frac{2\lambda \sin(\frac{\pi h}{2\lambda})}{\pi} \tag{26}$$

The trigonometric finite difference approximation for the second derivative can be obtained in a similar manner. Equation (19) can be evaluated at $x = x_o + h$ and $x = x_o - h$ to provide

$$v_{+1} = T_1 + T_2 \sin \frac{\pi h}{\lambda} + T_3 \cos \frac{\pi h}{\lambda} \tag{27}$$

$$v_{-1} = T_1 - T_2 \sin \frac{\pi h}{\lambda} + T_3 \cos \frac{\pi h}{\lambda} \tag{28}$$

By adding equations (27) and (28) and subtracting two times equation (19) evaluated at $x = x_o$, the following expression is obtained:

$$v_{+1} - 2v_o + v_{-1} = 2T_3 (\cos \frac{\pi h}{\lambda} - 1) \tag{29}$$

The second derivative of equation (19) with respect to x evaluated at $x = x_o$ is

$$v_o'' = -T_3 \frac{\pi^2}{\lambda^2} \cos \frac{\pi(x_o - x_o)}{\lambda} \tag{30}$$

Solving for T_3 yields

$$T_3 = -\left(\frac{\lambda}{\pi}\right)^2 v_o'' \tag{31}$$

If equation (31) is substituted in equation (29) and the resulting expression solved for v_o'', the following expression is obtained:

$$v_o'' = \frac{\pi^2}{4\lambda^2 \sin^2(\frac{\pi h}{2\lambda})} (v_{+1} - 2v_o + v_{-1}) \tag{32}$$

or

$$v_o'' = \frac{1}{\hat{h}^2} (v_{+1} - 2v_o + v_{-1}) \tag{33}$$

Note that the trigonometric and conventional finite difference expressions are similar with the mesh spacing, h, simply replaced by $\hat{h}$.

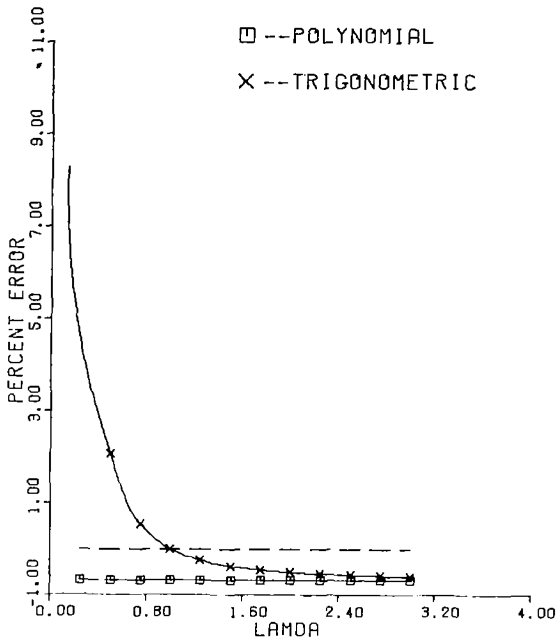

FIG. 1 ACCURACY COMPARISONS FOR P_{CR} USING THE VIRTUAL WORK EQUATION, PINNED-PINNED BOUNDARIES.

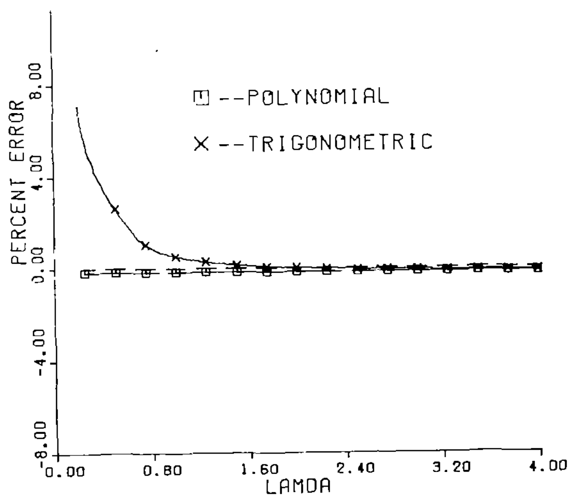

FIG. 2 ACCURACY COMPARISONS FOR P_{CR} USING THE VIRTUAL WORK EQUATION, FREE-GUIDED BOUNDARIES.

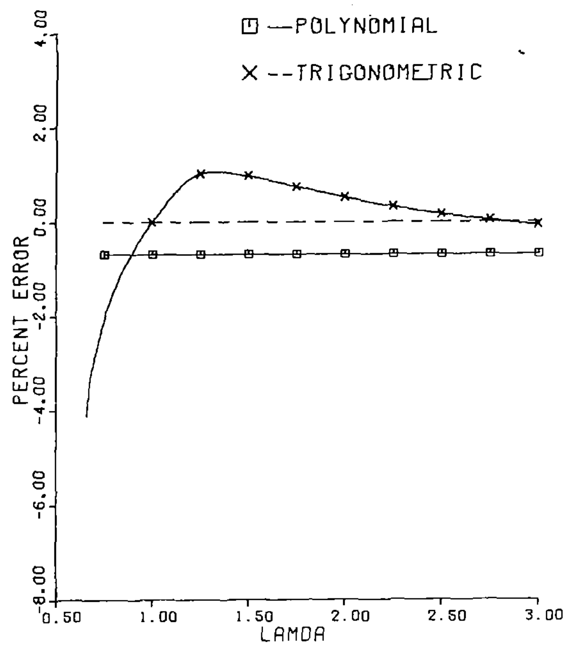

FIG. 3 ACCURACY COMPARISONS FOR P_{CR} USING THE EQUILIBRIUM EQUATION. PINNED-PINNED BOUNDARIES.

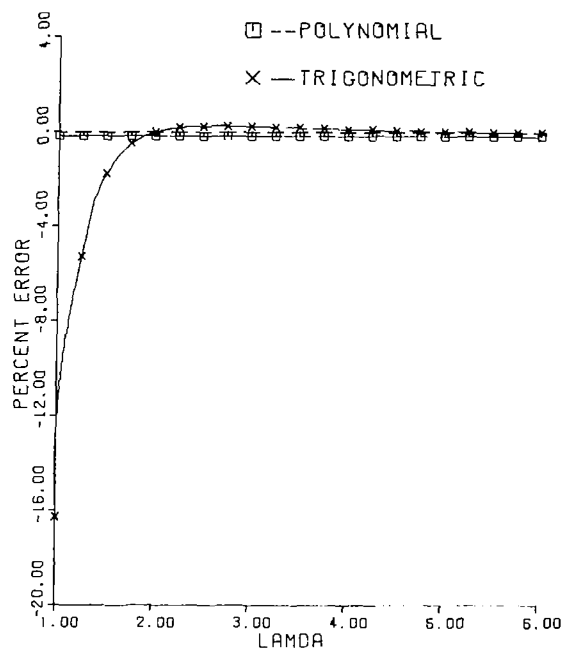

FIG. 4 ACCURACY COMPARISONS FOR P_{CR} USING THE EQUILIBRIUM EQUATION. FREE-GUIDED BOUNDARIES.