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NONLINEAR DYNAMIC RESPONSE OF WIND TURBINE ROTORS

by

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N O T I C E

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ABSTRACT

The nonlinear equations of motion for a rigid rotor restrained by three flexible springs representing, respectively, the flapping lagging and feathering motions are derived using Lagrange's equations, for arbitrary angular rotations. These are reduced to a consistent set of nonlinear equations using nonlinear terms up to third order. The complete analysis is divided into three parts A, B and C.

Part A consists of forced response of two-degree flapping-lagging rotor under the excitation of pure gravitational field (i.e., no aerodynamic forces). Both forced oscillations as well as parametric resonance are investigated using the Harmonic Balance method and solving the resulting nonlinear algebraic equations numerically by Newton-Raphson iterative technique. Effects of initial coning angle and flapping to lagging frequency ratio are discussed. For relatively small initial coning angle (about 9°) the nonlinearity becomes softening spring type and large coupled responses are possible for rotational frequencies significantly lower than the lagging frequency.

In Part B, the effect of aerodynamic forces on the dynamic response of two-degree flapping-lagging rotor is investigated. Significant aerodynamic effects are found for some of the previous forced oscillations and parametric resonances. Also, self-excited aerodynamic flutter instabilities are obtained after neglecting the gravity forces. Effects of vaious parameters like Lock number, inflow ratio, initial coning angle, structural damping, etc. are discussed. Also, the effect of a wind shear velocity gradient is investigated, and is found to produce little effect on the lagging response but appreciable effect on the flapping amplitude.

In Part C, the effect of third degree of motion, feathering, is considered. First, the forced response of flapping-lagging-feathering rotor under gravitational field and with wind shear flow is studied. It is found that even for relatively torsionally stiff rotor, the flapping amplitude is increased and the feathering response is appreciable. For the self-excited aerodynamic flutter instability, it is found that the feathering motion can reduce the linear instability speed appreciably. Also, the limit cycle flutter solution of a typical configuration shows a substantial nonlinear softening spring behavior. This reveals the possibility of sustained limit cycle flutter oscillations occurring well below the linear instability speed if large enough disturbances are given to the rotor.

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CONTENTS

LIST OF SYMBOLS

a	Section lift curve slope, $\frac{dC_L}{d\alpha}$
a_1,b_1	Vibratory components of β
$\hat{a}_1,\hat{b}_1$	Small perturbations of a_1,b_1
a_2,b_2	Vibratory Components of ϕ
$\hat{a}_2,\hat{b}_2$	Small perturbation of a_2,b_2
a_3,b_3	Vibratory components of θ
$\hat{a}_3,\hat{b}_3$	Small perturbation of a_3,b_3
A_β	Flapping amplitude, $\sqrt{a_1^2+b_1^2}$
A_ϕ	Lagging amplitude, $\sqrt{a_2^2+b_2^2}$
A_θ	Feathering amplitude, $\sqrt{a_3^2+b_3^2}$
B_0 to B_{15}	Constants defined in Appendix I
$\tilde{B}_0$, $\tilde{B}_5$-$\tilde{B}_{15}$	Constants defined in Appendix IV
c	Blade chord
C	Aerodynamic force along η direction
C_0 to C_{15}	Constants defined in Appendix I
$\tilde{C}_0$, $\tilde{C}_5$-$\tilde{C}_{15}$	Constants defined in Appendix IV
C_{do}	Blade profile drag coefficient
D	Aerodynamic drag

D_0, D_5-D_{15}	Constants defined in Appendix IV
E_0, E_5-E_{15}	Constants defined in Appendix IV
e	Hinge offset
$\bar{e}$	$= \frac{3}{2}\frac{e}{\ell}$
F_0-F_{45}	Constants defined in Appendix VI
g	Gravity
G	$= (\omega_{pend}/\Omega)^2$
G_0-G_{46}	Constants defined in Appendix VI
h	Height of a point above ground level
h_o	Height of hub from ground level
I_ξ, I_η, I_ζ	Moment of inertia
ℓ	Length of blade
L	Aerodynamic lift
m	Mass per unit length
N	Aerodynamic force along ζ direction
p	Velocity gradient constant, Eq. 116
$Q_\beta, Q_\phi, Q_\theta$	Generalized forces
r	Same as ξ
R_0-R_{33}	Constants defined in Appendix VI
R	Length of blade
R_m	I_ξ/I_η, feathering inertia/lagging inertia
S_0-S_{33}	Constants defined in Appendix VI

S_ζ	Static unbalance of blade
t	Time
u_i	Inflow velocity at the blade
u_h	Velocity hinge point of blade
U	Flow velocity in ξ direction
v	Flow velocity in η direction
V	Wind velocity at height h
V_T	Resultant velocity, $\sqrt{U^2+v^2}$
x,y,z	Fixed axes for no offset hinge case
X,Y,Z	Fixed axes for offset hinge case
Z	$= 1/(\frac{\Omega}{\omega_\phi})^2$
ψ,θ,β,ϕ	Angular coordinates of blade
β_s,ϕ_s,θ_s	Initial settings for β,ϕ,θ
β_o,ϕ_o,θ_o	Static components of β,ϕ,θ
β_c,ϕ_c,θ_c	Center shifts for β,ϕ,θ
$\tilde{\beta},\tilde{\phi},\tilde{\theta}$	Small perturbations of β,ϕ,θ
β_{DW},ϕ_{DW}	Angular deflections under dead weight, $(\frac{\omega_{pend}}{\omega_\beta})^2$ and $(\frac{\omega_{pend}}{\omega_\phi})^2$ respectively
$\nu_\beta,\nu_\phi,\nu_\theta$	$= \omega_\beta/\Omega,\ \omega_\phi/\Omega,\ \omega_\theta/\Omega$
ξ,η,ζ	Coordinates on blade
Ω	Rotation speed of blade

ω	Frequency of oscillation
ω_N	Natural frequency of oscillation
$\omega_\beta, \omega_\phi, \omega_\theta$	$= \sqrt{k_\beta/I_\eta}, \sqrt{k_\phi, I_\eta}, \sqrt{k_\theta/I_\xi}$
ω_{pend}	Pendulum frequency $= \sqrt{3g/2\ell}$
$\bar{\omega}_\beta$	$= \omega_\beta/\omega_\phi$
$\bar{\omega}_\theta$	$= \omega_\theta/\omega_\phi$
ρ	Air density
α	Ratio of response frequency to rotation frequency, ω/Ω
α_i	Inflow angle, $\tan^{-1} \frac{v}{U}$
θ_i	Blade built-in incidence
θ_1	Constant incidence along blade length
θ_2	Ideal twist distribution angle at tip
λ	Inflow velocity ratio, $\frac{u_i}{\Omega R}$
λ_o	λ at hub
λ_1, λ_2	Defined in Eq. 119
γ	$= \rho a c R^4/I_\eta$, Lock number
σ	$= \frac{nc}{\pi R}$, rotor solidity, (n = number of blades)
$\zeta_\beta, \zeta_\phi, \zeta_\theta$	Structural damping coefficients for flapping, lagging and feathering degrees, respectively
ζ	Effective damping coefficient

ζ_i	$\zeta_i=\zeta_\beta=\zeta_\phi=\zeta_\theta$
$\bar{\eta}$	(.75 chord - aerodynamic center)
$(\dot{\ })$	$= \frac{d}{dt}$
$(\overset{\circ}{\ })$	$= \frac{d}{d\psi}$

SECTION 1
INTRODUCTION

For centuries, wind energy systems have been used as sources of power in different forms like sailing of ships, pumping of water, grinding of grain, generation of electricity, etc. In the middle of the twentieth century, the interest in these systems declined because of the poor economic viability with other power generation systems. However, with the recent energy crisis and also because of readily available technology of fixed and rotary wing aircrafts, wind power is being considered as one of the potential sources of clean nondepleting energy.

Many types of wind power systems have been tried in the past to trap the thinly distributed wind energy. Most of the wind machines can be broadly classified into three categories depending upon the orientation of the axis of rotation, as Horizontal-axis wind rotor, Vertical-axis wind rotor and Cross-wind rotor. The Majority of these machines are based on the principle of Airfoil lift. At the present time, more attention has been given to the Horizontal-axis wind turbine partially because of its better theoretical understanding. So all the latter discussion and analysis is related mainly to this type of machine. (For general material see Refs. 1-8.)

Because of the low density of air, large amounts of air must be tapped to provide an appreciable amount of power. It is generally true that the cost of power produced is reduced with increasing size of wind driven plant, i.e., increasing the size of the rotor. However, one of the prime design problems of the big wind turbine system is the dynamics of rotor blades and the supporting tower and this has direct bearing on the operating life of the rotor. To keep low the cost of power production it is essential that the rotor should have long efficient life, i.e., subjected to less vibratory fatigue loads. Thus, there is need to understand the dynamic characteristics of the wind turbine system.

In some aspects, the dynamics of the big wind turbine is quite similar to that of the rotary wing aircraft. In the past three decades, a lot of research has undergone to understand the various dynamic or aeroelastic problems of helicopter and tilt rotor aircrafts, see for example Loewy's review paper [9] and other Refs. 10-23. A good deal of techniques developed in the formulation and analysis of the aeroelastic problems of rotary wing aircrafts can be used to study the dynamics of wind turbine rotors. However, all the results of rotary wing aircraft cannot be transformed directly for wind turbine because of differences in some of the parameters like rotational speed, tip velocity ratio, stiffnesses and weight properties, etc. In addition there are certain specific aspects of wind turbine dynamics which have to be looked into individually. For example, forced response of wind turbine blade under periodic forces due to gravitational field and sheared flow effect and also impulsive forces due to tower shadow effect are quite important problems, particularly for big wind turbines. A good general picture of various dynamic problems concerning wind turbines is given in Refs. 24-26.

There is little literature available related directly to the dynamics of wind turbine. Ormiston [27] has made a simple linear analysis from the uncoupled flap and lag equations for the forced response of wind turbine rotor under the excitation of gravity forcing function and also due to velocity gradient effect. The influence of blade number and hub articulation on the blade and tower stresses is examined and also the basic scaling relationships with respect to the length of the blade are discussed. Kaza and Hammond [28] has formulated the general linear flap-lag equations for flutter stability applicable both to the wind turbine rotor with velocity gradient as well as helicopter rotor in forward flight. Two types of hinge sequences for flap-lag motions are used and the equations with the periodic functions are solved using the Floquet-Liapunov method as well as the approximate method (time averaging of periodic functions). It was seen that the velocity gradient has little effect on the flutter boundary where as hinge sequence for flap-lag motions has a strong influence on the flutter stability of this two degree of freedom system. Friedmann [29] has derived the general coupled nonlinear flap-lag-torsional equations of motion for moderately large deflections of a pretwisted cantilevered wind turbine blade with the incoming wind having velocity gradient as well as gust components in all the three directions. The methods to solve these equations are mentioned. Miller [30] has obtained the linearized version of the nonlinear flapping-lagging-feathering flutter equations of rotor by considering the motion to be small perturbations about possibly large static solution. The importance of various physical quantities involved in the flutter and divergence of windmill blade is discussed. The effective damping plots are obtained for various configurations from the eigen analysis of the flutter equations. Dugundji [31] has given a good review of the whirl stability problem of wind turbine rotor mounted on a flexible tower. The general linear coupled equations of motions are derived for flapping-lagging rotor with two degrees of motion of tower head. The solution of these equations containing periodic coefficients using Floquet theory, for two bladed rotor particularly, is discussed. Some experimental results of small windmill model are given. In Refs. (26,32,33), the authors discuss the various aspects like design, fabrication, analysis, testing, etc. of 100 KW NASA Wind Turbine also discuss some of the dynamic problems pertianing to this wind turbine.

For most of the aeroelastic analysis of rotors, the basically nonlinear equations of motion are linearized by retaining only important static terms. Then it becomes much easier to work on the linearly coupled equations. However, there are some nonlinear analyses in the literature. Young [34] has made a qualitative analysis of the second order nonlinear equations of flapping-lagging rotor by the approximate method. Hohenemser and Heaton [35] have used the stepwise numerical integration scheme to solve the second order flap-lag equations. Tong and Friedmann [14] has made an exhaustive nonlinear analysis of flap-lag as well as Flap-Lag-Feathering rotor by the multiple time scales perturbation method. Another method, Harmonic Balancing, is quite widely used in the linear dynamic analysis of the rotor mainly because of its simplicity (e.g., Refs. 15,22,36). Dugundji, etc. [37,38] have used Harmonic Balance Method to solve the nonlinear panel flutter equations as well as to obtain the nonlinear forced oscillations response of the beams.

In the present report, nonlinear dynamic analysis is made for an isolated blade of wind turbine with no tower interaction. The blade is assumed to be completely rigid and is restrained by three flexible springs at the hinge point representing, respectively, the flapping, lagging and feathering degrees of motion. It is further assumed that the blade c.g., aerodynamic center and elastic axis lies at the quarter chord point and there is no variation of any of these along the blade axis. A particular hinge sequence of feathering first (from rotation axis), flapping second, and lagging motion last is followed. However, one can expect different results with changed hinge sequence [17,28]. The equations of motions are derived using the energy approach (i.e. Lagrange's equations). Keeping nonlinearity up to third order, the consistent nonlinear differential equations are obtained. The complete analysis is divided into three parts, A, B and C.

Part A consists of forced response of flapping-lagging rotor under the excitation of pure gravitational field. No aerodynamic forces are considered here. The blade can, however, have initial feathering angle setting. Both forced oscillations as well as parametric resonance are investigated. The forced oscillations response takes place at the frequency of the forcing function (i.e. rotational freq.) where as for parametric resonance the response frequency is one half the forcing frequency. First, simple linear solutions are worked out from the uncoupled flap and lag equation to get some basic understanding of the possible response of the blade. Then nonlinear limit cycle solutions are obtained for the flapping-lagging equations by applying the Harmonic Balance method and solving the resulting nonlinear algebraic equations numerically by Newton-Raphson iterative technique. These solutions are checked for stability to see whether they are physically existent or not. The stability check is made by giving small perturbations to these steady solutions and studying the growth rate of these disturbances with time under the assumption of slowly changing functions. If the perturbations grow with time means solution is unstable. The effect of initial coning angle and flapping to lagging frequency ratio on both forced response as well parametric resonance is investigated. The comparison of linear and nonlinear solutions near and away from resonance conditions is discussed.

In Part B, the effect of aerodynamic forces on the two-degree flapping-lagging rotor is investigated. Quasi-steady airfoil theory is used to obtain the aerodynamic forces. First forced response of rotor is studied under the excitation of gravitational forcing field and in the presence of aerodynamic forces. Again, the nonlinear analysis of flapping and lagging equations is made like Part A for both forced oscillations as well as parametric resonance. The effect of various parameters like Lock number, inflow ratio, coning angle, structural damping, flapping to lagging frequencies ratio etc. on response amplitude is investigated. Then the self-excited flutter response of this torsionally rigid rotor is studied in the absence of gravitational forces. The equations of motion are the same as the first case except that all periodic terms are absent in these equations because of neglecting gravity forces. First simple linear analysis is made and then more rigorous nonlinear solutions are obtained by using the Harmonic Balance method. The nonlinear flutter solution is slightly different from that of forced response, here, for a known lagging amplitude the solution is worked to obtain the corresponding flapping amplitude, the flutter frequency and the stiffness of the configuration. The results are presented in the form of stability envelopes. The effect of inflow ratio, Lock number, coning angle, structural damping, hinge offset, etc. on the ciritcal flutter boundary are discussed. The behavior of limit cycle flutter amplitude with changing rotational speeds is also studied. In the end of Part B the effect of sheared flow on the forced response of flapping-lagging rotor is investigated. Expanding the velocity profile power law relation and retaining terms up to the second order, and comparing the elemental thrusts obtained from momentum theory and blade element theory, the inflow at any point is expressed in terms of inflow at the hub, blade azimuthal angle and the radial distance of the point. The equations of motion here get modified and these contain periodic aerodynamic terms. Again, by nonlinear analysis the effect of velocity gradient on the forced response of the blade is studied with and without the gravity forces.

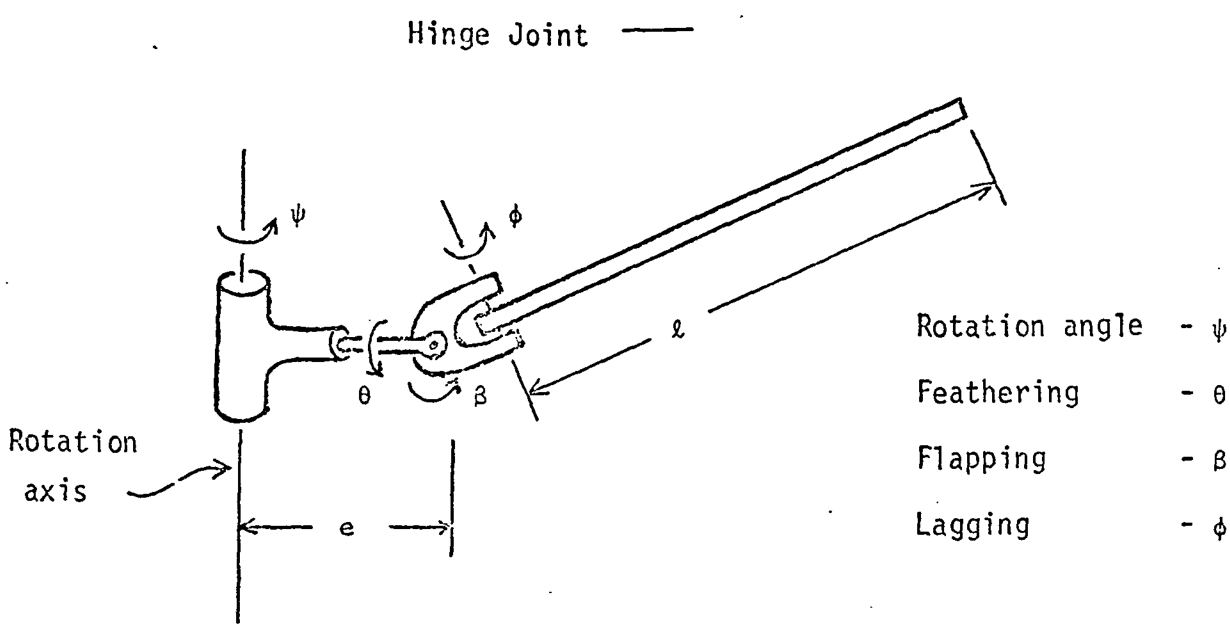
In Part C, the effect of third degree of motion, feathering, normal to the axis of rotation is considered, thus making the rotor a three degree of freedom system. The general equations of motion for this flapping-lagging-feathering rotor in the presence of gravity forces and with sheared flow are worked out. First simple solutions are obtained, then nonlinear limit cycle solutions are obtained for the forced response of the rotor by the Harmonic Balance Method. The nonlinear solutions are again checked for their well-posedness. The response amplitudes of a typical rotor configuration with the three degrees of motions, for two cases of with and without sheared flow, are compared with those of the same rotor with the feathering degree of motion locked. Then the self-excited flutter solutions for flapping-lagging-feathering rotor are investigated after neglecting gravity forces and also considering the uniform inflow. First the linear flutter analysis is made by assuming the motion to be small perturbations about some possibly large static solution. The damping plots are obtained for various configurations from the eigen analysis of the linearly coupled equations. Then nonlinear flutter analysis is made by the Harmonic Balance method. The behavior of the limit cycle flutter amplitude with changing rotational speeds is studied.

PART A: NO AERODYNAMIC FORCES

SECTION 2

NONLINEAR EQUATIONS OF MOTION

The rotor blade will be considered rigid with root hinges as shown below. The flapping and lagging hinges have the same offset, 'e', and the C.G. of each blade cross-section is assumed to lie on the longitudinal ξ axis of the blade. No aerodynamic forces will be considred at this time, since the main purpose here will be to assess the effects of the gravity forces on the rotating windmill rotor blade.

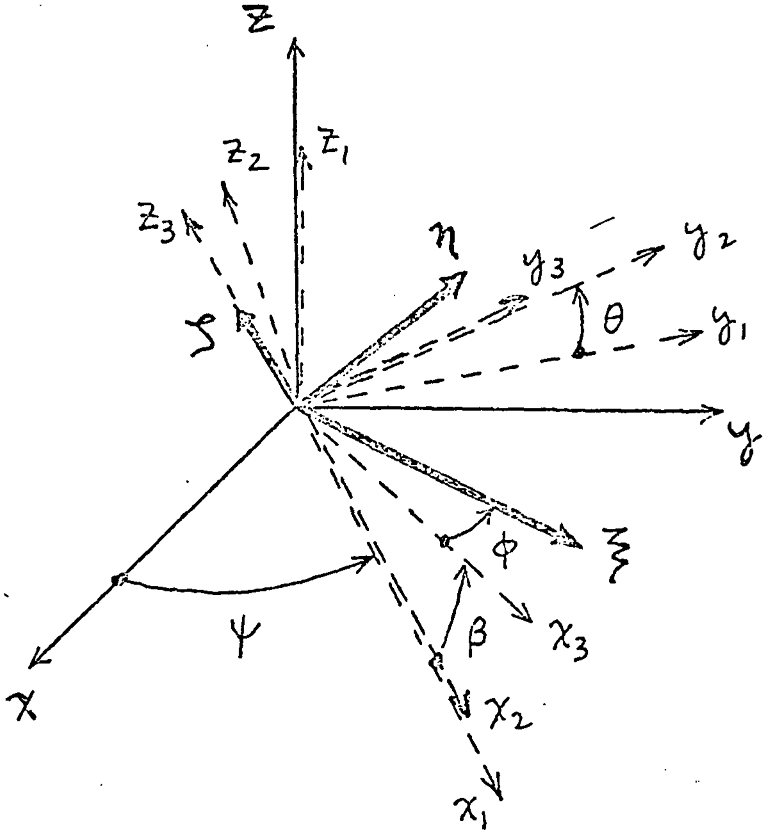


For convenience in setting up the nonlinear equations of motion, the no-offset-case (e=0) will be derived first, then the effect of the offset, e, will be added later.

All equations and all subsequent calculations in this section will apply for the hinge sequence shown above, i.e., feathering θ first, flapping β second, and lagging ϕ last.

2.1 No Hinge Offset Present

When no offset is present, the absolute location x, y, z of any point on the blade, ξ, η, ζ, can be defined in terms of four axis rotations involving the Euler angles ψ, θ, β, and φ respectively. These are shown in the sketch below.



Fixed Axes → x y z

Blade lies along ξ axis

Rotates ψ around z

" θ " x_1

" β " y_2

" φ " z_3

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} \qquad \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}$$

$$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} \qquad \begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \\ \zeta \end{Bmatrix}$$

Multiplying out the various rotation matrices above, gives the following relation between the fixed axes x, y, z and the blade axes ξ, η, ζ namely,

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \begin{array}{l} \cos\psi\cos\beta\cos\phi \\ +\sin\psi\sin\theta\sin\beta\cos\phi \\ -\sin\psi\cos\theta\sin\phi \end{array} & \begin{array}{l} -\cos\psi\cos\beta\sin\phi \\ -\sin\psi\sin\theta\sin\beta\sin\phi \\ -\sin\psi\cos\theta\cos\phi \end{array} & \begin{array}{l} -\cos\psi\sin\beta \\ +\sin\psi\sin\theta\cos\beta \end{array} \\ \begin{array}{l} \sin\psi\cos\beta\cos\phi \\ -\cos\psi\sin\theta\sin\beta\cos\phi \\ +\cos\psi\cos\theta\sin\phi \end{array} & \begin{array}{l} -\sin\psi\cos\beta\sin\phi \\ +\cos\psi\sin\theta\sin\beta\sin\phi \\ +\cos\psi\cos\theta\cos\phi \end{array} & \begin{array}{l} -\sin\psi\sin\beta \\ -\cos\psi\sin\theta\cos\beta \end{array} \\ \begin{array}{l} \cos\theta\sin\beta\cos\phi \\ +\sin\theta\sin\phi \end{array} & \begin{array}{l} -\cos\theta\sin\beta\sin\phi \\ +\sin\theta\cos\phi \end{array} & \cos\theta\cos\beta \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \\ \zeta \end{Bmatrix} \qquad (1)$$

It should be noted that the inverse of the above square matrix is equal to its transpose.

From the above relationship, the absolute velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ of any point ξ, η, ζ, can be found by differentiation to be,

$$\dot{x} = \Big[\left(-\sin\psi\cos\beta\cos\phi + \cos\psi\sin\theta\sin\beta\cos\phi - \cos\psi\cos\theta\sin\phi\right)\dot{\psi}$$

$$+\left(\sin\psi\sin\theta\sin\phi + \sin\psi\cos\theta\sin\beta\cos\phi\right)\dot{\theta} \qquad (2)$$

$$+\left(-\cos\psi\sin\beta\cos\phi + \sin\psi\sin\theta\cos\beta\cos\phi\right)\dot{\beta}$$

$$+\left(-\cos\psi\cos\beta\sin\phi - \sin\psi\sin\theta\sin\beta\sin\phi - \sin\psi\cos\theta\cos\phi\right)\dot{\phi}\Big]\xi$$

$$+\Big[\quad \text{etc.} \quad\Big]\eta + \Big[\quad \text{etc.} \quad\Big]\zeta$$

$$\dot{y} = \Big[\left(\cos\psi\cos\beta\cos\phi + \sin\psi\sin\theta\sin\beta\cos\phi - \sin\psi\cos\theta\sin\phi\right)\dot{\psi}$$

$$+\left(-\cos\psi\sin\theta\sin\phi - \cos\psi\cos\theta\sin\beta\cos\phi\right)\dot{\theta} \qquad (3)$$

$$+\left(-\sin\psi\sin\beta\cos\phi - \cos\psi\sin\theta\cos\beta\cos\phi\right)\dot{\beta}$$

$$+\left(-\sin\psi\cos\beta\sin\phi + \cos\psi\sin\theta\sin\beta\sin\phi + \cos\psi\cos\theta\cos\phi\right)\dot{\phi}\Big]\xi$$

$$+\Big[\quad \text{etc.} \quad\Big]\eta + \Big[\quad \text{etc.} \quad\Big]\zeta$$

$$\dot{z} = \Big[\left(-\sin\theta\sin\beta\cos\phi + \cos\theta\sin\phi\right)\dot{\theta}$$

$$+\left(\cos\theta\cos\beta\cos\phi\right)\dot{\beta} \qquad (4)$$

$$+\left(-\cos\theta\sin\beta\sin\phi + \sin\theta\cos\phi\right)\dot{\phi}\Big]\xi$$

$$+\Big[\quad \text{etc.} \quad\Big]\eta + \Big[\quad \text{etc.} \quad\Big]\zeta$$

The kinetic energy T of this rotating blade is,

$$T = \frac{1}{2} \int \left\{ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right\} dm \tag{5}$$

The evaluation of Eq. (5) is extremely tedious as it involves many trigonometric terms in the squaring process which later combine together. To circumvent this, an alternative expression is used for kinetic energy T based on the fact that the blade is in pure rotation about the origin. It can then be shown that the kinetic energy T is also,

$$T = \frac{1}{2} I_\xi \omega_\xi^2 + \frac{1}{2} I_\eta \omega_\eta^2 + \frac{1}{2} I_\zeta \omega_\zeta^2 \tag{6}$$

$$- I_{\xi\eta} \omega_\xi \omega_\eta - I_{\eta\zeta} \omega_\eta \omega_\zeta - I_{\zeta\xi} \omega_\zeta \omega_\xi$$

where ω_ξ, ω_η, ω_ζ are the angular velocities of the blade about the ξ, η, ζ axes respectively. For a flat blade with the section C.G. on the ξ axis, the product of inertia terms $I_{\xi\eta} \approx I_{\eta\zeta} \approx I_{\zeta\xi} \approx 0$, hence only the first three terms need be retained.

To evaluate the angular velocities ω_ξ, ω_η, ω_ζ, in terms of the coordinate velocities $\dot{\psi}$, $\dot{\theta}$, $\dot{\beta}$, $\dot{\phi}$, one notes that the total vector velocity $\overline{\omega}$ can be written in two ways,

$$\overline{\omega} = \omega_\xi \overline{i}_\xi + \omega_\eta \overline{i}_\eta + \omega_\zeta \overline{i}_\zeta \tag{7}$$

$$\overline{\omega} = \dot{\psi}\, \overline{i}_{z1} + \dot{\theta}\, \overline{i}_{x2} - \dot{\beta}\, \overline{i}_{y3} + \dot{\phi}\, \overline{i}_\zeta \tag{8}$$

From the rotation transformations between the various axes, the following relations exist between the unit vectors,

$$
\begin{aligned}
\bar{i}_{z1} &= \sin\theta\ \bar{i}_{y2} + \cos\theta\ \bar{i}_{z2} \\
\bar{i}_{x2} &= \cos\beta\ \bar{i}_{x3} - \sin\beta\ \bar{i}_{z3} \\
\bar{i}_{y2} &= \bar{i}_{y3} \\
\bar{i}_{z2} &= \sin\beta\ \bar{i}_{x3} + \cos\beta\ \bar{i}_{z3} \\
\bar{i}_{x3} &= \cos\phi\ \bar{i}_{\xi} - \sin\phi\ \bar{i}_{\eta} \\
\bar{i}_{y3} &= \sin\phi\ \bar{i}_{\xi} + \cos\phi\ \bar{i}_{\eta} \\
\bar{i}_{z3} &= \bar{i}_{\zeta}
\end{aligned}
\tag{9}
$$

Placing these into (8), reducing all unit vectors to $\bar{i}_{\xi}$, $\bar{i}_{\eta}$, $\bar{i}_{\zeta}$, and comparing with (7) gives the angular velocities as,

$$
\begin{aligned}
\omega_{\xi} &= \Omega(\sin\theta\sin\phi + \cos\theta\sin\beta\cos\phi) + \dot{\theta}\cos\beta\cos\phi - \dot{\beta}\sin\phi \\
\omega_{\eta} &= \Omega(\sin\theta\cos\phi - \cos\theta\sin\beta\sin\phi) - \dot{\theta}\cos\beta\sin\phi - \dot{\beta}\cos\phi \\
\omega_{\zeta} &= \Omega\cos\theta\cos\beta - \dot{\theta}\sin\beta + \dot{\phi}
\end{aligned}
\tag{10}
$$

where $\dot{\psi} \equiv \Omega$ is the constant rotation speed of the rotor. Using the above expressions for angular velocities, the kinetic energy (6) for this blade with $I_{\xi\eta} \approx I_{\eta\zeta} \approx I_{\zeta\xi} \approx 0$ becomes,

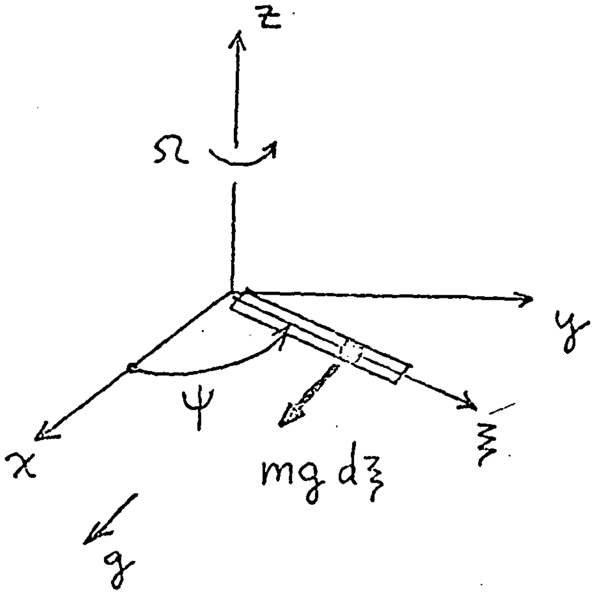
$$
\begin{aligned}
T = {} & \tfrac{1}{2}\dot{\theta}^2\left[\cos^2\beta\left(I_\xi\cos^2\phi + I_\eta\sin^2\phi\right) + I_\zeta\sin^2\beta\right] \\
& + \tfrac{1}{2}\dot{\beta}^2\left(I_\xi\sin^2\phi + I_\eta\cos^2\phi\right) + \tfrac{1}{2}\dot{\phi}^2 I_\zeta \\
& + \dot{\theta}\dot{\beta}\left(I_\eta - I_\xi\right)\tfrac{1}{2}\cos\beta\sin 2\phi - \dot{\theta}\dot{\phi}\, I_\zeta\sin\beta \\
& + \Omega\dot{\theta}\Big[\left(I_\xi\cos^2\phi + I_\eta\sin^2\phi\right)\tfrac{1}{2}\cos\theta\sin 2\beta \\
& \qquad - \left(I_\eta - I_\xi\right)\tfrac{1}{2}\sin\theta\cos\beta\sin 2\phi - I_\zeta\tfrac{1}{2}\cos\theta\sin 2\beta\Big] \\
& + \Omega\dot{\beta}\left[-\left(I_\xi\sin^2\phi + I_\eta\cos^2\phi\right)\sin\theta + \left(I_\eta - I_\xi\right)\tfrac{1}{2}\cos\theta\sin\beta\sin 2\phi\right] \\
& + \Omega\dot{\phi}\, I_\zeta\cos\theta\cos\beta \\
& + \tfrac{1}{2}\Omega^2\Big[\left(I_\xi\sin^2\phi + I_\eta\cos^2\phi\right)\sin^2\theta \\
& \qquad + \left(I_\xi\cos^2\phi + I_\eta\sin^2\phi\right)\cos^2\theta\sin^2\beta \\
& \qquad - \left(I_\eta - I_\xi\right)\tfrac{1}{2}\sin 2\theta\sin\beta\sin 2\phi + I_\zeta\cos^2\theta\cos^2\beta\Big]
\end{aligned}
\tag{11}
$$

The internal potential energy U of this rotor blade arises from torsional springs k_θ, k_β, k_ϕ which are placed at the three hinges of the hub. This gives,

$$U = \tfrac{1}{2}k_\theta\left(\theta - \theta_s\right)^2 + \tfrac{1}{2}k_\beta\left(\beta - \beta_s\right)^2 + \tfrac{1}{2}k_\phi\left(\phi - \phi_s\right)^2 \tag{12}$$

where θ_s, β_s, ϕ_s are the initial settings for no spring moments.

The gravity forces give rise to an incremental work δW as shown in the sketch below,



Gravity acts along x axis

$$\delta W = \int_0^{\ell} m g \, \delta x \, d\xi \qquad (13)$$

$$m = \text{mass/in}$$

The incremental displacement δx can be expressed in terms of the incremental variables $\delta\theta$, $\delta\beta$, $\delta\phi$ using (2) as,

$$\begin{aligned}
\delta x &= \dot{x}\,\delta t \\
&= \Big[(\sin\psi \sin\theta \sin\phi + \sin\psi \cos\theta \sin\beta \cos\phi)\,\delta\theta \\
&\quad + (-\cos\psi \sin\beta \cos\phi + \sin\psi \sin\theta \cos\beta \cos\phi)\,\delta\beta \\
&\quad + (-\cos\psi \cos\beta \sin\phi - \sin\psi \sin\theta \sin\beta \sin\phi - \sin\psi \cos\theta \cos\phi)\,\delta\phi \Big]\,\xi \\
&\quad + \Big[\quad \text{etc.} \quad \Big]\,\eta + \Big[\quad \text{etc.} \quad \Big]\,\zeta
\end{aligned} \qquad (14)$$

Since the blade section C.G. is assumed to lie on the ξ axis, then $\eta = \zeta = 0$ in (14). Placing (14) into (13) and integrating gives

$$\delta W = Q_\theta \delta\theta + Q_\beta \delta\beta + Q_\phi \delta\phi \tag{15}$$

where,

$$\begin{aligned}
Q_\theta &= g S_\zeta \sin\psi \left(\cos\theta \sin\beta \cos\phi + \sin\theta \sin\phi\right) \\
Q_\beta &= g S_\zeta \left[\sin\psi \left(\sin\theta \cos\beta \cos\phi\right) - \cos\psi \left(\sin\beta \cos\phi\right)\right] \\
Q_\phi &= g S_\zeta \left[-\sin\psi \left(\sin\theta \sin\beta \sin\phi + \cos\theta \cos\phi\right) \right. \\
&\qquad \left. - \cos\psi \left(\cos\beta \sin\phi\right)\right]
\end{aligned} \tag{16}$$

$$S_\zeta = \begin{array}{c}\text{static unbalance} \\ \text{about } \zeta \text{ axis}\end{array} = \int_0^\ell m \xi \, d\xi$$

If aerodynamic forces were also present, additional aerodynamic moments Q_θ, Q_β, Q_ϕ would be added to (15).

Gathering together the T, U, Q_i from (11), (12), (16), and placing into Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \tag{17}$$

gives, after some algebra, the nonlinear large deflection equations of motion as,

$\underline{\theta \text{ Equa:}}$

$$
\begin{aligned}
&\ddot{\theta}\left[\left(I_\xi \cos^2\phi + I_\eta \sin^2\phi\right)\cos^2\beta + I_\zeta \sin^2\beta\right] \\
&+ \ddot{\beta}\left(I_\eta - I_\xi\right)\tfrac{1}{2}\cos\beta \sin 2\phi - \ddot{\phi}\, I_\zeta \sin\beta \\
&- \dot{\theta}\dot{\beta}\left(I_\xi \cos^2\phi + I_\eta \sin^2\phi - I_\zeta\right)\sin 2\beta \\
&+ \dot{\theta}\dot{\phi}\left(I_\eta - I_\xi\right)\cos^2\beta \sin 2\phi + \dot{\beta}\dot{\phi}\left[\left(I_\eta - I_\xi\right)\cos 2\phi - I_\zeta\right]\cos\beta \\
&- \dot{\beta}^2\left(I_\eta - I_\xi\right)\tfrac{1}{2}\sin\beta \sin 2\phi + \Omega\dot{\beta}\Big[\left(I_\eta - I_\xi\right)\sin\theta \sin\beta \sin 2\phi \\
&\qquad + \left(I_\xi \sin^2\phi + I_\eta \cos^2\phi\right)\cos\theta + \left(I_\xi \cos^2\phi + I_\eta \sin^2\phi - I_\zeta\right)\cos\theta \cos 2\beta\Big] \\
&+ \Omega\dot{\phi}\left[\left(I_\eta - I_\xi\right)\left(\tfrac{1}{2}\cos\theta \sin 2\beta \sin 2\phi - \sin\theta \cos\beta \cos 2\phi\right) + I_\zeta \sin\theta \cos\beta\right] \\
&+ \tfrac{1}{2}\Omega^2\Big[\left(I_\xi \cos^2\phi + I_\eta \sin^2\phi\right)\sin 2\theta \sin^2\beta - \left(I_\xi \sin^2\phi + I_\eta \cos^2\phi\right)\sin 2\theta \\
&\qquad + \left(I_\eta - I_\xi\right)\cos 2\theta \sin\beta \sin 2\phi + I_\zeta \cos^2\beta \sin 2\theta\Big] + k_\theta\left(\theta - \theta_s\right) \\
&= g\, S_\zeta \sin\psi \left(\cos\theta \sin\beta \cos\phi + \sin\theta \sin\phi\right)
\end{aligned}
\tag{18}
$$

β Equa:

$$\ddot{\beta}\left(I_\xi \sin^2\phi + I_\eta \cos^2\phi\right) + \ddot{\theta}\left(I_\eta - I_\xi\right)\tfrac{1}{2}\cos\beta\sin 2\phi$$
$$-\dot{\beta}\dot{\phi}\left(I_\eta - I_\xi\right)\sin 2\phi + \dot{\theta}\dot{\phi}\left[\left(I_\eta - I_\xi\right)\cos\beta\cos 2\phi + I_\zeta\cos\beta\right]$$
$$+\dot{\theta}^2\left(I_\xi\cos^2\phi + I_\eta\sin^2\phi - I_\zeta\right)\tfrac{1}{2}\sin 2\beta$$
$$-\Omega\dot{\theta}\Big[\left(I_\xi\sin^2\phi + I_\eta\cos^2\phi\right)\cos\theta + \left(I_\xi\cos^2\phi + I_\eta\sin^2\phi - I_\zeta\right)\cos\theta\cos 2\beta$$
$$+ \left(I_\eta - I_\xi\right)\sin\theta\sin\beta\sin 2\phi\Big]$$
$$+\Omega\dot{\phi}\left[\left(I_\eta - I_\xi\right)\left(\sin 2\phi\sin\theta + \cos 2\phi\cos\theta\sin\beta\right) + I_\zeta\cos\theta\sin\beta\right]$$
$$+\tfrac{1}{2}\Omega^2\Big[-\left(I_\xi\cos^2\phi + I_\eta\sin^2\phi\right)\cos^2\theta\sin 2\beta + \left(I_\eta - I_\xi\right)\tfrac{1}{2}\sin 2\theta\cos\beta\sin 2\phi$$
$$+ I_\zeta\cos^2\theta\sin 2\beta\Big] + k_\beta\left(\beta - \beta_s\right)$$
$$= g S_\zeta\left[\sin\psi\left(\sin\theta\cos\beta\cos\phi\right) - \cos\psi\left(\sin\beta\cos\phi\right)\right] \qquad (19)$$

ϕ Equa:

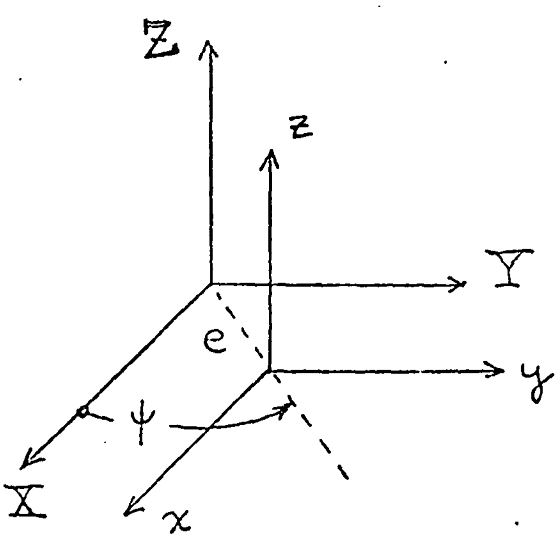
$$\ddot{\phi}\, I_\zeta - \ddot{\theta}\, I_\zeta\sin\beta - \dot{\theta}\dot{\beta}\left[\left(I_\eta - I_\xi\right)\cos\beta\cos 2\phi + I_\zeta\cos\beta\right]$$
$$-\dot{\theta}^2\left(I_\eta - I_\xi\right)\tfrac{1}{2}\cos^2\beta\sin 2\phi + \dot{\beta}^2\left(I_\eta - I_\xi\right)\tfrac{1}{2}\sin 2\phi$$
$$+\Omega\dot{\theta}\left[\left(I_\eta - I_\xi\right)\left(\sin\theta\cos\beta\cos 2\phi - \tfrac{1}{2}\cos\theta\sin 2\beta\sin 2\phi\right) - I_\zeta\sin\theta\cos\beta\right]$$

$$
\begin{aligned}
&-\Omega\dot{\beta}\Big[(I_\eta - I_\xi)(\sin\theta\sin 2\phi + \cos\theta\sin\beta\cos 2\phi) + I_\xi\cos\theta\sin\beta\Big] \\
&+\tfrac{1}{2}\Omega^2(I_\eta - I_\xi)\big(\sin^2\theta\sin 2\phi - \cos^2\theta\sin^2\beta\sin 2\phi \\
&\qquad + \sin 2\theta\sin\beta\cos 2\phi\big) + k_\phi(\phi - \phi_s) \\
&= g S_\xi\Big[-\sin\psi(\sin\theta\sin\beta\sin\phi + \cos\theta\cos\phi) - \cos\psi(\cos\beta\sin\phi)\Big]
\end{aligned}
\tag{20}
$$

The preceding equations apply to arbitrarily large angular deflections of a rotating blade with no offset.

2.2 Effect of Hinge Offset

When an offset, e, is present, one introduces an additional axis system X,Y,Z which now represents the fixed axis system. The origin of the previous x,y,z system now circles about the Z axis at the offset distance e, while maintaining its axes parallel to the XYZ axes.



Fixed axes → X Y Z

$$X = x + e\cos\psi$$

$$Y = y + e\sin\psi$$

$$Z = z$$

The velocities of the two axes systems are related by

$$\begin{aligned} \dot{X} &= \dot{x} - e \sin\psi \, \dot{\psi} \\ \dot{Y} &= \dot{y} + e \cos\psi \, \dot{\psi} \\ \dot{Z} &= \dot{z} \end{aligned} \tag{21}$$

The kinetic energy T of the rotor blade is given by,

$$\begin{aligned} T &= \frac{1}{2} \int_0^{\ell} \left\{ \dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 \right\} dm \\ &= \frac{1}{2} \int_0^{\ell} \left\{ \dot{x}^2 - 2e \sin\psi \, \dot{x} \dot{\psi} + e^2 \sin^2\psi \, \dot{\psi}^2 \right. \\ &\qquad \left. + \dot{y}^2 + 2e \cos\psi \, \dot{y} \dot{\psi} + e^2 \cos^2\psi \, \dot{\psi}^2 + \dot{z}^2 \right\} dm \end{aligned} \tag{22}$$

This can be regrouped into the form,

$$\begin{aligned} T = {} & \frac{1}{2} \int_0^{\ell} \left\{ \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right\} dm \\ & + e \Omega \int_0^{\ell} \left\{ \dot{y} \cos\psi - \dot{x} \sin\psi \right\} dm + \frac{1}{2} e^2 \Omega^2 \int_0^{\ell} dm \end{aligned} \tag{23}$$

The first term above represents the kinetic energy of the blade found previously and given by (11). The third term can be discarded since it is a constant, and will not contribute in Lagrange's equations (17). The second term represents the additional kinetic energy T_A due to the hinge offset. This can be further simplified by taking $\psi = 0$ in these equations to give,

$$T_A = e\Omega \int_0^{\ell} (\dot{y})_{\psi=0}\, dm \tag{24}$$

Introducing $\dot{y}$ from (3) into the above gives,

$$\begin{aligned} T_A = e S_\zeta \Big[&(\cos\beta \cos\phi)\,\Omega^2 + (-\sin\theta \sin\phi - \cos\theta \sin\beta \cos\phi)\,\Omega\dot{\theta} \\ &+ (-\sin\theta \cos\beta \cos\phi)\,\Omega\dot{\beta} \\ &+ (\sin\theta \sin\beta \sin\phi + \cos\theta \cos\phi)\,\Omega\dot{\phi} \Big] \end{aligned} \tag{25}$$

where S_ζ is the static unbalance as in (16).

The additional terms in Lagrange's equations (17) coming from this T_A are,

$$\begin{aligned} &\underline{\theta \text{ Equa :}} \quad && 0 \\ &\underline{\beta \text{ Equa :}} \quad && \Omega^2 e S_\zeta \sin\beta \cos\phi \\ &\underline{\phi \text{ Equa :}} \quad && \Omega^2 e S_\zeta \cos\beta \sin\phi \end{aligned} \tag{26}$$

These terms are to be added to the left hand sides of (18), (19) and (20) respectively, and represent the sole contribution of the hinge offset. It can be seen from (21) and (14) that no additional gravity force terms will result from the hinge offset.

2.3 Summary of Equations of Motion

The nonlinear equations of motion given by (18), (19), (20), (26) for the rotating blade can be simplified somewhat by expanding the trigonometric terms to 3rd order, i.e.,

$$\begin{aligned} \sin\alpha &\approx \alpha - \tfrac{1}{6}\alpha^3 \\ \cos\alpha &\approx 1 - \tfrac{1}{2}\alpha^2 \end{aligned} \tag{27}$$

and also by noting that for these blades,

$$\begin{aligned} I_\xi &\ll I_\eta \\ I_\zeta &\approx I_\eta + I_\xi \end{aligned} \tag{28}$$

Then considering only terms to 3rd order, the nonlinear equations (18), (19), (20), (26) reduce to,

$\underline{\theta \text{ Equa:}}$

$$\begin{aligned} &\ddot{\theta}\left[I_\xi + I_\eta(\beta^2+\phi^2)\right] + \ddot{\beta}\, I_\eta \phi - \ddot{\phi}\, I_\eta \beta \\ &+ 2\dot{\theta}\dot{\beta}\, I_\eta \beta + 2\dot{\theta}\dot{\phi}\, I_\eta \phi - 2\dot{\beta}\dot{\phi}\, I_\xi + 2\Omega\dot{\beta}\, I_\eta \beta^2 \\ &+ 2\Omega\dot{\phi}\left[I_\eta \beta\phi + I_\xi \theta\right] + \Omega^2\left[I_\eta(\beta\phi + \theta\phi^2 - \theta\beta^2) + I_\xi\theta\right] \\ &+ k_\theta(\theta - \theta_s) = g S_\zeta \sin\psi\left(\beta - \frac{\beta^3}{6} - \beta\frac{\theta^2}{2} - \beta\frac{\phi^2}{2} + \theta\phi\right) \end{aligned} \tag{29}$$

$\underline{\beta \text{ Equa:}}$

$$
\begin{aligned}
&\ddot{\beta}\, I_\eta (1-\phi^2) + \ddot{\theta}\, I_\eta \phi - 2\dot{\beta}\dot{\phi}\, I_\eta \phi + 2\dot{\theta}\dot{\phi}\, I_\eta \\
&-\dot{\theta}^2 I_\eta \beta - 2\Omega\dot{\theta}\, I_\eta \beta^2 + 2\Omega\dot{\phi}\, I_\eta (\beta + \theta\phi) \\
&+\Omega^2 \left[I_\eta \left(\beta - \tfrac{2}{3}\beta^3 - \beta\phi^2 - \beta\theta^2 + \theta\phi \right) + e S_\xi \beta \right] \\
&+ k_\beta (\beta - \beta_s) = g S_\xi \sin\psi \left(\theta - \frac{\theta^3}{6} - \theta\frac{\beta^2}{2} - \theta\frac{\phi^2}{2} \right) \\
&\qquad + g S_\xi \cos\psi \left(-\beta + \frac{\beta^3}{6} + \beta\frac{\phi^2}{2} \right)
\end{aligned}
\tag{30}
$$

$\underline{\phi \text{ Equa:}}$

$$
\begin{aligned}
&\ddot{\phi}\, I_\eta - \ddot{\theta}\, I_\eta \beta - 2\dot{\theta}\dot{\beta}\, I_\eta - \dot{\theta}^2 I_\eta \phi + \dot{\beta}^2 I_\eta \phi \\
&-2\Omega\dot{\theta} \left[I_\eta \beta\phi + I_\xi \theta \right] - 2\Omega\dot{\beta}\, I_\eta (\beta + \theta\phi) \\
&+\Omega^2 \left[I_\eta \left(\theta\beta + \theta^2\phi - \beta^2\phi \right) + e S_\xi \phi \right] \\
&+ k_\phi (\phi - \phi_s) = g S_\xi \sin\psi \left(-1 + \frac{\theta^2}{2} + \frac{\phi^2}{2} - \theta\beta\phi \right) \\
&\qquad + g S_\xi \cos\psi \left(-\phi + \frac{\phi^3}{6} + \phi\frac{\beta^2}{2} \right)
\end{aligned}
\tag{31}
$$

Equations (29), (30), (31) represent the final nonlinear 3rd order equations to be investigated. It should be noted that in reducing the offset terms (26) to 3rd order quantities, the offset e itself was considered a first order quantity.

SECTION 3

LINEAR ANALYSIS OF FLAPPING-LAGGING ROTOR

To gain understanding, the simpler flapping-lagging case will be examined before going to the full three degree of freedom case. A linear, small deflection analysis about a large static deflected position will first be conducted before examining the complete nonlinear analysis in the next section.

The basic equations of motion for this flapping-lagging case are obtained from (30) and (31) by setting $\dot{\theta}$ and $\ddot{\theta}$ equal to zero. The θ itself is retained as a constant initial hinge setting for the blade. It will also be convenient to nondimensionalzie the time variable in the standard way by introducing ψ as,

$$\psi \equiv \Omega t, \qquad (\dot{\ }) = \Omega \frac{d}{d\psi} = \Omega (\mathring{\ }) \tag{32}$$

Under these assumptions the basic nondimensionalized equations become

<u>β Equa :</u>

$$\begin{aligned} &\mathring{\mathring{\beta}}(1-\phi^2) - 2\mathring{\beta}\mathring{\phi}\phi + 2\mathring{\phi}(\beta+\theta\phi) \\ &+ \beta(1+\bar{e}+\nu_\beta^2) + \theta\phi - \tfrac{2}{3}\beta^3 - \beta\theta^2 - \beta\phi^2 = \\ &= \nu_\beta^2 \beta_s + G\left(\theta - \frac{\theta^3}{6} - \theta\frac{\beta^2}{2} - \theta\frac{\phi^2}{2}\right)\sin\psi \\ &\qquad + G\left(-\beta + \frac{\beta^3}{6} + \frac{\beta\phi^2}{2}\right)\cos\psi \end{aligned} \tag{33}$$

<u>ϕ Equa :</u>

$$\begin{aligned} &\mathring{\mathring{\phi}} + \mathring{\beta}^2\phi - 2\mathring{\beta}(\beta+\theta\phi) \\ &+ \phi(\bar{e}+\nu_\phi^2) + \theta\beta + \theta^2\phi - \beta^2\phi = \\ &= \nu_\phi^2\phi_s + G\left(-1 + \frac{\theta^2}{2} + \frac{\phi^2}{2} - \theta\beta\phi\right)\sin\psi \\ &\qquad + G\left(-\phi + \frac{\phi^3}{6} + \phi\frac{\beta^2}{2}\right)\cos\psi \end{aligned} \tag{34}$$

where the following nondimensional parameters have been introduced,

$$
\begin{aligned}
\bar{e} &\equiv \frac{e S_s}{I_\eta} \approx \frac{e m \ell^2/2}{m \ell^3/3} \approx \frac{3}{2}\frac{e}{\ell} \quad \left(\begin{array}{c}\text{for uniform}\\ \text{blade}\end{array}\right) \\
\nu_\beta &\equiv \frac{1}{\Omega}\sqrt{\frac{k_\beta}{I_\eta}} = \frac{\omega_\beta}{\Omega} \\
\nu_\phi &\equiv \frac{1}{\Omega}\sqrt{\frac{k_\phi}{I_\eta}} = \frac{\omega_\phi}{\Omega} \\
G &\equiv \frac{g S_s}{I_\eta \Omega^2} = \left(\frac{\omega_{Pend}}{\Omega}\right)^2 \approx \frac{3g}{2\ell\,\Omega^2} \quad \left(\begin{array}{c}\text{for uniform}\\ \text{blade}\end{array}\right)
\end{aligned}
\tag{35}
$$

In the G above, ω_{Pend} represents physically the natural frequency of the blade hanging as a pendulum with no stiffness k_ϕ present.

Equations (33) and (34) represent the basic nonlinear equations to be investigated for the flapping-lagging case.

3.1 Static Solution

A static solution to (33) and (34) can be obtained by neglecting all derivatives terms and by setting G = 0. Under these conditions, the equations reduce to,

$$
\begin{aligned}
\left(1 + \bar{e} + \nu_\beta^2 - \theta^2 - \tfrac{2}{3}\beta^2 - \phi^2\right)\beta + \theta\phi &= \nu_\beta^2 \beta_s \\
\theta\beta + \left(\bar{e} + \nu_\phi^2 + \theta^2 - \beta^2\right)\phi &= \nu_\phi^2 \phi_s
\end{aligned}
\tag{36}
$$

For a given $\bar{e}$, θ, ϕ_s, β_s, ν_β, ν_ϕ, these nonlinear equations can be solved to give the static solution $\phi = \phi_o$, $\beta = \beta_o$. The solution is most easily accomplished by iteration, i.e., first setting β^2 and ϕ^2 equal zero in the paranthesis terms, then solving these linear simultaneous equations for β and ϕ, then correcting the paranthesis terms with the previous β^2 and ϕ^2, then solving for β and ϕ again, etc.

3.2 Small Perturbation Equations

Having found the above static solutions β_o and ϕ_o, one can then investigate small perturbations of the nonlinear equations about these static positions by assuming solutions in the form,

$$\begin{aligned} \beta &= \beta_o + \tilde{\beta} \\ \phi &= \phi_o + \tilde{\phi} \end{aligned} \tag{37}$$

Here, $\tilde{\beta}$ and $\tilde{\phi}$ represent small perturbations about possibly large static positions β_o and ϕ_o. Placing (37) into (33) and (34), retaining only linear terms in the perturbations $\tilde{\beta}$, $\tilde{\phi}$, and cancelling out the previous static solution (36), results in the linear equations,

$\tilde{\beta}$ Equa:

$$\begin{aligned} (1-\phi_o^2)\,\mathring{\mathring{\tilde{\beta}}} &+ 2(\beta_o+\theta\phi_o)\,\mathring{\tilde{\phi}} + (1+\bar{e}+\nu_\beta^2-\theta^2-2\beta_o^2-\phi_o^2)\,\tilde{\beta} \\ &+ (\theta - 2\beta_o\phi_o)\,\tilde{\phi} = \\ = &\left[G\theta\left(1-\frac{\beta_o^2}{2}-\frac{\phi_o^2}{2}-\frac{\theta^2}{6}\right) - (G\theta\beta_o)\,\tilde{\beta} - (G\theta\phi_o)\,\tilde{\phi}\right]\sin\psi \\ &+ \left[-G\beta_o\left(1-\frac{\beta_o^2}{6}-\frac{\phi_o^2}{2}\right) - G\left(1-\frac{\beta_o^2}{2}-\frac{\phi_o^2}{2}\right)\tilde{\beta} + (G\beta_o\phi_o)\,\tilde{\phi}\right]\cos\psi \end{aligned}$$

(38)

$\tilde{\phi}$ Equa:

$$\ddot{\tilde{\phi}} - 2(\beta_o + \theta\phi_o)\dot{\tilde{\beta}} + (\theta - 2\beta_o\phi_o)\tilde{\beta}$$
$$+ (\bar{e} + \nu_\phi^2 + \theta^2 - \beta_o^2)\tilde{\phi} =$$
$$= \left[-G\left(1 - \frac{\theta^2}{2} - \frac{\phi_o^2}{2} + \theta\beta_o\phi_o\right) - (G\theta\phi_o)\tilde{\beta} + G(\phi_o - \theta\beta_o)\tilde{\phi}\right]\sin\psi$$
$$+ \left[-G\phi_o\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{6}\right) + (G\phi_o\beta_o)\tilde{\beta} - G\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)\tilde{\phi}\right]\cos\psi \quad (39)$$

These equations in $\tilde{\beta}$ and $\tilde{\phi}$ represent linear coupled equations with both forced excitation and parametric excitation present.

3.3 Simple Linear Solutions

One can obtain some simple linear solutions of the perturbation equations (38) and (39) by arbitrarily uncoupling them. This will give an indication of the source and rough magnitude of $\tilde{\beta}$ and $\tilde{\phi}$. Later a more accurate coupled linear analysis can be made. And of course, later still, in the next section, a complete nonlinear analysis will be made.

The $\tilde{\beta}$ equation, when uncoupled from the $\tilde{\phi}$ equation by setting all $\tilde{\phi} = 0$ in (38), is,

$$(1 - \phi_o^2)\ddot{\tilde{\beta}} + (1 + \bar{e} + \nu_\beta^2 - \theta^2 - 2\beta_o^2 - \phi_o^2)\tilde{\beta} =$$
$$= \left[G\theta\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2} - \frac{\theta^2}{6}\right) - (G\theta\beta_o)\tilde{\beta}\right]\sin\psi \quad (40)$$
$$+ \left[-G\beta_o\left(1 - \frac{\beta_o^2}{6} - \frac{\phi_o^2}{2}\right) - G\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)\tilde{\beta}\right]\cos\psi$$

Two types of strong oscillations are possible for this type of equation, a forced oscillation near $\Omega \approx \omega_N$ and a parametric instability in the neighborhood of the first parametric instability region, $\Omega \approx 2\omega_N$. In here, ω_N represents the natural frequency of this equation in flapping, namely,

$$\omega_N \approx \Omega\sqrt{\left(1+\bar{e}+\nu_\beta^2-\theta^2-2\beta_0^2-\phi_0^2\right)/\left(1-\phi_0^2\right)} \tag{41}$$

For the forced oscillation, one assumes approximately the steady state solution,

$$\tilde{\beta} \approx a_1 \sin\psi + b_1 \cos\psi \tag{42}$$

Placing into (40) and matching sin ψ and cos ψ terms gives,

$$a_1 \approx \frac{G\theta\left(1-\frac{\beta_0^2}{2}-\frac{\phi_0^2}{2}-\frac{\theta^2}{6}\right)}{\bar{e}+\nu_\beta^2-\theta^2-2\beta_0^2}$$

$$b_1 \approx \frac{-G\beta_0\left(1-\frac{\beta_0^2}{6}-\frac{\phi_0^2}{2}\right)}{\bar{e}+\nu_\beta^2-\theta^2-2\beta_0^2} \tag{43}$$

The coefficients above can also be rewritten using the definitions of G and ν_β in (35) as,

$$a_1 \approx \left(\frac{\omega_{Pend}}{\omega_\beta}\right)^2 \theta \, \frac{1-\frac{\beta_0^2}{2}-\frac{\phi_0^2}{2}-\frac{\theta^2}{6}}{1-\left(\frac{\Omega}{\omega_\beta}\right)^2\left(\theta^2+2\beta_0^2-\bar{e}\right)}$$

$$b_1 \approx -\left(\frac{\omega_{Pend}}{\omega_\beta}\right)^2 \beta_0 \, \frac{1-\frac{\beta_0^2}{6}-\frac{\phi_0^2}{2}}{1-\left(\frac{\Omega}{\omega_\beta}\right)^2\left(\theta^2+2\beta_0^2-\bar{e}\right)} \tag{44}$$

Equations (44) give simple estimates of the forced oscillation amplitudes in flapping. It is to be recalled that total amplitude, $A_\beta = \sqrt{a_1^2 + b_1^2}$. It can be seen from (44), that resonance will occur when the rotation speed reaches,

$$\frac{\Omega}{\omega_\beta} \approx \sqrt{\frac{1}{\theta^2 + 2\beta_0^2 - \bar{e}}} \tag{45}$$

Since the denominator in (45) must be positive this resonance in flapping can only occur if

$$\theta^2 + 2\beta_0^2 > \bar{e} \tag{46}$$

This may well occur for small hinge offsets $\bar{e}$.

For parametric instability in the first (and strongest) instability region, one assumes approximately the steady state solution,

$$\tilde{\beta} \approx a_1 \sin\frac{\psi}{2} + b_1 \cos\frac{\psi}{2} \tag{47}$$

Placing into (40) and matching sin $\psi/2$ and cos $\psi/2$ terms gives the equations,

$$\left[-\tfrac{1}{4}(1-\phi_0^2) + 1 + \bar{e} + \nu_\beta^2 - \theta^2 - 2\beta_0^2 - \phi_0^2 - \frac{G}{2}\left(1 - \frac{\beta_0^2}{2} - \frac{\phi_0^2}{2}\right)\right] a_1 + \left[\tfrac{1}{2} G\theta\beta_0\right] b_1 = 0 \tag{48}$$

$$\left[\tfrac{1}{2} G\theta\beta_0\right] a_1 + \left[-\tfrac{1}{4}(1-\phi_0^2) + 1 + \bar{e} + \nu_\beta^2 - \theta^2 - 2\beta_0^2 - \phi_0^2 + \frac{G}{2}\left(1 - \frac{\beta_0^2}{2} - \frac{\phi_0^2}{2}\right)\right] b_1 = 0$$

These homogeneous equations in a_1 and b_1 have the solution given by (47) only if their determinant equals zero, i.e., if,

$$\left[\frac{3}{4} - \frac{3}{4}\phi_o^2 + \bar{e} + \nu_\beta^2 - \theta^2 - 2\beta_o^2\right]^2 - \left(\frac{G}{2}\right)^2\left[\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)^2 + (\theta\beta_o)^2\right] = 0 \tag{49}$$

This equation can be solved for ν_β^2 by bringing the second term to the right hand side, then taking the square root of both sides, and rearranging to give,

$$\nu_\beta^2 = -\frac{3}{4} - \bar{e} + \theta^2 + \frac{3}{4}\phi_o^2 + 2\beta_o^2 \pm \frac{G}{2}\sqrt{\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)^2 + (\theta\beta_o)^2} \tag{50}$$

This gives two solutions corresponding to the two boundaries of the first instability region. The boundary with + G is the one first reached here. Noting from (35) that G can be expressed as,

$$G = \left(\frac{\omega_{Pend}}{\omega_\beta}\right)^2 \nu_\beta^2 \tag{51}$$

one may rearrange the criterion (50) to give,

$$\frac{\Omega}{\omega_\beta} \approx \sqrt{\frac{1 \mp \frac{1}{2}\left(\frac{\omega_{Pend}}{\omega_\beta}\right)^2 \sqrt{\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)^2 + (\theta\beta_o)^2}}{-\frac{3}{4} - \bar{e} + \theta^2 + \frac{3}{4}\phi_o^2 + 2\beta_o^2}} \tag{52}$$

This defines the rotation speeds at which parametric resonance in flapping occur. Since generally $(\omega_{Pend}/\omega_\beta)^2 << 1$ for windmills, this can only occur if,

$$\theta^2 + \frac{3}{4}\phi_0^2 + 2\beta_0^2 > \frac{3}{4} + \bar{e} \tag{52A}$$

which is generally impossible to meet. Hence no parametric resonance is expected in flapping.

Looking next at the lagging case, the $\tilde{\phi}$ equation, when uncoupled from the $\tilde{\beta}$ equation by setting all $\tilde{\beta} = 0$ in (39) is,

$$\begin{aligned}\ddot{\tilde{\phi}} + \left(\bar{e} + \nu_\phi^2 + \theta^2 - \beta_0^2\right)\tilde{\phi} &= \\ = \left[-G\left(1 - \frac{\theta^2}{2} - \frac{\phi_0^2}{2} + \theta\beta_0\phi_0\right) + G\left(\phi_0 - \theta\beta_0\right)\tilde{\phi}\right]\sin\psi \\ + \left[-G\phi_0\left(1 - \frac{\beta_0^2}{2} - \frac{\phi_0^2}{6}\right) - G\left(1 - \frac{\beta_0^2}{2} - \frac{\phi_0^2}{2}\right)\tilde{\phi}\right]\cos\psi\end{aligned} \tag{53}$$

The natural frequency of this equation in lagging is,

$$\omega_N \approx \Omega\sqrt{\bar{e} + \nu_\phi^2 + \theta^2 - \beta_0^2} \tag{54}$$

and strong forced oscillations occur near $\Omega \approx \omega_N$, while the first parametric instability region occurs near $\Omega \approx 2\,\omega_N$.

For the forced oscillation, one assumes approximately the steady state solution,

$$\tilde{\phi} \approx a_2 \sin\psi + b_2 \cos\psi \tag{55}$$

Placing this into (53) and carrying through as previously for the flapping $\tilde{\beta}$ case gives,

$$a_2 \approx -\left(\frac{\omega_{Pend}}{\omega_\phi}\right)^2 \frac{1 - \frac{\theta^2}{2} - \frac{\phi_o^2}{2} + \theta\beta_o\phi_o}{1 - \left(\frac{\Omega}{\omega_\phi}\right)^2\left[1 - \bar{e} + \beta_o^2 - \theta^2\right]}$$

$$b_2 \approx -\left(\frac{\omega_{Pend}}{\omega_\phi}\right)^2 \phi_o \frac{1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{6}}{1 - \left(\frac{\Omega}{\omega_\phi}\right)^2\left[1 - \bar{e} + \beta_o^2 - \theta^2\right]} \tag{56}$$

Equation (56) give simple estimates of the forced oscillation amplitudes in lagging. It is recalled that the total amplitude, $A_\phi = \sqrt{a_2^2 + b_2^2}$. From (56), resonance will occur when

$$\frac{\Omega}{\omega_\phi} \approx \sqrt{\frac{1}{1 - \bar{e} + \beta_o^2 - \theta^2}} \tag{57}$$

This resonance in lagging will occur if

$$1 + \beta_o^2 > \bar{e} + \theta^2 \tag{58}$$

This condition is always met in practice, hence large lagging forced oscillations are distinctly possible at the rotation speed given by (57).

For parametric instability in the first instability region, one assumes approximately the steady state solution,

$$\tilde{\phi} \approx a_2 \sin\frac{\psi}{2} + b_2 \cos\frac{\psi}{2} \tag{59}$$

Placing into (53) and matching sin $\psi/2$ and cos $\psi/2$ terms gives the equations,

$$\left[-\frac{1}{4} + \bar{e} + \nu_\phi^2 + \theta^2 - \beta_o^2 - \frac{G}{2}\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)\right] a_2 + \left[-\frac{G}{2}\left(\phi_o - \theta\beta_o\right)\right] b_2 = 0 \tag{60}$$

$$\left[-\frac{G}{2}\left(\phi_o - \theta\beta_o\right)\right] a_2 + \left[-\frac{1}{4} + \bar{e} + \nu_\phi^2 + \theta^2 - \beta_o^2 + \frac{G}{2}\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)\right] b_2 = 0$$

Setting the determinant of these homogeneous equations equal to zero as previously for the flapping $\tilde{\beta}$ case gives,

$$\nu_\phi^2 = \frac{1}{4} - \bar{e} - \theta^2 + \beta_o^2 \pm \frac{G}{2}\sqrt{\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)^2 + \left(\phi_o - \theta\beta_o\right)^2} \tag{61}$$

Using the stability boundary above and expressing G as in (51) previously, (but with reference to ω_ϕ now), the criterion (61) may be rearranged to give the rotation speeds for parametric resonance as,

$$\frac{\Omega}{\omega_\phi} \approx \sqrt{\frac{1 \mp \frac{1}{2}\left(\frac{\omega_{Pend}}{\omega_\phi}\right)^2 \sqrt{\left(1 - \frac{\beta_o^2}{2} - \frac{\phi_o^2}{2}\right)^2 + \left(\phi_o - \theta\beta_o\right)^2}}{\frac{1}{4} - \bar{e} - \theta^2 + \beta_o^2}} \tag{62}$$

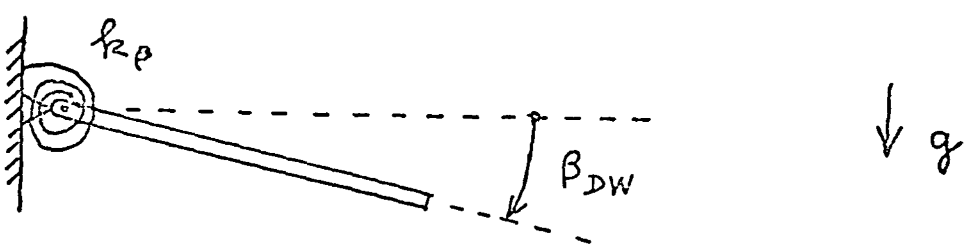
Since generally $(\omega_{Pend}/\omega_\phi)^2 \ll 1$ for windmills, this parametric resonance will occur if,

$$\frac{1}{4} + \beta_o^2 > \bar{e} + \theta^2 \tag{63}$$

This condition is generally met for small offsets $\bar{e}$, hence parametric resonance in lagging is distinctly possible at the rotation speeds given by (62).

To end this brief discussion of simple linear solutions, it may be of interest to introduce some simple physical interpretations to some of the results, equations (41) to (63). The criteria for strong forced oscillations in flapping (46) and lagging (58), and the rotation speeds at which these forced oscillations occur (45) and (57), come simply from the requirement $\Omega = \omega_N$. This can easily be shown by setting $\Omega = \omega_N$ in (41) and (54) respectively. Similarly, the criterion for the first parametric instability region in flapping (52A) and lagging (63) can be shown to come from the requirement $\Omega = 2\omega_N$. The rotation speeds at which these parametric instability regions first occur (52) and (62) are slightly less because of the $(\omega_{Pend}/\omega_\beta)^2$ and $(\omega_{Pend}/\omega_\phi)^2$ factors. If $(\omega_{Pend}/\omega_\beta)^2 \to 0$ and $(\omega_{Pend}/\omega_\phi)^2 \to 0$, then these rotation speeds also occur at $\Omega = 2\omega_N$.

Another quantity can be given a simple physical interpretation, namely, the $(\omega_{Pend}/\omega_\beta)^2$ factor appearing in (44) and (52). Consider a blade held horizontally against gravity and restrained at the β hinge by the spring k_β as shown below.



The angular deflection β_{DW} under its own dead weight is given by,

$$\beta_{DW} = \frac{g S_\eta}{k_\beta} \tag{64}$$

where $S_\eta \equiv \int \xi \, dm$ is the static unbalance about the η axis. The natural frequency ω_β of this blade is defined as

$$\omega_\beta = \sqrt{\frac{k_\beta}{I_\eta}} \tag{65}$$

Solving (65) for k_β, placing it into (64), using the definition $(\omega_{Pend})^2 = g\, S_\zeta / I_\eta$ given in (35), and noting that $S_\zeta \approx S_\eta$, results finally in

$$\beta_{DW} = \left(\frac{\omega_{Pend}}{\omega_\beta} \right)^2 \tag{66}$$

Thus, the $(\omega_{Pend}/\omega_\beta)^2$ factor simply represents the static deflection β of a horizontal blade under its own dead weight. Similarly it can be shown that,

$$\phi_{DW} = \left(\frac{\omega_{Pend}}{\omega_\phi} \right)^2 \tag{67}$$

These β_{DW} and ϕ_{DW} give simple interpretations of the forced amplitude oscillations (44) and (56).

In summary, the simple linear analysis in this section has indicated that gravity effects are likely to be more important for lagging motions than for flapping motions, and that both strong forced oscillations and parametric instabilities are possible for lagging at the rotation speeds given by (57) and (62).

3.4 Complete Linear Solution

complete linear solution of the coupled small perturbation equations (38) and (39) can be made for more accuracy. For the case of forced oscillations, one would assume the approximate steady state solution,

$$\begin{aligned} \tilde{\beta} &\approx \beta_c + a_1 \sin\psi + b_1 \cos\psi \\ \tilde{\phi} &\approx \phi_c + a_2 \sin\psi + b_2 \cos\psi \end{aligned} \tag{68}$$

The higher harmonics, $\sin 2\psi$, $\cos 2\psi$ will be neglected here. Then placing these into (38) and (39) and matching the constant, the $\sin\psi$, and the $\cos\psi$ terms of each equation results in six equations in six unknowns, β_c, a_1, b_1, ϕ_c, a_2, b_2. The β_c and ϕ_c are included in (68) to allow for small centershifts from the static values β_o and ϕ_o. The solution (68) will now include the effects of the small second parametric instability region near $\Omega \approx \omega_N$ in addition to the dominant forced oscillation resonance there. The six linear equations can be readily solved by inversion to give the forced oscillation amplitudes in (68). Also, the determinant of these equations can be evaluated numerically for different values of rotation speed Ω to find what Ω makes the determinant equal zero. This would then represent the boundary of the small second instability region.

Instead of developing and presenting these coupled linear equations here, it will be more convenient to present and solve them later as a subcase of the complete nonlinear equations to be given in the next section.

For the case of parametric instability in the first instability region, which is always the strongest instability region, (see Bolotin [40]), one would assume the approximate solution,

$$\tilde{\beta} \approx a_1 \sin\frac{\Psi}{2} + b_1 \cos\frac{\Psi}{2}$$
$$\tilde{\phi} \approx a_2 \sin\frac{\Psi}{2} + b_2 \cos\frac{\Psi}{2} \quad (68A)$$

and obtain the appropriate four equations in the four unknowns. The higher harmonics $\sin 3\psi/2$, $\cos 3\psi/2$ are neglected here. The determinant would then be examined numerically to find what rotation speeds Ω make it equal zero, thereby determining the stability boundaries. Again, this linear solution will be obtained as a subcase of the complete nonlinear equations in the next section.

SECTION 4

NONLINEAR ANALYSIS OF FLAPPING-LAGGING ROTOR

The linear solutions given in the previous section serve only as a guide to the small amplitude behavior of the rotor blade. When the amplitudes become large, the nonlinear terms in the equations will serve to limit the predictions of linear theory. This is particularly true of the forced oscillation resonances and the parametric instability regions where infinite amplitudes are predicted by linear theory. Accordingly, the complete nonlinear equations (33) and (34) will now be examined.

4.1 Forced Oscillations

For forced oscillations of equations (33) and (34), one seeks limit cycle solutions in the form

$$\begin{aligned} \beta &\approx \beta_c + a_1 \sin\psi + b_1 \cos\psi \\ \phi &\approx \phi_c + a_2 \sin\psi + b_2 \cos\psi \end{aligned} \tag{69}$$

The higher harmonics, $\sin 2\psi$, $\cos 2\psi$, etc. will be neglected here. The β_c and ϕ_c above represent the total centershift from zero and would now include the static solutions β_o and ϕ_o plus any additional centershift due to the oscillation amplitudes, a_1 b_1 a_2 b_2. Placing these expressions (69) into (33) and (34) and matching the constant, sin ψ, and cos ψ terms of each equation and discarding the higher harmonic terms, gives after much algebra and trigonometric reduction, the following six nonlinear equations,

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\ F_{21} & F_{22} & F_{23} & \cdot & \cdot & \cdot \\ F_{31} & F_{32} & F_{33} & \cdot & \cdot & \cdot \\ F_{41} & \cdot & \cdot & \cdot & \cdot & \cdot \\ F_{51} & \cdot & \cdot & \cdot & \cdot & \cdot \\ F_{61} & \cdot & \cdot & \cdot & \cdot & F_{66} \end{bmatrix} \begin{Bmatrix} \beta_c \\ a_1 \\ b_1 \\ \phi_c \\ a_2 \\ b_2 \end{Bmatrix} + \begin{Bmatrix} F_{10} \\ F_{20} \\ F_{30} \\ F_{40} \\ F_{50} \\ F_{60} \end{Bmatrix} = 0 \tag{70}$$

where the elements F_{ij} are given as,

$$F_{11} = \bar{\omega}_\beta^2 Z + 1 + \bar{e} - \theta^2 - \tfrac{2}{3}\beta_c^2 - \phi_c^2 - N\left\{(a_1^2 + b_1^2) + \tfrac{1}{2}(a_2^2 + b_2^2)\right\}$$

$$F_{12} = G\tfrac{1}{2}\theta\beta_c$$

$$F_{13} = G\left(\tfrac{1}{2} - \tfrac{\beta_c^2}{4} - \tfrac{\phi_c^2}{4}\right) - NG\left\{\tfrac{1}{16}(a_1^2 + b_1^2) + \tfrac{1}{16}(a_2^2 + 3b_2^2)\right\}$$

$$F_{14} = \theta - N\left\{a_1 a_2 + b_1 b_2\right\}$$

$$F_{15} = G\tfrac{1}{2}\theta\phi_c$$

$$F_{16} = -G\tfrac{1}{2}\beta_c\phi_c - NG\left\{\tfrac{1}{8}a_1 a_2\right\}$$

$$F_{21} = -NG\left\{\tfrac{1}{4}(a_1 b_1 + a_2 b_2)\right\}$$

$$F_{22} = \bar{\omega}_\beta^2 Z + \bar{e} - \theta^2 - 2\beta_c^2 - N\left\{\tfrac{1}{2}(a_1^2 + b_1^2) + \tfrac{1}{2}(a_2^2 - b_2^2)\right\}$$

$$F_{23} = 0$$

$$F_{24} = -NG\left\{\tfrac{1}{4}(a_1 b_2 + a_2 b_1)\right\}$$

$$F_{25} = \theta - 2\beta_c\phi_c - N\left\{b_1 b_2\right\}$$

$$F_{26} = -2(\beta_c + \theta\phi_c)$$

$$F_{31} = G\left(1 - \frac{\beta_c^2}{6} - \frac{\phi_c^2}{2}\right) - NG\left\{\frac{1}{8}(a_1^2 + 3b_1^2) + \frac{1}{8}(a_2^2 + 3b_2^2)\right\}$$

$$F_{32} = 0$$

$$F_{33} = \bar{\omega}_\beta^2 Z + \bar{e} - \theta^2 - 2\beta_c^2 - N\left\{\frac{1}{2}(a_1^2 + b_1^2) - \frac{1}{2}(a_2^2 - b_2^2)\right\}$$

$$F_{34} = -NG\left\{\frac{1}{4}(a_1 a_2 + 3b_1 b_2)\right\}$$

$$F_{35} = 2(\beta_c + \theta\phi_c)$$

$$F_{36} = \theta - 2\beta_c\phi_c - N\{a_1 a_2\}$$

$$F_{41} = \theta - N\{a_1 a_2 + b_1 b_2\}$$

$$F_{42} = G\frac{1}{2}\theta\phi_c$$

$$F_{43} = -G\frac{1}{2}\beta_c\phi_c - NG\left\{\frac{1}{8}a_1 a_2\right\}$$

$$F_{44} = Z + \bar{e} + \theta^2 - \beta_c^2$$

$$F_{45} = -G\frac{1}{2}(\phi_c - \theta\beta_c)$$

$$F_{46} = G\left(\frac{1}{2} - \frac{\beta_c^2}{4} - \frac{\phi_c^2}{4}\right) - NG\left\{\frac{1}{16}(a_1^2 + 3b_1^2) + \frac{1}{16}(a_2^2 + b_2^2)\right\}$$

$$F_{51} = -NG\left\{\tfrac{1}{4}(a_1 b_2 + a_2 b_1)\right\}$$

$$F_{52} = \theta - 2\beta_c \phi_c - N\left\{b_1 b_2\right\}$$

$$F_{53} = 2(\beta_c + \theta\phi_c)$$

$$F_{54} = -NG\left\{\tfrac{1}{4}(a_1 b_1 + a_2 b_2)\right\}$$

$$F_{55} = Z + \bar{e} - 1 + \theta^2 - \beta_c^2 - N\left\{\tfrac{1}{2}(a_1^2 - b_1^2)\right\}$$

$$F_{56} = 0$$

$$F_{61} = -NG\left\{\tfrac{1}{4}(a_1 a_2 + 3 b_1 b_2)\right\}$$

$$F_{62} = -2(\beta_c + \theta\phi_c)$$

$$F_{63} = \theta - 2\beta_c\phi_c - N\left\{a_1 a_2\right\}$$

$$F_{64} = G\left(1 - \frac{\beta_c^2}{2} - \frac{\phi_c^2}{6}\right) - NG\left\{\tfrac{1}{8}(a_1^2 + 3b_1^2) + \tfrac{1}{8}(a_2^2 + 3b_2^2)\right\}$$

$$F_{65} = 0$$

$$F_{66} = Z + \bar{e} - 1 + \theta^2 - \beta_c^2 + N\left\{\tfrac{1}{2}(a_1^2 - b_1^2)\right\}$$

$$F_{10} = -\bar{\omega}_\beta^2 Z \beta_s - N\left\{a_1 b_2 - a_2 b_1\right\}$$

$$F_{20} = -G\theta\left(1 - \frac{\theta^2}{6} - \frac{\beta_c^2}{2} - \frac{\phi_c^2}{2}\right) + NG\left\{\theta\left[\frac{1}{8}\left(3a_1^2 + b_1^2\right) + \frac{1}{8}\left(3a_2^2 + b_2^2\right)\right]\right\}$$

$$F_{30} = NG\left\{\theta\,\frac{1}{4}\left(a_1 b_1 + a_2 b_2\right)\right\}$$

$$F_{40} = -Z\phi_s - N\left\{\theta\left(a_1 b_2 - a_2 b_1\right)\right\}$$

$$F_{50} = G\left(1 - \frac{\theta^2}{2} - \frac{\phi_c^2}{2} + \theta\beta_c\phi_c\right) - NG\left\{\frac{1}{8}\left(3a_2^2 + b_2^2\right) - \frac{1}{4}\theta\left(3a_1 a_2 + b_1 b_2\right)\right\}$$

$$F_{60} = -NG\left\{\frac{a_2 b_2}{4} - \frac{1}{4}\theta\left(a_1 b_2 + a_2 b_1\right)\right\}$$

The nonlinear terms appearing in the F_{ij} expressions have been grouped to provide a reasonable symmetry. Other groupings could also have been chosen.

In the above elements the following definitions have been introduced for convenience,

$$Z \equiv \nu_\phi^2 = \frac{1}{\left(\Omega/\omega_\phi\right)^2}$$

$$\bar{\omega}_\beta \equiv \frac{\omega_\beta}{\omega_\phi} \qquad (71)$$

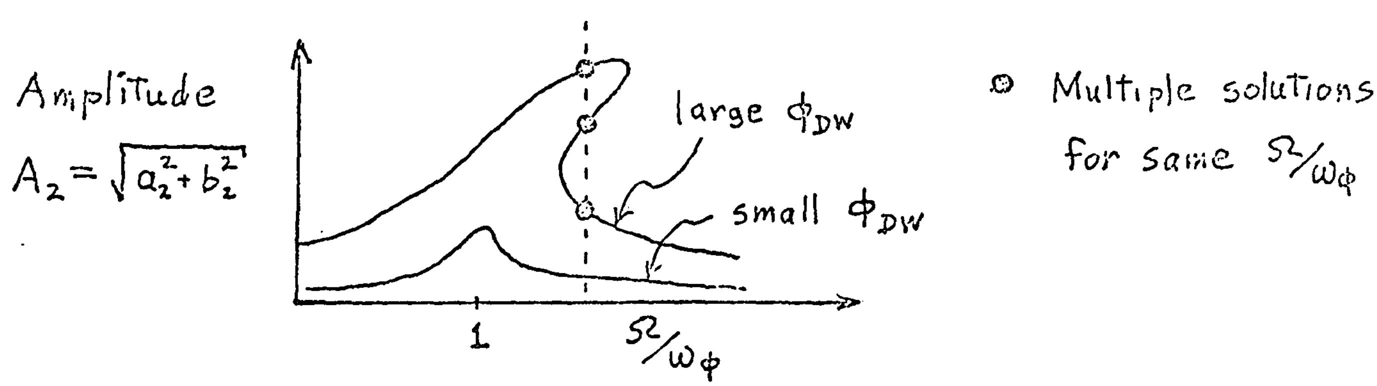
$$N \equiv \begin{cases} 0 & \text{for linear case} \\ 1 & \text{for nonlinear case} \end{cases}$$

For a given configuration defined by the six parameters θ, β_s, ϕ_s, $\bar{e}$, $\bar{\omega}_\beta$, and $(\omega_{pend}/\omega_\phi)^2 = \phi_{DW}$, one solves the nonlinear equations (70) for various values of rotation frequency ratio Ω/ω_ϕ. In evaluating the coefficients, it is convenient to use the relations,

$$Z = \frac{1}{(\Omega/\omega_\phi)^2} \tag{72}$$

$$G = \left(\frac{\omega_{pend}}{\omega_\phi}\right)^2 Z = \phi_{DW} Z$$

The solution of the nonlinear equations (70) is best accomplished numerically by using an iterative Newton-Raphson technique, which uses some initial estimate of the solution to begin the process. As a start, one may use the trial solution $\beta_c = \beta_s$, $\phi_c = \phi_s$, $a_1 = b_1 = a_2 = b_2 = 0$. One may be further guided by the simple linear solutions (56) and (44) given in Section 3.3 However, one must realize that for nonlinear equations, multiple solutions may be found in certain cases. The particular solution obtained would depend on the initial estimate in the iterative computation process. Solutions would probably vary with frequency as shown below,



The effects of various parameters such as ϕ_{DW}, β_s, θ etc. could be assessed as desired. Also, the complete linear case mentioned in the previous section 3.4 could be worked out by simply setting $N = 0$.

Finally, it should be mentioned that some of the limit cycle solutions obtained will be unstable and as such, have no physical reality. These unstable solutions can be checked formally by a stability analysis or may be inferred by experience.

4.2 Parametric Resonance

For parametric instability in the first instability region, one assumes a limit cycle solution of (33) and (34) in the form,

$$\begin{aligned} \beta &\approx \beta_c + a_1 \sin\frac{\psi}{2} + b_1 \cos\frac{\psi}{2} \\ \phi &\approx \phi_c + a_2 \sin\frac{\psi}{2} + b_2 \cos\frac{\psi}{2} \end{aligned} \tag{73}$$

The higher harmonics $\sin\psi$, $\sin 3\psi/2$, etc. will be neglected here. Placing these into (33) and (34) and matching the constant, $\sin\psi/2$, and $\cos\psi/2$ terms of each equation and discarding the higher harmonics, gives again six nonlinear equations,

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ P_{21} & P_{22} & P_{23} & \cdot & \cdot & \cdot \\ P_{31} & P_{32} & P_{33} & \cdot & \cdot & \cdot \\ P_{41} & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{51} & \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{61} & \cdot & \cdot & \cdot & \cdot & P_{66} \end{bmatrix} \begin{Bmatrix} \beta_c \\ a_1 \\ b_1 \\ \phi_c \\ a_2 \\ b_2 \end{Bmatrix} + \begin{Bmatrix} P_{10} \\ P_{20} \\ P_{30} \\ P_{40} \\ P_{50} \\ P_{60} \end{Bmatrix} = 0 \tag{74}$$

where the elements P_{ij} are given as,

$$P_{11} = \bar{\omega}_\beta^2 Z + 1 + \bar{e} - \theta^2 - \frac{2}{3}\beta_c^2 - \phi_c^2$$
$$- N\left\{ (a_1^2+b_1^2) + \frac{1}{2}(a_2^2+b_2^2) - G\left[\frac{1}{8}(a_1^2-b_1^2) + \frac{1}{8}(a_2^2-b_2^2)\right]\right\}$$

$$P_{12} = 0$$

$$P_{13} = 0$$

$$P_{14} = \theta - N\left\{ a_1 a_2 + b_1 b_2 - G\,\frac{1}{4}(a_1 a_2 - b_1 b_2)\right\}$$

$$P_{15} = 0$$

$$P_{16} = 0$$

$$P_{21} = 0$$

$$P_{22} = \bar{\omega}_\beta^2 Z + \frac{3}{4} + \bar{e} - \theta^2 - 2\beta_c^2 - \frac{3}{4}\phi_c^2 - G\left(\frac{1}{2} - \frac{\beta_c^2}{4} - \frac{\phi_c^2}{4}\right)$$
$$- N\left\{ \frac{1}{2}(a_1^2+b_1^2) + \frac{1}{16}(11\,a_2^2+b_2^2) - G\,\frac{1}{12}(a_1^2+3a_2^2)\right\}$$

$$P_{23} = G\,\frac{1}{2}\,\theta\,\beta_c$$

$$P_{24} = 0$$

$$P_{25} = \theta - 2\beta_c\phi_c + G\,\frac{1}{2}\,\beta_c\phi_c - N\left\{\frac{5}{8}\,b_1 b_2\right\}$$

$$P_{26} = -\beta_c - \theta\,\phi_c + G\,\frac{1}{2}\,\theta\,\phi_c$$

$$P_{31} = 0$$

$$P_{32} = G\,\tfrac{1}{2}\,\theta\,\beta_c$$

$$P_{33} = \bar{\omega}_\beta^{\,2}\,Z + \tfrac{3}{4} + \bar{e} - \theta^2 - 2\beta_c^2 - \tfrac{3}{4}\phi_c^2 + G\left(\tfrac{1}{2} - \tfrac{\beta_c^2}{4} - \tfrac{\phi_c^2}{4}\right)$$
$$- N\left\{\tfrac{1}{2}(a_1^2 + b_1^2) + \tfrac{1}{16}(a_2^2 + 11\,b_2^2) + G\,\tfrac{1}{12}(b_1^2 + 3b_2^2)\right\}$$

$$P_{34} = 0$$

$$P_{35} = \beta_c + \theta\phi_c + G\,\tfrac{1}{2}\,\theta\,\phi_c$$

$$P_{36} = \theta - 2\beta_c\phi_c - G\,\tfrac{1}{2}\,\beta_c\phi_c - N\left\{\tfrac{5}{8}\,a_1 a_2\right\}$$

$$P_{41} = \theta - N\left\{a_1 a_2 + b_1 b_2 - G\,\tfrac{1}{4}(a_1 a_2 - b_1 b_2)\right\}$$

$$P_{42} = 0$$

$$P_{43} = 0$$

$$P_{44} = Z + \bar{e} + \theta^2 - \beta_c^2 - N\left\{\tfrac{3}{8}(a_1^2 + b_1^2) - G\left[\tfrac{1}{8}(a_1^2 - b_1^2)\right.\right.$$
$$\left.\left. + \tfrac{1}{8}(a_2^2 - b_2^2)\right]\right\}$$

$$P_{45} = 0$$

$$P_{46} = 0$$

$$P_{51} = 0$$

$$P_{52} = \theta - 2\beta_c\phi_c + G\tfrac{1}{2}\beta_c\phi_c - N\left\{\tfrac{5}{8}\, b_1 b_2\right\}$$

$$P_{53} = \beta_c + \theta\phi_c + G\tfrac{1}{2}\theta\phi_c$$

$$P_{54} = 0$$

$$P_{55} = Z - \tfrac{1}{4} + \bar{e} + \theta^2 - \beta_c^2 - G\left(\tfrac{1}{2} - \tfrac{\beta_c^2}{4} - \tfrac{\phi_c^2}{4}\right)$$
$$- N\left\{\tfrac{1}{16}\left(11 a_1^2 + b_1^2\right) - G\tfrac{1}{12}\left(3a_1^2 + a_2^2\right)\right\}$$

$$P_{56} = -G\tfrac{1}{2}\left(\phi_c - \theta\beta_c\right)$$

$$P_{61} = 0$$

$$P_{62} = -\beta_c - \theta\phi_c + G\tfrac{1}{2}\theta\phi_c$$

$$P_{63} = \theta - 2\beta_c\phi_c - G\tfrac{1}{2}\beta_c\phi_c - N\left\{\tfrac{5}{8}\, a_1 a_2\right\}$$

$$P_{64} = 0$$

$$P_{65} = -G\tfrac{1}{2}\left(\phi_c - \theta\beta_c\right)$$

$$P_{66} = Z - \tfrac{1}{4} + \bar{e} + \theta^2 - \beta_c^2 + G\left(\tfrac{1}{2} - \tfrac{\beta_c^2}{4} - \tfrac{\phi_c^2}{4}\right)$$
$$- N\left\{\tfrac{1}{16}\left(a_1^2 + 11 b_1^2\right) + G\tfrac{1}{12}\left(3b_1^2 + b_2^2\right)\right\}$$

$$P_{10} = -\bar{\omega}_\beta^2 Z \beta_s - N\left\{\frac{1}{2}(a_1 b_2 - a_2 b_1) - \frac{G\theta}{4}(a_1 b_1 + a_2 b_2)\right\}$$

$$P_{20} = 0$$

$$P_{30} = 0$$

$$P_{40} = -Z\phi_s - N\left\{\frac{1}{2}\theta(a_1 b_2 - a_2 b_1) + G\left[\frac{1}{4}a_2 b_2 - \frac{\theta}{4}(a_1 b_2 + a_2 b_1)\right]\right\}$$

$$P_{50} = 0$$

$$P_{60} = 0$$

In these elements, the same definitions (71) and (72) are used as before. The equations (74) are solved as previously for the forced oscillation case, only now, one is guided by the simple linear parametric instability solutions (52) and (62), and one seeks solutions near $\Omega/\omega_\phi \approx 2$. These steady state limit cycles, appearing at roughly twice the rotational speeds Ω of the forced oscillation resonances, may be equally as severe as the forced oscillations. It is to be noted that the actual vibrations themselves occur at roughly $\omega \approx \omega_\phi$ even though the rotational speed here is roughly $\Omega \approx 2\omega_\phi$.

Other nonlinear subharmonic and superharmonic solutions can be investigated in a similar manner by introducing equations (69) or (73) with additional harmonic terms present into the basic nonlinear equations (33) and (34). One would then harmonically balance these additional harmonics which would lead to larger size nonlinear algebraic equations in place of (70) and (74). Such subharmonic and superharmonic solutions for simple beams were examined by Tseng and Dugundji [38].

SECTION 5

NUMERICAL RESULTS FOR NONLINEAR FLAPPING-LAGGING ROTOR

Numerical results using the previous nonlinear analysis of Section 3 are obtained for the following configurations:

CASE I: $\beta_s = 0$ $\phi_s = 0$ $\bar{\omega}_\beta = .71$ $\phi_{DW} = .088$ $\theta = 0$ $\bar{e} = .1$

CASE II: $\beta_s = .15$ $\phi_s = 0$ $\bar{\omega}_\beta = .71$ $\phi_{DW} = .088$ $\theta = 0$ $\bar{e} = .1$

CASE III: $\beta_s = .15$ $\phi_s = 0$ $\bar{\omega}_\beta = 1.4$ $\phi_{DW} = .088$ $\theta = 0$ $\bar{e} = .1$

The $\phi_{DW} = .088$ represents a relatively flexible lag rotor, i.e., the case for which $\omega_\phi = 3.37\ \omega_{pend}$.

5.1 Forced Oscillations

First, simple uncoupled solutions are calculated for the different cases using Eqs. (36) and (44). Then the linear and nonlinear solutions are obtained from Eq. (70) by computer. For the compution of nonlinear solutions, the corresponding linear solutions are used as initial guess as far as possible. The results for the above three cases are presented in Tables 1-3. From these results, one can see that the simple solutions give reasonable estimate of linear solutions, particularly, where coupling motion is weak. Also, linear solutions agree well with nonlinear solutions away from the resonance region. This is quite apparent from the fact that the linear theory is good for small amplitudes and becomes inadequate for large amplitudes which take place near resonance condition. All these solutions are checked for stability by giving linear perturbations to the steady solution and then studying the growth of these perturbations (discussed later in Section 5.3).

In Fig. 1, the solutions are plotted for Case I. For this configuration of zero coning angle, the response is uncoupled and one gets only lagging amplitude. Here it can be seen that the linear and nonlinear solutions are quite close except near resonance (i.e., $\Omega/\omega_\phi \simeq .95$ to 1.15). Arrow mark on the graph is a resonance point obtained by linear theory. The nonlinear solutions become unstable for A_ϕ greater than about 1.

In Figs. 2a, 2b and 2c are respectively plotted lagging amplitude, flapping amplitude and center shift β_c for coupled response of Case II. Here, one can find a large shift of nonlinear behavior from the linear one particularly after the amplitude started increasing. At large amplitudes, the nonlinear resonance curves bend towards decreasing frequencies depicting a typical nonlinear softening spring type system. From Fig. 2a, one finds that for lagging amplitude, the higher frequency branch becomes almost flat in the overhang whereas lower frequency branch becomes unstable after maximum slope. Similar strong nonlinear characteristics are also visible for flapping motion in Fig. 2b. Further it can be seen in Fig. 2c that β_c the mean angle setting about which limit cycle oscillations take place increases with increasing amplitude. These curves show that small initial coning angle of the order of 9^o can produce an appreciable change in the nonlinear forced response of the blade.

In Figs. 3a to 3c, the results are plotted for Case III. This physically signifies a system in which lagging hinge stiffness is lower than that of flapping. Here, also, like Case II, one sees the softening spring characteristics for large amplitudes. In fact, the nonlinear response for case III appear to be more violent than Case II.

5.2 Parametric Resonance

The numerical results are obtained by solving Eq. (74) by computer, for the same configurations for which forced oscillation response results were obtained. These results are plotted in Figs. 4-6.

In Fig. 4, corresponding to Case I, results are plotted for linear and nonlinear solutions. Like forced response of Case I, (i.e. zero coning angle) one gets here also only uncoupled lagging motion. It is found that there is distinct unstable band for Ω/ω_ϕ of 2.52 to 2.64. The linear solution predicts infinite amplitude in this band whereas nonlinear solutions give two limited amplitude branches which try to bend toward each other in this region. However, for higher lagging amplitude (~1.4), the nonlinear solutions become unstable. The trivial solution of $\beta_c=a_1=b_1=\phi_c=a_2=b_2=0$ exists for every Ω/ω_ϕ except for this instability band where trivial solutions becomes unstable.

In Figs. 5a to 5c, results are plotted for Case II. With a small coning angle (~9^o), there is hardly any change in the linear response but there is distinct change in the nonlinear parametric response of the blade. Nonlinear response is coupled one with lagging motion more dominant as compared to flapping motion. Like forced oscillations of Case II, the nonlinear parametric resonance curves for large amplitudes bend towards decreasing frequencies, again, depicting a typical "Softening Springs" type system. The higher frequency branch for lagging motion, A_ϕ becomes flat and extends to lower frequency region and lower frequency branch becomes completely unstable. Again the trivial solution is stable except for Ω/ω_ϕ of 2.52 to 2.64.

In Figs. 6a to 6c, the results for Case III are plotted. The results are just similar to those plotted for Case in II in Figs. 5. Again one gets the coupled response with strong lagging motion.

5.3 Stability Analysis

The nonlinear solutions obtained by Harmonic Balance method for forced oscillations as well as parametric excitations are not always physically existent. One has to make stability check on these solutions to find out whether any of these can be a physical reality. So, these solutions are further investigated here for stability by giving small perturbations to the steady solutions and studying the growth rate of these disturbances under the assumption of slowly changing functions. The solution will be unstable if the growth rate of perturbations with time is positive. By slowly changing is meant here that the increase of a function during a period is small as compared to the average value for this period, i.e.,

$$\left| \frac{\mathring{a}_1}{a_1} 2\pi \right| << 1 , \quad \left| \frac{\mathring{\mathring{a}}_1}{\mathring{a}_1} 2\pi \right| << 1 \quad \text{etc} \tag{75}$$

See Bolotin [40] for details.

First, considering the stability of forced oscillations, one can write perturbed solution in the form

$$\begin{aligned} \beta &= \beta_{c0} + \hat{\beta}_c(\psi) + \left[a_{10} + \hat{a}_1(\psi)\right] \sin\psi + \left[b_{10} + \hat{b}_1(\psi)\right] \cos\psi \\ \phi &= \phi_{c0} + \hat{\phi}_c(\psi) + \left[a_{20} + \hat{a}_2(\psi)\right] \sin\psi + \left[b_{20} + \hat{b}_2(\psi)\right] \cos\psi \end{aligned} \tag{76}$$

where β_{c0}, a_{10}, b_{10}, ϕ_{c0}, a_{20}, b_{20} represent the steady-state solution for which stability check is being made, and $\hat{\beta}_c, \hat{a}_1, \hat{b}_1, \hat{\phi}_c, \hat{a}_2, \hat{b}_2$ are the time dependent perturbations given to respective steady components. To study the growth of these disturbances, the above equations are substituted in the basic flapping and lagging equations (Eqs. 33 and 34). Then retaining only linear terms in perturbations and their first order derivatives and subtracting the steady-state solution, once again on matching the constant, $\sin\psi$ and $\cos\psi$ terms from these two equations, one gets six linear algebraic equations which can be concisely put into matrix form as

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} & \cdot & \cdot & D_{16} \\ D_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_{31} & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_{41} & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_{51} & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_{61} & \cdot & \cdot & \cdot & \cdot & D_{66} \end{bmatrix} \begin{bmatrix} \hat{\beta}_c \\ \hat{a}_1 \\ \hat{b}_1 \\ \hat{\phi}_c \\ \hat{a}_2 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & \cdot & \cdot & E_{16} \\ E_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ E_{31} & \cdot & \cdot & \cdot & \cdot & \cdot \\ E_{41} & \cdot & \cdot & \cdot & \cdot & \cdot \\ E_{51} & \cdot & \cdot & \cdot & \cdot & \cdot \\ E_{61} & \cdot & \cdot & \cdot & \cdot & E_{66} \end{bmatrix} \begin{bmatrix} \mathring{\hat{\beta}}_c \\ \mathring{\hat{a}}_1 \\ \mathring{\hat{b}}_1 \\ \mathring{\hat{\phi}}_c \\ \mathring{\hat{a}}_2 \\ \mathring{\hat{b}}_2 \end{bmatrix} \tag{77}$$

Putting,

$$\begin{bmatrix} \hat{\beta}_c \\ \hat{a}_1 \\ \hat{b}_1 \\ \hat{\phi}_c \\ \hat{a}_2 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} \bar{\beta}_c \\ \bar{a}_1 \\ \bar{b}_1 \\ \bar{\phi}_c \\ \bar{a}_2 \\ \bar{b}_2 \end{bmatrix} e^{\bar{\lambda}\psi} \tag{78}$$

Substituting this into the above Eq. (77), results in an algebraic eigenvalue problem which can be easily solved using any standard eigenvalue subroutine. The solution will give six eigenvalues -- if any of the eigenvalues has positive real part, this means perturbation will grow with time and this marks the solution as an unstable one.

On similar lines, the stability of parametric excitation solutions are checked by putting,

$$\begin{aligned} \beta &= \beta_{co} + \hat{\beta}_c(\psi) + [a_{10} + \hat{a}_1(\psi)]\sin\tfrac{\psi}{2} + [b_{10} + \hat{b}_1(\psi)]\cos\tfrac{\psi}{2} \\ \phi &= \phi_{co} + \hat{\phi}_c(\psi) + [a_{20} + \hat{a}_2(\psi)]\sin\tfrac{\psi}{2} + [b_{20} + \hat{b}_2(\psi)]\cos\tfrac{\psi}{2} \end{aligned} \tag{79}$$

Following the same steps like Forcing Solution we get here also an algebraic eigenvalue matrices.

$$\begin{bmatrix} A_{11} & A_{12} & - & - & - & A_{16} \\ A_{21} & A_{22} & - & - & - & A_{26} \\ - & - & . & . & . & . \\ - & - & . & . & . & . \\ - & - & . & . & . & . \\ A_{61} & - & . & . & . & A_{66} \end{bmatrix} \begin{bmatrix} \bar{\beta}_c \\ \bar{a}_1 \\ \bar{b}_1 \\ \bar{\phi}_c \\ \bar{a}_2 \\ \bar{b} \end{bmatrix} = \bar{\lambda} \begin{bmatrix} B_{11} & B_{12} & - & - & - & B_{16} \\ B_{21} & B_{22} & - & - & - & B_{26} \\ - & - & . & . & . & . \\ - & - & . & . & . & . \\ - & - & . & . & . & . \\ B_{61} & - & . & . & . & B_{66} \end{bmatrix} \begin{bmatrix} \bar{\beta}_c \\ \bar{a}_1 \\ \bar{b}_1 \\ \bar{\phi}_c \\ \bar{a}_2 \\ \bar{b}_2 \end{bmatrix} \tag{80}$$

Again the nature of the roots $\bar{\lambda}$ will indicate whether the solution is stable or not.

PART B: EFFECT OF AERODYNAMIC FORCES

SECTION 6

FORMULATION OF AERODYNAMIC FORCES

The aerodynamic forces are obtained using quasi-steady airfoil theory. The elemental lift and drag forces acting on airfoil section can be written,

$$
\begin{aligned}
dL &= \tfrac{1}{2}\rho V_T^2\, a\, c\, dr\, (\theta_i - \alpha_i) \\
dD &= \tfrac{1}{2}\rho V_T^2\, a\, c\, dr \left(\frac{C_{d_0}}{a}\right)
\end{aligned}
\qquad (81)
$$

where

a = section lift curve slope, $\frac{dC_L}{d\alpha}$

C_{d_0} = blade profite drag coefficient

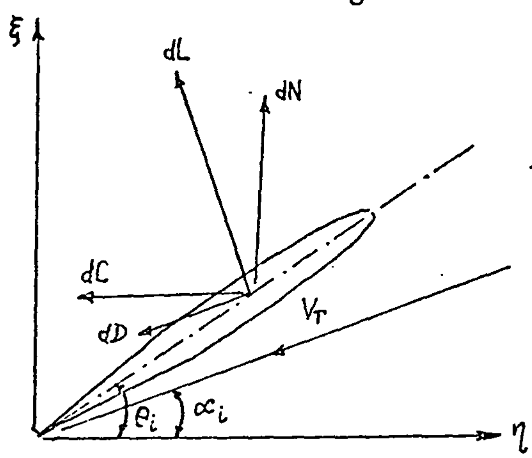
c = blade chord

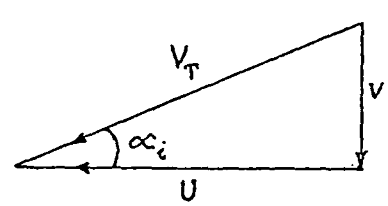
ρ = air density [r is same as ξ]

V_T = resultant velocity, $\sqrt{U^2+v^2}$

θ_i = local blade built-in incidence

α_i = inflow angle, $\tan^{-1}\frac{v}{U}$





As shown in the above figure, U,v are flow velocity components along blade axes η and ζ and $\theta_i - \alpha_i$ is the effective angle of incidence. Resolving the aerodynamic forces along the blade axes, one gets

$$
\begin{aligned}
dN &= dL\cos\alpha_i - dD\sin\alpha_i \\
dC &= dL\sin\alpha_i + dD\cos\alpha_i
\end{aligned}
\qquad (82)
$$

Assuming the effective angle of incidence is small (i.e., below stalling angle), such that

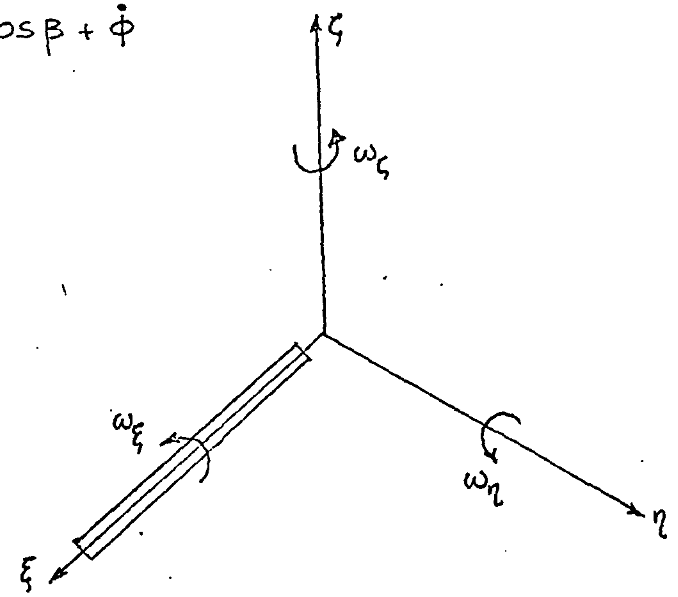
$$\theta_i - \alpha_i \approx \sin(\theta_i - \alpha_i)$$

and also considering $\frac{1}{2}\,\frac{Cd_o}{a}\,\frac{v^2}{U^2} << 1$, the forces dN and dC can be written as

$$dN \approx \tfrac{1}{2}\rho\, a\, c\, dr\left[U^2\sin\theta_i - vU\cos\theta_i - \frac{Cd_o}{a}vU\right]$$
$$dC \approx \tfrac{1}{2}\rho\, a\, c\, dr\left[U^2\frac{Cd_o}{a} + vU\sin\theta_i - v^2\left(\cos\theta_i - \tfrac{1}{2}\frac{Cd_o}{a}\right)\right] \tag{83}$$

From Eq. (10) of part A, the angular blade velocities about the three axes ξ, η, ζ are given as,

$$\omega_\xi = \Omega(\sin\theta\,\sin\phi + \cos\theta\,\sin\beta\,\cos\phi) - \dot{\beta}\sin\phi$$
$$\omega_\eta = \Omega(\sin\theta\,\cos\phi - \cos\theta\,\sin\beta\,\sin\phi) - \dot{\beta}\cos\phi \tag{84}$$
$$\omega_\zeta = \Omega\cos\theta\cos\beta + \dot{\phi}$$



The inflow velocity through the rotor is,

$$u_i = -\lambda\Omega R\,\vec{i_z} \tag{85}$$

which can be resolved along three blade axes as

$$u_i = u_{i\xi}\,\vec{i_\xi} + u_{i\eta}\,\vec{i_\eta} + u_{i\zeta}\,\vec{i_\zeta} \tag{85a}$$

where

$$u_{i\xi} = -\lambda\Omega R(\cos\theta \sin\beta \cos\phi + \sin\theta \sin\phi)$$

$$u_{i\eta} = -\lambda\Omega R(-\cos\theta \sin\beta \sin\phi + \sin\theta \cos\phi)$$

$$u_{i\zeta} = -\lambda\Omega R \cos\theta \cos\beta$$

In the above, λ is the inflow ratio parameter and is assumed here constant over the disk area for a uniform approaching wind stream.*

Introducing the blade hinge-offset effect, the velocity of the hinge-point is given as

$$u_h = e\Omega \vec{i}_{y_1} \qquad (86)$$

which can again be resolved along the three blade axes as,

$$u_h = u_{h\xi} \vec{i}_\xi + u_{h\eta} \vec{i}_\eta + u_{h\zeta} \vec{i}_\zeta \qquad (86a)$$

where

$$u_{h\xi} = e\Omega(-\sin\theta \sin\beta \cos\phi + \cos\theta \sin\phi)$$

$$u_{h\eta} = e\Omega(\sin\theta \sin\beta \sin\phi + \cos\theta \cos\phi)$$

$$u_{h\zeta} = -e\Omega \sin\theta \cos\beta.$$

The resultant flow velocities U and v for any point ξ on the blade is then,

$$\begin{aligned} U &= r\omega_\zeta - u_{i\eta} + u_{h\eta} \\ &= r(\Omega\cos\theta\cos\beta + \dot{\phi}) + \lambda\Omega R(-\cos\theta\sin\beta\sin\phi + \sin\theta\cos\phi) \\ &\quad + e\Omega(\sin\theta\sin\beta\sin\phi + \cos\theta\cos\phi) \end{aligned} \qquad (87)$$

$$\begin{aligned} V &= -r\omega_\eta - u_{i\zeta} + u_{h\zeta} \\ &= \Omega r(\cos\theta\sin\beta\sin\phi - \sin\theta\cos\phi) + r\dot{\beta}\cos\phi + \lambda\Omega R\cos\theta\cos\beta \\ &\quad - e\Omega\sin\theta\cos\beta. \end{aligned} \qquad (88)$$

*This is strictly true only for an ideally twisted rotor. For other rotors, there may be some variation with r.

Like part A (Eq. 27), expanding the trignometric terms to third order in above equations we get

$$U = \Omega r\left(1 - \frac{\theta^2}{2} - \frac{\beta^2}{2}\right) + r\dot{\phi} + \lambda\Omega R\left(\theta - \beta\phi - \frac{\theta^3}{6} - \frac{\theta\phi^2}{2}\right) + e\Omega\left(1 + \theta\beta\phi - \frac{\theta^2}{2} - \frac{\phi^2}{2}\right)$$

$$V = -\Omega r\left(\theta - \frac{\theta^3}{6} - \frac{\theta\phi^2}{2} - \beta\phi\right) + r\dot{\beta}\left(1 - \frac{\phi^2}{2}\right) + \lambda\Omega R\left(1 - \frac{\theta^2}{2} - \frac{\beta^2}{2}\right) - e\Omega\left(\theta - \frac{\theta^3}{6} - \frac{\theta\beta^2}{2}\right) \qquad (89)$$

With the inclusion of aerodynamic forces, the generalized forces Q_β and Q_ϕ in Lagrange's equations (Eq. 17) have the additional contributions,

$$(Q_\beta)_{Aero.} = \cos\phi \int_{span} r\,dN$$

$$(Q_\phi)_{Aero.} = -\int_{span} r\,dC \qquad (90)$$

One may assume the built-in incidence along the blade span is

$$\theta_i = \theta_1 + \frac{R+e}{r+e}\,\theta_2 \qquad (91)$$

where θ_1 is constant incidence along blade length and θ_2 alone represents the ideal twist distribution.

After performing various integrations and combining with Eqs. 33 and 34 we get the nonlinear equations of motion for a flapping-lagging rigid rotor with flexible springs at the hinge-point in the presence of airflow as

<u>Flapping Equation</u>:

$$\overset{\circ\circ}{\beta}(1-\phi^2) - 2\mathring{\beta}\mathring{\phi}\phi + 2\mathring{\phi}(\beta + \theta\phi) + \beta(1 + \bar{e} + \nu_\beta^2) + \theta\phi - \frac{2}{3}\beta^2 - \beta\theta^2$$

$$-\beta\phi^2 + 2\zeta_\beta\nu_\beta\mathring{\beta} - \nu_\beta^2\beta_s + G\left(-\theta + \frac{\theta^3}{6} + \frac{\theta\beta^2}{2} + \frac{\theta\phi^2}{2}\right)\sin\psi + G\left(\beta - \frac{\beta^3}{6} - \frac{\beta\phi^2}{2}\right)\cos\psi$$

$$= \frac{\gamma}{8}\Big[C_0 + \beta^2 C_5 + \phi^2 C_6 + \beta\phi C_7 + \mathring{\beta} C_8 + \mathring{\phi} C_9 + \mathring{\beta}\beta\phi\, C_{10} + \mathring{\beta}\beta^2 C_{11}$$

$$\mathring{\beta}\phi^2 C_{12} + \mathring{\phi}\beta\phi\, C_{13} + \mathring{\phi}\beta^2 C_{14} + \mathring{\phi}\phi^2 C_{15} - \mathring{\beta}\mathring{\phi}\, C\theta_1 + \mathring{\phi}^2 C_1\Big] \qquad (92)$$

Lagging Equation:

$$
\begin{aligned}
&\mathring{\mathring{\phi}} + \mathring{\beta}^2\phi - 2\mathring{\beta}(\beta + \theta\phi) + \phi(\bar{e} + \nu_\phi^2) + \theta\beta + \theta^2\phi - \beta^2\phi + 2\zeta_\phi \nu_\phi \mathring{\phi} \\
&- \nu_\phi^2 \phi_s + G\left(1 - \frac{\theta^2}{2} - \frac{\phi^2}{2} + \theta\beta\phi\right)\sin\psi + G\left(\phi - \frac{\phi^3}{6} - \frac{\theta\beta^2}{2}\right)\cos\psi \\
&= -\frac{\gamma}{8}\Big[B_0 + \beta^2 B_5 + \phi^2 B_6 + \beta\phi B_7 + \mathring{\beta} B_8 + \mathring{\phi} B_9 + \mathring{\beta}\beta\phi B_{10} + \mathring{\beta}\beta^2 B_{11} \\
&+ \mathring{\beta}\phi^2 B_{12} + \mathring{\phi}\beta\phi B_{13} + \mathring{\phi}\beta^2 B_{14} + \mathring{\phi}\phi^2 B_{15} + \mathring{\beta}\mathring{\phi} C_1 + \mathring{\phi}^2 \frac{C_{d_0}}{a} - \mathring{\beta}^2 BC_1\Big]
\end{aligned}
\qquad (93)
$$

In above equations, $\gamma = \rho acR^4/I_\eta$ is the Lock number and, ζ_β and ζ_ϕ represent structural damping coefficients for flapping and lagging motions, respectively. The other constants are defined in Appendix I.

SECTION 7

NONLINEAR RESPONSE OF FLAPPING-LAGGING ROTOR WITH AERODYNAMIC FORCES

Seeking a general limit cycle solution of nonlinear Eqs. 92 and 93 for both forced oscillations as well as parametric resonance in the form of

$$\begin{aligned} \beta &\approx \beta_c + a_1 \sin \alpha\psi + b_1 \cos \alpha\psi \\ \phi &\approx \phi_c + a_2 \sin \alpha\psi + b_2 \cos \alpha\psi \end{aligned} \tag{94}$$

where α is the ratio of response frequency to forcing frequency. Here $\alpha=1$ represents forced oscillations and $\alpha = \frac{1}{2}$ gives parametric resonance. Substituting these expressions (94) into Eqs. (92) and (93) and balancing out the constant, sin $\alpha\psi$ and cos $\alpha\psi$ terms of each equation and neglecting higher harmonics, gives six nonlinear algebraic equations,

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} \\ R_{21} & R_{22} & R_{23} & - & - & - \\ R_{31} & R_{32} & - & - & - & - \\ R_{41} & - & - & - & - & - \\ R_{51} & - & - & - & - & - \\ R_{61} & R_{62} & - & - & - & R_{66} \end{bmatrix} \begin{bmatrix} \beta_c \\ a_1 \\ b_1 \\ \phi_c \\ a_2 \\ b_2 \end{bmatrix} + \begin{bmatrix} R_{10} \\ R_{20} \\ R_{30} \\ R_{40} \\ R_{50} \\ R_{60} \end{bmatrix} = 0 \tag{95}$$

The various terms in the above matrices are defined in Appendix II.

The solution of these nonlinear algebraic equations (95) can be again obtained numerically by Newton-Raphson technique.

Stability Check of Solution

As discussed in Part A, the nonlinear solution got by Harmonic Balance method has to be checked for stability to prove its well-posedness. Once again, here the solutions are checked for stability by giving small perturbations to these steady solutions and studying the growth rate of these perturbations with time under the assumptions of slowly changing functions. (See Section 5.3 for more details.)

Writing the perturbed solution as

$$\beta = \beta_{c0} + \hat{\beta}_c(\psi) + [a_{10} + \hat{a}_1(\psi)]\sin\alpha\psi + [b_{10} + \hat{b}_1(\psi)]\cos\alpha\psi$$

$$\phi = \phi_{c0} + \hat{\phi}_c(\psi) + [a_{20} + \hat{a}_2(\psi)]\sin\alpha\psi + [b_{20} + \hat{b}_2(\psi)]\cos\alpha\psi \tag{96}$$

where β_{c0}, a_{10}, b_{10}, ϕ_{c0}, a_{20}, b_{20} is steady solution for which stability check is being made and $\hat{\beta}_c$, $\hat{a}_1$, $\hat{b}_1$, $\hat{\phi}_c$, $\hat{a}_2$, $\hat{b}_2$ are the time dependent perturbations given to respective steady solution components.

These equations (96) are put into basic nonlinear equations of motion (92) and (93). Retaining only the linear terms in perturbations and their first order derivatives and filtering out the steady solution, then again on matching the constant, sin $\alpha\psi$ and cos $\alpha\psi$, one gets six linear algebraic equations.

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & - & - & - \\ S_{31} & - & - & - & - & - \\ S_{41} & - & - & - & - & - \\ S_{51} & - & - & - & - & - \\ S_{61} & - & - & - & - & S_{66} \end{bmatrix} \begin{bmatrix} \hat{\beta}_c \\ \hat{a}_1 \\ \hat{b}_1 \\ \hat{\phi}_c \\ \hat{a}_2 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & - & - & - \\ T_{31} & - & - & - & - & - \\ T_{41} & - & - & - & - & - \\ T_{51} & - & - & - & - & - \\ T_{61} & - & - & - & - & T_{66} \end{bmatrix} \begin{bmatrix} \mathring{\hat{\beta}}_c \\ \mathring{\hat{a}}_1 \\ \mathring{\hat{b}}_1 \\ \mathring{\hat{\phi}}_c \\ \mathring{\hat{a}}_2 \\ \mathring{\hat{b}}_2 \end{bmatrix} \tag{97}$$

writing

$$\begin{bmatrix} \hat{\beta}_c \\ \hat{a}_1 \\ \hat{b}_1 \\ \hat{\phi}_c \\ \hat{a}_2 \\ \hat{b}_2 \end{bmatrix} = \begin{bmatrix} \bar{\beta}_c \\ \bar{a}_1 \\ \bar{b}_1 \\ \bar{\phi}_c \\ \bar{a}_2 \\ \bar{b}_2 \end{bmatrix} e^{\bar{\lambda}\psi} \tag{98}$$

substituting (98) into (97) results into an algebraic eigenvalue problem which can be solved by using one of the standard subroutines. Solution will give six eigenvalues $\bar{\lambda}$ and the nature of the eigenvalues will decide whether solution is stable or not. If any of the these eigenvalues has positive real part means perturbation will grow with time and makes the solution unstable.

Numerical Results

Numerical calculations for forced oscillations as well as parametric resonance with aerodynamic forces are repeated for most of the configurations of Part A. Once again

Case I.	$\beta_s = 0$	$\phi_s = 0$	$\bar{\omega}_\beta = .71$	$\phi_{DW} = .088$	$\theta = 0$	$\bar{e} = .1$
Case II.	$\beta_s = .15$	$\phi_s = 0$	$\bar{\omega}_\beta = .71$	$\phi_{DW} = .088$	$\theta = 0$	$\bar{e} = .1$
Case III.	$\beta_s = .15$	$\phi_s = 0$	$\bar{\omega}_\beta = 1.4$	$\phi_{DW} = .088$	$\theta = 0$	$\bar{e} = .1$

For aerodynamic characteristics, Lock number γ of 12 is used for most of the results and the values of 'a' and C_{d_o} are taken respectively of 6.0 and .012.

7.1 Forced Oscillations Results ($\alpha=1$)

In Fig. 7 results for Case I of zero initial coning angle with aerodynamic forces are plotted. Here $\lambda=0$ represents a configuration with zero inflow velocity. It can be seen that for this case of no inflow, response is uncoupled and one gets only lagging amplitude, which is quite similar to that of no aerodynamic case (Fig. 1). Here also the amplitude shoots up near resonance ($\frac{\Omega}{\omega_\phi} \sim 1.05$). However, with the inclusion of small inflow ratio of $\lambda \triangleq .1$, the amplitude is well bounded. Also, with inflow, response is coupled, though the amplitude of flapping motion is comparatively much smaller except near resonance (Fig. 7b). Further, in Fig. 7b, β_c the mean angle about which limit cycle oscillations take place, is plotted and β_c has negative value because of thrust direction for this positive inflow.

In Fig. 8, the lagging amplitude response for Case II with aerodynamic forces is plotted. Here positive λ represents configuration with initial coning angle facing into flow direction and negative λ the case of initial coning angle facing along the flow direction. For $\lambda=0$, the response amplitude is entirely different from that of no aerodynamic case, Fig. 2a. With the presence of aerodynamic forces, the instability overhang region completely vanishes. Though the amplitude increases near $\frac{\Omega}{\omega_\phi} \approx 1.05$, but it is bounded with flat peak. The peak value is higher for negative λ as compared to positive λ. This can be explained with the help of Fig. 7b that with positive λ, the coupling is reduced by decreased β_c and for negative λ the coupling is amplified by increased β_c. It is found that the flapping amplitude response though smaller in comparison to lagging amplitude, the peak for negative λ is quite higher than that for positive λ (not shown in figure). Further, it is seen that the solution for $\lambda = -0.1$ becomes unstable in the peak value region.

In Fig. 9, the results for Case III are plotted. For this configuration of softer lagging hinge stiffness than flapping stiffness, the behavior is quite similar to that of Case II in eliminating the violent overhang instability region with the presence of aerodynamic forces. Here the peaks for lagging amplitude are comparatively sharp but are of nearly same magnitude for the three different cases of $\lambda=-.1$, 0, .1. This is mainly because of higher flapping stiffness, the average coning angle β_c is not too much effected by the small value of inflow, $\lambda=\pm.1$.

Figure 10 illustrates the lagging response for a typical configuration of Case II with $\lambda=-0.1$ for different Lock numbers, γ. Here, one can see that by reducing Lock number, the behavior of forcing response tends towards that of no aerodynamic case (Fig. 2). For Lock number of 3, there is little overhang towards left depicting a typical nonlinear softening effect. Again, the higher frequency branch remains stable in overhang region whereas lower frequency branch becomes unstable.

In Fig. 11, the effect of structural damping on forced oscillations with no aerodynamic forces for Case II is presented. For small value of damping coefficients $\zeta_\beta=\zeta_\phi=.01$ encloses the stability branches (see for comparison Fig. 2a). The behavior is quite similar to that shown in Fig. 10 for low Lock number. This means that the aerodynamic forces can be looked as if they add the equivalent structural damping to the system depending upon Lock number γ and inflow λ.

7.2 Parametric Resonance Results ($\alpha=\frac{1}{2}$)

In Fig. 12, the results for Parametric Resonance of Case I in the presence of aerodynamic forces are plotted. There is a distinct band of instability for zero inflow and the response is uncoupled lagging motion. From stability analysis one can see that the trivial solution of $\beta_c=a_1=b_1=\phi_c=a_2=b_2=0$ is stable for every $\frac{\Omega}{\omega_\phi}$ except in the instability band where trivial solution becomes unstable. This behavior is quite similar to that for no aerodynamic forces (Fig. 4). The two instability branches bend towards each other and the higher frequency branch also becomes unstable for large lagging amplitude ($A_\phi \simeq .75$). However, with the inclusion of small inflow $\lambda=.05$, this instability band completely vanishes.

Figure 13 presents the response for Case II. With the inclusion of aerodynamic forces with no inflow, one can see that the behavior of rotor is quite different from that of no aerodynamic solution (Fig. 5). The overhang instability region gets completely eliminated and the two stable solution branches joins in the instability band resulting in the limit cycle amplitude. The response is coupled but the flapping amplitude is of an order of magnitude lower than the lagging amplitude. Again with the inclusion of small inflow, the instability band vanishes.

Figure 14 shows the effect of Lock number on Parametric response of Case II with $\lambda=-0.1$. For Lock number $\gamma=12$, there is no instability region, but for $\gamma=6$ and 3 there is instability region with overhang towards left examplifying a typical nonlinear softening effect. Out of the two branches of solution, the higher frequency branch is stable while the other branch is unstable from the very initiation. The other noticeable thing about these graphs is that with the increase of Lock number, not only the instability overhang and peak amplitude reduces but also the instability band shifts towards the right.

Figure 15 illustrates the effect of structural damping on parametric oscillations of Case II with no aerodynamic forces. Here the solution is not very different from that of no-damping case, Fig. 5a, except that the two branches of solutions shrinks in the overhang region.

SECTION 8

SELF-EXCITED FLUTTER RESPONSE OF FLAPPING-LAGGING ROTOR

In the earlier sections it has been shown that a rigid rotor restrained by two springs at hinge point representing flapping and lagging motions can get into large nonlinear response due to gravity forcing excitation. Here, it is intended to discuss that this torsionally rigid rotor, in the absence of gravity forces can also lead to self-sustained oscillations caused by the interaction of aerodynamic forces with structural vibrations of blade.

The equations of motion for flapping-lagging rotor for flutter can be rewritten from Eqs. (92) and (93) by eliminating gravitational effects.

Flapping:

$$\overset{\circ\circ}{\beta}(1-\phi^2) - 2\mathring{\beta}\mathring{\phi}\phi + 2\mathring{\phi}(\beta+\theta\phi) + \beta(1+\bar{e}+\nu_\beta^2) + \theta\phi$$
$$-\frac{2}{3}\beta^3 - \beta\theta^2 - \beta\phi^2 + 2\zeta_\beta\nu_\beta\mathring{\beta} - \nu_\beta^2\beta_s$$
$$= \frac{\gamma}{8}\Big[C_0 + \beta^2 C_5 + \phi^2 C_6 + \beta\phi C_7 + \mathring{\beta} C_8 + \mathring{\phi} C_9 + \mathring{\beta}\beta\phi C_{10} + \mathring{\beta}\beta^2 C_{11}$$
$$+ \mathring{\beta}\phi^2 C_{12} + \mathring{\phi}\beta\phi C_{13} + \mathring{\phi}\beta^2 C_{14} + \mathring{\phi}\phi^2 C_{15} - \mathring{\beta}\mathring{\phi} C B_1 + \mathring{\phi}^2 C_1\Big] \tag{99}$$

Lagging:

$$\overset{\circ\circ}{\phi} + \mathring{\beta}^2\phi - 2\mathring{\beta}(\beta+\theta\phi) + \phi(\bar{e}+\nu_\phi^2) + \theta\beta + \theta^2\phi - \beta^2\phi$$
$$+ 2\zeta_\phi\nu_\phi\mathring{\phi} - \nu_\phi^2\phi_s$$
$$= -\frac{\gamma}{8}\Big[B_0 + \beta^2 B_5 + \phi^2 B_6 + \beta\phi B_7 + \mathring{\beta} B_8 + \mathring{\phi} B_9 + \mathring{\beta}\beta\phi B_{10} + \mathring{\beta}\beta^2 B_{11}$$
$$+ \mathring{\beta}\phi^2 B_{12} + \mathring{\phi}\beta\phi B_{13} + \mathring{\phi}\beta^2 B_{14} + \mathring{\phi}\phi^2 B_{15} + \mathring{\beta}\mathring{\phi} C_1 + \mathring{\phi}^2\frac{C_{d_0}}{a} - \mathring{\beta}^2 B C_1\Big] \tag{100}$$

8.1 Simple Linear Solution

It is sometimes advantageous to deal with simple solutions to get the feel of the nature of phenomena and then later on using these simple solutions as a guide one can make more rigorous analysis. Here also first simple linear solution for self-sustained oscillation is worked out by assuming that the small amplitude motion is taking place about possible large static positions β_0 and ϕ_0.

$$\beta = \beta_0 + \tilde{\beta}$$
$$\phi = \phi_0 + \tilde{\phi} \tag{101}$$

where $\tilde{\beta}$ and $\tilde{\phi}$ represents small perturbations. Placing (101) into Eqs. (99) and (100) and keeping only linear terms in perturbations and their derivatives and subtracting the static solution results in

$\tilde{\beta}$ Equation:

$$\begin{aligned}&\overset{\circ\circ}{\tilde{\beta}}(1-\phi_0^2) + 2\overset{\circ}{\tilde{\phi}}(\beta_0+\theta\phi_0) + 2\zeta_\beta \nu_\beta \overset{\circ}{\tilde{\beta}} + \tilde{\beta}(1+\bar{e}+\nu_\beta^2-\theta^2-2\beta_0^2-\phi_0) \\ &+\tilde{\phi}(\theta-2\beta_0\phi_0) = \frac{\gamma}{8}\Big[\tilde{\beta}(2\beta_0 C_5+\phi_0 C_7) + \tilde{\phi}(2\phi_0 C_6+\beta_0 C_7) \\ &\overset{\circ}{\tilde{\beta}}(C_8+\beta_0\phi_0 C_{10}+\beta_0^2 C_{11}+\phi_0^2 C_{12}) + \overset{\circ}{\tilde{\phi}}(C_9+\beta_0\phi_0 C_{13}+\beta_0^2 C_{14}+\phi_0^2 C_{15})\Big]\end{aligned} \tag{102}$$

$\tilde{\phi}$ Equation:

$$\begin{aligned}&\overset{\circ\circ}{\tilde{\phi}} + 2\zeta_\phi \nu_\phi \overset{\circ}{\tilde{\phi}} - 2\overset{\circ}{\tilde{\beta}}(\beta_0+\theta\phi_0) + \tilde{\beta}(\theta-2\beta_0\phi_0) + \tilde{\phi}(\bar{e}+\nu_\phi^2+\theta^2-\beta_0^2) \\ &= -\frac{\gamma}{8}\Big[\tilde{\beta}(2\beta_0 B_5+\phi_0 B_7) + \tilde{\phi}(2\phi_0 B_6+\beta_0 B_7) + \overset{\circ}{\tilde{\beta}}(B_8+\beta_0\phi_0 B_{10}+\beta_0^2 B_{11}+\phi_0^2 B_{12}) \\ &\quad + \overset{\circ}{\tilde{\phi}}(B_9+\beta_0\phi_0 B_{13}+\beta_0^2 B_{14}+\phi_0^2 B_{15})\Big]\end{aligned} \tag{103}$$

and static solution equations are

$$\begin{aligned}&\beta_0(1+\bar{e}+\nu_\beta^2-\theta^2-\tfrac{2}{3}\beta_0^2-\phi_0^2) + \phi_0\theta = \nu_\beta^2\beta_S + \frac{\gamma}{8}(C_0+\beta_0^2 C_5+\phi_0^2 C_6+\beta_0\phi_0 C_7) \\ &\beta_0\theta + \phi_0(\bar{e}+\nu_\phi^2+\theta^2-\beta_0^2) = \nu_\phi^2\phi_S - \frac{\gamma}{8}(B_0+\beta_0^2 B_5+\phi_0^2 B_6+\beta_0\phi_0 B_7)\end{aligned} \tag{104}$$

These equations are further simplified by considering a rotor with ideal twist distribution (only θ_2) and with no hinge offset (e=0) and also assuming that the static equilibrium angles β_0<1 and ϕ_0<<1 so that β_0^2 and ϕ_0 can be neglected. Equations 102-104 reduces to

$\tilde{\beta}$ Equation:

$$\overset{\circ\circ}{\tilde{\beta}} + \overset{\circ}{\tilde{\beta}}\left(\frac{\gamma}{8} + 2\zeta_\beta \nu_\beta\right) + \overset{\circ}{\tilde{\phi}}\left(2\beta_0 - \frac{\gamma\theta_2}{3} + \frac{\gamma\lambda}{6}\right) + \tilde{\beta}(1+\nu_\beta^2) + \tilde{\phi}\left(\frac{\gamma\beta_0}{8}\right) = 0 \tag{105}$$

$\tilde{\phi}$ Equation:

$$\overset{\circ\circ}{\tilde{\phi}} + \overset{\circ}{\tilde{\beta}}\left(-2\beta_0 + \frac{\gamma\theta_2}{6} - \frac{\gamma\lambda}{3}\right) + \overset{\circ}{\tilde{\phi}}\left[\frac{\gamma}{8}\left(\frac{2C_{d_0}}{a} + 2\theta_2\lambda\right) + 2\zeta_\phi \nu_\phi\right] + \left(\nu_\phi^2 - \frac{\gamma\lambda}{3}\beta_0\right)\tilde{\phi} = 0 \tag{106}$$

and static equilibrium equation

$$\beta_0(1+\nu_\beta^2) = \nu_\beta^2 \beta_s + \frac{\gamma}{6}(\theta_2 - \lambda) \tag{107}$$

For the critical condition of flutter

$$\begin{Bmatrix} \tilde{\beta} \\ \tilde{\phi} \end{Bmatrix} = e^{i\omega t} \begin{Bmatrix} \bar{\beta} \\ \bar{\phi} \end{Bmatrix} \tag{108}$$

substituting in Eqs. (105) and (106), gives characteristic equation

$$\begin{bmatrix} (i\omega)^2 + i\omega\, m_{\dot{\beta}} + m_\beta & i\omega\, m_{\dot{\phi}} + m_\phi \\ i\omega\, n_{\dot{\beta}} & (i\omega)^2 + i\omega\, n_{\dot{\phi}} + n_\phi \end{bmatrix} \begin{Bmatrix} \bar{\beta} \\ \bar{\phi} \end{Bmatrix} = 0 \tag{109}$$

For non-trivial solution, expanding the determinate and comparing the real and imaginary parts separately to zero, one gets

Real part:

$$\omega^4 - \omega^2\left(m_\beta + n_\phi + m_{\dot{\beta}}\, n_{\dot{\phi}} - m_{\dot{\phi}}\, n_{\dot{\beta}}\right) + m_\beta\, n_\phi = 0 \tag{110}$$

Imaginary part:

$$\omega^2\left(n_{\dot{\phi}} + m_{\dot{\beta}}\right) - \left(m_{\dot{\beta}}\, n_\phi + m_\beta\, n_{\dot{\phi}} - m_\phi\, n_{\dot{\beta}}\right) = 0 \tag{111}$$

where

$$m_{\dot\beta} = \frac{\gamma}{8} + 2\nu_\beta \zeta_\beta \qquad m_\beta = 1 + \nu_\beta^2$$

$$m_{\dot\phi} = 2\beta_0 - \frac{\gamma\theta_2}{3} + \frac{\gamma\lambda}{6} \qquad m_\phi = \frac{\gamma}{8}\beta_0$$

$$n_{\dot\beta} = -2\beta_0 + \frac{\gamma\theta_2}{6} - \frac{\gamma\lambda}{3} \qquad n_\phi = \nu_\phi^2 - \frac{\gamma\lambda}{3}\beta_0$$

$$n_{\dot\phi} = \frac{\gamma}{8}\left(2\frac{C_{d_0}}{a} + 2\theta_2\lambda\right) + 2\nu_\phi\zeta_\phi$$

$$\beta_0 = \frac{1}{1+\nu_\beta^2}\left[\frac{\gamma}{6}(\theta_2 - \lambda) + \nu_\beta^2 \beta_s\right]$$

Equations 110 and 111 can be solved for flutter boundary by simple iterating scheme.

8.2 Nonlinear Solution

Assuming the limit cycle solution of nonlinear Equations (99) and (100) for self-sustained oscillations of the form

$$\beta \approx \beta_c + a_1 \sin\alpha\psi + b_1 \cos\alpha\psi$$
$$\phi \approx \phi_c + a_2 \sin\alpha\psi + b_2 \cos\alpha\psi \qquad (112)$$

where α represents the ratio of flutter frequency to rotational frequency.

Placing these expressions (112) into Eqs. (99) and (100) and matching the constant, sin $\alpha\psi$ and cos $\alpha\psi$ terms of each equations and neglecting the higher harmonics yields six nonlinear algebraic equations. Since we are interested in the limit cycle flutter solution, this can be done by assuming a known lagging amplitude of flutter and finding the other associated unknowns to get this condition. Thus putting $b_2=0$, and a_2 which now represents lagging amplitude, a known quantity, into these nonlinear algebraic equations,

$$
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\
A_{21} & A_{22} & - & - & - & A_{26} \\
A_{31} & - & - & - & - & - \\
A_{41} & - & - & - & - & - \\
A_{51} & - & - & - & - & - \\
A_{61} & - & - & - & - & A_{66}
\end{bmatrix}
\begin{bmatrix}
\beta_c \\ a_1 \\ b_1 \\ \phi_c \\ Z \\ \alpha
\end{bmatrix}
+
\begin{bmatrix}
A_{10} \\ A_{20} \\ A_{30} \\ A_{40} \\ A_{50} \\ A_{60}
\end{bmatrix}
= 0 \tag{113}
$$

where $Z = (\omega_\phi/\Omega)^2$ and the other terms in the above matrices are defined in Appendix III.

The solution of these nonlinear Equations (113) are obtained numerically by iterative Newton-Raphson technique.

8.3 Stability Check of Solution

The nonlinear solution got above using Harmonic Balance method is checked for stability again by giving small perturbations to the steady solution

$$
\begin{aligned}
\beta &= \beta_{c0} + \hat{\beta}_c(\psi) + [a_{10} + \hat{a}_1(\psi)]\sin\alpha\psi + [b_{10} + \hat{b}_1(\psi)]\cos\alpha\psi \\
\phi &= \phi_{c0} + \hat{\phi}_c(\psi) + [a_{20} + \hat{a}_2(\psi)]\sin\alpha\psi + [\hat{b}_2(\psi)]\cos\alpha\psi
\end{aligned} \tag{114}
$$

where β_{c0}, a_1, b_1, ϕ_{c0}, a_2 and $b_2=0$ with corresponding Z and α is the steady solution and $\hat{\beta}_c$, $\hat{a}_1$, $\hat{b}_1$, $\hat{\phi}_c$, $\hat{a}_2$, $\hat{b}_2$ with same Z and α are the time dependent perturbations.

Following the same steps like forcing response, placing (114) into Eqs. (99) and (100), keeping only linear terms in perturbations and subtracting the steady solution, once again on matching the constant, $\sin\alpha\psi$ and $\cos\alpha\psi$ arranging appropriately one gets in the form of algebraic eigenvalue

$$
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{21} & S_{22} & S_{23} & - & - & - \\
S_{31} & S_{32} & - & - & - & - \\
S_{41} & - & - & - & - & - \\
S_{51} & - & - & - & - & - \\
S_{61} & - & - & - & - & S_{66}
\end{bmatrix}
\begin{bmatrix}
\bar{\beta}_c \\ \bar{a}_1 \\ \bar{b}_1 \\ \bar{\phi}_c \\ a_2 \\ b_2
\end{bmatrix}
= \bar{\lambda}
\begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\
T_{21} & T_{22} & - & - & - & - \\
T_{31} & - & - & - & - & - \\
T_{41} & - & - & - & - & - \\
T_{51} & - & - & - & - & - \\
T_{61} & - & - & - & - & T_{66}
\end{bmatrix}
\begin{bmatrix}
\bar{\beta}_c \\ \bar{a}_1 \\ \bar{b}_1 \\ \bar{\phi}_c \\ \bar{a}_2 \\ \bar{b}_2
\end{bmatrix}
\tag{115}
$$

This is just the same as Eq. (97). Again, the nature of roots $\bar{\lambda}$ will decide whether the solution is stable or not. If any one of the roots has positive real part, points out that the solution is unstable.

8.4 Numerical Results

For most of the flutter calculations, Lock number of 12 is used and values of a and C_{d_o} are taken respectively of 6.0 and .012. It is also assumed here for flutter calculations that initial lagging angle and initial built-in incidence are zero. $\lambda=0.1$ is taken for most of the results as this inflow represents nearly the maximum power extraction condition of the rotor. The critical flutter boundary is obtained from nonlinear analysis for very small lagging amplitude of .05.

Figure 16 presents the critical flutter boundary for centrally hinged blade with no preconing (i.e. $\bar{e}=0$. $\beta_s=0$) for different inflow λ. Both, simple linear solution and more accurate Harmonic Balance solution are plotted for comparison. The region of instability lies inside the respective contours. It can be seen that simple solution gives reasonable estimate of flutter boundary for higher values of lagging stiffness, ν_ϕ, and it starts deviating more and more at low ν_ϕ. This is, mainly because in simple solution, ϕ_0, the equilibrium lagging angle is neglected and it becomes appreciable at low values of ν_ϕ. One can also find that the instability envelop increases in size with increasing inflow ratio λ. This can be explained by looking at linear Eqs. (105) and (106) that λ effects the potential destabilizing flap-lag coupling terms.

Figure 17 shows the effect of preconing and hinge offset on the flutter solution of rotor with a typical inflow λ of 0.1. On comparison with the corresponding instability graph for $\lambda=0.1$ from Fig. 16, one can easily visualize that with the hinge offset the instability envelop not only expands in size but also opens up at lower end from the elliptic shape. Preconing also has an important effect on the flutter boundary - positive preconing, coning facing flow direction shrinks instability envelop whereas negative preconing expands the instability envelop. This is because the blade preconing effects blade equilibrium coning angle and thereby effect mainly the cross coupling terms.

Figure 18 illustrates the effect of Lock number γ on the stability envelop for rotor with no preconing and no hinge offset. Unlike the forcing response, decreasing Lock number reduces the instability region. It can also be seen that the solution becomes unstable on the part of the instability envelop for Lock numbers of 6 and 3.

Figure 19 presents the effect of structural damping of the flutter boundary. With the inclusion of small structural damping coefficients of $\zeta_\beta=\zeta_\phi=.005$ for centrally hinged rotor with no preconing, the instability region is reduced drastically. Also for $\zeta_\beta=\zeta_\phi=.01$, this rotor becomes completely stable. This can be seen from simple linear solution (Eq. (106) that the structural damping coefficient has a very strong effect on the comparatively low direct damping term for lagging motion.

In Fig. 20, the penetration lines of increasing rotational speed Ω into instability envelop for different $\bar{\omega}_\beta$ are plotted. Figure 21a represents the limit cycle flutter solution for $\bar{\omega}_\beta=0.8$. Points A and B here correspond to the boundary points A and B on instability envelop (Fig. 20). One gets large fluttering oscillations at the very initiation of flutter and then the amplitude reduces with increasing rotational speed. The solution becomes unstable for part of the speed range. It can also be seen that the flapping response amplitude is lower than the lagging amplitude mainly because of, comparatively high aerodynamic damping in flapping motion. In Fig. 21b, the limit cycle lagging amplitudes are plotted for different $\bar{\omega}_\beta$ at the lower flutter speed side. The solution shows nonlinear stiffening effects for $\bar{\omega}_\beta$ of .6 and .8 and softening effects for higher $\bar{\omega}_\beta$ of 1.0, 1.2 and 1.4.

SECTION 9

EFFECT OF VELOCITY GRADIENT

In the earlier analysis, it is assumed that the incoming wind velocity is uniform and perpendicular to the plane of rotation. In reality, wind is not uniform but has velocity gradient because of earth's boundary layer. Generally, the velocity profile near the surface of the earth is approximated by a power law relation [8,13]

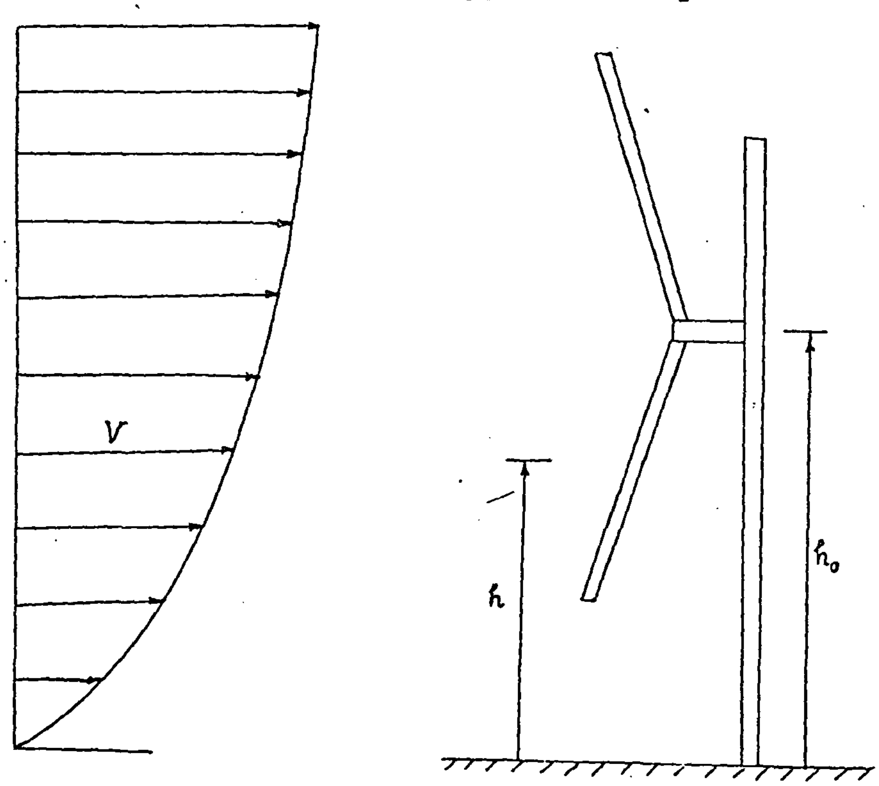
$$\frac{V}{V_o} = \left(\frac{h}{H_o}\right)^p \tag{116}$$

where

V = wind velocity at height h

V_o = wind velocity at height H_o

p = constant quantity depends on topographical conditions of the place, between .167 to .40 approximately.



Putting the above relation for the blade spanwise position r with azimuthal angle ψ

$$\frac{V}{V_o} = \left(\frac{h_o - h\cos\psi}{H_o}\right)^p \tag{117}$$

ψ is zero at the lower end position of the blade.

Using Taylor's series and keeping terms up to second order, one can expand Eq. (117) as

$$V \approx V_m \left\{ 1 + \frac{1}{4}(p)(p-1)\left(\frac{h}{h_o}\right)^2 \right\} - V_m\, p\, \frac{h}{h_o} \cos\psi \tag{118}$$

where V_m represents the velocity at the hub, $V\left(\frac{h_o}{H_o}\right)^p$

The induced velocity at the blade is calculated by equating elemental thrusts obtained from momentum theory and blade element theory. For small built in twist θ_i, the inflow ratio parameter λ is approximated as

$$\lambda(h,\psi) = \lambda_o + \lambda_1\left(\frac{h}{R}\right)^2 + \lambda_2\, \frac{h}{R} \cos\psi \tag{119}$$

where λ_o corresponds to the inflow ratio at the hub and

$$\lambda_1 = \frac{1}{4}\, p(p-1)\left(\frac{R}{h_o}\right)^2\left(\lambda_o + \frac{\sigma a}{8}\right)$$

$$\lambda_2 = -p\, \frac{R}{h_o}\left(\lambda_o + \frac{\sigma a}{8}\right) \qquad (\sigma = \text{rotor solidity} = \frac{nc}{\pi R})$$

Thus, the inflow at a particular station on the blade varies as the blade rotates, causing periodic variation in airloads. This acts mainly as forcing function in the flapping degree of motion.

Substituting this new value of λ in Eq. (85) to obtain the inflow velocity at the rotor and modifying accordingly the resultant flow velocities U and v in Eqs. 87 and 88, and once again, on performing the various integrations for generalized aerodynamic forces one gets the new versions of Eqs. 92 and 93 representing the equations of motion for flapping-lagging rotor with sheared flow.

Flapping Equation:

$$
\begin{aligned}
&\overset{\circ\circ}{\beta}(1-\phi^2) - 2\mathring{\beta}\mathring{\phi}\phi + 2\mathring{\phi}(\beta+\theta\phi) + \beta(1+\bar{e}+\nu_\beta^2) + \theta\phi - \tfrac{2}{3}\beta^3 - \beta\theta^2 \\
&-\beta\phi^2 + 2\zeta_\beta\nu_\beta\mathring{\beta} - \nu_\beta^2\beta_s + G\sin\psi\left(-\theta+\tfrac{1}{6}\theta^3+\tfrac{1}{2}\theta\beta^2+\tfrac{1}{2}\theta\phi^2\right) + G\cos\psi\left(\beta-\tfrac{1}{6}\beta^3-\tfrac{1}{2}\beta\phi^2\right) \\
&= \frac{\gamma}{8}\Big\{\tilde{C}_0 + \beta^2\tilde{C}_5 + \phi^2\tilde{C}_6 + \beta\phi\tilde{C}_7 + \mathring{\beta}\tilde{C}_8 + \mathring{\phi}\tilde{C}_9 + \mathring{\beta}\beta\phi\tilde{C}_{10} + \mathring{\beta}\beta^2\tilde{C}_{11} \\
&\qquad + \mathring{\beta}\phi^2\tilde{C}_{12} + \mathring{\phi}\beta\phi\tilde{C}_{13} + \mathring{\phi}\beta^2\tilde{C}_{14} + \mathring{\phi}\phi^2\tilde{C}_{15} - \mathring{\beta}\mathring{\phi}CB_1 + \mathring{\phi}^2C_1\Big\} \\
&\quad + \frac{\gamma}{8}\cos\psi\Big\{D_0 + \beta^2D_5 + \phi^2D_6 + \beta\phi D_7 + \mathring{\beta}D_8 + \mathring{\phi}D_9 + \mathring{\beta}\beta\phi D_{10} + \mathring{\beta}\beta^2D_{11} \\
&\qquad + \mathring{\beta}\phi^2D_{12} + \mathring{\phi}\beta\phi D_{13} + \mathring{\phi}\beta^2D_{14} + \mathring{\phi}\phi^2D_{15}\Big\}
\end{aligned}
\tag{120}
$$

Lagging Equation:

$$
\begin{aligned}
&\overset{\circ\circ}{\phi} + \mathring{\beta}^2\phi - 2\mathring{\beta}(\beta+\theta\phi) + \phi(\bar{e}+\nu_\phi^2) + \theta\beta + \theta^2\phi - \beta^2\phi + 2\zeta_\phi\nu_\phi\mathring{\phi} \\
&-\nu_\phi^2\phi_s + G\sin\psi\left(1-\tfrac{1}{2}\theta^2-\tfrac{1}{2}\phi^2+\theta\beta\phi\right) + G\cos\psi\left(\phi-\tfrac{1}{6}\phi^3-\tfrac{1}{2}\theta\beta^2\right) \\
&= -\frac{\gamma}{8}\Big\{\tilde{B}_0 + \beta^2\tilde{B}_5 + \phi^2\tilde{B}_6 + \beta\phi\tilde{B}_7 + \mathring{\beta}\tilde{B}_8 + \mathring{\phi}\tilde{B}_9 + \mathring{\beta}\beta\phi\tilde{B}_{10} + \mathring{\beta}\beta^2\tilde{B}_{11} \\
&\qquad + \mathring{\beta}\phi^2\tilde{B}_{12} + \mathring{\phi}\beta\phi\tilde{B}_{13} + \mathring{\phi}\beta^2\tilde{B}_{14} + \mathring{\phi}\phi^2\tilde{B}_{15} + \mathring{\beta}\mathring{\phi}C_1 + \mathring{\phi}^2\frac{C_{d0}}{a} - \mathring{\beta}^2BC_1\Big\} \\
&\quad - \frac{\gamma}{8}\cos\psi\Big\{E_0 + \beta^2E_5 + \phi^2E_6 + \beta\phi E_7 + \mathring{\beta}E_8 + \mathring{\phi}E_9 + \mathring{\beta}\beta\phi E_{10} \\
&\qquad + \mathring{\beta}\beta^2E_{11} + \mathring{\beta}\phi^2E_{12} + \mathring{\phi}\beta\phi E_{13} + \mathring{\phi}\beta^2E_{14} + \mathring{\phi}\phi^2E_{15}\Big\}
\end{aligned}
\tag{121}
$$

The various constants in the above equations are defined in Appendix IV.

The important difference between these equations and those representing uniform inflow case, Eqs. 92 and 93, is that here in Eqs. 120 and 121, periodic aerodynamic terms are present because of velocity gradient (the D and E coefficients terms above).

The nonlinear response of flapping-lagging rotor in the presence of gravity and aerodynamic forces with sheared flow is obtained by following the similar steps as applied for uniform inflow case. On substituting general limit cycle solution (Eq. 94) into nonlinear Eqs. 120 and 121, and balancing out various terms one gets modified versions of R (Eq. 95), which now consists of four separate parts.

(a) contribution from inertia and stiffness, RI

(b) Contribution from gravity forces, RG

(c) Contribution from direct aerodynamic forces, RA

(d) Contribution from periodic aerodynamic forces, RV

such that

$$R_{ij} = RI(i,j) + RG(i,j) + \frac{\gamma}{8} RA(i,j) + \frac{\gamma}{8} RV(i,j)$$
$$R_{i0} = RIO(i) + RGO(i) + \frac{\gamma}{8} RAO(i) + \frac{\gamma}{8} RVO(i) \quad (122)$$

RI and RG are same as given in Appendix II for uniform inflow case and RA gets modified by replacing old C's and D's in Appendix II with new $\tilde{C}$'s and $\tilde{D}$'s of sheared flow. The RV which appears because of inflow variation at the blade is expressed in Appendix V.

The stability analysis here is also quite similar to that of earlier case except that the periodic aerodynamic terms have to be treated like gravity terms.

9.1 Numerical Results

Numerical calculations have been made for some specific configurations of Case IV to demonstrate the effect of sheared flow on the dynamic response of the rotor.

Case IV.

$\beta_s = -.12$	$\phi_s = 0$	$\bar{\omega}_\beta = .693$	$\phi_{DW} = .0035$	$\bar{e} = .075$
$\gamma = 10.6$	$\lambda = .1$	$\sigma = .023$	$\theta_1 = -.14$	$\theta_2 = .105$
$\zeta_\beta = .005$	$\zeta_\phi = .005$	$Cd_o = .012$	$a = 6.0$	$\theta = 0$

This case approximately simulates the characteristics of NASA Plumbrook 100 KW windmill blade with the assumption of complete rigidity in the torsional degree of freedom.

The forced response results are plotted in Fig 22.

Configuration A: Represents the condition of the rotor being excited only by gravity forces with uniform inflow.
(.e. no velocity gradient, p=0)

Configuration B: Represents the rotor being excited by only aerodynamic forces because of velocity gradient, p=.167, R/h_o=.625.
(i.e. no gravity forces, ϕ_{DW}=0)

Configuration C: Represents the condition of rotor under combined action of gravity and aerodynamic forces.
(p=.167, R/h_o=.625, ϕ_{DW}=.0035)

In Fig. 22(a), one can find that lagging amplitude is little effected by the velocity gradient and it is infact not possible to differentiate the results of configurations A and C. Fig. 22(b) shows that away from the resonance condition, the flapping amplitude is predominantly excited by the velocity gradient, and near the resonance condition the increased amplitude is mainly due to coupling from the gravity forces.

It should be noted that the presence of any tower shadow effect would combine with the velocity gradient effect here to produce a greater excitation of the flapping amplitude.

PART C: INCLUSION OF FEATHERING DEGREE OF FREEDOM

SECTION 10

RESPONSE OF FLAPPING-LAGGING-FEATHERING ROTOR

In the earlier analysis, it is assumed that the rotor has only two degrees of motion, i.e., flapping and lagging. In the present chapter, the third degree of motion, feathering, normal to the axis of rotation is also considered thus making the rotor a three degree of freedom system.

The fomulation of general equations of motion is very similar to that obtained earlier for flapping-lagging rotor. The inertia part is already derived as a general case in part A (Eqs. 29-31) and the aerodynamic forces are obtained after modifying the flow components of part B to include the feathering motion θ.

The motion of the blade as angular velocities about the three axes ξ, η, ζ from Eq. 10 are,

$$\begin{aligned}
\omega_\xi &= \Omega(\sin\theta \sin\phi + \cos\theta \sin\beta \cos\phi) + \dot{\theta}\cos\beta\cos\phi - \dot{\beta}\sin\phi \\
\omega_\eta &= \Omega(\sin\theta \cos\phi - \cos\theta \sin\beta \sin\phi) - \dot{\theta}\cos\beta \sin\phi - \dot{\beta}\cos\phi \\
\omega_\zeta &= \Omega\cos\theta\cos\beta - \dot{\theta}\sin\beta + \dot{\phi}
\end{aligned} \tag{123}$$

Modifying Equations 87 and 88 for resultant flow velocities U and v

$$\begin{aligned}
U &= r(\Omega\cos\theta\cos\beta + \dot{\phi} - \dot{\theta}\sin\beta) + \lambda\Omega R(-\cos\theta\sin\beta\sin\phi + \sin\theta\cos\phi) \\
&\quad + e\Omega(\sin\theta\sin\beta\sin\phi + \cos\theta\cos\phi) \\
v &= -\Omega r(\sin\theta\cos\phi - \cos\theta\sin\beta\sin\phi) + r\dot{\beta}\cos\phi + r\dot{\theta}\cos\beta\sin\phi \\
&\quad + \lambda\Omega R\cos\theta\cos\beta - e\Omega\sin\theta\cos\beta + \bar{\eta}\{\Omega(\sin\theta\sin\phi + \cos\theta\sin\beta\cos\phi) \\
&\quad - \dot{\beta}\sin\phi + \dot{\theta}\cos\beta\cos\phi\}
\end{aligned} \tag{124}$$

The λ is inflow velocity parameter which for sheared flow is given in Eq. 119 as

$$\lambda(r,\psi) = \lambda_0 + \lambda_1\left(\frac{r}{R}\right)^2 + \lambda_2\frac{r}{R}\cos\psi \tag{125}$$

and $\bar{\eta}$ is the distance of 75% chord position from aerodynamic center.

The generalized aerodynamic forces are

$$(Q_\beta)_{Aero.} = \cos\phi \int r\, dN$$
$$(Q_\phi)_{Aero.} = -\int r\, dC \qquad (126)$$
$$(Q_\theta)_{Aero.} = \cos\beta \sin\phi \int r\, dN + \sin\beta \int r\, dC$$

(r is the same as ξ and integration is on the span of the blade)

Again, taking the built-in incidence distribution (from Eq. 91)

$$\theta_i = \theta_1 + \frac{R+e}{r+e}\theta_2 \qquad (127)$$

Like earlier analysis, expanding trignometric terms to the third order and after performing various integrations and putting together inertia, aerodynamic and stiffness parts, we get the nonlinear equations of motion for a flappin-lagging-feathering rigid rotor with flexible springs at the hinge-point in the presence of sheared flow.

Flapping Equation:

$$
\begin{aligned}
&\overset{\circ\circ}{\beta}(1-\phi^2) + \overset{\circ\circ}{\theta}\phi - 2\mathring{\beta}\mathring{\phi}\phi + 2\mathring{\phi}\mathring{\theta} - \mathring{\theta}^2\beta - 2\mathring{\theta}\beta^2 + 2\mathring{\phi}(\beta+\theta\phi) + \beta(1+\bar{e}+\nu_\beta^2) + \theta\phi \\
&-\tfrac{2}{3}\beta^3 - \beta\theta^2 - \beta\phi^2 + 2\zeta_\beta\nu_\beta\mathring{\beta} - \nu_\beta^2\beta_s + G\sin\psi\left(-\theta+\tfrac{1}{6}\theta^3+\tfrac{1}{2}\theta\beta^2+\tfrac{1}{2}\theta\phi^2\right) + G\cos\psi\left(\beta-\tfrac{1}{6}\beta^3-\tfrac{1}{2}\beta\phi^2\right) \\
&= \frac{\gamma}{8}\Big\{ F_0 + F_1\beta + F_2\theta + F_3\beta^2 + F_4\phi^2 + F_5\theta^2 + F_6\beta\phi + F_7\beta\theta + F_8\phi\theta + F_9\beta\phi\theta + F_{10}\beta^2\theta \\
&\quad + F_{11}\phi^2\theta + F_{12}\theta^3 + F_{13}\mathring{\beta} + F_{14}\mathring{\phi} + F_{15}\mathring{\theta} + F_{16}\mathring{\beta}\phi + F_{17}\mathring{\beta}\theta + F_{18}\mathring{\phi}\beta + F_{19}\mathring{\phi}\theta + F_{20}\mathring{\theta}\beta \\
&\quad + F_{21}\mathring{\theta}\phi + F_{22}\mathring{\theta}\theta + F_{23}\mathring{\beta}\beta\phi + F_{24}\mathring{\beta}\beta^2 + F_{25}\mathring{\beta}\phi^2 + F_{26}\mathring{\beta}\theta^2 + F_{27}\mathring{\phi}\beta\phi + F_{28}\mathring{\phi}\beta^2 \\
&\quad + F_{29}\mathring{\phi}\phi^2 + F_{30}\mathring{\phi}\theta^2 + F_{31}\mathring{\theta}\beta\phi + F_{32}\mathring{\theta}\beta\theta + F_{33}\mathring{\theta}\phi\theta + F_{34}\mathring{\beta}\mathring{\phi} + F_{35}\mathring{\phi}\mathring{\theta} + F_{36}\mathring{\beta}\mathring{\theta}\beta \\
&\quad + F_{37}\mathring{\phi}\mathring{\theta}\phi + F_{38}\mathring{\phi}^2 + F_{39}\mathring{\theta}\phi^2 + F_{44}\mathring{\phi}\mathring{\theta}\beta + F_{45}\phi^2\beta \Big\} \\
&+ \frac{\gamma}{8}\cos\psi\Big\{ R_0 + R_1\beta + R_2\theta + R_3\beta^2 + R_4\phi^2 + R_5\theta^2 + R_6\beta\phi + R_7\beta\theta + R_8\phi\theta + R_9\beta\phi\theta \\
&\quad + R_{10}\beta^2\theta + R_{11}\phi^2\theta + R_{12}\theta^3 + R_{13}\mathring{\beta} + R_{14}\mathring{\phi} + R_{15}\mathring{\theta} + R_{16}\mathring{\beta}\phi + R_{17}\mathring{\beta}\theta + R_{18}\mathring{\beta}\phi \\
&\quad + R_{19}\mathring{\phi}\theta + R_{20}\mathring{\theta}\beta + R_{21}\mathring{\theta}\phi + R_{22}\mathring{\theta}\theta + R_{23}\mathring{\beta}\beta\phi + R_{24}\mathring{\beta}\beta^2 + R_{25}\mathring{\beta}\phi^2 + R_{26}\mathring{\beta}\theta^2 \\
&\quad + R_{27}\mathring{\phi}\beta\phi + R_{28}\mathring{\phi}\beta^2 + R_{29}\mathring{\phi}\phi^2 + R_{30}\mathring{\phi}\theta^2 + R_{31}\mathring{\theta}\beta\phi + R_{32}\mathring{\theta}\beta\theta + R_{33}\mathring{\theta}\phi\theta \Big\}
\end{aligned}
\tag{128}
$$

Lagging Equation:

$$
\begin{aligned}
&\overset{\circ\circ}{\phi} - \overset{\circ\circ}{\theta}\beta - 2\mathring{\theta}\mathring{\beta} - \mathring{\theta}^2\phi + \mathring{\beta}^2\phi - 2\mathring{\theta}(\beta\phi + R_m\theta) - 2\mathring{\beta}(\beta+\theta\phi) + \phi(\bar{e}+\nu_\phi^2) + \theta\beta + \theta^2\phi \\
&-\beta^2\phi + 2\zeta_\phi\nu_\phi\mathring{\phi} - \nu_\phi^2\phi_s + G\sin\psi\left(1-\tfrac{1}{2}\theta^2-\tfrac{1}{2}\phi^2+\theta\beta\phi\right) + G\cos\psi\left(\phi-\tfrac{1}{6}\phi^3-\tfrac{1}{2}\theta\beta^2\right) \\
&= -\frac{\gamma}{8}\Big\{ G_0 + G_1\beta + G_2\theta + G_3\beta^2 + G_4\phi^2 + G_5\theta^2 + G_6\beta\phi + G_7\beta\theta + G_8\phi\theta + G_9\beta\phi\theta + G_{10}\beta^2\theta \\
&\quad + G_{11}\phi^2\theta + G_{12}\theta^3 + G_{13}\mathring{\beta} + G_{14}\mathring{\phi} + G_{15}\mathring{\theta} + G_{16}\mathring{\beta}\phi + G_{17}\mathring{\beta}\theta + G_{18}\mathring{\phi}\beta + G_{19}\mathring{\phi}\theta + G_{20}\mathring{\theta}\beta \\
&\quad + G_{21}\mathring{\theta}\phi + G_{22}\mathring{\theta}\theta + G_{23}\mathring{\beta}\beta\phi + G_{24}\mathring{\beta}\beta^2 + G_{25}\mathring{\beta}\phi^2 + G_{26}\mathring{\beta}\theta^2 + G_{27}\mathring{\phi}\beta\phi + G_{28}\mathring{\phi}\beta^2 \\
&\quad + G_{29}\mathring{\phi}\phi^2 + G_{30}\mathring{\phi}\theta^2 + G_{31}\mathring{\theta}\beta\phi + G_{32}\mathring{\theta}\beta\theta + G_{33}\mathring{\theta}\phi\theta + G_{34}\mathring{\beta}\mathring{\phi} + G_{35}\mathring{\phi}\mathring{\theta} + G_{36}\mathring{\beta}\mathring{\theta}\beta \\
&\quad + G_{37}\mathring{\phi}\mathring{\theta}\phi + G_{38}\mathring{\phi}^2 + G_{40}\mathring{\theta}^2 + G_{41}\mathring{\beta}\beta + G_{42}\mathring{\beta}\mathring{\theta} + G_{43}\mathring{\beta}\mathring{\theta}\phi + G_{44}\mathring{\phi}\mathring{\theta}\beta + G_{46}\mathring{\beta}^2 \Big\} \\
&- \frac{\gamma}{8}\cos\psi\Big\{ S_0 + S_1\beta + S_2\theta + S_3\beta^2 + S_4\phi^2 + S_5\theta^2 + S_6\beta\phi + S_7\beta\theta + S_8\phi\theta + S_9\beta\phi\theta \\
&\quad + S_{10}\beta^2\theta + S_{11}\phi^2\theta + S_{12}\theta^3 + S_{13}\mathring{\beta} + S_{14}\mathring{\phi} + S_{15}\mathring{\theta} + S_{16}\mathring{\beta}\phi + S_{17}\mathring{\beta}\theta + S_{18}\mathring{\beta}\phi \\
&\quad + S_{19}\mathring{\phi}\theta + S_{20}\mathring{\theta}\beta + S_{21}\mathring{\theta}\phi + S_{22}\mathring{\theta}\theta + S_{23}\mathring{\beta}\beta\phi + S_{24}\mathring{\beta}\beta^2 + S_{25}\mathring{\beta}\phi^2 + S_{26}\mathring{\beta}\theta^2 \\
&\quad + S_{27}\mathring{\phi}\beta\phi + S_{28}\mathring{\phi}\beta^2 + S_{29}\mathring{\phi}\phi^2 + S_{30}\mathring{\phi}\theta^2 + S_{31}\mathring{\theta}\beta\phi + S_{32}\mathring{\theta}\beta\theta + S_{33}\mathring{\theta}\phi\theta \Big\}
\end{aligned}
\tag{129}
$$

Feathering Equation:

$$
\begin{aligned}
&\overset{\circ\circ}{\theta}(R_m+\beta^2+\phi^2)+\overset{\circ\circ}{\beta}\phi-\overset{\circ\circ}{\phi}\beta+2\mathring{\theta}\mathring{\beta}\beta+2\mathring{\theta}\mathring{\phi}\phi-2\mathring{\beta}\mathring{\phi}R_m+\mathring{\beta}\beta^2+2\mathring{\phi}(\beta\phi+R_m\theta)\\
&+\beta\phi+\theta(R_m-\beta^2+\phi^2+R_m\nu_\theta^2)+2\zeta_\theta\nu_\theta R_m\mathring{\theta}-\nu_\theta^2R_m\theta_s-G\sin\psi(\beta-\tfrac{1}{6}\beta^3-\tfrac{1}{2}\beta\theta^2-\tfrac{1}{2}\beta\phi^2+\theta\phi)\\
&=\frac{\gamma}{8}\Big\{G_0\beta+F_0\phi+G_1\beta^2+F_1\beta\phi+G_2\beta\theta+F_2\phi\theta+(F_7+G_8)\beta\phi\theta+G_7\beta^2\theta+F_8\phi^2\theta\\
&\quad+(F_3+G_6-\tfrac{1}{2}F_0)\beta^2\phi+(F_4'-\tfrac{1}{6}F_0)\phi^3+F_5\theta^2\phi+(F_6+G_4)\beta\phi^2+(G_3-\tfrac{1}{6}G_0)\beta^3\\
&\quad+G_5\beta\theta^2+F_{13}\mathring{\beta}\phi+G_{14}\mathring{\phi}\beta+G_{15}\mathring{\theta}\beta+F_{15}\mathring{\theta}\phi+G_{13}\mathring{\beta}\beta+F_{14}\mathring{\phi}\phi+G_{16}\mathring{\beta}\beta\phi\\
&\quad+F_{16}\mathring{\beta}\phi^2+F_{17}\mathring{\beta}\phi\theta+G_{17}\mathring{\beta}\beta\theta+F_{18}\mathring{\phi}\beta\phi+G_{18}\mathring{\phi}\beta^2+F_{19}\mathring{\phi}\phi\theta+G_{19}\mathring{\phi}\beta\theta\\
&\quad+(F_{20}+G_{21})\mathring{\theta}\beta\phi+G_{20}\mathring{\theta}\beta^2+F_{21}\mathring{\theta}\phi^2+G_{22}\mathring{\theta}\beta\theta+F_{22}\mathring{\theta}\phi\theta+G_{34}\mathring{\beta}\mathring{\phi}\beta\\
&\quad+F_{34}\mathring{\beta}\mathring{\phi}\phi+F_{35}\mathring{\phi}\mathring{\theta}\phi+G_{35}\mathring{\phi}\mathring{\theta}\beta+F_{38}\mathring{\phi}^2\phi+G_{35}\mathring{\phi}\mathring{\theta}\beta+G_{38}\mathring{\phi}^2\beta+G_{40}\mathring{\theta}^2\beta\\
&\quad+G_{41}\mathring{\beta}\beta^2+G_{42}\mathring{\beta}\mathring{\theta}\beta+G_{46}\mathring{\beta}^2\beta\Big\}\\
&+\frac{\gamma}{8}\cos\psi\Big\{S_0\beta+R_0\phi+S_1\beta^2+S_2\beta\theta+R_2\phi\theta+(S_3-\tfrac{1}{6}S_0)\beta^3+(R_3-\tfrac{1}{2}R_0+S_6)\beta^2\phi\\
&\quad+(S_4+R_6)\phi^2\beta+(R_4'-\tfrac{1}{6}R_0)\phi^3+S_5\theta^2\beta+R_5\theta^2\phi+S_7\beta^2\theta+(R_7+S_8)\beta\phi\theta\\
&\quad+S_{13}\mathring{\beta}\beta+S_{14}\mathring{\phi}\beta+R_{14}\mathring{\phi}\phi+S_{15}\mathring{\theta}\beta+S_{16}\mathring{\beta}\beta\phi+S_{17}\mathring{\beta}\beta\theta+R_{17}\mathring{\beta}\phi\theta\\
&\quad+S_{18}\mathring{\phi}\beta^2+S_{19}\mathring{\phi}\beta\theta+R_{19}\mathring{\phi}\phi\theta+S_{20}\mathring{\theta}\beta^2+(R_{20}+S_{21})\mathring{\theta}\beta\phi+S_{22}\mathring{\theta}\beta\theta\\
&\quad+R_{22}\mathring{\theta}\phi\theta\Big\}
\end{aligned}
\tag{130}
$$

The various constants in the above equations are defined in Appendix VI. In the derivation of the above equations, it is assumed that the aerodynamic center, the c.g. and the elastic axis lies at the same chordwise position and there is no variation of this position along the length of the blade.

10.1 Simple Solution

Considering small perturbation solution about some static solution

$$\begin{aligned} \beta &= \beta_0 + \tilde{\beta} \\ \phi &= \phi_0 + \tilde{\phi} \\ \theta &= \theta_0 + \tilde{\theta} \end{aligned} \tag{131}$$

Substituting this in Eqs. 128-130, retaining only the linear terms in the perturbations $\tilde{\beta}$, $\tilde{\phi}$, $\tilde{\theta}$, one gets three linear coupled equations. To simplify these equations further, one arbitrarily uncouples these equations and considers simple forced response of a rotor with zero twist distribution (θ_i=0) and with uniform inflow

Flapping Equation:

$$\overset{\circ\circ}{\tilde{\beta}} + \overset{\circ}{\tilde{\beta}}\left\{2\zeta_\beta \nu_\beta + \frac{\gamma}{8}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\} + \beta\left\{1+\bar{e}+\nu_\beta^2\right\} = G\theta_0 \sin\psi - G\beta_0 \cos\psi \tag{132}$$

Lagging Equation:

$$\overset{\circ\circ}{\tilde{\phi}} + \overset{\circ}{\tilde{\phi}}\left\{2\zeta_\phi \nu_\phi + \frac{\gamma}{4}\frac{C_{do}}{a}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\} + \phi\left(\bar{e}+\nu_\phi^2\right) = -G\sin\psi - G\phi_0 \cos\psi \tag{132b}$$

Feathering Equation:

$$\begin{aligned} \overset{\circ\circ}{\theta}(R_m+\beta_0^2+\phi_0^2) + \overset{\circ}{\theta}\left\{2\zeta_\theta \nu_\theta R_m + \frac{\gamma}{2}\frac{\bar{\eta}}{R}\left(\beta_0\lambda+\frac{\phi_0}{3}-\frac{2}{3}\beta_0\theta_0\right)+\frac{\gamma}{8}\phi_0^2\right\} \\ + \theta\left\{(1+\nu_\theta^2)R_m-\beta_0^2+\phi_0^2\right\} = G\beta_0 \sin\psi \end{aligned} \tag{132c}$$

and static solutions are

$$\beta_0 \approx \frac{\nu_\beta^2\beta_s - \frac{\gamma\lambda}{4}\left(\frac{2}{3}+\frac{e}{R}\right)}{1+\bar{e}+\nu_\beta^2}$$

$$\phi_0 \approx \frac{\nu_\phi^2\phi_s - \frac{\gamma}{8}\left\{\frac{C_{do}}{a}\left(1+\frac{8}{3}\frac{e}{R}\right)-2\lambda^2\right\}}{\bar{e}+\nu_\phi^2}$$

$$\theta_0 \approx \frac{R_m\nu_\theta^2\theta_s + \frac{\gamma}{8}\left\{\frac{C_{do}}{a}\left(1+\frac{8}{3}\frac{e}{R}\right)-2\lambda^2\right\}\beta_0 - \frac{\gamma\lambda}{4}\left(\frac{2}{3}+\frac{e}{R}\right)\phi_0}{R_m(1+\nu_\theta^2)-\beta_0^2+\phi_0^2} \tag{133}$$

For the forced oscillations assuming steady solution

$$\tilde{\beta} \approx a_1 \sin\psi + b_1 \cos\psi$$
$$\tilde{\Phi} \approx a_2 \sin\psi + b_2 \cos\psi$$
$$\tilde{\Theta} \approx a_3 \sin\psi + b_3 \cos\psi \qquad (134)$$

By matching and solving equations, one gets

$$a_1 \approx \frac{G\theta_0(\bar{e}+\nu_\beta^2) - G\beta_0\left\{2\zeta_\beta\nu_\beta + \frac{\gamma}{8}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\}}{(\bar{e}+\nu_\beta^2)^2 + \left\{2\zeta_\beta\nu_\beta + \frac{\gamma}{8}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\}^2}$$

$$b_1 \approx \frac{-G\theta_0\left\{2\zeta_\beta\nu_\beta + \frac{\gamma}{8}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\} - G\beta_0(\bar{e}+\nu_\beta^2)}{(\bar{e}+\nu_\beta^2)^2 + \left\{2\zeta_\beta\nu_\beta + \frac{\gamma}{8}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\}^2}$$

$$a_2 \approx \frac{-G(\bar{e}+\nu_\phi^2-1) - G\phi_0\left\{2\zeta_\phi\nu_\phi + \frac{\gamma}{4}\frac{C_{d0}}{a}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\}}{(\bar{e}+\nu_\phi^2-1)^2 + \left\{2\zeta_\phi\nu_\phi + \frac{\gamma}{4}\frac{C_{d0}}{a}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\}^2}$$

$$b_2 \approx \frac{G\left\{2\nu_\phi\zeta_\phi + \frac{\gamma}{4}\frac{C_{d0}}{a}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\} - G\phi_0(\bar{e}+\nu_\phi^2-1)}{(\bar{e}+\nu_\phi^2-1)^2 + \left\{2\zeta_\phi\nu_\phi + \frac{\gamma}{4}\frac{C_{d0}}{a}\left(1+\frac{4}{3}\frac{e}{R}\right)\right\}^2}$$

$$a_3 \approx \frac{G\beta_0(R_m\nu_\theta^2 - 2\beta_0^2)}{(R_m\nu_\theta^2 - 2\beta_0^2)^2 + \left\{2\zeta_\theta\nu_\theta R_m + \frac{\gamma}{2}\frac{\bar{\eta}}{R}\left(\beta_0\lambda + \frac{1}{3}\phi_0 - \frac{2}{3}\beta_0\theta_0\right) + \frac{\gamma}{8}\phi_0^2\right\}^2}$$

$$b_3 \approx \frac{-G\beta_0\left\{2\zeta_\theta\nu_\theta R_m + \frac{\gamma}{2}\frac{\bar{\eta}}{R}\left(\beta_0\lambda + \frac{\phi_0}{3} - \frac{2}{3}\beta_0\theta_0\right) + \frac{\gamma}{8}\phi_0^2\right\}}{(R_m\nu_\theta^2 - 2\beta_0^2)^2 + \left\{2\zeta_\theta\nu_\theta R_m + \frac{\gamma}{2}\frac{\bar{\eta}}{R}\left(\beta_0\lambda + \frac{1}{3}\phi_0 - \frac{2}{3}\beta_0\theta_0\right) + \frac{\gamma}{8}\phi_0^2\right\}}$$

(135)

From this simple analysis one can find that the flapping response is effected appreciably by the feathering angle θ_o. It is found that this simple solution gives reasonable estimate of the solution, but more than that, this solution is quite useful in giving an initial trial vector for nonlinear solution.

10.2 Nonlinear Solution

Assuming a limit cycle solution of Eqs. 128-130 for both forced oscillations as well as parametric resonance

$$\begin{aligned} \beta &\approx \beta_c + a_1 \sin \alpha\psi + b_1 \cos \alpha\psi \\ \phi &\approx \phi_c + a_2 \sin \alpha\psi + b_2 \cos \alpha\psi \\ \theta &\approx \theta_c + a_3 \sin \alpha\psi + b_3 \cos \alpha\psi \end{aligned} \tag{136}$$

Here, $\alpha=1$ represents forced oscillations and $\alpha=\frac{1}{2}$ gives parametric resonance. Substituting these expressions (136) into Eqs. 128-130 and balancing out the constant, $\sin\alpha\psi$ and $\cos\alpha\psi$ terms from each equation gives nine nonlinear algebraic equation;

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} & R_{17} & R_{18} & R_{19} \\ R_{21} & R_{22} & R_{23} & - & - & - & - & - & - \\ R_{31} & R_{32} & - & - & - & - & - & - & - \\ R_{41} & - & - & - & - & - & - & - & - \\ R_{51} & - & - & - & - & - & - & - & - \\ R_{61} & - & - & - & - & - & - & - & - \\ R_{71} & - & - & - & - & - & - & - & - \\ R_{81} & - & - & - & - & - & - & - & - \\ R_{91} & - & - & - & - & - & - & - & R_{99} \end{bmatrix} \begin{bmatrix} \beta_c \\ a_1 \\ b_1 \\ \phi_c \\ a_2 \\ b_2 \\ \theta_c \\ a_3 \\ b_3 \end{bmatrix} + \begin{bmatrix} R_{10} \\ R_{20} \\ R_{30} \\ R_{40} \\ R_{50} \\ R_{60} \\ R_{70} \\ R_{80} \\ R_{90} \end{bmatrix} = 0 \tag{137}$$

The various terms in the above matrices are expressed in Appendix VII.

The solution of these nonlinear equations (137) is again obtained numerically by the Newton-Raphson technique.

10.3 Stability Check of Solution

The stability check for the solution obtained through the Harmonic Balance Method is made as done for the flapping-lagging rotor by giving small perturbations to the steady solution and studying the growth rate of these perturbations with time (see details in Section 5.3).

The pertubed solution is

$$
\begin{aligned}
\beta &= \beta_{co} + \hat{\beta}_c(\psi) + [a_{10} + \hat{a}_1(\psi)] \sin \alpha\psi + [b_{10} + \hat{b}_1(\psi)] \cos \alpha\psi \\
\phi &= \phi_{co} + \hat{\phi}_c(\psi) + [a_{20} + \hat{a}_2(\psi)] \sin \alpha\psi + [b_{20} + \hat{b}_2(\psi)] \cos \alpha\psi \\
\theta &= \theta_{co} + \hat{\theta}_c(\psi) + [a_{30} + \hat{a}_3(\psi)] \sin \alpha\psi + [b_{30} + \hat{b}_3(\psi)] \cos \alpha\psi
\end{aligned}
\tag{138}
$$

Since β_{co}, a_{10} ..... a_{30}, b_{30} is the steady solution for which the stability check is being made and $\hat{\beta}_c$, $\hat{a}_1$, $\hat{b}_1$ ..... $\hat{b}_3$ are the time dependent small perturbations given to respective steady solution components.

Putting Eq. (93) into basic governing Eqs. 128-130, keeping only linear terms in perturbations and their first order derivatives, filtering out the steady solution, once again on matching various terms one gets nine linear algebraic equations

$$
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & \cdots & S_{19} \\
S_{21} & S_{22} & \cdot & \cdots & \cdot & S_{29} \\
S_{31} & \cdot & \cdot & \cdots & \cdot & \cdot \\
S_{41} & \cdot & \cdot & \cdots & \cdot & \cdot \\
S_{51} & \cdot & \cdot & \cdots & \cdot & \cdot \\
S_{61} & \cdot & \cdot & \cdots & \cdot & \cdot \\
S_{71} & \cdot & \cdot & \cdots & \cdot & \cdot \\
S_{81} & \cdot & \cdot & \cdots & \cdot & \cdot \\
S_{91} & \cdot & \cdot & \cdots & \cdot & S_{99}
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}_c \\ \hat{a}_1 \\ \hat{b}_1 \\ \hat{\phi}_c \\ \hat{a}_2 \\ \hat{b}_2 \\ \hat{\theta}_c \\ \hat{a}_3 \\ \hat{b}_3
\end{bmatrix}
=
\begin{bmatrix}
T_{11} & T_{12} & T_{13} & T_{14} & \cdots & T_{19} \\
T_{21} & T_{22} & \cdot & \cdots & \cdot & \cdot \\
T_{31} & \cdot & \cdot & \cdots & \cdot & \cdot \\
T_{41} & \cdot & \cdot & \cdots & \cdot & \cdot \\
T_{51} & \cdot & \cdot & \cdots & \cdot & \cdot \\
T_{61} & \cdot & \cdot & \cdots & \cdot & \cdot \\
T_{71} & \cdot & \cdot & \cdots & \cdot & \cdot \\
T_{81} & \cdot & \cdot & \cdots & \cdot & \cdot \\
T_{91} & \cdot & \cdot & \cdots & \cdot & T_{99}
\end{bmatrix}
\begin{bmatrix}
\mathring{\hat{\beta}}_c \\ \mathring{\hat{a}}_1 \\ \mathring{\hat{b}}_1 \\ \mathring{\hat{\phi}}_c \\ \mathring{\hat{a}}_2 \\ \mathring{\hat{b}}_2 \\ \mathring{\hat{\theta}}_c \\ \mathring{\hat{a}}_3 \\ \mathring{\hat{b}}_3
\end{bmatrix}
$$

This is standard algebraic eigenvalue problem, the nature of the roots explains whether the solution is stable or not. If any one of the roots has positive real value, means perturbation will grow with time and thus makes the solution unstable.

10.4 Numerical Results

Numerical calculations for forced response are made for case IV (see Section 9.1) with $\bar{\omega}_\theta$ of 14.3 (i.e., $\omega_\theta = 14.3\ \omega_\phi$) for two conditions of with and without sheared flow. In Fig. 23, the response amplitudes represening flapping, lagging and feathering degrees of freedom are plotted for different rotational speeds, for a rotor with uniform inflow. Comparing with the results of same configuration with the torsion degree of freedom locked (i.e., $\omega_\theta \to \infty$) from Figs. 22(a) and 22(b), one finds that there is relatively less effect on the lagging response but flapping response is appreciably effected by the inclusion of the feathering degree of freedom. The flapping amplitude is increased near as well as away from the resonance condition and this is quite evident from the simple linear flapping equation (132a) where one finds that the forcing function for flapping is dependent on feathering angle. Also, even for this relatively torsionally stiff rotor ($\bar{\omega}_\theta$=14.3) the feathering response is quite significant and it is of the same order of magnitude as the lagging response or flapping response. The resonance condition, however, takes place at almost the same rotational speed as for two degrees of freedom case.

In Fig. 24, the response amplitudes are plotted for a rotor with sheared flow but with the same inflow λ of 0.1 at the hub as the first case. Once again one finds that the velocity gradient has more prominant influence on the flapping response as compared to other two degrees of freedom. Away from the resonance condition, the flapping amplitude is generally increased and lagging and feathering response amplitudes remain nearly the same whereas near the resonance condition all the three response amplitudes are reduced because of the sheared flow.

SECTION 11

SELF-EXCITED FLUTTER RESPONSE OF FLAPPING-LAGGING-FEATHERING ROTOR

It is primarily intended to show here that the blade representing the three degrees of motion, namely flapping, lagging and feathering, in the absence of gravity forces and with uniform inflow can get into self-sustained oscillations by the interaction of unsteady aerodynamic forces with structural vibrations. The equations of motion for flutter of this system are the same as Eqs. 128-130 except that all periodic terms are absent (because G=0, $\lambda_1=0$, $\lambda_2=0$).

11.1 Linear Analysis

Linear flutter analysis is worked out by assuming that the blade response consists of small perturbation motion ($\tilde{\beta}$, $\tilde{\phi}$, $\tilde{\theta}$) about some possibly large static positions (β_o, ϕ_o, θ_o)

$$\begin{aligned} \beta &= \beta_o + \tilde{\beta} \\ \phi &= \phi_o + \tilde{\phi} \\ \theta &= \theta_o + \tilde{\theta} \end{aligned} \tag{140}$$

Substituting this in Eqs. 129-131 and keeping only linear terms and derivatives and after filtering out the static part, the characteristic equation of motion can be put in standard spring-mass-damper form

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \mathring{\mathring{\tilde{\beta}}} \\ \mathring{\mathring{\tilde{\phi}}} \\ \mathring{\mathring{\tilde{\theta}}} \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \mathring{\tilde{\beta}} \\ \mathring{\tilde{\phi}} \\ \mathring{\tilde{\theta}} \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} \tilde{\beta} \\ \tilde{\phi} \\ \tilde{\theta} \end{bmatrix} = 0 \tag{141}$$

and the static equation is

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} \begin{bmatrix} \beta_o \\ \phi_o \\ \theta_o \end{bmatrix} = \begin{bmatrix} \nu_\beta^2 \beta_s + \frac{\gamma}{8} F_o \\ \nu_\phi^2 \phi_s - \frac{\gamma}{8} G_o \\ \nu_\theta^2 \theta_s R_m \end{bmatrix} \tag{142}$$

The coefficients of the above matrices are given as

$$M_{11} = 1.0$$
$$M_{12} = 0$$
$$M_{13} = \phi_o$$
$$M_{21} = 0$$
$$M_{22} = 1.0$$
$$M_{23} = -\beta_o$$
$$M_{31} = \phi_o$$
$$M_{32} = -\beta_o$$
$$M_{33} = R_m + \beta_o^2 + \phi_o^2$$

$$C_{11} = 2\zeta_\beta \nu_\beta - \frac{\gamma}{8}(F_{13} + \phi_o F_{16} + \theta_o F_{17} + \beta_o^2 F_{24} + \phi_o^2 F_{26})$$
$$C_{12} = 2\beta_o - \frac{\gamma}{8}(F_{14} + \beta_o F_{18} + \theta_o F_{19})$$
$$C_{13} = -\frac{\gamma}{8}(F_{15} + \beta_o F_{20} + \phi_o F_{21} + \theta_o F_{22})$$
$$C_{21} = -2\beta_o + \frac{\gamma}{8}(G_{13} + \phi_o G_{16} + \theta_o G_{17} + \beta_o G_{41})$$
$$C_{22} = 2\zeta_\phi \nu_\phi + \frac{\gamma}{8}(G_{14} + \beta_o G_{18} + \theta_o G_{19} + \beta_o \phi_o G_{27} + \beta_o^2 G_{28})$$
$$C_{23} = \frac{\gamma}{8}(G_{15} + \beta_o G_{20} + \phi_o G_{21} + \theta_o G_{22})$$
$$C_{31} = 2\beta_o^2 - \frac{\gamma}{8}(\phi_o F_{13} + \beta_o G_{13})$$

$$C_{32} = 2\beta_0\phi_0 + 2R_m\theta_0 - \frac{\gamma}{8}(\beta_0 G_{14} + \phi_0 F_{14})$$

$$C_{33} = 2\zeta_\theta \nu_\theta R_m - \frac{\gamma}{8}\{\beta_0 G_{15} + \phi_0 F_{15} + \beta_0\phi_0(F_{20}+G_{21}) + \beta_0^2 G_{20} + \phi_0^2 F_{21} + \beta_0\theta_0 G_{22} + \phi_0\theta_0 F_{22}\}.$$

$$k_{11} = 1 + \bar{e} + \nu_\beta^2 - \frac{\gamma}{8}(F_1 + 2\beta_0 F_3 + \phi_0 F_6 + \theta_0 F_7)$$

$$k_{12} = \theta_0 - \frac{\gamma}{8}(2\phi_0 F_4 + \beta_0 F_6 + \theta_0 F_8)$$

$$k_{13} = \phi_0 - \frac{\gamma}{8}(F_2 + 2\theta_0 F_5 + \beta_0 F_7 + \phi_0 F_8)$$

$$k_{21} = \theta_0 + \frac{\gamma}{8}(G_1 + 2\beta_0 G_3 + \phi_0 G_6 + \theta_0 G_7)$$

$$k_{22} = \bar{e} + \nu_\phi^2 + \frac{\gamma}{8}(2\phi_0 G_4 + \beta_0 G_6 + \theta_0 G_8)$$

$$k_{23} = \beta_0 + \frac{\gamma}{8}(G_2 + 2\theta_0 G_5 + \beta_0 G_7 + \phi_0 G_8)$$

$$k_{31} = \phi_0 - 2\theta_0\beta_0 - \frac{\gamma}{8}(G_0 + 2\beta_0 G_1 + \phi_0 F_1 + \theta_0 G_2)$$

$$k_{32} = \beta_0 - \frac{\gamma}{8}(F_0 + \beta_0 F_1 + \theta_0 F_2) + 2\phi_0\theta_0$$

$$k_{33} = R_m(1+\nu_\theta^2) + \phi_0^2 - \beta_0^2 - \frac{\gamma}{8}\{\beta_0 G_2 + \phi_0 F_2 + \beta_0\phi_0(F_7+G_8) + \beta_0^2 G_7 + \phi_0^2 F_8 + 2\phi_0\theta_0 F_5 + 2\beta_0\theta_0 G_5\}$$

$$H_{11} = 1 + \bar{e} + \nu_\beta^2 - \frac{\gamma}{8}(F_1 + \beta_0 F_3 + \phi_0 F_6 + \theta_0 F_7)$$

$$H_{12} = \frac{1}{2}\theta_0 - \frac{\gamma}{8}(\phi_0 F_4 + \theta_0 F_8)$$

$$H_{13} = \frac{1}{2}\phi_0 - \frac{\gamma}{8}(F_2 + \theta_0 F_5)$$

$$H_{21} = \frac{1}{2}\theta_0 + \frac{\gamma}{8}(G_1 + \beta_0 G_3 + \theta_0 G_7)$$

$$H_{22} = \bar{e} + \nu_\phi^2 + \frac{\gamma}{8}(\phi_0 G_4 + \beta_0 G_6 + \theta_0 G_8)$$

$$H_{23} = \frac{1}{2}\beta_0 + \frac{\gamma}{8}(G_2 + \theta_0 G_5)$$

$$H_{31} = \frac{1}{2}\phi_0 - \frac{\gamma}{8}\, G_0 + \beta_0 G_1 + \phi_0 F_1 + \phi_0^2(F_6 + G_4) + \beta_0^2(G_3 - \frac{G_0}{6})\}$$

$$H_{32} = \frac{1}{2}\beta_0 - \frac{\gamma}{8}\{F_0 + \beta_0^2(F_3 + G_6 - \frac{1}{2}F_0)\}$$

$$H_{33} = \phi_0^2 - \beta_0^2 + R_m(1+\nu_\theta^2) - \frac{\gamma}{8}\{\beta_0 G_2 + \phi_0 F_2 + \beta_0\phi_0(F_7+G_8) + \beta_0^2 G_7 + \phi_0^2 F_8 + \phi_0\theta_0 F_5 + \beta_0\theta_0 G_5\}$$

Coefficients F_i's and G_i's are given in Appendix VI.

Equation (141) can be rewritten as

$$\begin{bmatrix} [M] & \vdots & [C] \\ \cdots & & \cdots \\ [0] & \vdots & [M] \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \cdots \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} [0] & \vdots & [K] \\ \cdots & & \cdots \\ -[M] & \vdots & [0] \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \cdots \\ q \end{Bmatrix} = 0 \tag{143}$$

where [M], [C] and [K] are respectively inertia, damping and stiffness matrices, each of order 3x3 and q is column matrix

$$\{q\} = \begin{Bmatrix} \tilde{\beta} \\ \tilde{\Phi} \\ \tilde{\theta} \end{Bmatrix} \tag{144}$$

writing

$$\{q\} = \{\bar{q}\} e^{\bar{\lambda}\psi} \tag{145}$$

Substituting into Eq. (143) results in an algebraic eigenvalue problem which can be solved numerically by using any one of the standard subroutines. The solution will give six eigenvalues, in fact, three pair of complex conjugate roots,

$$\bar{\lambda}^{(n)} = \lambda_R^{(n)} + i\ \lambda_I^{(n)} \tag{146}$$

$\lambda_I^{(n)}$ gives the frequency of the rotor in the presence of airflow for the nth mode and $\lambda_R^{(n)}$ represents the effective damping of the nth mode. The damping coefficient of the rotor for any mode in the presence of airflow can be obtained from Eq. (146)

$$\zeta = \frac{\lambda_R}{\sqrt{\lambda_R^2 + \lambda_I^2}} \tag{147}$$

The mode shape for each one of these natural frequencies can be had from the corresponding lower half of the eigenvector. Also, one can obtain the phase plot for these three degrees of motion from the Argand diagram of complex eigenvector.

It should be noted that $\zeta=0$ defines the critical flutter condition. ζ positive gives damped oscillations and ζ negative results in unstable oscillations.

11.2 Nonlinear Flutter Analysis

Again assuming the limit cycle flutter solution of nonlinear equations 128-130 (with G=0, $\lambda_1=\lambda_2=0$) of the form

$$\begin{aligned}\beta &\approx \beta_c + a_1 \sin\alpha\psi + b_1 \cos\alpha\psi \\ \phi &\approx \phi_c + a_2 \sin\alpha\psi + b_2 \cos\alpha\psi \\ \theta &\approx \theta_c + a_3 \sin\alpha\psi + b_3 \cos\alpha\psi\end{aligned} \tag{148}$$

where α represents the ratio of flutter frequency to rotational frequency.

Once again, trying to find the limit cycle flutter solution for a known lagging amplitude by setting $b_2=0$ and a_2 as known quantity. Placing these expressions (148) into Eqs. (128-130) and balancing out the constant, $\sin\alpha\psi$ and $\cos\alpha\psi$ terms and putting appropriately one can get

$$\begin{bmatrix} A_{11} & A_{12} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} & A_{19} \\ A_{21} & A_{22} & A_{23} & - & - & - & - & - \\ A_{31} & A_{32} & - & - & - & - & - & - \\ A_{41} & - & - & - & - & - & - & - \\ A_{51} & - & - & - & - & - & - & - \\ A_{61} & - & - & - & - & - & - & - \\ A_{71} & - & - & - & - & - & - & - \\ A_{81} & - & - & - & - & - & - & - \\ A_{91} & - & - & - & - & - & - & A_{99} \end{bmatrix} \begin{bmatrix} \beta_c \\ a_1 \\ b_1 \\ \phi_c \\ Z \\ \alpha \\ \theta_c \\ a_3 \\ b_3 \end{bmatrix} + \begin{bmatrix} A_{10} \\ A_{20} \\ A_{30} \\ A_{40} \\ A_{50} \\ A_{60} \\ A_{70} \\ A_{80} \\ A_{90} \end{bmatrix} = 0 \tag{149}$$

Here $Z = (\omega_\phi/\Omega)^2$ and the other terms in the above matrices are defined in Appendix VIII.

The solution of these nonlinear algebraic equations (149) is obtained numerically by Newton-Raphson technique.

11.3 Stability Check of Solution

The stability check of the nonlinear flutter solution is made by giving small perturbations to the steady solution

$$
\begin{aligned}
\beta &= \beta_{co} + \hat{\beta}_c(\psi) + [a_{10} + \hat{a}_1(\psi)]\sin\alpha\psi + [b_{10} + \hat{b}_1(\psi)]\cos\alpha\psi \\
\phi &= \phi_{co} + \hat{\phi}_c(\psi) + [a_{20} + \hat{a}_2(\psi)]\sin\alpha\psi + [b_{20} + \hat{b}_2(\psi)]\cos\alpha\psi \\
\theta &= \theta_{co} + \hat{\theta}_c(\psi) + [a_{30} + \hat{a}_3(\psi)]\sin\alpha\psi + [b_{30} + \hat{b}_3(\psi)]\cos\alpha\psi
\end{aligned}
\tag{150}
$$

where β_{co}, a_{10}, b_{10}, ϕ_{co}, a_{20}, θ_{co}, a_{30}, b_{30} and $b_{20} = 0$ with corresponding Z and α is the steady solution and $\hat{\beta}_c$, $\hat{a}_1$, $\hat{b}_1$, $\hat{\phi}_c$, $\hat{a}_2$, $\hat{b}_2$, $\hat{\phi}_c$, $\hat{a}_3$, $\hat{b}_3$ with same Z and α are the time dependent perturbations.

Placing (150) into Eqs. (128-130), linearizing and subtracting the steady solution, again on balancing the constant, $\sin\alpha\psi$ and $\cos\alpha\psi$ terms one gets

$$
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & \cdots & S_{19} \\
S_{21} & S_{22} & \cdot & \cdots & \cdot \\
S_{31} & S_{32} & \cdot & \cdots & \cdot \\
S_{41} & \cdot & \cdot & \cdots & \cdot \\
S_{51} & \cdot & \cdot & \cdots & \cdot \\
\vdots & & & & \vdots \\
S_{91} & \cdot & \cdot & \cdots & S_{99}
\end{bmatrix}
\begin{bmatrix}
\hat{\beta}_c \\ \hat{a}_1 \\ \hat{b}_1 \\ \hat{\phi}_c \\ \hat{a}_2 \\ \hat{b}_2 \\ \hat{\theta}_c \\ \hat{a}_3 \\ \hat{b}_3
\end{bmatrix}
=
\begin{bmatrix}
T_{11} & T_{12} & T_{13} & \cdots & T_{19} \\
T_{21} & T_{22} & \cdot & \cdots & \cdot \\
T_{31} & \cdot & \cdot & \cdots & \cdot \\
T_{41} & \cdot & \cdot & \cdots & \cdot \\
\vdots & & & & \vdots \\
T_{91} & \cdot & \cdot & \cdots & T_{99}
\end{bmatrix}
\begin{bmatrix}
\dot{\hat{\beta}}_c \\ \dot{\hat{a}}_1 \\ \dot{\hat{b}}_1 \\ \dot{\hat{\phi}}_c \\ \vdots \\ \dot{\hat{b}}_3
\end{bmatrix}
\tag{151}
$$

Again the nature of the eigenvalues will decide whether the solution is stable or not. If any one of the root has positive real part shows that the solution is unstable.

11.4 Numerical Results

In Fig. 25, the total damping coefficient ζ is plotted for different rotational speeds for Case IV with $\bar{\omega}_\theta$=6.0 (i.e., ω_θ=6.0 ω_ϕ). The dotted lines correspond to the zero structural damping configurations where as the full lines represent the configurations with structural damping coefficients of $\zeta_\beta=\zeta_\phi=\zeta_\theta=.005$. For any particular rotational speed, one gets three damping coefficients corresponding to three different vibration modes, but in the diagram only two branches are shown since the third mode is comparatively highly damped. In Fig. 26, the relative response amplitude for these two branches for the damped case are plotted. From these two figures, one can find that the high frequency branch I which primarily represents feathering mode with small coupling from the lagging, is unstable from the very beginning for zero structural damping, and becomes completely stable with the inclusion of small amount of structural damping. Here, the lagging response is roughly 180° out of phase with the feathering amplitude. Low frequency branch II which is predominantly feathering-flapping branch with comparatively small coupling from lagging is not much effected by the small amount of structural damping except that the instability boundary shifts to slightly higher rotational speed. This instability is like classical bending-torsion flutter, and it is also found that at the critical flutter condition, the flapping response lags the feathering amplitude by about 90°.

Fig. 27 represents the damping coefficient ζ of Case IV with more torsionally stiff rotor i.e., $\bar{\omega}_\theta$=14.3. Here one finds that the high frequency feathering branch I does not become completely stable with the addition of small structural damping but only the instability region shifts to higher rotational speed. Low frequency branch II which is now evenly coupled between flapping, lagging and feathering modes becomes completely stable with the addition of small amounts of structural damping.

Fig. 28 shows the nonlinear limit cycle flutter amplitudes for flapping, lagging and feathering motions for Case IV with $\bar{\omega}_\theta$=6.0. The bending over of the response amplitude curves towards decreasing rotational speeds depict a typical nonlinear softening spring type system. For $\frac{\Omega}{\omega_\phi}$ less than about 0.3, solutions becomes unstable. These nonlinear solutions indicate the possibility of sustained limit cycle flutter oscillations occurring well below the critical speed predicted by linear theory. These might be initiated by a sufficiently large finite distrubance, and as such may be potentially dangerous.

SECTION 12

CONCLUSIONS AND SUGGESTED FURTHER WORK

In Part A, the nonlinear response of two-degree flapping-lagging rotor in the absence of aerodynamic forces is obtained. The analysis is made for forced oscillations as well as parametric resonance under the action of periodic gravity forces. The effects of various parameters like initial coning angle, flapping to lagging frequencies ratio, hinge offset, etc. on the nonlinear response amplitude is discussed.

In Part B, the analysis is extended to include the effect of aerodynamic forces on the nonlinear response of the flapping-lagging rotor. The inclusion of aerodynamic forces produce quite significant effects on some of the configurations for both forced oscillations as well as parametric resonance. Also, the self-excited flutter solution for flapping-lagging rotor is obtained after neglecting the gravity forces. The effect of the various parameters associated with aerodynamic forces like inflow ratio, Lock number, initial coning angle, etc. on the forced response as well as critical flutter boundary is discussed. The effect of wind shear on the forced response of flapping-lagging rotor is also investigated and it is seen that the velocity gradient produces little effect on the lagging response but it has appreciable influence on the flapping amplitude.

In Part C, the third degree of motion, feathering, is considered thus making the rotor a three degree of freedom system. First, the forced response of flapping-lagging-feathering rotor under gravitational field and with sheared flow is studied. It is seen that even for relatively torsionally stiff rotor, the flapping amplitude is very much increased with the inclusion of third degree of freedom and also the feathering response is appreciable. Then the self-excited flutter solution of this three-degree rotor is investigated and it is found that the feathering degree of motion is very important for the flutter analysis. The effect of rotational speed on the linear as well as nonlinear flutter solutions is discussed.

One may further extend these analyses to include

1. Tower motion
2. Blade flexibility
3. Interaction of forced response and flutter response
4. Subharmonic and Superharmonic response
5. Improved Aerodynamic Model

The inclusion of tower flexibility would involve the interaction of the overall vibration levels at the tower hub with the supporting structure which may give rise to instabilities similar to the ground resonance problems in helicopter rotors. The inclusion of blade flexibility would bring a more realistic model of long, thin windmill blades into the picture. The interaction of forced response and self-excited flutter response makes the analysis a more involved one and this interaction can become particularly significant when flutter frequency is near the resonance frequency. Because of low structural damping, the subharmonic and superharmonic response may be important. The variable inflow model taking effects of tower shadow, and yawed flow, etc. and also the inclusion of returning wake and other unsteady aerodynamic effects will expand the domain of the dynamic problems of the wind turbine rotor.

The investigation of these problems would contribute to a better and more realistic understanding of the aeroelastic behavior of thin rotating windmill blade.

REFERENCES

1. Putnam, P.C., Power from the Wind, D. Van Nostrand Company, Inc., New York, 1948.
2. Golding, E.W., The Generation of Electricity by Wind Power, Philosophical Library, New York, 1956.
3. Simmons, D.M., Wind Power, Energy Technology Review No. 6, Noyes Data Corporation, New Jersey, 1975.
4. Eldridge, F.R., Wind Machines, The Mitre Corporation, Oct. 1975 (also NSF-RA-N-75-051).
5. Savino, J.N., ed., "Wind Energy Conversion Systems", First Workshop Proceedings, Washington, D.C., June 11-13, 1973 (also NASA TM X-69, 786, 1973).
6. Eldridge, F.R., ed., "Wind Energy Conversion Systems", Second Workshop Proceedings, Washington, D.C., June 9-11, 1975 (aslo NSF-RA-N-75-050).
7. Glauert, H., Aerodynamic Theory (W.F. Durand, Editor-in-Chief), Vol. 6, Division L, Julius Springer, Berlin, 1935.
8. Wilson, R.E. and Lissaman, P.B.S., "Applied Aerodynamics of Wind Power Machines", Oregon State University, July 1974.
9. Loewy, R.G., "Review of Rotary-Wing V/STOL Dynamic and Aeroelastic Problems", J. American Helicopter Society, Vol. 14, No. 3, July 1969, pp. 3-24.
10. Hohenemser, K.H., "Hingeless Rotorcraft Flight Dynamics", AGARDograph No. 197, Sept. 1974.
11. "Rotorcraft Dynamics", NASA Special Publication, NASA SP-352, 1974.
12. Ham, N.D., "Helicopter Blade Flutter", AGARD Report No. 607, Jan. 1973.
13. Ormiston, R.A. and Hodges, D.H., "Linear Flap-Lag Dynamics of Hingeless Helicopter Rotor Blades in Hover", J. American Helicopter Society, Vol. 17, No. 2, April 1972, pp. 2-14.

14. Friedmann, P. and Tong, P., "Dynamic Nonlinear Elastic Stability of Helicopter Rotor Blades in Hover and Forward Flight", NASA CR-114485, May 1972 (also M.I.T., ASRL TR 166-3).
15. Peters, D.A., "Flap-Lag Stability of Helicopter Rotor Blades in Forward Flight", J. American Helicopter Society, Vol. 20, No. 4, Oct. 1975, pp. 2-13.
16. Hodges, D.H. and Dowell, E.H., "Nonlinear Equations of Motion for the Elastic Bending and Torsion of Twisted Nonuniform Rotor Blades", NASA TND-7818, Dec. 1974.
17. Kaza, K.R.V. and Kvaternik, R.G., "A Critical Examination of the Flap-Lag Dynamics of Helicopter Rotor Blades in Hover and in Forward Flight", Preprint No. 1034, 32nd Annual National V/STOL Forum, American Helicopter Society, Washington, D.C., May 1976.
18. Friedmann, P., "Influence of Structural Damping, Preconing, Offsets and Large Deflections on the Flap-Lag-Torsional Stability of a Cantilevered Rotor Blade", AIAA Paper No. 75-780, AIAA/ASME/SAE 16th Structures, Structural Dynamics and Materials Conference, Colorado, May 1975.
19. Hodges, D.H. and Ormiston, R.A., "Stability of Hingeless Rotor Blades in Hover with Pitch Link Flexibility", Proceedings of AIAA/ASME/SAE 17th Structures, Structural Dynamics and Materials Conference, Pennsylvania, May 5-7, 1976, pp. 412-420.
20. Reed, W.H., III, "Review of Propeller-Rotor Whirl Flutter", NASA TR R-264, July 1967.
21. Johnson, W., "Dynamics of Tilting Proprotor Aircraft in Cruise Flight", NASA TN D-7677, May 1974.
22. Yasue, M., "A Study of Gust Response for a Rotor-Propeller in Cruising Flight", M.I.T., ASRL TR 174-1, August 1974.
23. Johnson, W., "Analytical Model for Tilting Proprotor Aircraft Dynamics including Blade Torsion and Coupled Bending Modes and Conversion Mode Operation", NASA TM X-62,369, August 1974.

24. Ormiston, R.A., "Rotor Dynamic Considerations for Large Wind Power Generator Systems", Wind Energy Conversion Systems, First Workshop Proceedings, J.M. Savino, Ed., NASA TM X-69, 786, 1973.

25. Ham, N.D., "Notes on the Aerodynamics, Dynamics, and Aeroelasticity of Wind Turbines and Rotorcraft, M.I.T., Course Notes 16.88, Sept 1976.

26. Donham, R.E., Schmidt, J. and Linscott, B.S., "100-KW Hingeless Metal Wind Turbine Blade Design, Analysis and Fabrication", Preprint No. S-998, 31st Annual National Forum, American Helicopter Society, Washington, D.C., May 1975.

27. Ormiston, R.A., "Dynamic Response of Wind Turbine Rotor Systems", Preprint No. S-993, 31st Annual National Forum, American Helicopter Society, Washington, D.C., May 1975.

28. Kaza, K.R.V. and Hammond, C.E., "An Investigation of Flap-Lag Stability of Wind Turbine Rotors in the Presence of Velocity Gradients and Helicopter Rotors in Forward Flight", Proceedings of AIAA/ASME/SAE 17th Structures, Structural Dynamics and Materials Conference, Pennsylvania, May 5-7, 1976, pp. 421-431.

29. Friedmann, P., "Aeroelastic Modeling of Large Wind Turbines", J. American Helicopter Society, Vol. 21, No. 4, Oct. 1976, pp. 17-27.

30. Miller, R.H., "Linear Aeroelastic Stability of Rotors", Chapter 5, Wind Energy Conversion, Final Report, NSF Grant No. AER 75-00826, Dec. 1976.

31. Dugundji, J., "Some Dynamic Problems of Rotating Windmill Systems", Proceedings of 13th Annual Meeting of Society of Engineering Science, Hampton, Va., Nov. 1-3, 1976 (also NASA CP-2001, Vol. 2, p. 439).

32. Spera, D.A., "Structural Analysis of Wind Turbine Rotors for NSF-NASA MOD-0 Wind Power System", NASA TM X-3198, March 1975.

33. Linscott, B.S. and Shapton, W.R., "Tower and Rotor Blade Vibration Test Results for 100-Kilowatt Wind Turbine", NASA TM X-3426, Oct. 1976.

34. Young, M.I., "A Theory of Rotor Blade Motion Stability in Powered Flight", J. American Helicopter Society, Vol. 9, No. 3, July 1964, pp. 12-25.

35. Hohenemser, K.H. and Heaton, P.W., Aeroelastic Instability of Torsionally Rigid Helicopter Blades", J. American Helicopter Society, Vol. 12, No. 2, April 1967, pp. 1-13.

36. Peters, D.A. and Ormiston, R.A., "Flapping Response Characteristics of Hingeless Rotor Blades by a Generalized Harmonic Balance Method", NASA TN D-7856, Feb. 1975.

37. Kuo, C.C., Morino, L. and Dugundji, J., "Perturbation and Harmonic Balance Methods for Nonlinear Flutter Analysis", AIAA J., Vol. 10, No. 11, Nov. 1972, pp. 1479-1484.

38. Tseng, W.V. and Dugundji, J., "Nonlinear Vibrations of Beam under Harmonic Excitation", J. Applied Mechanics, Trans. ASME, Vol. 37, No. 2, June 1970, pp. 292-297.

39. Davenport, A.G., "The Treatment of Wind Loading on Tall Buildings", Proceedings of a Symposium on Tall Buildings, Southampton, April 1966, (Pergamon Press, 1967).

40. Bolotin, V.V., Dynamic Stability of Elastic Systems, Holden-Day Inc., San Francisco, California, 1964.

APPENDIX I

The various constants used in the definition of aerodynamic forces are given below.

$$C_1 = \frac{4}{R^4}\int_0^R r^3 \sin\theta_0 \, dr$$

$$C_2 = \frac{4}{R^3}\int_0^R r^2 \sin\theta_0 \, dr$$

$$C_3 = \frac{4}{R^2}\int_0^R r \sin\theta_0 \, dr$$

$$C_4 = 1 + \frac{C_{d_0}}{a}$$

$$C_5 = -C_1 - C_2\left(\lambda\theta + \frac{e}{R}\right) - \frac{1}{2}CB_1\theta + CB_2\left(\lambda - \frac{e}{R}\theta\right) + \frac{1}{2}CB_3\left(\lambda^2\theta + \frac{e}{R}\lambda - \frac{e^2}{R^2}\theta\right)$$

$$C_6 = -\frac{C_1}{2} - 2C_2\left(\lambda\theta + \frac{e}{R}\right) - \frac{3}{2}C_3\frac{e}{R}\left(2\lambda\theta + \frac{e}{R}\right) - CB_1\theta + \frac{1}{2}CB_2\left(\lambda - 4\frac{e}{R}\theta\right) + CB_3\left(\lambda^2\theta + \lambda\frac{e}{R} - \frac{e^2}{R^2}\theta\right)$$

$$C_7 = -2C_2\left(\lambda - \frac{e}{R}\theta\right) - 2C_3\left(\lambda^2\theta + \lambda\frac{e}{R} - \frac{e^2}{R^2}\theta\right) - CB_1 - CB_2\left(2\lambda\theta + \frac{e}{R}\right) + CB_3\lambda\left(\lambda - 2\frac{e}{R}\theta\right)$$

$$C_8 = -CB_1\left(1 - \frac{\theta^2}{2}\right) - CB_2\lambda\theta - \frac{e}{R}CB_2\left(1 - \frac{\theta^2}{2}\right)$$

$$C_9 = 2C_1\left(1 - \frac{\theta^2}{2}\right) + 2\lambda C_2\theta + 2C_2\frac{e}{R}\left(1 - \frac{\theta^2}{2}\right) + CB_1\theta - CB_2\lambda\left(1 - \frac{\theta^2}{2}\right) + CB_2\frac{e}{R}\theta$$

$$C_{10} = CB_2\lambda$$

$$C_{11} = \frac{1}{2}CB_1$$

$$C_{12} = CB_1 + \frac{3}{2}CB_2\frac{e}{R}$$

$$C_{13} = -CB_1 - 2\lambda C_2$$

$$C_{14} = -C_1 + \frac{1}{2}CB_2\lambda$$

$$C_{15} = -C_1 - 2C_2\frac{e}{R} + \frac{1}{2}CB_2\lambda$$

$$C_0 = C_1(1-\theta^2) + 2C_2\lambda\left(\theta - \frac{2}{3}\theta^3\right) + 2C_2\frac{e}{R}(1-\theta^2) + C_3\lambda^2\theta^2 + 2C_3\lambda\frac{e}{R}\left(\theta - \frac{2}{3}\theta^3\right) + C_3\frac{e^2}{R^2}(1-\theta^2)$$
$$+ CB_1\left(\theta - \frac{2}{3}\theta^3\right) - CB_2\lambda(1-2\theta^2) + 2CB_2\frac{e}{R}\left(\theta - \frac{2}{3}\theta^3\right) - CB_3\lambda^2\left(\theta - \frac{2}{3}\theta^3\right) + CB_3\lambda\frac{e}{R}(2\theta^2-1) + CB_3\frac{e^2}{R^2}\left(\theta - \frac{2}{3}\theta^3\right)$$

$$B_1 = \frac{4}{R^4}\int_0^R r^3 \cos\theta_0 \, dr$$

$$B_2 = \frac{4}{R^3}\int_0^R r^2 \cos\theta_0 \, dr$$

$$B_3 = \frac{4}{R^2}\int_0^R r \cos\theta_0 \, dr$$

$$B_5 = -\frac{C_{d_0}}{a}\left(1 + \frac{4}{3}\lambda\theta + \frac{4}{3}\frac{e}{R}\right) + \frac{1}{2}C_1\theta - C_2\left(\lambda - \frac{e}{R}\theta\right) - \frac{1}{2}C_3\left(\lambda^2\theta + \lambda\frac{e}{R} - \frac{e^2}{R^2}\theta\right) - BC_2\lambda\theta$$
$$+ BC_3\lambda\left(\lambda - 2\frac{e}{R}\theta\right)$$

$$B_6 = -2\frac{C_{d_0}}{a}\left(\frac{2}{3}\lambda\theta + \frac{2}{3}\frac{e}{R} + 2\lambda\frac{e}{R}\theta + \frac{e^2}{R^2}\right) + \frac{1}{2}C_1\theta + C_2\frac{e}{R}\theta - \frac{1}{2}C_3\left(\lambda^2\theta + \lambda\frac{e}{R} - \frac{e^2}{R^2}\theta\right) - BC_2\lambda\theta$$

$$B_7 = -4\frac{C_{d_0}}{a}\left(\lambda^2\theta + \frac{2}{3}\lambda - \frac{2}{3}\frac{e}{R}\theta + \lambda\frac{e}{R} - \frac{e^2}{R^2}\theta\right) + C_1 + C_2\left(2\lambda\theta + \frac{e}{R}\right) - C_3\lambda\left(\lambda - 2\frac{e}{R}\theta\right)$$
$$+ 2BC_1\theta - 2BC_2\left(\lambda - \frac{e}{R}\theta\right)$$

$$B_8 = C_1\left(1 - \frac{\theta^2}{2}\right) + C_2\lambda\theta + C_2\frac{e}{R}\left(1 - \frac{\theta^2}{2}\right) + 2BC_1\theta - 2BC_2\left(\lambda - \frac{\lambda\theta^2}{2} - \frac{e}{R}\theta\right)$$

$$B_9 = \frac{C_{d_0}}{a}\left(2 - \theta^2 + \frac{8}{3}\lambda\theta + \frac{8}{3}\frac{e}{R} - \frac{4}{3}\frac{e}{R}\theta^2\right) - C_1\theta + C_2\left(\lambda - \frac{\lambda\theta^2}{2} - \frac{e}{R}\theta\right)$$

$$B_{10} = -C_2\lambda - 2BC_1$$

$$B_{11} = -\frac{1}{2}C_1 + BC_2\lambda$$

$$B_{12} = -\frac{1}{2}C_1 - C_2\frac{e}{R} + BC_2\lambda$$

$$B_{13} = -\frac{8}{3}\frac{C_{d_0}}{a}\lambda + C_1$$

$$B_{14} = -\frac{C_{d_0}}{a} - \frac{1}{2}C_2\lambda$$

$$B_{15} = -\frac{4}{3}\frac{C_{d_0}}{a}\frac{e}{R}$$

$$B_0 = \frac{C_{d_0}}{a}\left\{1 - \theta^2 + 2\lambda^2\theta^2 + \frac{8}{3}\lambda\left(\theta - \frac{2}{3}\theta^3\right) + \frac{8}{3}\frac{e}{R}\left(1 - \theta^2\right) + 4\frac{e}{R}\lambda\left(\theta - \frac{2}{3}\theta^3\right) + 2\frac{e^2}{R^2}\left(1 - \theta^2\right)\right\}$$
$$- C_1\left(\theta - \frac{2}{3}\theta^3\right) + C_2\left\{\lambda\left(1 - 2\theta^2\right) - 2\frac{e}{R}\left(\theta - \frac{2}{3}\theta^3\right)\right\} + C_3\left\{\lambda^2\left(\theta - \frac{2}{3}\theta^3\right) + \lambda\frac{e}{R}\left(1 - 2\theta^2\right) - \frac{e^2}{R^2}\left(\theta - \frac{2}{3}\theta^3\right)\right\}$$
$$- BC_1\theta^2 + BC_2\left\{2\lambda\left(\theta - \frac{2}{3}\theta^3\right) - 2\frac{e}{R}\theta^2\right\} + BC_3\left\{2\frac{e}{R}\lambda\left(\theta - \frac{2}{3}\theta^3\right) - \lambda^2\left(1 - \theta^2\right) - \frac{e^2}{R^2}\theta^2\right\}$$

$$CB_1 = B_1 + \frac{Cd_0}{a}$$

$$CB_2 = B_2 + \frac{4}{3}\frac{Cd_0}{a}$$

$$CB_3 = B_3 + 2\frac{Cd_0}{a}$$

$$BC_1 = B_1 - \frac{1}{2}\frac{Cd_0}{a}$$

$$BC_2 = B_2 - \frac{2}{3}\frac{Cd_0}{a}$$

$$BC_3 = B_3 - \frac{Cd_0}{a}$$

APPENDIX II

The elements of response matrix 'R' consist of three separate parts, i.e.,

(a) contribution from inertia and stiffness, RI

(b) contribution from gravity forces, RG

(c) contribution from aerodynamic forces, RA

so that

$$R_{ij} = RI(i,j) + RG(i,j) + \frac{\gamma}{8} RA(i,j)$$

$$R_{io} = RIO(i) + RGO(i) + \frac{\gamma}{8} RAO(i)$$

The various elements are given as

$$RI(1,1) = \bar{\omega}_\beta^2 \bar{z} + 1 + \bar{e} - \theta^2 - \frac{2}{3}\beta_c^2 - \phi_c^2 - N_I\left\{(a_1^2 + b_1^2) + \frac{1}{2}(a_2^2 + b_2^2)\right\}$$

$$RI(1,2) = 0$$

$$RI(1,3) = 0$$

$$RI(1,4) = \theta - N_I\left\{a_1 a_2 + b_1 b_2\right\}$$

$$RI(1,5) = 0$$

$$RI(1,6) = 0$$

$RI(2,1)=0$

$RI(2,2) = \bar{\omega}_\beta^2 Z + (1-\alpha^2)(1-\phi_c^2) - 2\beta_c^2 - \theta^2 + \bar{e} - N_I\left\{\frac{1}{2}(a_1^2+b_1^2) + \frac{1}{4}(1-\alpha^2)(3a_2^2+b_2^2) + \frac{\alpha^2}{2}(a_2^2-b_2^2)\right\}$

$RI(2,3) = -2\alpha\,\bar{\omega}_\beta\sqrt{Z}\,\zeta_\beta$

$RI(2,4)=0$

$RI(2,5) = \theta - 2\phi_c\beta_c - N_I\left\{\frac{1}{2}b_1 b_2(1+\alpha^2)\right\}$

$RI(2,6) = -2\alpha(\beta_c + \theta\phi_c)$

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$RI(3,1) = 0$

$RI(3,2) = 2\alpha\,\bar{\omega}_\beta\sqrt{Z}\,\zeta_\beta$

$RI(3,3) = \bar{\omega}_\beta^2 Z + (1-\alpha^2)(1-\phi_c^2) - 2\beta_c^2 - \theta^2 + \bar{e} - N_I\left\{\frac{1}{2}(a_1^2+b_1^2) + \frac{1}{4}(1-\alpha^2)(a_2^2+3b_2^2) - \frac{\alpha^2}{2}(a_2^2-b_2^2)\right\}$

$RI(3,4)=0$

$RI(3,5) = 2\alpha(\beta_c + \theta\phi_c)$

$RI(3,6) = \theta - 2\beta_c\phi_c - N_I\left\{\alpha(a_1 b_2 - a_2 b_1)\right\}$

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$RI(4,1) = \theta - N_I\left\{a_1 a_2 + b_1 b_2\right\}$

$RI(4,2)=0$

$RI(4,3)=0$

$RI(4,4) = Z + \bar{e} + \theta^2 - \beta_c^2 - N_I\left\{\frac{1}{2}(1-\alpha^2)(a_1^2+b_1^2)\right\}$

$RI(4,5)=0$

$RI(4,6)=0$

$RI(5,1) = 0$

$RI(5,2) = \theta - 2\beta_c \phi_c - N_I \left\{ \frac{1}{2} b_1 b_2 (1+\alpha^2) \right\}$

$RI(5,3) = 2\alpha(\beta_c + \theta\phi_c)$

$RI(5,4) = 0$

$RI(5,5) = Z + \bar{e} - \alpha^2 + \theta^2 - \beta_c^2 + N_I \left\{ \frac{1}{4}\alpha^2(a_1^2 + 3b_1^2) - \frac{1}{4}(3a_1^2 + b_1^2) \right\}$

$RI(5,6) = -2\alpha\sqrt{Z}\,\zeta_\phi$

$RI(6,1) = 0$

$RI(6,2) = -2\alpha(\beta_c + \theta\phi_c)$

$RI(6,3) = \theta - 2\beta_c\phi_c - N_I \left\{ \frac{1}{2} a_1 a_2 (1+\alpha^2) \right\}$

$RI(6,4) = 0$

$RI(6,5) = 2\alpha\sqrt{Z}\,\zeta_\phi$

$RI(6,6) = Z + \bar{e} - \alpha^2 + \theta^2 - \beta_c^2 + N_I \left\{ \frac{1}{4}\alpha^2(3a_1^2 + b_1^2) - \frac{1}{4}(a_1^2 + 3b_1^2) \right\}$

$RI\Phi(1) = -\bar{\omega}_\beta^2 Z \beta_s - N_I \left\{ \frac{1}{2} a_1 a_2 (1+\alpha^2) \right\}$

$RIO(2) = 0$

$RIO(3) = 0$

$RIO(4) = -Z\phi_s - N_I \left\{ \alpha\theta(a_1 b_2 - a_2 b_1) \right\}$

$RIO(5) = 0$

$RI\Phi(6) = 0$

Gravity terms for forced oscillations

$RG(1,1) = 0$

$RG(1,2) = \frac{1}{2} G \theta \beta_c$

$RG(1,3) = G\left(\frac{1}{2} - \frac{\beta_c^2}{4} - \frac{\phi_c^2}{4}\right) - N_I\left\{\frac{G}{16}\left(a_1^2 + b_1^2 + a_2^2 + 3b_2^2\right)\right\}$

$RG(1,4) = 0$

$RG(1,5) = \frac{1}{2} G \theta \phi_c$

$RG(1,6) = -\frac{1}{2} G \beta_c \phi_c - N_I\left\{\frac{G}{8} a_1 a_2\right\}$

$RG(2,1) = -N_I\left\{\frac{G}{4}\left(a_1 b_2 + a_2 b_1\right)\right\}$

$RG(2,2) = 0$

$RG(2,3) = 0$

$RG(2,4) = -N_I\left\{\frac{G}{4}\left(a_1 b_2 + a_2 b_1\right)\right\}$

$RG(2,5) = 0$

$RG(2,6) = 0$

$RG(3,1) = G\left(1 - \frac{\beta_c^2}{6} - \frac{\phi_c^2}{2}\right) - N_I\left\{\frac{G}{8}\left(a_1^2 + 3b_1^2 + a_2^2 + 3b_2^2\right)\right\}$

$RG(3,2) = 0$

$RG(3,3) = 0$

$RG(3,4) = -N_I\left\{\frac{G}{4}\left(a_1 a_2 + 3b_1 b_2\right)\right\}$

$RG(3,5) = 0$

$RG(3,6) = 0$

$RG(4,1) = 0$

$RG(4,2) = \frac{1}{2} G \beta_c \phi_c$

$RG(4,3) = -\frac{1}{2} G \beta_c \phi_c - N_I \left\{ \frac{G}{8} a_1 a_2 \right\}$

$RG(4,4) = 0$

$RG(4,5) = -\frac{1}{2} G (\phi_c - \theta \beta_c)$

$RG(4,6) = G\left(\frac{1}{2} - \frac{\beta_c^2}{4} - \frac{\phi_c^2}{4}\right) - N_I \left\{ \frac{G}{16} (a_1^2 + 3b_1^2 + a_2^2 + b_2^2) \right\}$

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$RG(5,1) = -N_I \left\{ \frac{G}{4} (a_1 b_2 + a_2 b_1) \right\}$

$RG(5,2) = 0$

$RG(5,3) = 0$

$RG(5,4) = -N_I \left\{ \frac{G}{4} (a_1 b_1 + a_2 b_2) \right\}$

$RG(5,5) = 0$

$RG(5,6) = 0$

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$RG(6,1) = -N_I \left\{ \frac{G}{4} (a_1 a_2 + 3 b_1 b_2) \right\}$

$RG(6,2) = 0$

$RG(6,3) = 0$

$RG(6,4) = G\left(1 - \frac{\beta_c^2}{2} - \frac{\phi_c^2}{6}\right) - N_I \left\{ \frac{G}{8} (a_1^2 + 3b_1^2 + a_2^2 + 3b_2^2) \right\}$

$RG(6,5) = 0$

$RG(6,6) = 0$

$$RGO(1) = 0$$

$$RGO(2) = -G\theta\left(1 - \frac{\theta^2}{6} - \frac{\beta_c^2}{2} - \frac{\phi_c^2}{2}\right) + N_I\left\{ \frac{G\theta}{8}\left(3a_1^2 + b_1^2 + 3a_2^2 + b_2^2\right)\right\}$$

$$RGO(3) = N_I\left\{ \frac{G\theta}{4}\left(a_1 b_1 + a_2 b_2\right)\right\}$$

$$RGO(4) = 0$$

$$RGO(5) = G\left(1 - \frac{\theta^2}{2} - \frac{\phi_c^2}{2} + \theta \beta_c \phi_c\right) - N_I\left\{ \frac{G}{8}\left(3a_2^2 + b_2^2\right) - \frac{G\theta}{4}\left(3a_1 a_2 + b_1 b_2\right)\right\}$$

$$RGO(6) = -N_I\left\{ \frac{G}{4} a_2 b_2 - \frac{G\theta}{4}\left(a_1 b_2 + a_2 b_1\right)\right\}$$

Gravity terms for Parametric Resonance

$$RG(1,1) = N_I \left\{ \frac{G}{8} (a_1^2 - b_1^2 + a_2^2 - b_2^2) \right\}$$

$$RG(1,2) = 0$$

$$RG(1,3) = 0$$

$$RG(1,4) = N_I \left\{ \frac{G}{4} (a_1 a_2 - b_1 b_2) \right\}$$

$$RG(1,5) = 0$$

$$RG(1,6) = 0$$

- - - - - - - - - - - - - - - -

$$RG(2,1) = 0$$

$$RG(2,2) = -G\left(\frac{1}{2} - \frac{\beta_c^2}{4} - \frac{\phi_c^2}{4} \right) + N_I \left\{ \frac{G}{12} (a_1^2 + 3a_2) \right\}$$

$$RG(2,3) = \frac{1}{2} G \beta_c \theta$$

$$RG(2,4) = 0$$

$$RG(2,5) = \frac{1}{2} G \beta_c \phi_c$$

$$RG(2,6) = \frac{1}{2} G \phi_c \theta$$

- - - - - - - - - - - - - - - -

$$RG(3,1) = 0$$

$$RG(3,2) = \frac{1}{2} G \beta_c \theta$$

$$RG(3,3) = G\left(\frac{1}{2} - \frac{\beta_c^2}{4} - \frac{\phi_c^2}{4} \right) - N_I \left\{ \frac{G}{12} (b_1^2 + 3b_2^2) \right\}$$

$$RG(3,4) = 0$$

$$RG(3,5) = \frac{1}{2} G \phi_c \theta$$

$$RG(3,6) = -\frac{1}{2} G \beta_c \phi_c$$

$$RG(4,1) = N_I \left\{ \frac{G}{4}(a_1 a_2 - b_1 b_2) \right\}$$

$$RG(4,2) = 0$$

$$RG(4,3) = 0$$

$$RG(4,4) = N_I \left\{ \frac{G}{8}(a_1^2 - b_1^2 + a_2^2 - b_2^2) \right\}$$

$$RG(4,5) = 0$$

$$RG(4,6) = 0$$

- - - - - - - - - - - - - - -

$$RG(5,1) = 0$$

$$RG(5,2) = \frac{1}{2} G \beta_c \phi_c$$

$$RG(5,3) = \frac{1}{2} G \phi_c \theta$$

$$RG(5,4) = 0$$

$$RG(5,5) = -G\left(\frac{1}{2} - \frac{\beta_c^2}{4} - \frac{\phi_c^2}{4}\right) + N_I \left\{ \frac{G}{12}(3a_1^2 + a_2^2) \right\}$$

$$RG(5,6) = -\frac{1}{2} G(\phi_c - \theta \beta_c)$$

- - - - - - - - - - - - - - -

$$RG(6,1) = 0$$

$$RG(6,2) = \frac{1}{2} G \phi_c \theta$$

$$RG(6,3) = -\frac{1}{2} G \beta_c \phi_c$$

$$RG(6,4) = 0$$

$$RG(6,5) = -\frac{1}{2} G(\phi_c - \theta \beta_c)$$

$$RG(6,6) = G\left(\frac{1}{2} - \frac{\beta_c^2}{4} - \frac{\phi_c^2}{4}\right) - N_I \left\{ \frac{G}{12}(3b_1^2 + b_2^2) \right\}$$

$$RGO(1) = N_I \left\{ \frac{G\theta}{4}(a_1 b_1 + a_2 b_2) \right\}$$

$$RGO(2) = 0$$

$$RGO(3) = 0$$

$$RGO(4) = -N_I \left\{ \frac{G}{4} a_2 b_2 - \frac{G\theta}{4}(a_1 b_2 + a_2 b_1) \right\}$$

$$RGO(5) = 0$$

$$RGO(6) = 0$$

In the above elements

$$N_I \equiv \begin{cases} 0 & \text{for linear case} \\ 1 & \text{for nonlinear case} \end{cases}$$

Aerodynamic contribution terms

$$RA(1,1) = -N_A \left\{ \alpha (a_1 b_2 - a_2 b_1)\left(\frac{C_{10}}{2} - C_{14} \right) \right\}$$

$$RA(1,2) = 0$$

$$RA(1,3) = 0$$

$$RA(1,4) = -N_A \left\{ \alpha (a_1 b_2 - a_2 b_1)\left(C_{12} - \frac{C_{13}}{2} \right) \right\}$$

$$RA(1,5) = 0$$

$$RA(1,6) = 0$$

- - - - - - - - - - - - -

$$RA(2,1) = 0$$

$$RA(2,2) = -2\beta_c C_5 - \phi_c C_7 - N_A \left\{ \tfrac{1}{2} \alpha a_2 b_2 C_{12} \right\}$$

$$RA(2,3) = \alpha C_8 + \alpha \beta_c \phi_c C_{10} + \alpha \beta_c^2 C_{11} + \alpha \phi_c^2 C_{12} + N_A \Big\{ \frac{\alpha}{4}(a_1^2 + b_1^2) C_{11} + \frac{\alpha}{4}(3a_2^2 + b_2^2) C_{12}$$
$$+ \frac{\alpha}{4}(a_2^2 + b_2^2) C_{15} \Big\}$$

$$RA(2,4) = 0$$

$$RA(2,5) = -2\phi_c C_6 - \beta_c C_7 + N_A \left\{ \frac{\alpha}{2} a_1 b_1 (C_{10} - C_{14}) \right\}$$

$$RA(2,6) = \alpha C_9 + \alpha \beta_c \phi_c C_{13} + \alpha \beta_c^2 C_{14} + \alpha \phi_c^2 C_{15} + N_A \Big\{ -\frac{\alpha}{4}(a_1^2 - b_1^2) C_{10} + \frac{\alpha}{2} a_1 a_2 C_{13} + \frac{\alpha}{4}(3a_1^2 + b_1^2) C_{14}$$
$$+ \frac{\alpha}{4}(a_2^2 + b_2^2) C_{15}$$

- - - - - - - - - - - - - - - - -

$$RA(3,1) = 0$$

$$RA(3,2) = -\alpha C_8 - \alpha \beta_c \phi_c C_{10} - \alpha \beta_c^2 C_{11} - \alpha \phi_c^2 C_{12} - N_A \left\{ \frac{\alpha}{4}(a_1^2 - b_1^2) C_{11} + \frac{\alpha}{4}(a_2^2 + 3b_2^2) C_{12} + \frac{\alpha}{4}(a_2^2 - b_2^2) C_{13} \right\}$$

$$RA(3,3) = -2\beta_c C_5 - \phi_c C_7 + N_A \left\{ \tfrac{1}{2} \alpha a_2 b_2 C_{12} \right\}$$

$$RA(3,4) = 0$$

$$RA(3,5) = -\alpha C_9 - \alpha \beta_c \phi_c C_{13} - \alpha \beta_c^2 C_{14} - \alpha \phi_c^2 C_{15} - N_A \Big\{ \frac{\alpha}{4}(a_1^2 - b_1^2) C_{10} + \frac{\alpha}{2} b_1 b_2 C_{13} + \frac{\alpha}{4}(a_1^2 + 3b_1^2) C_{14}$$
$$+ \frac{\alpha}{4}(a_2^2 + b_2^2) C_{15} \Big\}$$

$$RA(3,6) = -2\phi_c C_6 - \beta_c C_7 - N_A \left\{ \frac{\alpha}{2} a_1 b_1 (C_{10} - C_{14}) \right\}$$

$$RA(4,1) = N_A \left\{ \alpha (a_1 b_2 - a_2 b_1) \left(\frac{B_{10}}{2} - B_{14} \right) \right\}$$

$$RA(4,2) = 0$$

$$RA(4,3) = 0$$

$$RA(4,4) = N_A \left\{ \alpha (a_1 b_2 - a_2 b_1) \left(B_{12} - \frac{B_{13}}{2} \right) \right\}$$

$$RA(4,5) = 0$$

$$RA(4,6) = 0$$

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$$RA(5,1) = 0$$

$$RA(5,2) = 2\beta_c B_5 + \phi_c B_7 + N_A \left\{ \tfrac{1}{2} \alpha a_2 b_2 B_{12} \right\}$$

$$RA(5,3) = -\alpha B_8 - \alpha \beta_c \phi_c B_{10} - \alpha \beta_c^2 B_{11} - \alpha \phi_c^2 B_{12} + N_A \left\{ -\tfrac{\alpha}{4}(a_1^2 + b_1^2) B_{11} - \tfrac{\alpha}{4}(3a_2^2 + b_2^2) B_{12} + \tfrac{\alpha}{4}(a_2^2 - b_2^2) B_{13} \right\}$$

$$RA(5,4) = 0$$

$$RA(5,5) = 2\phi_c B_6 + \beta_c B_7 + N_A \left\{ -\tfrac{1}{2} \alpha a_1 b_1 (B_{10} - B_{14}) \right\}$$

$$RA(5,6) = -\alpha B_9 - \alpha \beta_c \phi_c B_{13} - \alpha \beta_c^2 B_{14} - \alpha \phi_c^2 B_{15} + N_A \left\{ \tfrac{\alpha}{4}(a_1^2 - b_1^2) B_{10} - \tfrac{\alpha}{2} a_1 a_2 B_{13} - \tfrac{\alpha}{4}(3a_1^2 + b_1^2) B_{14} - \tfrac{\alpha}{4}(a_2^2 + b_2^2) B_{15} \right\}$$

- - - - - - - - - - - - - - - - -

$$RA(6,1) = 0$$

$$RA(6,2) = \alpha B_8 + \alpha \beta_c \phi_c B_{10} + \alpha \beta_c^2 B_{11} + \alpha \phi_c^2 B_{12} + N_A \left\{ \tfrac{\alpha}{4}(a_1^2 + b_1^2) B_{11} + \tfrac{\alpha}{4}(a_2^2 + 3b_2^2) B_{12} + \tfrac{\alpha}{4}(a_2^2 - b_2^2) B_{13} \right\}$$

$$RA(6,3) = 2\beta_c B_5 + \phi_c B_7 - N_A \left\{ \tfrac{1}{2} \alpha a_2 b_2 B_{12} \right\}$$

$$RA(6,4) = 0$$

$$RA(6,5) = \alpha B_9 + \alpha \beta_c \phi_c B_{13} + \alpha \beta_c^2 B_{14} + \alpha \phi_c^2 B_{15} + N_A \left\{ \tfrac{\alpha}{4}(a_1^2 - b_1^2) B_{10} + \tfrac{\alpha}{2} b_1 b_2 B_{13} + \tfrac{\alpha}{4}(a_1^2 + 3b_1^2) B_{14} + \tfrac{\alpha}{4}(a_2^2 + b_2^2) B_{15} \right\}$$

$$RA(6,6) = 2\phi_c B_6 + \beta_c B_7 + N_A \left\{ \tfrac{\alpha}{2} a_1 b_1 (B_{10} - B_{14}) \right\}$$

$$RAO(1) = -C_0 - \beta_c^2 C_5 - \phi_c^2 C_6 - \beta_c \phi_c C_7 - N_A \left\{ \tfrac{1}{2}(a_1^2 + b_1^2) C_5 + \tfrac{1}{2}(a_2^2 + b_2^2) C_6 + \tfrac{1}{2}(a_1 a_2 + b_1 b_2) C_7 \right.$$
$$\left. - \tfrac{1}{2}\alpha^2 (a_1 a_2 + b_1 b_2) CB_1 + \tfrac{1}{2}\alpha^2 (a_2^2 + b_2^2) C_1 \right\}$$

$$RAO(2) = 0$$

$$RAO(3) = 0$$

$$RAO(4) = B_0 + \beta_c^2 B_5 + \phi_c^2 B_6 + \beta_c \phi_c B_7 + N_A \left\{ \tfrac{1}{2}(a_1^2 + b_1^2) B_5 + \tfrac{1}{2}(a_2^2 + b_2^2) B_6 + \tfrac{1}{2}(a_1 a_2 + b_1 b_2) B_7 \right.$$
$$\left. + \tfrac{1}{2}\alpha^2 (a_1 a_2 + b_1 b_2) C_1 + \tfrac{1}{2}\alpha^2 (a_2^2 + b_2^2) \frac{C_{d_0}}{a} - \tfrac{1}{2}\alpha^2 (a_1^2 + b_1^2) BC_1 \right\}$$

$$RAO(5) = 0$$

$$RAO(6) = 0$$

In the above elements

$$N_A \equiv \begin{cases} 0 \text{ for linear aerodynamic case} \\ 1 \text{ for nonlinear aerodynamic case} \end{cases}$$

APPENDIX III

The elements A_{ij} of Eq. (113) are divided into two parts

(a) contribution from inertia and stiffness, AI

(b) contribution from aerodynamic forces, AA

Thus

$$A_{ij} = AI(i,j) + \frac{\gamma}{8} AA(i,j)$$

$$A_{i0} = AIO(i) + \frac{\gamma}{8} AAO(i)$$

The various elements are defined below

$$AI(1,1) = 1 + \bar{e} - \theta^2 - \frac{2}{3}\beta_c^2 - \phi_c^2 - N_I\left\{\frac{1}{2}a_2^2 + a_1^2 + b_1^2\right\}$$

$$AI(1,2) = 0$$

$$AI(1,3) = 0$$

$$AI(1,4) = \theta - N_I\left\{a_1 a_2\right\}$$

$$AI(1,5) = \bar{\omega}_\beta^2 (\beta_c - \beta_s)$$

$$AI(1,6) = N_I\left\{a_2 b_1\right\}$$

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$$AI(2,1) = 0$$

$$AI(2,2) = 1 + \bar{e} - \alpha^2 - \theta^2 - \phi_c^2(1-\alpha^2) - 2\beta_c^2 - N_I\left\{\frac{a_2^2}{4}(3-\alpha^2) + \frac{1}{2}(a_1^2 + b_1^2)\right\}$$

$$AI(2,3) = -2\alpha\, \bar{\omega}_\beta \sqrt{\bar{z}}\; \zeta_\beta$$

$$AI(2,4) = 0$$

$$AI(2,5) = \bar{\omega}_\beta^2 a_1$$

$$AI(2,6) = 0$$

$AI(3,1) = 0$

$AI(3,2) = 2\alpha\,\bar{\omega}_\beta\sqrt{Z}\,\zeta_{1\beta}$

$AI(3,\ ) = 1 + \bar{e} - \alpha^2 - \theta^2 - \phi_c^2(1-\alpha^2) - 2\beta_c^2 - N_I\left\{\frac{a_2^2}{4}(1-3\alpha^2) + \frac{1}{2}(a_1^2 + b_1^2)\right\}$

$AI(3,4) = 0$

$AI(3,5) = \bar{\omega}_\beta^2 b_1$

$AI(3,6) = 0$

- - - - - - - - - - - - -

$AI(4,1) = \theta - N_I\left\{a_1 a_2\right\}$

$AI(4,2) = 0$

$AI(4,3) = 0$

$AI(4,4) = \bar{e} + \theta^2 - \beta_c^2 - N_I\left\{\frac{1}{2}(1-\alpha^2)(a_1^2 + b_1^2)\right\}$

$AI(4,5) = \phi_c - \phi_s$

$AI(4,6) = N_I\left\{\theta\, a_2 b_1\right\}$

- - - - - - - - - - - - - -

$AI(5,1) = 0$

$AI(5,2) = \theta - 2\beta_c\phi_c$

$AI(5,3) = 2\alpha(\beta_c + \theta\phi_c)$

$AI(5,4) = 0$

$AI(5,5) = a_2$

$AI(5,6) = 0$

$AI(6,1) = 0$

$AI(6,2) = -2\alpha(\beta_c + \theta\phi_c)$

$AI(6,3) = \theta - 2\beta_c\phi_c$

$AI(6,4) = 0$

$AI(6,5) = 0$

$AI(6,6) = 2\sqrt{\bar{E}}\ a_2\zeta_\phi$

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$AIO(1) = 0$

$AIO(2) = \theta a_2 - 2\beta_c\phi_c a_2$

$AIO(3) = 2\alpha a_2(\beta_c + \theta\phi_c)$

$AIO(4) = 0$

$AIO(5) = a_2(\bar{E} - \alpha^2 + \theta^2 - \beta_c^2) + N_I\left\{\frac{\alpha^2}{4}a_2(a_1^2 + 3b_1^2) - a_2(3a_1^2 + b_1^2)\right\}$

$AIO(6) = -N_I\left\{\frac{1}{2}a_1b_1a_2(1+\alpha^2)\right\}$

In the above elements

$$N_I \equiv \begin{cases} 0 \text{ for linear case} \\ 1 \text{ for nonlinear case} \end{cases}$$

Aerodynamic contribution terms

$AA(1,1) = 0$

$AA(1,2) = -N_A\left\{\frac{a_2}{2}(C_7 - \alpha^2 CB_1)\right\}$

$AA(1,3) = -N_A\left\{\alpha\beta_c a_2\left(-\frac{C_{10}}{2} + C_{14}\right) + \alpha\phi_c a_2\left(-C_{12} + \frac{C_{13}}{2}\right)\right\}$

$AA(1,4) = 0$

$AA(1,5) = 0$

$AA(1,6) = 0$

- - - - - - - - - - - - - -

$AA(2,1) = -a_2 C_7$

$AA(2,2) = -2\beta_c C_5 - \phi_c C_7$

$AA(2,3) = \alpha(C_8 + \beta_c\phi_c C_{10} + \beta_c^2 C_{11} + \phi_c^2 C_{12}) + N_A\left\{\frac{\alpha}{4}(a_1^2 + b_1^2)C_{11} + \frac{\alpha}{4}a_2^2(3C_{12} - C_{13})\right\}$

$AA(2,4) = -2a_2 C_6$

$AA(2,5) = 0$

$AA(2,6) = N_A\left\{\frac{1}{2}a_1 b_1 a_2(C_{10} - C_{14})\right\}$

- - - - - - - - - - - - - - - -

$AA(3,1) = 0$

$AA(3,2) = -\alpha(C_8 + \beta_c\phi_c C_{10} + \beta_c^2 C_{11} + \phi_c^2 C_{12}) - N_A\left\{\frac{\alpha}{4}(a_1^2 + b_1^2)C_{11} + \frac{\alpha}{4}a_2^2(C_{12} + C_{13})\right\}$

$AA(3,3) = -2\beta_c C_5 - \phi_c C_7$

$AA(3,4) = 0$

$AA(3,5) = 0$

$AA(3,6) = -a_2(C_9 + \beta_c\phi_c C_{13} + \beta_c^2 C_{14} + \phi_c^2 C_{15}) - N_A\left\{\frac{a_2}{4}(a_1^2 - b_1^2)C_{10} + \frac{a_2}{4}(a_1^2 + 3b_1^2)C_{14} + \frac{a_2^3}{4}C_{15}\right\}$

$$AA(4,1) = 0$$

$$AA(4,2) = N_A \left\{ \frac{a_2}{2}(B_7 + \alpha^2 C_1) \right\}$$

$$AA(4,3) = N_A \left\{ \alpha \beta_c a_2 \left(-\frac{B_{10}}{2} + B_{14}\right) + \alpha \phi_c a_2 \left(-B_{12} + \frac{B_{13}}{2}\right) \right\}$$

$$AA(4,4) = 0$$

$$AA(4,5) = 0$$

$$AA(4,6) = 0$$

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$$AA(5,1) = a_2 B_7$$

$$AA(5,2) = 2\beta_c B_5 + \phi_c B_7$$

$$AA(5,3) = -\alpha \left(B_8 + \beta_c \phi_c B_{10} + \beta_c^2 B_{11} + \phi_c^2 B_{12}\right) - N_A \left\{ \frac{\alpha}{4}(a_1^2 + b_1^2) B_{11} + \frac{\alpha}{4} a_2^2 (3B_{12} - B_{13}) \right\}$$

$$AA(5,4) = 2 a_2 B_6$$

$$AA(5,5) = 0$$

$$AA(5,6) = -N_A \left\{ \frac{1}{2} a_1 b_1 a_2 (B_{10} - B_{14}) \right\}$$

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$$AA(6,1) = 0$$

$$AA(6,2) = \alpha \left(B_8 + \beta_c \phi_c B_{10} + \beta_c^2 B_{11} + \phi_c^2 B_{12}\right) + N_A \left\{ \frac{\alpha}{4} a_2^2 (B_{12} + B_{13}) + \frac{\alpha}{4}(a_1^2 + b_1^2) B_{11} \right\}$$

$$AA(6,3) = 2\beta_c B_5 + \phi_c B_7$$

$$AA(6,4) = 0$$

$$AA(6,5) = 0$$

$$AA(6,6) = a_2 \left(B_9 + \beta_c \phi_c B_{13} + \beta_c^2 B_{14} + \phi_c^2 B_{15}\right) + N_A \left\{ \frac{a_2^3}{4} B_{15} + \frac{a_2}{4}(a_1^2 - b_1^2) B_{10} + \frac{a_2}{4}(a_1^2 + 3b_1^2) B_{14} \right\}$$

$$AAO(1) = -(C_0 + \beta_c^2 C_5 + \phi_c^2 C_6 + \beta_c \phi_c C_7) - N_A \left\{ \frac{1}{2}(a_1^2 + b_1^2) C_5 + \frac{1}{2} a_2^2 C_6 + \alpha^2 a_2^2 C_1 \right\}$$

$$AAO(2) = 0$$

$$AAO(3) = 0$$

$$AAO(4) = B_0 + \beta_c^2 B_5 + \phi_c^2 B_6 + \beta_c \phi_c B_7 + N_A \left\{ \frac{1}{2} a_2^2 B_6 + \alpha^2 a_2^2 \frac{C_{d_0}}{a} - \frac{\alpha^2}{2}(a_1^2 + b_1^2) + (a_1^2 + b_1^2) \frac{B_5}{2} \right\}$$

$$AAO(5) = 0$$

$$AAO(6) = 0$$

In the above elements

$$N_A \equiv \begin{cases} 0 & \text{for linear aerodynamic case} \\ 1 & \text{for nonlinear aerodynamic case} \end{cases}$$

APPENDIX IV

The various constants used in the definition of aerodynamic forces with shear flow are given below:

$$\tilde{C}_0 = C_0 - \lambda_1(CB_4 + \tfrac{e}{R}CB_1) + 2\lambda_1\theta(CH_1 + \tfrac{e}{R}C_1) - EB_3\theta + E_1\theta^2 + 2\lambda_1\theta^2(CB_4 + \tfrac{e}{R}CB_1) + \tfrac{2}{3}EB_3\theta^3$$
$$- \tfrac{4}{3}\lambda_1\theta^3(CH_1 + \tfrac{e}{R}C_1)$$

$$\tilde{C}_5 = C_5 + \lambda_1(CB_4 + \tfrac{1}{2}\tfrac{e}{R}CB_1) - \theta(\lambda_1 CH_1 - \tfrac{1}{2}EB_3)$$

$$\tilde{C}_6 = C_6 + \lambda_1(\tfrac{e}{R}CB_1 + \tfrac{1}{2}CB_4) - \lambda_1\theta(2CH_1 + 3\tfrac{e}{R}C_1) + EB_3\theta$$

$$\tilde{C}_7 = C_7 - 2\lambda_1(CH_1 + \tfrac{e}{R}C_1) + EB_3 - 2E_1\theta - 2\lambda_1\theta(CB_4 + \tfrac{e}{R}CB_1)$$

$$\tilde{C}_8 = C_8 - \lambda_1 CB_4\theta$$

$$\tilde{C}_9 = C_9 + \lambda_1(-CB_4 + 2CH_1\theta + \tfrac{1}{2}CB_4\theta^2)$$

$$\tilde{C}_{10} = C_{10} + \lambda_1 CB_4$$

$$\tilde{C}_{11} = C_{11}$$

$$\tilde{C}_{12} = C_{12}$$

$$\tilde{C}_{13} = C_{13} - 2\lambda_1 CH_1$$

$$\tilde{C}_{14} = C_{14} + \tfrac{1}{2}\lambda_1 CB_4$$

$$\tilde{C}_{15} = C_{15} + \tfrac{1}{2}\lambda_1 CB_4$$

$$\tilde{B}_0 = B_0 + \lambda_1(CH_1 + \tfrac{e}{R}C_1) - EC_3 + 2\lambda_1\tfrac{C_{do}}{a}\theta(\tfrac{4}{5} + \tfrac{e}{R}) + E_1\theta + 2\lambda_1\theta(BC_4 + \tfrac{e}{R}BC_1)$$
$$+\theta^2(\tfrac{C_{do}}{a}\bar{E}_5 + EC_3) - 2\lambda_1\theta^2(CH_1 + \tfrac{e}{R}C_1) - \tfrac{4}{3}\tfrac{C_{do}}{a}\lambda_1\theta^3(\tfrac{4}{5} + \tfrac{e}{R}) - \tfrac{2}{3}E_1\theta^3$$
$$- \tfrac{4}{3}\lambda_1\theta^3(BC_4 + \tfrac{e}{R}BC_1)$$

$$\tilde{B}_5 = B_5 - \lambda_1\left(CH_1 + \frac{1}{2}\frac{e}{R}C_1\right) + EC_3 - \frac{1}{2}E_1\theta - \lambda_1\theta\left(BC_4 + 2\frac{e}{R}BC_1 + \frac{4}{5}\frac{C_{do}}{a}\right).$$

$$\tilde{B}_6 = B_6 - \frac{1}{2}\frac{e}{R}\lambda_1 C_1 - 2\frac{C_{do}}{a}\lambda_1\theta\left(\frac{2}{5} + \frac{e}{R}\right) - \theta\left(\frac{1}{2}E_1 + \lambda_1 BC_4\right)$$

$$\tilde{B}_7 = B_7 - 2\frac{C_{do}}{a}\lambda_1\left(\frac{4}{5} + \frac{e}{R}\right) - \left(E_1 + 2\lambda_1 BC_4\right) - 2\frac{C_{do}}{a}\bar{E}_5\theta + 2\lambda_1\theta\left(CH_1 + \frac{e}{R}C_1\right)$$

$$\tilde{B}_8 = B_8 + \lambda_1\left(-2BC_4 + \theta CH_1 + \theta^2 BC_4\right)$$

$$\tilde{B}_9 = B_9 + \lambda_1\left(CH_1 + \frac{8}{5}\frac{C_{do}}{a}\theta - \frac{1}{2}CH_1\theta^2\right)$$

$$\tilde{B}_{10} = B_{10} - \lambda_1 CH_1$$

$$\tilde{B}_{11} = B_{11} + \lambda_1 BC_4$$

$$\tilde{B}_{12} = B_{12} + \lambda_1 BC_4$$

$$\tilde{B}_{13} = B_{13} - \frac{8}{5}\frac{C_{do}}{a}\lambda_1$$

$$\tilde{B}_{14} = B_{14} - \frac{1}{2}\lambda_1 CH_1$$

$$\tilde{B}_{15} = B_{15}$$

$$D_0 = -\lambda_2\left(CB_1 + \frac{e}{R}CB_2\right) + 2\lambda_2\theta\left(C_1 + \frac{e}{R}C_2\right) - EB_4\theta + E_2\theta^2 + 2\lambda_2\theta^2\left(CB_1 + \frac{e}{R}CB_2\right)$$
$$-\frac{4}{3}\lambda_2\theta^3\left(C_1 + \frac{e}{R}C_2\right) + \frac{2}{3}EB_4\theta^3$$

$$D_5 = \lambda_2\left(CB_1 + \frac{1}{2}\frac{e}{R}CB_2\right) + \theta\left(-\lambda_2 C_1 + \frac{1}{2}EB_4\right)$$

$$D_6 = \frac{1}{2}\lambda_2\left(CB_1 + 2\frac{e}{R}CB_2\right) - \lambda_2\theta\left(2C_1 + 3\frac{e}{R}C_2\right) + EB_4\theta$$

$$D_7 = -2\lambda_2\left(C_1 + \frac{e}{R}C_2\right) + EB_4 - 2E_2\theta - 2\lambda_2\theta\left(CB_1 + \frac{e}{R}CB_2\right)$$

$$D_8 = -\lambda_2 CB_1\theta$$

$$D_9 = \lambda_2\left(-CB_1 + 2C_1\theta + \frac{1}{2}CB_1\theta^2\right)$$

$$D_{10} = \lambda_2 \, CB_1$$

$$D_{11} = 0$$

$$D_{12} = 0$$

$$D_{13} = -2\lambda_2 C_1$$

$$D_{14} = \tfrac{1}{2}\lambda_2 \, CB_1$$

$$D_{15} = \tfrac{1}{2}\lambda_2 \, CB_1$$

$$E_0 = \lambda_2\left(C_1 + \frac{e}{R}C_2\right) - EC_4 + 2\frac{C_{d_0}}{a}\lambda_2\theta\left(1 + \frac{4}{3}\frac{e}{R}\right) + E_2\theta + 2\lambda_2\theta\left(BC_1 + \frac{e}{R}BC_2\right)$$
$$+ \theta^2\left(\frac{C_{d_0}}{a}\bar{E}_6 + EC_4\right) - 2\lambda_2\theta^2\left(C_1 + \frac{e}{R}C_2\right) - \frac{4}{3}\frac{C_{d_0}}{a}\lambda_2\theta^3\left(1 + \frac{4}{3}\frac{e}{R}\right) - \frac{2}{3}E_2\theta^3$$
$$- \frac{4}{3}\lambda_2\theta^3\left(BC_1 + \frac{e}{R}BC_2\right)$$

$$E_5 = -\lambda_2\left(C_1 + \frac{1}{2}\frac{e}{R}C_2\right) + EC_4 - \frac{1}{2}E_2\theta - \lambda_2\theta\left(\frac{C_{d_0}}{a} + BC_1 + 2\frac{e}{R}BC_2\right)$$

$$E_6 = -\frac{1}{2}\frac{e}{R}\lambda_2 C_2 - \frac{C_{d_0}}{a}\lambda_2\theta\left(1 + \frac{8}{3}\frac{e}{R}\right) - \theta\left(\frac{1}{2}E_2 + \lambda_2 BC_1\right)$$

$$E_7 = -2\frac{C_{d_0}}{a}\lambda_2\left(1 + \frac{4}{3}\frac{e}{R}\right) - \left(E_2 + 2\lambda_2 BC_1\right) - 2\frac{C_{d_0}}{a}\bar{E}_6\theta + 2\lambda_2\theta\left(C_1 + \frac{e}{R}C_2\right)$$

$$E_8 = \lambda_2\left(-2BC_1 + C_1\theta + BC_1\theta^2\right)$$

$$E_9 = \lambda_2\left(C_1 + 2\frac{C_{d_0}}{a}\theta - \frac{1}{2}C_1\theta^2\right)$$

$$E_{10} = -\lambda_2 C_1$$

$$E_{11} = \lambda_2 \, BC_1$$

$$E_{12} = \lambda_2 \, BC_1$$

$$E_{13} = -2\frac{C_{d_0}}{a}\lambda_2$$

$$E_{14} = -\frac{1}{2}\lambda_2 C_1$$

$$E_{15} = 0$$

$$CH_1 = \frac{4}{R^5}\int_0^R r^4 \sin\theta_0 \, dr$$

$$CH_2 = \frac{4}{R^6}\int_0^R r^5 \sin\theta_0 \, dr$$

$$BH_1 = \frac{4}{R^5}\int_0^R r^4 \cos\theta_0 \, dr$$

$$BH_2 = \frac{4}{R^6}\int_0^R r^5 \cos\theta_0 \, dr$$

$$CB_4 = BH_1 + \frac{4}{5}\frac{C_{d_0}}{a}$$

$$BC_4 = BH_1 - \frac{2}{5}\frac{C_{d_0}}{a}$$

$$E_1 = \frac{4}{R^2}\int_0^R r\left\{\lambda_1^2\left(\frac{r}{R}\right)^4 + \frac{1}{2}\lambda_2^2\left(\frac{r}{R}\right)^2 + 2\lambda_0\lambda_1\left(\frac{r}{R}\right)^2\right\}\sin\theta_0 \, dr$$

$$E_2 = \frac{4}{R^2}\int_0^R r\left\{2\lambda_2\lambda_0\left(\frac{r}{R}\right) + 2\lambda_2\lambda_1\left(\frac{r}{R}\right)^3\right\}\sin\theta_0 \, dr$$

$$E_3 = \frac{4}{R^2}\int_0^R r\left\{\lambda_1^2\left(\frac{r}{R}\right)^4 + \frac{1}{2}\lambda_2^2\left(\frac{r}{R}\right)^2 + 2\lambda_0\lambda_1\left(\frac{r}{R}\right)^2\right\}\cos\theta_0 \, dr$$

$$E_4 = \frac{4}{R^2}\int_0^R r\left\{2\lambda_0\lambda_2\frac{r}{R} + 2\lambda_1\lambda_2\left(\frac{r}{R}\right)^3\right\}\cos\theta_0 \, dr$$

$$\bar{E}_5 = \frac{4}{R^2}\int_0^R r\left\{\lambda_1^2\left(\frac{r}{R}\right)^4 + \frac{1}{2}\lambda_2^2\left(\frac{r}{R}\right)^2 + 2\lambda_0\lambda_1\left(\frac{r}{R}\right)^2\right\}dr$$

$$\bar{E}_6 = \frac{4}{R^2}\int_0^R r\left\{2\lambda_0\lambda_2\frac{r}{R} + 2\lambda_1\lambda_2\left(\frac{r}{R}\right)^3\right\}dr$$

$$EB_3 = E_1 + \frac{C_{d_0}}{a}E_5$$

$$EB_4 = E_2 + \frac{C_{d_0}}{a}E_6$$

$$EC_3 = E_1 - \frac{1}{2}\frac{C_{d_0}}{a}E_5$$

$$EC_4 = E_2 - \frac{1}{2}\frac{C_{d_0}}{a}E_6$$

APPENDIX V

Element Matrix RV for FORCED RESPONSE

$$RV(1,1) = 0$$

$$RV(1,2) = -\tfrac{1}{2}\left(D_8 + \beta_c \phi_c D_{10} + \beta_c^2 D_{11} + \phi_c^2 D_{12}\right) - N_A \tfrac{1}{8}\left\{(a_1 a_2 + 2 b_1 b_2) D_{10} + (a_1^2 + b_1^2) D_{11} + (a_2^2 + 3 b_2^2) D_{12} - b_2^2 D_{13} - 2 b_1 b_2 D_{14}\right\}$$

$$RV(1,3) = -\beta_c D_5 - \tfrac{1}{2}\phi_c D_7$$

$$RV(1,4) = 0$$

$$RV(1,5) = -\tfrac{1}{2}\left(D_9 + \beta_c \phi_c D_{13} + \beta_c^2 D_{14} + \phi_c^2 D_{15}\right) - N_A \tfrac{1}{8}\left\{(a_1 a_2 + 2 b_1 b_2) D_{13} + (a_1^2 + 3 b_1^2) D_{14} + (a_2^2 + b_2^2) D_{15} - b_1^2 D_{10} - 2 b_1 b_2 D_{12}\right\}$$

$$RV(1,6) = -\phi_c D_6 - \tfrac{1}{2}\beta_c D_7$$

$$RV(2,1) = -N_A\left\{\tfrac{1}{4}(a_1 a_2 - b_1 b_2)(D_{10} + 2 D_{14}) + \tfrac{1}{2}(a_1^2 - b_1^2) D_{11} + \tfrac{1}{4}(a_2^2 - b_2^2) D_{13}\right\}$$

$$RV(2,2) = 0$$

$$RV(2,3) = 0$$

$$RV(2,4) = -N_A\left\{\tfrac{1}{4}(a_1 a_2 - b_1 b_2)(D_{13} + 2 D_{12}) + \tfrac{1}{4}(a_1^2 - b_1^2) D_{10} + \tfrac{1}{2}(a_2^2 - b_2^2) D_{15}\right\}$$

$$RV(2,5) = 0$$

$$RV(2,6) = 0$$

$$RV(3,1) = -N_A\left\{\tfrac{1}{4}(3 a_1 b_2 - a_2 b_1) D_{10} + a_1 b_1 D_{11} + \tfrac{1}{2} a_2 b_2 D_{13} + \tfrac{1}{2}(3 a_2 b_1 - a_1 b_2) D_{14}\right\}$$

$$RV(3,2) = 0$$

$$RV(3,3) = 0$$

$$RV(3,4) = -N_A\left\{\tfrac{1}{2} a_1 b_1 D_{10} + \tfrac{1}{2}(3 a_1 b_2 - a_2 b_1) D_{12} + \tfrac{1}{4}(3 a_2 b_1 - a_1 b_2) D_{13} + a_2 b_2 D_{15}\right\}$$

$$RV(3,5) = 0$$

$$RV(3,6) = 0$$

$$RV(4,1) = 0$$

$$RV(4,2) = \frac{1}{2}(E_8 + \beta_c \phi_c E_{10} + \beta_c^2 E_{11} + \phi_c^2 E_{12}) + \frac{1}{8} N_A \{ (a_1 a_2 + 2 b_1 b_2) E_{10} + (a_1^2 + b_1^2) E_{11} + (a_2^2 + 3b_2^2) E_{12} - b_2^2 E_{13} - 2 b_1 b_2 E_{14} \}$$

$$RV(4,3) = \beta_c E_5 + \frac{1}{2} \phi_c E_7$$

$$RV(4,4) = 0$$

$$RV(4,5) = \frac{1}{2}(E_9 + \beta_c \phi_c E_{13} + \beta_c^2 E_{14} + \phi_c^2 E_{15}) + N_A \frac{1}{8} \{ (a_1 a_2 + 2 b_1 b_2) E_{13} + (a_1^2 + 3b_1^2) E_{14} + (a_2^2 + b_2^2) E_{15} - b_1^2 E_{10} - 2 b_1 b_2 E_{12} \}$$

$$RV(4,6) = \phi_c E_6 + \frac{1}{2} \beta_c E_7$$

$$RV(5,1) = N_A \{ \frac{1}{4}(a_1 a_2 - b_1 b_2)(E_{10} + 2 E_{14}) + \frac{1}{2}(a_1^2 - b_1^2) E_{11} + \frac{1}{4}(a_2^2 - b_2^2) E_{13} \}$$

$$RV(5,2) = 0$$

$$RV(5,3) = 0$$

$$RV(5,4) = N_A \{ \frac{1}{4}(a_1 a_2 - b_1 b_2)(E_{13} + 2 E_{12}) + \frac{1}{4}(a_1^2 - b_1^2) E_{10} + \frac{1}{2}(a_2^2 - b_2^2) E_{15} \}$$

$$RV(5,5) = 0$$

$$RV(5,6) = 0$$

$$RV(6,1) = N_A \{ \frac{1}{4}(3 a_1 b_2 - a_2 b_1) E_{10} + a_1 b_1 E_{11} + \frac{1}{2} a_2 b_2 E_{13} + \frac{1}{2}(3 a_2 b_1 - a_1 b_2) E_{14} \}$$

$$RV(6,2) = 0$$

$$RV(6,3) = 0$$

$$RV(6,4) = N_A \{ \frac{1}{2} a_1 b_1 E_{10} + \frac{1}{2}(3 a_1 b_2 - a_2 b_1) E_{12} + \frac{1}{4}(3 a_2 b_1 - a_1 b_2) E_{13} + a_2 b_2 E_{15} \}$$

$$RV(6,5) = 0$$

$$RV(6,6) = 0$$

$$RVO(1) = 0$$

$$RVO(2) = -N_A \left\{ \tfrac{1}{2}(a_1 b_1 D_5 + a_2 b_2 D_6) + \tfrac{1}{4}(a_1 b_2 + a_2 b_1) D_7 \right\}$$

$$RVO(3) = -\left(D_0 + \beta_c^2 D_5 + \phi_{1c}^2 D_6 + \beta_{1c} \phi_{1c} D_7\right) - N_A \tfrac{1}{4} \left\{ (a_1^2 + 3 b_1^2) D_5 + (a_2^2 + 3 b_2^2) D_6 + (a_1 a_2 + 3 b_1 b_2) D_7 \right\}$$

$$RVO(4) = 0$$

$$RVO(5) = N_A \left\{ \tfrac{1}{2}(a_1 b_1 E_5 + a_2 b_2 E_6) + \tfrac{1}{4}(a_1 b_2 + a_2 b_1) E_7 \right\}$$

$$RVO(6) = E_0 + \beta_c^2 E_5 + \phi_c^2 E_6 + \beta_c \phi_c E_7 + N_A \tfrac{1}{4} \left\{ (a_1^2 + 3 b_1^2) E_5 + (a_2^2 + 3 b_2^2) E_6 + (a_1 a_2 + 3 b_1 b_2) E_7 \right\}$$

Elements of Matrix RV for PARAMETRIC RESONANCE

$$RV(1,1) = -N_A\left\{\frac{1}{8}(a_1b_2+a_2b_1)(D_{10}+2D_{14}) + \frac{1}{2}a_1b_1D_{11} + \frac{1}{4}a_2b_2D_{13}\right\}$$

$$RV(1,2) = 0$$

$$RV(1,3) = 0$$

$$RV(1,4) = -N_A\left\{\frac{1}{8}(a_1b_2+a_2b_1)(D_{13}+2D_{12}) + \frac{1}{4}a_1b_1D_{10} + \frac{1}{2}a_2b_2D_{15}\right\}$$

$$RV(1,5) = 0$$

$$RV(1,6) = 0$$

$$RV(2,1) = 0$$

$$RV(2,2) = \beta_c D_5 + \frac{1}{2}\phi_c D_7$$

$$RV(2,3) = -\frac{1}{4}\left(D_8 + \beta_c\phi_c D_{10} + \beta_c^2 D_{11} + \phi_c^2 D_{12}\right) - N_A\frac{1}{4}\left\{a_1a_2D_{10} + a_1^2D_{11} + a_2^2D_{12}\right\}$$

$$RV(2,4) = 0$$

$$RV(2,5) = \phi_c D_6 + \frac{1}{2}\beta_c D_7$$

$$RV(2,6) = -\frac{1}{4}\left(D_9 + \beta_c\phi_c D_{13} + \beta_c^2 D_{14} + \phi_c^2 D_{15}\right) - N_A\frac{1}{4}\left\{a_1a_2D_{13} + a_1^2D_{14} + a_2^2D_{15}\right\}$$

$$RV(3,1) = 0$$

$$RV(3,2) = -\frac{1}{4}\left\{D_8 + \beta_c\phi_c D_{10} + \beta_c^2 D_{11} + \phi_c^2 D_{12}\right\} - N_A\frac{1}{4}\left\{b_1b_2D_{10} + b_1^2D_{11} + b_2^2D_{12}\right\}$$

$$RV(3,3) = -\left(\beta_c D_5 + \frac{1}{2}\phi_c D_7\right)$$

$$RV(3,4) = 0$$

$$RV(3,5) = -\frac{1}{4}\left\{D_9 + \beta_c\phi_c D_{13} + \beta_c^2 D_{14} + \phi_c^2 D_{15}\right\} - N_A\frac{1}{4}\left\{b_1b_2D_{13} + b_1^2D_{14} + b_2^2D_{15}\right\}$$

$$RV(3,6) = -\left(\phi_c D_6 + \frac{1}{2}\beta_c D_7\right)$$

$$RV(4,1) = N_A\left\{ \tfrac{1}{8}(a_1b_2 + a_2b_1)(E_{10} + 2E_{14}) + \tfrac{1}{2}a_1b_1E_{11} + \tfrac{1}{2}a_2b_2E_{13} \right\}$$

$$RV(4,2) = 0$$

$$RV(4,3) = 0$$

$$RV(4,4) = N_A\left\{ \tfrac{1}{8}(a_1b_2 + a_2b_1)(E_{13} + 2E_{12}) + \tfrac{1}{4}a_1b_1E_{10} + \tfrac{1}{2}a_2b_2E_{15} \right\}$$

$$RV(4,5) = 0$$

$$RV(4,6) = 0$$

$$RV(5,1) = 0$$

$$RV(5,2) = -(\beta_c E_5 + \tfrac{1}{2}\phi_c E_7)$$

$$RV(5,3) = \tfrac{1}{4}(E_8 + \beta_c\phi_c E_{10} + \beta_c^2 E_{11} + \phi_c^2 E_{12}) + N_A \tfrac{1}{4}\left\{a_1a_2E_{10} + a_1^2E_{11} + a_2^2E_{12}\right\}$$

$$RV(5,4) = 0$$

$$RV(5,5) = -(\phi_c E_6 + \tfrac{1}{2}\beta_c E_7)$$

$$RV(5,6) = \tfrac{1}{4}(E_9 + \beta_c\phi_c E_{13} + \beta_c^2 E_{14} + \phi_c^2 E_{15}) + N_A \tfrac{1}{4}\left\{a_1a_2E_{13} + a_1^2E_{14} + a_2^2E_{15}\right\}$$

$$RV(6,1) = 0$$

$$RV(6,2) = \tfrac{1}{4}(E_8 + \beta_c\phi_c E_{10} + \beta_c^2 E_{11} + \phi_c^2 E_{12}) + N_A \tfrac{1}{4}\left\{b_1b_2E_{10} + b_1^2E_{11} + b_2^2E_{12}\right\}$$

$$RV(6,3) = \beta_c E_5 + \tfrac{1}{2}\phi_c E_7$$

$$RV(6,4) = 0$$

$$RV(6,5) = \tfrac{1}{4}(E_9 + \beta_c\phi_c E_{13} + \beta_c^2 E_{14} + \phi_c^2 E_{15}) + N_A \tfrac{1}{4}\left\{b_1b_2E_{13} + b_1^2E_{14} + b_2^2E_{15}\right\}$$

$$RV(6,6) = \phi_c E_6 + \tfrac{1}{2}\beta_c E_7$$

$$RVO(1) = N_A \frac{1}{4}\left\{ (a_1^2 - b_1^2) D_5 + (a_2^2 - b_2^2) D_6 + (a_1 a_2 - b_1 b_2) D_7 \right\}$$

$$RVO(2) = 0$$

$$RVO(3) = 0$$

$$RVO(4) = -N_A \frac{1}{4}\left\{ (a_1^2 - b_1^2) E_5 + (a_2^2 - b_2^2) E_6 + (a_1 a_2 - b_1 b_2) E_7 \right\}$$

$$RVO(5) = 0$$

$$RVO(6) = 0$$

APPENDIX VI

The various constants used in the definition of aerodynamic forces for flapping-lagging-feathering rotor are given below

$$F_0 = C_1 + 2\frac{e}{R}C_2 + \frac{e^2}{R^2}C_3 - (\lambda_0 CB_2 + \lambda_1 CB_4) - \frac{e}{R}(\lambda_0 CB_3 + \lambda_1 CB_1)$$

$$F_1 = -\frac{\bar{\eta}}{R}\left(CB_2 + \frac{e}{R}CB_3\right)$$

$$F_2 = 2(\lambda_0 C_2 + \lambda_1 CH_1) + 2\frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1) + CB_1 - (\lambda_0^2 CB_3 + EB_3) + 2\frac{e}{R}CB_2 + \frac{e^2}{R^2}CB_3$$

$$F_3 = -C_1 - \frac{e}{R}C_2 + (\lambda_0 CB_2 + \lambda_1 CB_4) + \frac{1}{2}\frac{e}{R}(\lambda_0 CB_3 + \lambda_1 CB_1)$$

$$F_4' = -\frac{e}{R}C_2 - \frac{e^2}{R^2}C_3 + \frac{1}{2}\frac{e}{R}(\lambda_0 CB_3 + \lambda_1 CB_1)$$

$$F_4 = F_4' - \frac{1}{2}F_0$$

$$F_5 = -C_1 + (\lambda_0^2 C_3 + E_1) - 2\frac{e}{R}C_2 - \frac{e^2}{R^2}C_3 + 2(\lambda_0 CB_2 + \lambda_1 CB_4) + 2\frac{e}{R}(\lambda_0 CB_3 + \lambda_1 CB_1)$$

$$F_6 = -2(\lambda_0 C_2 + \lambda_1 CH_1) - 2\frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1) - CB_1 + (\lambda_0^2 CB_3 + EB_3) - \frac{e}{R}CB_2$$

$$F_7 = -\frac{\bar{\eta}}{R}(\lambda_0 CB_3 + \lambda_1 CB_1)$$

$$F_8 = -\frac{\bar{\eta}}{R}\left(CB_2 + \frac{e}{R}CB_3\right)$$

$$F_9 = -2(\lambda_0^2 C_3 + E_1) + 2\frac{e}{R}C_2 + 2\frac{e^2}{R^2}C_3 - 2(\lambda_0 CB_2 + \lambda_1 CB_4) - 2\frac{e}{R}(\lambda_0 CB_3 + \lambda_1 CB_1)$$

$$F_{10} = -(\lambda_0 C_2 + \lambda_1 CH_1) - \frac{1}{2}CB_1 + \frac{1}{2}(\lambda_0^2 CB_3 + EB_3) - \frac{e}{R}CB_2 - \frac{1}{2}\frac{e^2}{R^2}CB_3$$

$$F_{11} = -(\lambda_0 C_2 + \lambda_1 CH_1) - 2\frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1) - \frac{1}{2}CB_1 + \frac{1}{2}(\lambda_0^2 CB_3 + EB_3) - \frac{e}{R}CB_2 - \frac{1}{2}\frac{e^2}{R^2}CB_3 - \frac{1}{2}F_2$$

$$F_{12} = -\frac{4}{3}(\lambda_0 C_2 + \lambda_1 CH_1) - \frac{4}{3}\frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1) - \frac{2}{3}CB_1 + \frac{2}{3}(\lambda_0^2 CB_3 + EB_3) - \frac{4}{3}\frac{e}{R}CB_2 - \frac{2}{3}\frac{e^2}{R^2}CB_3$$

$$F_{13} = -\left(CB_1 + \frac{e}{R}CB_2\right)$$

$$F_{14} = 2C_1 + 2\frac{e}{R}C_2 - (\lambda_0 CB_2 + \lambda_1 CB_4)$$

$$F_{15} = -\frac{\bar{\eta}}{R}\left(CB_2 + \frac{e}{R}CB_3\right)$$

$$F_{16} = \frac{\bar{\eta}}{R}\left(\frac{e}{R}CB_3 + CB_2\right)$$

$$F_{17} = -(\lambda_0 CB_2 + \lambda_1 CB_4)$$

$$F_{18} = -\frac{\bar{\eta}}{R} CB_2$$

$$F_{19} = 2(\lambda_0 C_2 + \lambda_1 CH_1) + CB_1 + \frac{e}{R} CB_2$$

$$F_{20} = -2\frac{e}{R} C_2 + (\lambda_0 CB_2 + \lambda_1 CB_4) - 2C_1$$

$$F_{21} = -CB_1 - \frac{e}{R} CB_2$$

$$F_{22} = -\frac{\bar{\eta}}{R}(\lambda_0 CB_3 + \lambda_1 CB_1)$$

$$F_{23} = (\lambda_0 CB_2 + \lambda_1 CB_4)$$

$$F_{24} = \frac{1}{2} CB_1$$

$$F_{25} = \frac{1}{2} CB_1 + \frac{e}{R} CB_2 - \frac{1}{2} F_{13}$$

$$F_{26} = \frac{1}{2}\frac{e}{R} CB_2 + \frac{1}{2} CB_1$$

$$F_{27} = -2(\lambda_0 C_2 + \lambda_1 CH_1) - CB_1$$

$$F_{28} = -C_1 + \frac{1}{2}(\lambda_0 CB_2 + \lambda_1 CB_4)$$

$$F_{29} = -\frac{e}{R} C_2 - \frac{1}{2} F_{14}$$

$$F_{30} = -C_1 - \frac{e}{R} C_2 + \frac{1}{2}(\lambda_0 CB_2 + \lambda_1 CB_4)$$

$$F_{31} = -2(\lambda_0 C_2 + \lambda_1 CH_1)$$

$$F_{32} = -CB_1 - \frac{e}{R} CB_2$$

$$F_{33} = -(\lambda_0 CB_2 + \lambda_1 CB_4)$$

$$F_{34} = -CB_1$$

$$F_{35} = -\frac{\bar{\eta}}{R} CB_2$$

$$F_{36} = CB_1$$

$$F_{37} = -CB_1$$

$$F_{38} = C_1$$

$$F_{39} = -\frac{1}{2} F_{15}$$

$$F_{44} = -2C_1$$

$$F_{45} = -\frac{1}{2} F_1$$

$$G_0 = \frac{C_{d_0}}{a}\left(1 + \frac{8}{3}\frac{e}{R} + 2\frac{e^2}{R^2}\right) + (\lambda_0 C_2 + \lambda_1 CH_1) + \frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1) - (\lambda_0^2 BC_3 + EC_3)$$

$$G_1 = \frac{\bar{\eta}}{R}\left\{\frac{e}{R}C_3 - 2(\lambda_0 BC_3 + \lambda_1 BC_1) + C_2\right\}$$

$$G_2 = \frac{C_{d_0}}{a}\left(\frac{8}{3}\lambda_0 + \frac{8}{5}\lambda_1\right) + 2\frac{C_{d_0}}{a}\frac{e}{R}(2\lambda_0 + \lambda_1) - C_1 + (\lambda_0^2 C_3 + E_1) - 2\frac{e}{R}C_2 - \frac{e^2}{R^2}C_3 + 2(\lambda_0 BC_2 + \lambda_1 BC_4)$$
$$+ 2\frac{e}{R}(\lambda_0 BC_3 + \lambda_1 BC_1)$$

$$G_3 = -\frac{C_{d_0}}{a}\left(1 + \frac{4}{3}\frac{e}{R}\right) - (\lambda_0 C_2 + \lambda_1 CH_1) - \frac{1}{2}\frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1) + (\lambda_0^2 BC_3 + EC_3) - \frac{\bar{\eta}^2}{R^2}BC_3$$

$$G_4 = -2\frac{C_{d_0}}{a}\frac{e}{R}\left(\frac{2}{3} + \frac{e}{R}\right) - \frac{1}{2}\frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1)$$

$$G_5 = -\frac{C_{d_0}}{a}\left(1 + \frac{8}{3}\frac{e}{R} + 2\frac{e^2}{R^2}\right) + \frac{C_{d_0}}{a}(2\lambda_0^2 + E_5) - 2(\lambda_0 C_2 + \lambda_1 CH_1) - 2\frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1) - BC_1$$
$$+ (\lambda_0^2 BC_3 + EC_3) - 2\frac{e}{R}BC_2 - \frac{e^2}{R^2}BC_3$$

$$G_6 = -\frac{C_{d_0}}{a}\left(\frac{8}{3}\lambda_0 + \frac{8}{5}\lambda_1\right) - 2\frac{C_{d_0}}{a}\frac{e}{R}(2\lambda_0 + \lambda_1) + C_1 - (\lambda_0^2 C_3 + E_1) + \frac{e}{R}C_2 - 2(\lambda_0 BC_2 + \lambda_1 BC_4)$$

$$G_7 = \frac{\bar{\eta}}{R}(\lambda_0 C_3 + \lambda_1 C_1) + 2\frac{\bar{\eta}}{R}\left(BC_2 + \frac{e}{R}BC_3\right)$$

$$G_8 = \frac{\bar{\eta}}{R}\left(C_2 + \frac{e}{R}C_3\right) - 2\frac{\bar{\eta}}{R}(\lambda_0 BC_3 + \lambda_1 BC_1)$$

$$G_9 = -2\frac{C_{d_0}}{a}(2\lambda_0^2 + E_5) + 4\frac{C_{d_0}}{a}\frac{e}{R}\left(\frac{2}{3} + \frac{e}{R}\right) + 2(\lambda_0 C_2 + \lambda_1 CH_1) + 2\frac{e}{R}(\lambda_0 C_3 + \lambda_1 C_1)$$
$$+ 2BC_1 + 2\frac{e}{R}BC_2$$

$$G_{10} = -\frac{1}{2}\frac{C_{d_0}}{a}\left(\frac{8}{3}\lambda_0 + \frac{8}{5}\lambda_1\right) + \frac{1}{2}C_1 - \frac{1}{2}(\lambda_0^2 C_3 + E_1) + \frac{e}{R}C_2 + \frac{1}{2}\frac{e^2}{R^2}C_3 - (\lambda_0 BC_2 + \lambda_1 BC_4)$$
$$-2\frac{e}{R}(\lambda_0 BC_3 + \lambda_1 BC_1)$$

$$G_{11} = -\frac{1}{2}\frac{C_{d_0}}{a}\left(\frac{8}{3}\lambda_0 + \frac{8}{5}\lambda_1\right) - 2\frac{C_{d_0}}{a}\frac{e}{R}(2\lambda_0 + \lambda_1) + \frac{1}{2}C_1 - \frac{1}{2}(\lambda_0^2 C_3 + E_1) + \frac{e}{R}C_2 + \frac{1}{2}\frac{e^2}{R^2}C_3$$
$$-(\lambda_0 BC_2 + \lambda_1 BC_4)$$

$$G_{12} = -\frac{2}{3}\frac{C_{d_0}}{a}\left(\frac{8}{3}\lambda_0 + \frac{8}{5}\lambda_1\right) - \frac{4}{3}\frac{e}{R}\frac{C_{d_0}}{a}(2\lambda_0 + \lambda_1) + \frac{2}{3}C_1 - \frac{2}{3}(\lambda_0^2 C_3 + E_1) + \frac{4}{3}\frac{e}{R}C_2$$
$$+ \frac{2}{3}\frac{e^2}{R^2}C_3 - \frac{4}{3}(\lambda_0 BC_2 + \lambda_1 BC_4) - \frac{4}{3}\frac{e}{R}(\lambda_0 BC_3 + \lambda_1 BC_1)$$

$$G_{13} = C_1 + \frac{e}{R}C_2 - 2(\lambda_0 BC_2 + \lambda_1 BC_4)$$

$$G_{14} = 2\frac{C_{d_0}}{a}\left(1 + \frac{4}{3}\frac{e}{R}\right) + (\lambda_0 C_2 + \lambda_1 CH_1)$$

$$G_{15} = \frac{\bar{\eta}}{R}\left(C_2 + \frac{e}{R}C_3\right) - 2\frac{\bar{\eta}}{R}(\lambda_0 BC_3 + \lambda_1 BC_1)$$

$$G_{16} = -\frac{e}{R}\frac{\bar{\eta}}{R}C_3 + 2\frac{\bar{\eta}}{R}(\lambda_0 BC_3 + \lambda_1 BC_1)$$

$$G_{17} = (\lambda_0 C_2 + \lambda_1 CH_1) + 2BC_1 + 2\frac{e}{R}BC_2 - \frac{\bar{\eta}}{R}C_2$$

$$G_{18} = \frac{\bar{\eta}}{R}C_2$$

$$G_{19} = \frac{Cd_0}{a}\left(\frac{8}{3}\lambda_0 + \frac{8}{5}\lambda_1\right) - C_1 - \frac{e}{R}C_2$$

$$G_{20} = -2\frac{Cd_0}{a}\left(1 + \frac{4}{3}\frac{e}{R}\right) - (\lambda_0 C_2 + \lambda_1 CH_1) - 2\frac{\bar{\eta}^2}{R^2}BC_3$$

$$G_{21} = C_1 + \frac{e}{R}C_2 - 2(\lambda_0 BC_2 + \lambda_1 BC_4)$$

$$G_{22} = \frac{\bar{\eta}}{R}(\lambda_0 C_3 + \lambda_1 C_1) + 2\frac{\bar{\eta}}{R}\left(BC_2 + \frac{e}{R}BC_3\right)$$

$$G_{23} = -(\lambda_0 C_2 + \lambda_1 CH_1) - 2BC_1$$

$$G_{24} = -\frac{1}{2}C_1 + (\lambda_0 BC_2 + \lambda_1 BC_4)$$

$$G_{25} = -\frac{1}{2}C_1 - \frac{e}{R}C_2 + (\lambda_0 BC_2 + \lambda_1 BC_4)$$

$$G_{26} = -\frac{1}{2}\left(C_1 + \frac{e}{R}C_2\right) + (\lambda_0 BC_2 + \lambda_1 BC_4)$$

$$G_{27} = -\frac{Cd_0}{a}\left(\frac{8}{3}\lambda_0 + \frac{8}{5}\lambda_1\right) + C_1$$

$$G_{28} = -\frac{Cd_0}{a} - \frac{1}{2}(\lambda_0 C_2 + \lambda_1 CH_1)$$

$$G_{29} = -\frac{4}{3}\frac{Cd_0}{a}\frac{e}{R}$$

$$G_{30} = -\frac{4}{3}\frac{Cd_0}{a}\frac{e}{R} - \frac{1}{2}(\lambda_0 C_2 + \lambda_1 CH_1) - \frac{Cd_0}{a}$$

$$G_{31} = \frac{e}{R}C_2$$

$$G_{32} = -\frac{Cd_0}{a}\left(\frac{8}{3}\lambda_0 + \frac{8}{5}\lambda_1\right) + C_1$$

$$G_{33} = (\lambda_0 C_2 + \lambda_1 CH_1) + 2BC_1 + 2\frac{e}{R}BC_2$$

$$G_{34} = C_1$$

$$G_{35} = \frac{\bar{\eta}}{R}C_2$$

$$G_{36} = -C_1$$

$$G_{37} = C_1$$

$$G_{38} = \frac{Cd_0}{a}$$

$$G_{39} = 0$$

$$G_{40} = -\frac{\bar{\eta}^2}{R^2} BC_3$$

$$G_{41} = -2\frac{\bar{\eta}}{R} BC_2$$

$$G_{42} = -2\frac{\bar{\eta}}{R} BC_2$$

$$G_{43} = -2BC_1$$

$$G_{44} = -2\frac{C_{d_0}}{a}$$

$$G_{45} = 0$$

$$G_{46} = -BC_1$$

$$R_0 = -\lambda_2 \left(CB_1 + \frac{e}{R} CB_2 \right)$$

$$R_1 = 0$$

$$R_2 = 2\lambda_2 C_1 + 2\frac{e}{R}\lambda_2 C_2 - EB_4$$

$$R_3 = \lambda_2 CB_1 + \frac{1}{2}\frac{e}{R}\lambda_2 CB_2$$

$$R_4' = \frac{1}{2}\lambda_2 \frac{e}{R} CB_2$$

$$R_4 = R_4' - \frac{1}{2} R_0$$

$$R_5 = E_2 + 2\lambda_2 CB_1 + 2\frac{e}{R}\lambda_2 CB_2$$

$$R_6 = -2\lambda_2 C_1 - 2\frac{e}{R}\lambda_2 C_2 + EB_4$$

$$R_7 = -\frac{\bar{\eta}}{R}\lambda_2 CB_2$$

$$R_8 = 0$$

$$R_9 = -2\left(E_2 + \lambda_2 CB_1 + \frac{e}{R}\lambda_2 CB_2 \right)$$

$$R_{10} = -\lambda_2 C_1 + \frac{1}{2} EB_4$$

$$R_{11} = -2\lambda_2 C_1 - 3\frac{e}{R}\lambda_2 C_2 + EB_4$$

$$R_{12} = -\frac{4}{3}\lambda_2 C_1 - \frac{4}{3}\frac{e}{R}\lambda_2 C_2 + \frac{2}{3} EB_4$$

$$R_{13} = 0$$

$R_{14} = -\lambda_2 C B_1$

$R_{15} = 0$

$R_{16} = 0$

$R_{17} = -\lambda_2 C B_1$

$R_{18} = 0$

$R_{19} = 2\lambda_2 C_1$

$R_{20} = \lambda_2 C B_1$

$R_{21} = 0$

$R_{22} = -\lambda_2 C B_2 \frac{\bar{\eta}}{R}$

$R_{23} = \lambda_2 C B_1$

$R_{24} = 0$

$R_{25} = 0$

$R_{26} = 0$

$R_{27} = -2\lambda_2 C_1$

$R_{28} = \frac{1}{2}\lambda_2 C B_1$

$R_{29} = \frac{1}{2}\lambda_2 C B_1$

$R_{30} = \frac{1}{2}\lambda_2 C B_1$

$R_{31} = -2\lambda_2 C_1$

$R_{32} = 0$

$R_{33} = -\lambda_2 C B_1$

$S_0 = \lambda_2 C_1 + \frac{e}{R}\lambda_2 C_2 - E C_4$

$S_1 = -2\frac{\bar{\eta}}{R}\lambda_2 B C_2$

$S_2 = 2\frac{C_{do}}{a}\lambda_2\left(1 + \frac{4}{3}\frac{e}{R}\right) + E_2 + 2\lambda_2 B C_1 + 2\frac{e}{R}\lambda_2 B C_2$

$$S_3 = -\lambda_2 C_1 - \frac{1}{2}\frac{e}{R}\lambda_2 C_2 + EC_4$$

$$S_4 = -\frac{1}{2}\frac{e}{R}\lambda_2 C_2$$

$$S_5 = \frac{C_{d_0}}{a} E_6 - 2\lambda_2 C_1 - 2\frac{e}{R}\lambda_2 C_2 + EC_4$$

$$S_6 = -2\frac{C_{d_0}}{a}\lambda_2 - \frac{8}{3}\frac{C_{d_0}}{a}\frac{e}{R}\lambda_2 - E_2 - 2\lambda_2 BC_1$$

$$S_7 = \frac{\bar{\eta}}{R}\lambda_2 C_2$$

$$S_8 = -2\frac{\bar{\eta}}{R}\lambda_2 BC_2$$

$$S_9 = -2\frac{C_{d_0}}{a} E_6 + 2\lambda_2 C_1 + 2\frac{e}{R}\lambda_2 C_2$$

$$S_{10} = -\frac{C_{d_0}}{a}\lambda_2 - 2\frac{e}{R}\lambda_2 BC_2 - \frac{1}{2}E_2 - \lambda_2 BC_1$$

$$S_{11} = -\frac{C_{d_0}}{a}\lambda_2 - \frac{8}{3}\frac{C_{d_0}}{a}\frac{e}{R}\lambda_2 - \frac{1}{2}E_2 - \lambda_2 BC_1$$

$$S_{12} = -\frac{4}{3}\frac{C_{d_0}}{a}\lambda_2 - \frac{16}{9}\frac{C_{d_0}}{a}\frac{e}{R}\lambda_2 - \frac{2}{3}E_2 - \frac{4}{3}\lambda_2\left(BC_1 + \frac{e}{R}BC_2\right)$$

$$S_{13} = -2\lambda_2 BC_1$$

$$S_{14} = \lambda_2 C_1$$

$$S_{15} = -2\frac{\bar{\eta}}{R}\lambda_2 BC_2$$

$$S_{16} = 2\frac{\bar{\eta}}{R}\lambda_2 BC_2$$

$$S_{17} = \lambda_2 C_1$$

$$S_{18} = 0$$

$$S_{19} = 2\frac{C_{d_0}}{a}\lambda_2$$

$$S_{20} = -\lambda_2 C_1$$

$$S_{21} = -2\lambda_2 BC_1$$

$$S_{22} = \frac{\bar{\eta}}{R}\lambda_2 C_2$$

$$S_{23} = -\lambda_2 C_1$$

$$S_{24} = \lambda_2 BC_1$$

$$S_{25} = \lambda_2 BC_1$$

$$S_{26} = \lambda_2 B C_1$$

$$S_{27} = -2 \frac{C_{do}}{a} \lambda_2$$

$$S_{28} = -\frac{1}{2} \lambda_2 C_1$$

$$S_{29} = 0$$

$$S_{30} = -\frac{1}{2} \lambda_2 C_1$$

$$S_{31} = 0$$

$$S_{32} = -2 \frac{C_{do}}{a} \lambda_2$$

$$S_{33} = \lambda_2 C_1$$

APPENDIX VII

The elements of response matrix 'R' for flapping-lagging-feathering rotor consist of four separate parts

(a) contribution from inertia and stiffness, RI

(b) contribution from gravity forces, RG

(c) contribution from aerodynamic forces, RA

(d) contribution from periodic aerodynamic forces, RV

such that

$$R_{ij} = RI(i,j) + G \quad RG(ij) + \frac{\gamma}{8} RA(i,j) + \frac{\gamma}{8} RV(i,j)$$

$$R_{i0} = RIO(i) + G \quad RGO(i) + \frac{\gamma}{8} RAO(i) + \frac{\gamma}{8} RVO(i)$$

The various elements are given as

$$RI(1,1) = 1 + \bar{\omega}_\beta^2 z + \bar{e} - \theta_c^2 - \frac{2}{3}\beta_c^2 - \phi_c^2 - N_I \left\{ (a_1^2 + b_1^2) + \frac{1}{2}(a_2^2 + b_2^2) + 2\alpha(a_3 b_1 - a_1 b_3) + \frac{1}{2}(1 + \alpha^2)(a_3^2 + b_3^2) \right\}$$

$$RI(1,2) = 0$$

$$RI(1,3) = 0$$

$$RI(1,4) = \theta_c - N_I \left\{ (a_1 a_2 + b_1 b_2) - \alpha(a_2 b_3 - a_3 b_2) \right\}$$

$$RI(1,5) = 0$$

$$RI(1,6) = 0$$

$$RI(1,7) = -N_I \left\{ a_1 a_3 + b_1 b_3 \right\}$$

$$RI(1,8) = 0$$

$$RI(1,9) = 0$$

$$RI(2,1) = 0$$

$$RI(2,2) = \bar{\omega}_\beta^2 Z + (1-\alpha^2)(1-\phi_c^2) - 2\beta_c^2 - \theta_c^2 + \bar{e} - N_I\left\{ \tfrac{1}{2}(a_1^2+b_1^2) + \tfrac{1}{4}(1-\alpha^2)(3a_2^2+b_2^2) + \tfrac{1}{2}\alpha^2(a_2^2-b_2^2) + \tfrac{1}{4}\alpha^2(a_3^2+3b_3^2) + \tfrac{1}{4}(3a_3^2+b_3^2)\right\}$$

$$RI(2,3) = -2\alpha\bar{\omega}_\beta\sqrt{Z}\,\zeta_\beta - N_I\left\{\alpha a_1 a_3\right\}$$

$$RI(2,4) = 0$$

$$RI(2,5) = \theta - 2\beta_c\phi_c - N_I\left\{\tfrac{1}{2} b_1 b_2(1+\alpha^2)\right\}$$

$$RI(2,6) = -2\alpha(\beta_c + \theta\phi_c) - N_I\left\{\alpha a_2 a_3\right\}$$

$$RI(2,7) = 0$$

$$RI(2,8) = \phi_c(1-\alpha^2) - 2\beta_c\theta_c - N_I\left\{\tfrac{1}{2} b_1 b_3(1-\alpha^2)\right\}$$

$$RI(2,9) = 2\alpha\beta_c^2 + N_I\left\{\tfrac{1}{2}\alpha(3a_1^2+b_1^2) + \tfrac{1}{2}\alpha(a_2^2-b_2^2)\right\}$$

$$RI(3,1) = 0$$

$$RI(3,2) = 2\alpha\bar{\omega}_\beta\sqrt{Z}\,\zeta_\beta + N_I\left\{\alpha b_1 b_3\right\}$$

$$RI(3,3) = \bar{\omega}_\beta^2 Z + (1-\alpha^2)(1-\phi_c^2) - 2\beta_c^2 - \theta_c^2 + \bar{e} - N_I\left\{\tfrac{1}{2}(a_1^2+b_1^2)\right\} + \tfrac{1}{4}(1-\alpha^2)(a_2^2+3b_2^2) - \tfrac{1}{2}\alpha^2(a_2^2-b_2^2) + \tfrac{1}{4}\alpha^2(3a_3^2+b_3^2) + \tfrac{1}{4}(a_3^2+3b_3^2)\}$$

$$RI(3,4) = 0$$

$$RI(3,5) = 2\alpha(\beta_c + \phi_c\theta_c) + N_I\left\{\alpha b_2 b_3\right\}$$

$$RI(3,6) = \theta - 2\beta_c\phi_c - N_I\left\{\tfrac{1}{2} a_1 a_2(1+\alpha^2)\right\}$$

$$RI(3,7) = 0$$

$$RI(3,8) = -2\alpha\beta_c^2 - N_I\left\{\tfrac{\alpha}{2}(a_1^2+3b_1^2) - \tfrac{\alpha}{2}(a_2^2-b_2^2)\right\}$$

$$RI(3,9) = \phi_c(1-\alpha^2) - 2\beta_c\theta_c - N_I\left\{\tfrac{1}{2}(1-\alpha^2) a_1 a_3\right\}$$

$$RI(4,1) = \theta_c - N_I\left\{(a_1a_2 + b_1b_2) + \alpha(a_3b_2 - a_2b_3)\right\}$$

$$RI(4,2) = 0$$

$$RI(4,3) = 0$$

$$RI(4,4) = Z + \bar{e} + \theta_c^2 - \beta_c^2 - N_I\left\{\tfrac{1}{2}(1-\alpha^2)(a_1^2 + b_1^2) - \tfrac{1}{2}(a_3^2 + b_3^2)(1-\alpha^2)\right\}$$

$$RI(4,5) = 0$$

$$RI(4,6) = 0$$

$$RI(4,7) = N_I\left\{(a_2a_3 + b_2b_3) - \alpha(a_1b_2 - a_2b_1)\right\}$$

$$RI(4,8) = 0$$

$$RI(4,9) = 0$$

$$RI(5,1) = 0$$

$$RI(5,2) = \theta_c - 2\beta_c\phi_c - N_I\left\{\tfrac{1}{2}b_1b_2(1+\alpha^2)\right\}$$

$$RI(5,3) = 2\alpha(\beta_c + \theta\phi_c) + N_I\left\{\tfrac{1}{2}\alpha(2a_2a_3 + b_2b_3)\right\}$$

$$RI(5,4) = 0$$

$$RI(5,5) = Z + \bar{e} - \alpha^2 + \theta_c^2 - \beta_c^2 + N_I\left\{\tfrac{1}{4}\alpha^2(a_1^2 + 3b_1^2) - \tfrac{1}{4}(3a_1^2 + b_1^2) - \tfrac{\alpha^2}{4}(a_3^2 + 3b_3^2) + \tfrac{1}{4}(3a_3^2 + b_3^2)\right\}$$

$$RI(5,6) = -2\alpha\sqrt{Z}\,\zeta_\phi - N_I\left\{\alpha a_1a_3\right\}$$

$$RI(5,7) = 0$$

$$RI(5,8) = \beta_c(1+\alpha^2) + 2\phi_c\theta_c + N_I\left\{\tfrac{1}{2}(1+\alpha^2)b_2b_3\right\}$$

$$RI(5,9) = 2\alpha(R_m\theta + \beta_c\phi_c) + N_I\left\{\tfrac{1}{2}\alpha(2a_1a_2 + b_1b_2)\right\}$$

$$RI(6,1) = 0$$

$$RI(6,2) = -2\alpha(\beta_c + \theta_c \phi_c) - N_I\left\{\tfrac{1}{2}\alpha(a_2a_3 + 2b_2b_3)\right\}$$

$$RI(6,3) = \theta - 2\beta_c\phi_c - N_I\left\{\tfrac{1}{2}a_1a_2(1+\alpha^2)\right\}$$

$$RI(6,4) = 0$$

$$RI(6,5) = 2\alpha\sqrt{Z}\,\zeta_\phi + N_I\left\{\alpha b_1 b_3\right\}$$

$$RI(6,6) = Z + \bar{e} - \alpha^2 + \theta^2 - \beta_c^2 + N_I\left\{\tfrac{1}{4}\alpha^2(3a_1^2 + b_1^2) - \tfrac{1}{4}(a_1^2 + 3b_1^2) - \tfrac{1}{4}\alpha^2(3a_3^2 + b_3^2) + \tfrac{1}{4}(a_3^2 + 3b_3^2)\right\}$$

$$RI(6,7) = 0$$

$$RI(6,8) = -2\alpha(R_M\theta_c + \beta_c\phi_c) - N_I\left\{\tfrac{1}{2}\alpha(a_1a_2 + 2b_1b_2)\right\}$$

$$RI(6,9) = \beta_c(1+\alpha^2) + 2\phi_c\theta_c + N_I\left\{\tfrac{1}{2}(1+\alpha^2)a_2a_3\right\}$$

$$RI(7,1) = -N_I\left\{a_1a_3 + b_1b_3\right\}$$

$$RI(7,2) = 0$$

$$RI(7,3) = 0$$

$$RI(7,4) = N_I\left\{\alpha(a_2b_1 - a_1b_2) + (a_2a_3 + b_2b_3)\right\}$$

$$RI(7,5) = 0$$

$$RI(7,6) = 0$$

$$RI(7,7) = R_M(1 + Z\tilde{\omega}_\theta^2) - \beta_c^2 + \phi_c^2 + N_I\left\{\tfrac{1}{2}(a_2^2 + b_2^2) - \tfrac{1}{2}(a_1^2 + b_1^2)\right\}$$

$$RI(7,8) = 0$$

$$RI(7,9) = 0$$

$$RI(8,1) = 0$$

$$RI(8,2) = \phi_c(1-\alpha^2) - 2\beta_c\theta_c + N_I\left\{\tfrac{1}{2} b_1 b_3(\alpha^2-1)\right\}$$

$$RI(8,3) = -2\alpha\beta_c^2 + N_I\left\{\tfrac{1}{2}\alpha(a_2^2-b_2^2) - \tfrac{1}{2}\alpha(a_1^2+b_1^2)\right\}$$

$$RI(8,4) = 0$$

$$RI(8,5) = \beta_c(1+\alpha^2) + 2\phi_c\theta_c + N_I\left\{\tfrac{1}{2} b_2 b_3(1+\alpha^2)\right\}$$

$$RI(8,6) = -2\alpha(R_m\theta_c + \beta_c\phi_c) - N_I\left\{\alpha a_1 a_2\right\}$$

$$RI(8,7) = 0$$

$$RI(8,8) = R_m(1+Z\bar{\omega}_\theta^2) - R_m\alpha^2 - \beta_c^2(1+\alpha^2) + \phi_c^2(1-\alpha^2) - N_I\left\{\tfrac{1}{4}(1+\alpha^2)(3a_1^2+b_1^2)\right.$$
$$\left. + \tfrac{1}{4}(3a_2^2+b_2^2)(\alpha^2-1) - \tfrac{1}{2}\alpha^2(a_1^2-b_1^2) - \tfrac{1}{2}\alpha^2(a_2^2-b_2^2)\right\}$$

$$RI(8,9) = -2\alpha\bar{\omega}_\theta\sqrt{Z}\,R_m\zeta_\theta$$

$$RI(9,1) = 0$$

$$RI(9,2) = 2\alpha\beta_c^2 + N_I\left\{\tfrac{1}{2}\alpha(a_2^2-b_2^2) + \tfrac{1}{2}\alpha(a_1^2+b_1^2)\right\}$$

$$RI(9,3) = \phi_c(1-\alpha^2) - 2\beta_c\theta_c + N_I\left\{\tfrac{1}{2} a_1 a_3(\alpha^2-1)\right\}$$

$$RI(9,4) = 0$$

$$RI(9,5) = 2\alpha(R_m\theta_c + \beta_c\phi_c) + N_I\left\{\alpha b_1 b_2\right\}$$

$$RI(9,6) = \beta_c(1+\alpha^2) + 2\phi_c\theta_c + N_I\left\{\tfrac{1}{2} a_2 a_3(1+\alpha^2)\right\}$$

$$RI(9,7) = 0$$

$$RI(9,8) = 2\alpha\bar{\omega}_\theta\sqrt{Z}\,R_m\zeta_\theta$$

$$RI(9,9) = R_m(1+Z\bar{\omega}_\theta^2) - R_m\alpha^2 - \beta_c^2(1+\alpha^2) + \phi_c^2(1-\alpha^2) + N_I\left\{-\tfrac{1}{4}(1+\alpha^2)(a_1^2+3b_1^2)\right.$$
$$\left. + \tfrac{1}{4}(1-\alpha^2)(a_2^2+3b_2^2) - \tfrac{1}{2}\alpha^2(a_1^2-b_1^2) - \tfrac{1}{2}\alpha^2(a_2^2-b_2^2)\right\}$$

$$RIO(1) = -\bar{\omega}_\beta^2 Z \beta_S + N_I \left\{ \alpha (a_2 b_1 - a_1 b_2) + (1 + \tfrac{\alpha^2}{2})(a_2 a_3 + b_2 b_3) \right\}$$

$$RIO(2) = 0$$

$$RIO(3) = 0$$

$$RIO(4) = -Z \phi_S + N_I \left\{ \tfrac{1}{2}(1 - \alpha^2)(a_1 a_3 + b_1 b_3) \right\}$$

$$RIO(5) = 0$$

$$RIO(6) = 0$$

$$RIO(7) = -Z \bar{\omega}_\theta^2 R_m \theta_S + \beta_c \phi_c + N_I \left\{ (\tfrac{1}{2} - R_m \alpha^2)(a_1 a_2 + b_1 b_2) + R_m \alpha (a_2 b_3 - a_3 b_2) \right\}$$

$$RIO(8) = 0$$

$$RIO(9) = 0$$

Gravity terms for forced oscillations

$$RG(1,1) = 0$$

$$RG(1,2) = \frac{1}{2}\beta_c\Theta_c + N_I\left\{\frac{1}{8}b_1b_3\right\}$$

$$RG(1,3) = \frac{1}{2} - \frac{1}{4}\beta_c^2 - \frac{1}{4}\Phi_c^2 - N_I\left\{\frac{1}{16}(a_1^2+b_1^2) + \frac{1}{16}(a_2^2+3b_2^2)\right\}$$

$$RG(1,4) = 0$$

$$RG(1,5) = \frac{1}{2}\Phi_c\Theta_c + N_I\left\{\frac{1}{8}b_2b_3\right\}$$

$$RG(1,6) = -\frac{1}{2}\beta_c\Phi_c - N_I\left\{\frac{1}{8}a_1a_2\right\}$$

$$RG(1,7) = 0$$

$$RG(1,8) = -\frac{1}{2} + \frac{1}{4}(\Theta_c^2+\beta_c^2+\Phi_c^2) + N_I\cdot\frac{1}{16}\left\{3a_1^2+b_1^2 + 3a_2^2+b_2^2 + a_3^2 + b_3^2\right\}$$

$$RG(1,9) = 0$$

$$RG(2,1) = N_I\left\{\frac{1}{4}(3a_1a_3+b_1b_3) - \frac{1}{4}(a_1b_1+a_2b_2)\right.$$

$$RG(2,2) = 0$$

$$RG(2,3) = 0$$

$$RG(2,4) = N_I\left\{\frac{1}{4}(3a_2a_3+b_2b_3) - \frac{1}{4}(a_1b_2+a_2b_1)\right\}$$

$$RG(2,5) = 0$$

$$RG(2,6) = 0$$

$$RG(2,7) = -1 + \frac{1}{2}\beta_c^2 + \frac{1}{2}\Phi_c^2 + \frac{1}{6}\Theta_c^2 + N_I\frac{1}{8}\left\{3a_1^2+b_1^2+3a_2^2+b_2^2+3a_3^2+b_3^2\right\}$$

$$RG(2,8) = 0$$

$$RG(2,9) = 0$$

$$RG(3,1) = 1 - \frac{1}{6}\beta_c^2 - \frac{1}{2}\phi_c^2 + N_I\left\{\frac{1}{4}(a_1b_3 + a_3b_1) - \frac{1}{8}(a_1^2 + 3b_1^2) - \frac{1}{8}(a_2^2 + 3b_2^2)\right\}$$

$$RG(3,2) = 0$$

$$RG(3,3) = 0$$

$$RG(3,4) = N_I\left\{\frac{1}{4}(a_2b_3 + a_3b_2) - \frac{1}{4}(a_1a_2 + 3b_1b_2)\right\}$$

$$RG(3,5) = 0$$

$$RG(3,6) = 0$$

$$RG(3,7) = N_I\left\{\frac{1}{4}(a_1b_1 + a_2b_2 + a_3b_3)\right\}$$

$$RG(3,8) = 0$$

$$RG(3,9) = 0$$

$$RG(4,1) = 0$$

$$RG(4,2) = \frac{1}{2}\phi_c\theta_c$$

$$RG(4,3) = -\frac{1}{2}\beta_c\phi_c - N_I\left\{\frac{1}{8}a_1a_2\right\}$$

$$RG(4,4) = 0$$

$$RG(4,5) = -\frac{1}{2}(\phi_c - \beta_c\theta_c)$$

$$RG(4,6) = \frac{1}{2} - \frac{1}{4}\beta_c^2 - \frac{1}{4}\phi_c^2 - N_I\frac{1}{16}\left\{a_1^2 + 3b_1^2 + a_2^2 + b_2^2\right\}$$

$$RG(4,7) = 0$$

$$RG(4,8) = -\frac{1}{2}(\theta_c - \beta_c\phi_c) + N_I\left\{\frac{1}{8}(3a_1a_2 + b_1b_2)\right\}$$

$$RG(4,9) = N_I\left\{\frac{1}{8}(a_1b_2 + a_2b_1)\right\}$$

$$RG(5,1) = N_I\left\{\tfrac{1}{4}(3a_2a_3+b_2b_3)-\tfrac{1}{4}(a_1b_2+a_2b_1)\right\}$$

$$RG(5,2) = 0$$

$$RG(5,3) = 0$$

$$RG(5,4) = N_I\left\{\tfrac{1}{4}(3a_1a_3+b_1b_3)-\tfrac{1}{4}(a_1b_1+a_2b_2)\right\}$$

$$RG(5,5) = 0$$

$$RG(5,6) = 0$$

$$RG(5,7) = \beta_c\phi_c + N_I\left\{\tfrac{1}{4}(3a_1a_2+b_1b_2)\right\}$$

$$RG(5,8) = 0$$

$$RG(5,9) = 0$$

$$RG(6,1) = N_I\left\{\tfrac{1}{4}(a_2b_3+a_3b_2)-\tfrac{1}{4}(a_1a_2+3b_1b_2)\right\}$$

$$RG(6,2) = 0$$

$$RG(6,3) = 0$$

$$RG(6,4) = 1-\tfrac{1}{2}\beta_c^2-\tfrac{1}{6}\phi_c^2+N_I\left\{\tfrac{1}{4}(a_3b_1+a_1b_3)-\tfrac{1}{8}(a_1^2+3b_1^2)-\tfrac{1}{8}(a_2^2+3b_2^2)\right\}$$

$$RG(6,5) = 0$$

$$RG(6,6) = 0$$

$$RG(6,7) = N_I\left\{\tfrac{1}{4}(a_1b_2+a_2b_1)\right\}$$

$$RG(6,8) = 0$$

$$RG(6,9) = 0$$

$RG(7,1) = 0$

$RG(7,2) = -\frac{1}{2} + \frac{1}{4}\left(\beta_c^2 + \phi_c^2 + \theta_c^2\right) + N_I\left\{\frac{1}{16}\left(a_1^2 + b_1^2\right) + \frac{1}{16}\left(3a_2^2 + b_2^2\right) + \frac{1}{16}\left(3a_3^2 + b_3^2\right)\right\}$

$RG(7,3) = 0$

$RG(7,4) = 0$

$RG(7,5) = -\frac{1}{2}\left(\theta_c - \beta_c \phi_c\right) + N_I\left\{\frac{1}{8} b_1 b_2\right\}$

$RG(7,6) = 0$

$RG(7,7) = 0$

$RG(7,8) = -\frac{1}{2}\left(\phi_c - \beta_c \theta_c\right) + N_I\left\{\frac{1}{8} b_1 b_3\right\}$

$RG(7,9) = 0$

$RG(8,1) = -1 + \frac{1}{2}\left(\frac{1}{3}\beta_c^2 + \phi_c^2 + \theta_c^2\right) + N_I\left\{\frac{1}{8}\left(3a_2^2 + b_2^2\right) + \frac{1}{8}\left(3a_1^2 + b_1^2\right) + \frac{1}{8}\left(3a_3^2 + b_3^2\right)\right\}$

$RG(8,2) = 0$

$RG(8,3) = 0$

$RG(8,4) = N_I\left\{\frac{1}{4}\left(3a_1 a_2 + b_1 b_2\right)\right\}$

$RG(8,5) = 0$

$RG(8,6) = 0$

$RG(8,7) = N_I\left\{\frac{1}{4}\left(3a_1 a_3 + b_1 b_3\right)\right\}$

$RG(8,8) = 0$

$RG(8,9) = 0$

$$RG(9,1) = N_I \left\{ \frac{1}{4}(a_1 b_1 + a_2 b_2 + a_3 b_3) \right\}$$

$$RG(9,2) = 0$$

$$RG(9,3) = 0$$

$$RG(9,4) = N_I \left\{ \frac{1}{4}(a_1 b_2 + a_2 b_1) \right\}$$

$$RG(9,5) = 0$$

$$RG(9,6) = 0$$

$$RG(9,7) = N_I \left\{ \frac{1}{4}(a_1 b_3 + a_3 b_1) \right\}$$

$$RG(9,8) = 0$$

$$RG(9,9) = 0$$

$$RGO(1) = 0$$

$$RGO(2) = 0$$

$$RGO(3) = 0$$

$$RGO(4) = 0$$

$$RGO(5) = 1 - \frac{1}{2}(\phi_c^2 + \theta_c^2) - N_I \left\{ \frac{1}{8}(3a_2^2 + b_2^2) + \frac{1}{8}(3a_3^2 + b_3^2) \right\}$$

$$RGO(6) = -N_I \left\{ \frac{1}{4}(a_2 b_2 + a_3 b_3) \right\}$$

$$RGO(7) = 0$$

$$RGO(8) = -\phi_c \theta_c - N_I \left\{ \frac{1}{4}(3a_2 a_3 + b_2 b_3) \right\}$$

$$RGO(9) = -N_I \left\{ \frac{1}{4}(a_2 b_3 + a_3 b_2) \right\}$$

Gravity terms for Parametric Resonance

$$RG(1,1) = N_I\left\{\tfrac{1}{4}(a_3 b_1 + a_1 b_3) + \tfrac{1}{8}(a_1^2 - b_1^2) + \tfrac{1}{8}(a_2^2 - b_2^2)\right\}$$

$$RG(1,2) = 0$$

$$RG(1,3) = 0$$

$$RG(1,4) = N_I\left\{\tfrac{1}{4}(a_3 b_2 + a_2 b_3) + \tfrac{1}{4}(a_1 a_2 - b_1 b_2)\right\}$$

$$RG(1,5) = 0$$

$$RG(1,6) = 0$$

$$RG(1,7) = N_I\left\{\tfrac{1}{4}(a_1 b_1 + a_2 b_2 + a_3 b_3)\right\}$$

$$RG(1,8) = 0$$

$$RG(1,9) = 0$$

$$RG(2,1) = 0$$

$$RG(2,2) = -\tfrac{1}{2} + \tfrac{1}{4}(\beta_c^2 + \phi_c^2) + N_I\left\{\tfrac{1}{12}a_1^2 + \tfrac{1}{4}a_2^2\right\}$$

$$RG(2,3) = \tfrac{1}{2}\beta_c \theta_c + N_I\left\{\tfrac{1}{4}a_1 a_3\right\}$$

$$RG(2,4) = 0$$

$$RG(2,5) = \tfrac{1}{2}\beta_c \phi_c$$

$$RG(2,6) = \tfrac{1}{2}\phi_c \theta_c + N_I\left\{\tfrac{1}{4}a_2 a_3\right\}$$

$$RG(2,7) = 0$$

$$RG(2,8) = 0$$

$$RG(2,9) = -\tfrac{1}{2} + \tfrac{1}{4}(\beta_c^2 + \phi_c^2 + \theta_c^2) + N_I\left\{\tfrac{1}{8}(a_1^2 + b_1^2) + \tfrac{1}{8}(a_2^2 + b_2^2) + \tfrac{1}{24}(3a_3^2 + b_3^2)\right\}$$

$$RG(3,1) = 0$$

$$RG(3,2) = \tfrac{1}{2}\beta_c \theta_c + N_I\left\{\tfrac{1}{4}b_1 b_3\right\}$$

$$RG(3,3) = \tfrac{1}{2} - \tfrac{1}{4}(\beta_c^2 + \phi_c^2) - N_I\left\{\tfrac{1}{12}b_1^2 + \tfrac{1}{4}b_2^2\right\}$$

$$RG(3,4) = 0$$

$$RG(3,5) = \tfrac{1}{2}\phi_c \theta_c + N_I\left\{\tfrac{1}{4}b_2 b_3\right\}$$

$$RG(3,6) = -\tfrac{1}{2}\beta_c \phi_c$$

$$RG(3,7) = 0$$

$$RG(3,8) = -\tfrac{1}{2} + \tfrac{1}{4}(\beta_c^2 + \phi_c^2 + \theta_c^2) + N_I\left\{\tfrac{1}{8}(a_1^2 + b_1^2) + \tfrac{1}{8}(a_2^2 + b_2^2) + \tfrac{1}{24}(a_3^2 + 3b_3^2)\right\}$$

$$RG(3,9) = 0.$$

$$RG(4,1) = N_I\left\{ \tfrac{1}{4}(a_2 b_3 + a_3 b_2) + \tfrac{1}{4}(a_1 a_2 - b_1 b_2) \right\}$$

$$RG(4,2) = 0$$

$$RG(4,3) = 0$$

$$RG(4,4) = N_I\left\{ \tfrac{1}{4}(a_1 b_3 + a_3 b_1) + \tfrac{1}{8}(a_1^2 - b_1^2) + \tfrac{1}{8}(a_2^2 - b_2^2) \right\}$$

$$RG(4,5) = 0$$

$$RG(4,6) = 0$$

$$RG(4,7) = N_I\left\{ \tfrac{1}{4}(a_1 b_2 + a_2 b_1) \right\}$$

$$RG(4,8) = 0$$

$$RG(4,9) = 0$$

$$RG(5,1) = 0$$

$$RG(5,2) = \tfrac{1}{2}\beta_c \phi_c$$

$$RG(5,3) = \tfrac{1}{2}\phi_c \Theta_c + N_I\left\{ \tfrac{1}{4}(a_2 a_3 + b_2 b_3) \right\}$$

$$RG(5,4) = 0$$

$$RG(5,5) = -\tfrac{1}{2} + \tfrac{1}{4}(\beta_c^2 + \phi_c^2) + N_I\left\{ \tfrac{1}{4}a_1^2 + \tfrac{1}{12}a_2^2 \right\}$$

$$RG(5,6) = -\tfrac{1}{2}(\phi_c - \beta_c \Theta_c) + N_I\left\{ \tfrac{1}{4}a_1 a_3 \right\}$$

$$RG(5,7) = 0$$

$$RG(5,8) = 0$$

$$RG(5,9) = -\tfrac{1}{2}(\Theta_c - \beta_c \phi_c) + N_I\left\{ \tfrac{1}{4}a_1 a_2 \right\}$$

$$RG(6,1) = 0$$

$$RG(6,2) = \tfrac{1}{2}\phi_c \Theta_c + N_I\left\{ \tfrac{1}{4}(a_2 a_3 + b_2 b_3) \right\}$$

$$RG(6,3) = -\tfrac{1}{2}\beta_c \phi_c$$

$$RG(6,4) = 0$$

$$RG(6,5) = -\tfrac{1}{2}(\phi_c - \beta_c \Theta_c) + N_I\left\{ \tfrac{1}{4} b_1 b_3 \right\}$$

$$RG(6,6) = \tfrac{1}{2} - \tfrac{1}{4}(\beta_c^2 + \phi_c^2) - N_I\left\{ \tfrac{1}{4} b_1^2 + \tfrac{1}{12} b_2^2 \right\}$$

$$RG(6,7) = 0$$

$$RG(6,8) = -\tfrac{1}{2}(\Theta_c - \beta_c \phi_c) + N_I\left\{ \tfrac{1}{4} b_1 b_2 \right\}$$

$$RG(6,9) = 0$$

$$RG(7,1) = N_I\left\{ \tfrac{1}{4}\left(a_1 b_1 + a_2 b_2 + a_3 b_3\right)\right\}$$

$$RG(7,2) = 0$$

$$RG(7,3) = 0$$

$$RG(7,4) = N_I\left\{ \tfrac{1}{4}\left(a_1 b_2 + a_2 b_1\right)\right\}$$

$$RG(7,5) = 0$$

$$RG(7,6) = 0$$

$$RG(7,7) = N_I\left\{ \tfrac{1}{4}\left(a_1 b_3 + a_3 b_1\right)\right\}$$

$$RG(7,8) = 0$$

$$RG(7,9) = 0$$

$$RG(8,1) = 0$$

$$RG(8,2) = 0$$

$$RG(8,3) = -\tfrac{1}{2} + \tfrac{1}{4}\left(\beta_c^2 + \phi_c^2 + \theta_c^2\right) + N_I\left\{ \tfrac{1}{24}\left(3a_1^2 + b_1^2\right) + \tfrac{1}{8}\left(a_2^2 + b_2^2\right) + \tfrac{1}{8}\left(a_3^2 + b_3^2\right)\right\}$$

$$RG(8,4) = 0$$

$$RG(8,5) = 0$$

$$RG(8,6) = -\tfrac{1}{2}\left(\theta_c - \beta_c \phi_c\right) + N_I\left\{ \tfrac{1}{4} a_1 a_2 \right\}$$

$$RG(8,7) = 0$$

$$RG(8,8) = 0$$

$$RG(8,9) = -\tfrac{1}{2}\left(\phi_c - \beta_c \theta_c\right) + N_I\left\{ \tfrac{1}{4} a_1 a_3 \right\}$$

$$RG(9,1) = 0$$

$$RG(9,2) = -\tfrac{1}{2} + \tfrac{1}{4}\left(\beta_c^2 + \phi_c^2 + \theta_c^2\right) + N_I\left\{ \tfrac{1}{24}\left(a_1^2 + 3b_1^2\right) + \tfrac{1}{8}\left(a_2^2 + b_2^2\right) + \tfrac{1}{8}\left(a_3^2 + b_3^2\right)\right\}$$

$$RG(9,3) = 0$$

$$RG(9,4) = 0$$

$$RG(9,5) = -\tfrac{1}{2}\left(\theta_c - \beta_c \phi_c\right) + N_I\left\{ \tfrac{1}{4} b_1 b_2 \right\}$$

$$RG(9,6) = 0$$

$$RG(9,7) = 0$$

$$RG(9,8) = -\tfrac{1}{2}\left(\phi_c - \beta_c \theta_c\right) + N_I\left\{ \tfrac{1}{4} b_1 b_3 \right\}$$

$$RG(9,9) = 0$$

$RGO(1) = 0$

$RGO(2) = 0$

$RGO(3) = 0$

$RGO(4) = -N_I\left\{ \frac{1}{4} (a_2 b_2 + a_3 b_3) \right\}$

$RGO(5) = 0$

$RGO(6) = 0$

$RGO(7) = -N_I\left\{ \frac{1}{4} (a_2 b_3 + a_3 b_2) \right\}$

$RGO(8) = 0$

$RGO(9) = 0$

Aerodynamic contribution terms

$$RA(1,1) = -\left(F_1 + \phi_c\theta_c F_9 + \phi_c^2 F_{45}\right) - N_A\Big\{\tfrac{1}{2}(a_2a_3 + b_2b_3)F_9 + (a_1a_3 + b_1b_3)F_{10} + \tfrac{\alpha}{2}(a_1b_2 - a_2b_1)F_{23}$$
$$+ \alpha(a_2b_1 - a_1b_2)F_{28} + \tfrac{\alpha}{2}(a_3b_2 - a_2b_3)F_{31} + \tfrac{\alpha^2}{2}(a_1a_3 + b_1b_3)F_{36} + \tfrac{\alpha^2}{2}(a_2a_3 + b_2b_3)F_{44}$$
$$+ \tfrac{1}{2}(a_2^2 + b_2^2)F_{45}\Big\}$$

$$RA(1,2) = 0$$

$$RA(1,3) = 0$$

$$RA(1,4) = -N_A\Big\{\tfrac{1}{2}(a_1a_3 + b_1b_3)F_9 + (a_2a_3 + b_2b_3)F_{11} + \alpha(a_1b_2 - a_2b_1)F_{25} + \tfrac{\alpha}{2}(a_2b_1 - a_1b_2)F_{27}$$
$$+ \tfrac{\alpha}{2}(a_3b_1 - a_1b_3)F_{31} + \tfrac{\alpha^2}{2}(a_2a_3 + b_2b_3)F_{37} + \alpha(a_3b_2 - a_2b_3)F_{39} + (a_1a_2 + b_1b_2)F_{45}\Big\}$$

$$RA(1,5) = 0$$

$$RA(1,6) = 0$$

$$RA(1,7) = -\left(F_2 + \beta_c^2 F_{10} + \phi_c^2 F_{11} + \theta_c^2 F_{12}\right) - N_A\Big\{\tfrac{1}{2}(a_1a_2 + b_1b_2)F_9 + \tfrac{1}{2}(a_1^2 + b_1^2)F_{10}$$
$$+ \tfrac{1}{2}(a_2^2 + b_2^2)F_{11} + \tfrac{3}{2}(a_3^2 + b_3^2)F_{12} + \alpha(a_1b_3 - a_3b_1)F_{26} + \alpha(a_2b_3 - a_3b_2)F_{30}$$
$$+ \tfrac{\alpha}{2}(a_3b_1 - a_1b_3)F_{32} + \tfrac{\alpha}{2}(a_3b_2 - a_2b_3)F_{33}\Big\}$$

$$RA(1,8) = 0$$

$$RA(1,9) = 0$$

$$RA(2,1) = 0$$

$$RA(2,2) = -\left(F_1 + 2\beta_c F_3 + \phi_c F_6 + \theta_c F_7 + \phi_c\theta_c F_9 + 2\beta_c\theta_c F_{10} + \phi_c^2 F_{45}\right) - N_A\Big\{\tfrac{1}{4}(3a_2a_3 + b_2b_3)F_9 + \tfrac{1}{2}b_1b_3F_{10}$$
$$+ \tfrac{1}{2}\alpha^2 b_1b_3F_{36} + \tfrac{1}{4}\alpha^2(a_2a_3 + 3b_2b_3)F_{44} + \tfrac{1}{4}(3a_2^2 + b_2^2)F_{45}\Big\}$$

$$RA(2,3) = \alpha\left(F_{13} + \phi_c F_{16} + \theta_c F_{17} + \beta_c\phi_c F_{23} + \beta_c^2 F_{24} + \phi_c^2 F_{25} + \theta_c^2 F_{26}\right) + N_A\Big\{\tfrac{\alpha}{2}a_1a_2F_{23} + \tfrac{\alpha}{4}(a_1^2 + b_1^2)F_{24}$$
$$+ \tfrac{\alpha}{4}(3a_2^2 + b_2^2)F_{25} + \tfrac{\alpha}{4}(3a_3^2 + b_3^2)F_{26} - \tfrac{\alpha}{4}(a_2^2 - b_2^2)F_{27} - \tfrac{\alpha}{2}a_1a_2F_{28} - \tfrac{\alpha}{4}(a_3^2 - b_3^2)F_{32}\Big\}$$

$$RA(2,4) = 0$$

$$RA(2,5) = -\left(2\phi_c F_4 + \beta_c F_6 + \theta_c F_8 + \beta_c\theta_c F_9 + 2\phi_c\theta_c F_{11} + 2\beta_c\phi_c F_{45}\right) - N_A\Big\{\tfrac{1}{4}b_1b_3F_9 + \tfrac{1}{2}b_2b_3F_{11}$$
$$+ \tfrac{1}{2}\alpha^2 b_2b_3F_{37} - \tfrac{1}{4}\alpha^2 b_1b_3F_{44} + \tfrac{1}{2}b_1b_2F_{45}\Big\}$$

$$RA(2,6) = \alpha\left(F_{14} + \beta_c F_{18} + \theta_c F_{19} + \beta_c\phi_c F_{27} + \beta_c^2 F_{28} + \phi_c^2 F_{29} + \theta_c^2 F_{30}\right) - N_A\Big\{\tfrac{\alpha}{4}(a_1^2 - b_1^2)F_{23} + \tfrac{\alpha}{2}a_1a_2F_{25}$$
$$- \tfrac{\alpha}{2}a_1a_2F_{27} - \tfrac{\alpha}{4}(3a_1^2 + b_1^2)F_{28} - \tfrac{\alpha}{4}(a_2^2 + b_2^2)F_{29} - \tfrac{\alpha}{4}(3a_3^2 + b_3^2)F_{30} + \tfrac{\alpha}{4}(a_3^2 - b_3^2)F_{33} + \tfrac{\alpha}{2}a_2a_3F_{39}\Big\}$$

$$RA(2,7) = 0$$

$$RA(2,8) = -\left(F_2 + 2\theta_c F_5 + \beta_c F_7 + \phi_c F_8 + \beta_c\phi_c F_9 + \beta_c^2 F_{10} + \phi_c^2 F_{11} + 3\theta_c^2 F_{12}\right) - N_A\Big\{\tfrac{1}{4}b_1b_2F_9 + \tfrac{1}{4}(3a_1^2 + b_1^2)F_{10}$$
$$+ \tfrac{1}{4}(3a_2^2 + b_2^2)F_{11} + \tfrac{3}{4}(a_3^2 + b_3^2)F_{12} + \tfrac{\alpha}{4}(a_1b_2 + a_2b_1)F_{31} + \tfrac{\alpha^2}{4}(a_1^2 - b_1^2)F_{36} + \tfrac{\alpha^2}{4}(a_2^2 - b_2^2)F_{37} - \tfrac{\alpha^2}{4}b_1b_2F_{44}\Big\}$$

$$RA(2,9) = \alpha\left(F_{15} + \beta_c F_{20} + \phi_c F_{21} + \theta_c F_{22} + \beta_c\phi_c F_{31} + \beta_c\theta_c F_{32} + \phi_c\theta_c F_{33} + \phi_c^2 F_{39}\right) - N_A\Big\{\tfrac{\alpha}{2}a_1a_3F_{26}$$
$$+ \tfrac{\alpha}{2}a_2a_3F_{30} - \tfrac{1}{4}\alpha(3a_1a_2 + b_1b_2)F_{31} - \tfrac{1}{2}\alpha a_1a_3F_{32} - \tfrac{\alpha}{2}a_2a_3F_{33} - \tfrac{\alpha}{4}(3a_2^2 + b_2^2)F_{39}\Big\}$$

$$RA(3,1)=0$$

$$RA(3,2)=-\alpha\left(F_{13}+\phi_c F_{16}+\theta_c F_{17}+\beta_c\phi_c F_{23}+\beta_c^2 F_{24}+\phi_c^2 F_{25}+\theta_c^2 F_{26}\right)-N_A\Big\{\frac{\alpha}{2}b_1b_2F_{23}+\frac{\alpha}{4}(a_1^2+b_1^2)F_{24}$$
$$+\frac{\alpha}{4}(a_2^2+3b_2^2)F_{25}+\frac{\alpha}{4}(a_3^2+3b_3^2)F_{26}+\frac{\alpha}{4}(a_2^2-b_2^2)F_{27}-\frac{\alpha}{2}b_1b_2F_{28}+\frac{\alpha}{4}(a_3^2-b_3^2)F_{32}\Big\}$$

$$RA(3,3)=-\left(F_1+2\beta_c F_3+\phi_c F_6+\theta_c F_7+\phi_c\theta_c F_9+2\beta_c\theta_c F_{10}+\phi_c^2 F_{45}\right)-N_A\Big\{\frac{1}{4}(a_2a_3+3b_2b_3)F_9+\frac{1}{2}a_1a_3F_{10}$$
$$+\frac{\alpha^2}{2}a_1a_3F_{36}+\frac{\alpha^2}{4}(3a_2a_3+b_2b_3)F_{44}+\frac{1}{4}(a_2^2+3b_2^2)F_{45}\Big\}$$

$$RA(3,4)=0$$

$$RA(3,5)=-\alpha\left(F_{14}+\beta_c F_{18}+\theta_c F_{19}+\beta_c\phi_c F_{27}+\beta_c^2 F_{28}+\phi_c^2 F_{29}+\theta_c^2 F_{30}\right)-N_A\Big\{\frac{\alpha}{4}(a_1^2-b_1^2)F_{23}-\frac{\alpha}{2}b_1b_2F_{25}$$
$$+\frac{\alpha}{2}b_1b_2F_{27}+\frac{\alpha}{4}(a_1^2+3b_1^2)F_{28}+\frac{\alpha}{4}(a_2^2+b_2^2)F_{29}+\frac{\alpha}{4}(a_3^2+3b_3^2)F_{30}+\frac{\alpha}{4}(a_3^2-b_3^2)F_{33}-\frac{\alpha}{2}b_2b_3F_{39}\Big\}$$

$$RA(3,6)=-\left(2\phi_c F_4+\beta_c F_6+\theta_c F_8+\beta_c\theta_c F_9+2\phi_c\theta_c F_{11}+2\beta_c\phi_c F_{45}\right)-N_A\Big\{\frac{a_1a_3}{4}F_9+\frac{a_2a_3}{2}F_{11}$$
$$+\frac{\alpha^2}{2}a_2a_3F_{37}-\frac{\alpha^2}{4}a_1a_3F_{44}+\frac{a_1a_2}{2}F_{45}\Big\}$$

$$RA(3,7)=0$$

$$RA(3,8)=-\alpha\left(F_{15}+\beta_c F_{20}+\phi_c F_{21}+\theta_c F_{22}+\beta_c\phi_c F_{31}+\beta_c\theta_c F_{32}+\phi_c\theta_c F_{33}+\phi_c^2 F_{39}\right)-N_A\Big\{-\frac{\alpha}{2}b_1b_3F_{26}$$
$$-\frac{\alpha}{2}b_2b_3F_{30}+\frac{\alpha}{4}(a_1a_2+3b_1b_2)F_{31}+\frac{\alpha}{2}b_1b_3F_{32}+\frac{\alpha}{2}b_2b_3F_{33}+\frac{\alpha}{4}(a_2^2+3b_2^2)F_{39}\Big\}$$

$$RA(3,9)=-\left(F_2+2\theta_c F_5+\beta_c F_7+\phi_c F_8+\beta_c\phi_c F_9+\beta_c^2 F_{10}+\phi_c^2 F_{11}+3\theta_c^2 F_{12}\right)-N_A\Big\{\frac{1}{4}a_1a_2F_9+\frac{1}{4}(a_1^2+3b_1^2)F_{10}$$
$$+\frac{1}{4}(a_2^2+3b_2^2)F_{11}+\frac{3}{4}(a_3^2+b_3^2)F_{12}-\frac{\alpha}{4}(a_1b_2+a_2b_1)F_{31}-\frac{\alpha^2}{4}(a_1^2-b_1^2)F_{36}-\frac{\alpha^2}{4}(a_2^2-b_2^2)F_{37}-\frac{\alpha^2}{4}a_1a_2F_{44}\Big\}$$

$$RA(4,1)=G_1+\phi_c\theta_c G_9+N_A\Big\{\frac{1}{2}(a_2a_3+b_2b_3)G_9+(a_1a_3+b_1b_3)G_{10}+\frac{\alpha}{2}(a_1b_2-a_2b_1)G_{23}+\alpha(a_2b_1-a_1b_2)G_{28}$$
$$+\frac{1}{2}\alpha(a_3b_2-a_2b_3)G_{31}+\frac{1}{2}\alpha^2(a_1a_3+b_1b_3)G_{36}+\frac{1}{2}\alpha^2(a_2a_3+b_2b_3)G_{44}\Big\}$$

$$RA(4,2)=0$$

$$RA(4,3)=0$$

$$RA(4,4)=N_A\Big\{\frac{1}{2}(a_1a_3+b_1b_3)G_9+(a_2a_3+b_2b_3)G_{11}+\alpha(a_1b_2-a_2b_1)G_{25}+\frac{\alpha}{2}(a_2b_1-a_1b_2)G_{27}$$
$$+\frac{\alpha}{2}(a_3b_1-a_1b_3)G_{31}+\frac{\alpha^2}{2}(a_2a_3+b_2b_3)G_{37}+\frac{\alpha^2}{2}(a_1a_3+b_1b_3)G_{43}\Big\}$$

$$RA(4,5)=0$$

$$RA(4,6)=0$$

$$RA(4,7)=G_2+\beta_c^2 G_{10}+\phi_c^2 G_{11}+\theta_c^2 G_{12}+N_A\Big\{\frac{1}{2}(a_1a_2+b_1b_2)G_9+\frac{1}{2}(a_1^2+b_1^2)G_{10}+\frac{1}{2}(a_2^2+b_2^2)G_{11}$$
$$+\frac{3}{2}(a_3^2+b_3^2)G_{12}+\alpha(a_1b_3-a_3b_1)G_{26}+\alpha(a_2b_3-a_3b_2)G_{30}+\frac{\alpha}{2}(a_3b_1-a_1b_3)G_{32}+\frac{\alpha}{2}(a_3b_2-a_2b_3)G_{33}\Big\}$$

$$RA(4,8)=0$$

$$RA(4,9)=0$$

$$RA(5,1) = 0$$

$$RA(5,2) = G_1 + 2\beta_c G_3 + \phi_c G_6 + \theta_c G_7 + \phi_c\theta_c G_9 + 2\beta_c\theta_c G_{10} + N_A\Big\{\tfrac{1}{4}(3a_2a_3 + b_2b_3)G_9 + \tfrac{1}{2}b_1b_3G_{10} + \tfrac{1}{2}\alpha^2 b_1b_3G_{36} + \tfrac{1}{4}\alpha^2(a_2a_3 + 3b_2b_3)G_{44} - \tfrac{\alpha^2}{4}b_2b_3G_{43}\Big\}$$

$$RA(5,3) = -\alpha(G_{13} + \phi_c G_{16} + \theta_c G_{17} + \beta_c\phi_c G_{23} + \beta_c^2 G_{24} + \phi_c^2 G_{25} + \theta_c^2 G_{26} + \beta_c G_{41}) - N_A\Big\{\tfrac{\alpha}{2}a_1a_2G_{23} + \tfrac{\alpha}{4}(a_1^2 + b_1^2)G_{24} + \tfrac{\alpha}{4}(3a_2^2 + b_2^2)G_{25} + \tfrac{\alpha}{4}(3a_3^2 + b_3^2)G_{26} - \tfrac{\alpha}{4}(a_2^2 - b_2^2)G_{27} - \tfrac{\alpha}{2}a_1a_2G_{28} + \tfrac{\alpha}{4}(a_3^2 - b_3^2)G_{32}\Big\}$$

$$RA(5,4) = 0$$

$$RA(5,5) = 2\phi_c G_4 + \beta_c G_6 + \theta_c G_8 + \beta_c\theta_c G_9 + 2\phi_c\theta_c G_{11} + N_A\Big\{\tfrac{1}{4}b_1b_3G_9 + \tfrac{1}{2}b_2b_3G_{11} + \tfrac{\alpha^2}{2}b_2b_3G_{37} + \tfrac{\alpha^2}{4}(a_1a_3 + 3b_1b_3)G_{43} - \tfrac{\alpha^2}{4}b_1b_3G_{44}\Big\}$$

$$RA(5,6) = -\alpha(G_{14} + \beta_c G_{18} + \theta_c G_{19} + \beta_c\phi_c G_{27} + \beta_c^2 G_{28} + \phi_c^2 G_{29} + \theta_c^2 G_{30}) + N_A\Big\{\tfrac{\alpha}{4}(a_1^2 - b_1^2)G_{23} + \tfrac{\alpha}{2}a_1a_2G_{25} - \tfrac{\alpha}{2}a_1a_2G_{27} - \tfrac{\alpha}{4}(3a_1^2 + b_1^2)G_{28} - \tfrac{\alpha}{4}(a_2^2 + b_2^2)G_{29} - \tfrac{\alpha}{4}(3a_3^2 + b_3^2)G_{30} + \tfrac{\alpha}{4}(a_3^2 - b_3^2)G_{33}\Big\}$$

$$RA(5,7) = 0$$

$$RA(5,8) = G_2 + 2\theta_c G_5 + \beta_c G_7 + \phi_c G_8 + \beta_c\phi_c G_9 + \beta_c^2 G_{10} + \phi_c^2 G_{11} + 3\theta_c^2 G_{12} + N_A\Big\{\tfrac{1}{4}b_1b_2G_9 + \tfrac{1}{4}(3a_1^2 + b_1^2)G_{10} + \tfrac{1}{4}(3a_2^2 + b_2^2)G_{11} + \tfrac{3}{4}(a_3^2 + b_3^2)G_{12} + \tfrac{\alpha}{4}(a_1b_2 + a_2b_1)G_{31} + \tfrac{\alpha^2}{4}(a_1^2 - b_1^2)G_{36} + \tfrac{\alpha^2}{4}(a_2^2 - b_2^2)G_{37} - \tfrac{\alpha^2}{4}b_1b_2(G_{43} + G_{44})$$

$$RA(5,9) = -\alpha(G_{15} + \beta_c G_{20} + \phi_c G_{21} + \theta_c G_{22} + \beta_c\phi_c G_{31} + \beta_c\theta_c G_{32} + \phi_c\theta_c G_{33}) + N_A\Big\{\tfrac{\alpha}{2}a_1a_3G_{26} + \tfrac{1}{2}\alpha a_2a_3G_{30} - \tfrac{\alpha}{4}(3a_1a_2 + b_1b_2)G_{31} - \tfrac{\alpha}{2}a_1a_3G_{32} - \tfrac{\alpha}{2}a_2a_3G_{33}\Big\}$$

$$RA(6,1) = 0$$

$$RA(6,2) = \alpha(G_{13} + \phi_c G_{16} + \theta_c G_{17} + \beta_c\phi_c G_{23} + \beta_c^2 G_{24} + \phi_c^2 G_{25} + \theta_c^2 G_{26} + \beta_c G_{41}) + N_A\Big\{\tfrac{\alpha}{2}b_1b_2G_{23} + \tfrac{\alpha}{4}(a_1^2 + b_1^2)G_{24} + \tfrac{\alpha}{4}(a_2^2 + 3b_2^2)G_{25} + \tfrac{\alpha}{4}(a_3^2 + 3b_3^2)G_{26} + \tfrac{\alpha}{4}(a_2^2 - b_2^2)G_{27} - \tfrac{\alpha}{2}b_1b_2G_{28} + \tfrac{\alpha}{4}(a_3^2 - b_3^2)G_{32}\Big\}$$

$$RA(6,3) = G_1 + 2\beta_c G_3 + \phi_c G_6 + \theta_c G_7 + \phi_c\theta_c G_9 + 2\beta_c\theta_c G_{10} + N_A\Big\{\tfrac{1}{4}(a_2a_3 + 3b_2b_3)G_9 + \tfrac{a_1a_3}{2}G_{10} + \tfrac{\alpha^2}{2}a_1a_3G_{36} + \tfrac{\alpha^2}{4}(3a_2a_3 + b_2b_3)G_{44} - \tfrac{\alpha^2}{4}a_2a_3G_{43}\Big\}$$

$$RA(6,4) = 0$$

$$RA(6,5) = \alpha(G_{14} + \beta_c G_{18} + \theta_c G_{19} + \beta_c\phi_c G_{27} + \beta_c^2 G_{28} + \phi_c^2 G_{29} + \theta_c^2 G_{30}) + N_A\Big\{\tfrac{\alpha}{4}(a_1^2 - b_1^2)G_{23} - \tfrac{\alpha}{2}b_1b_2G_{25} + \tfrac{\alpha}{2}b_1b_2G_{27} + \tfrac{\alpha}{4}(a_1^2 + 3b_1^2)G_{28} + \tfrac{\alpha}{4}(a_2^2 + b_2^2)G_{29} + \tfrac{\alpha}{4}(a_3^2 + 3b_3^2)G_{30} + \tfrac{\alpha}{4}(a_3^2 - b_3^2)G_{33}\Big\}$$

$$RA(6,6) = 2\phi_c G_4 + \beta_c G_6 + \theta_c G_8 + \beta_c\theta_c G_9 + 2\phi_c\theta_c G_{11} + N_A\Big\{\tfrac{1}{4}a_1a_3G_9 + \tfrac{1}{2}a_2a_3G_{11} + \tfrac{\alpha^2}{2}a_2a_3G_{37} + \tfrac{\alpha^2}{4}(3a_1a_3 + b_1b_3)G_{43} - \tfrac{\alpha^2}{4}a_1a_3G_{44}\Big\}$$

$$RA(6,7) = 0$$

$$RA(6,8) = \alpha(G_{15} + \beta_c G_{20} + \phi_c G_{21} + \theta_c G_{22} + \beta_c\phi_c G_{31} + \beta_c\theta_c G_{32} + \phi_c\theta_c G_{33}) + N_A\Big\{-\tfrac{\alpha}{2}b_1b_3G_{26} - \tfrac{\alpha}{2}b_2b_3G_{30} + \tfrac{\alpha}{4}(a_1a_2 + 3b_1b_2)G_{31} + \tfrac{\alpha}{2}b_1b_3G_{32} + \tfrac{\alpha}{2}b_2b_3G_{33}\Big\}$$

$$RA(6,9) = G_2 + 2\theta_c G_5 + \beta_c G_7 + \phi_c G_8 + \beta_c\phi_c G_9 + \beta_c^2 G_{10} + \phi_c^2 G_{11} + 3\theta_c^2 G_{12} + N_A\Big\{\tfrac{a_1a_2}{4}G_9 + \tfrac{1}{4}(a_1^2 + 3b_1^2)G_{10} + \tfrac{1}{4}(a_2^2 + 3b_2^2)G_{11} + \tfrac{3}{4}(a_3^2 + b_3^2)G_{12} - \tfrac{\alpha}{4}(a_1b_2 + a_2b_1)G_{31} - \tfrac{\alpha^2}{4}(a_1^2 - b_1^2)G_{36} - \tfrac{\alpha^2}{4}(a_2^2 - b_2^2)G_{37} - \tfrac{\alpha^2}{4}a_1a_2(G_{43} + G_{44})\Big\}$$

$$
\begin{aligned}
RA(7,1) = {} & -G_0 - \phi_c\theta_c(F_7+G_8) - \phi_c^2(F_6+G_4) - \beta_c^2\left(G_3-\tfrac{G_0}{6}\right) - \theta_c^2 G_5 - N_A\Big\{\tfrac{1}{2}(a_2a_3+b_2b_3)(F_7+G_8) \\
& + (a_1a_3+b_1b_3)G_7 + (a_1a_2+b_1b_2)\left(F_3+G_6-\tfrac{F_0}{2}\right) + \tfrac{1}{2}(a_2^2+b_2^2)(F_6+G_4) + \tfrac{3}{2}(a_1^2+b_1^2)\left(G_3-\tfrac{G_0}{6}\right) \\
& + \tfrac{1}{2}(a_3^2+b_3^2)G_5 + \tfrac{\alpha}{2}(a_1b_2-a_2b_1)G_{16} + \tfrac{\alpha}{2}(a_1b_3-a_3b_1)G_{17} + \alpha(a_2b_1-a_1b_2)G_{18} + \tfrac{\alpha}{2}(a_2b_3-a_3b_2)G_{19} \\
& + \tfrac{\alpha}{2}(a_3b_2-a_2b_3)(F_{20}+G_{21}) + \alpha(a_3b_1-a_1b_3)G_{20} + \tfrac{\alpha^2}{2}(a_1a_2+b_1b_2)G_{34} + \tfrac{\alpha^2}{2}(a_2a_3+b_2b_3)G_{35} \\
& + \tfrac{\alpha^2}{2}(a_2^2+b_2^2)G_{38} + \tfrac{\alpha^2}{2}(a_3^2+b_3^2)G_{40} + \tfrac{\alpha^2}{2}(a_1a_3+b_1b_3)G_{42} + \tfrac{\alpha^2}{2}(a_1^2+b_1^2)G_{46}\Big\}
\end{aligned}
$$

$RA(7,2) = 0$

$RA(7,3) = 0$

$$
\begin{aligned}
RA(7,4) = {} & -F_0 - \phi_c^2\left(F_4'-\tfrac{F_0}{6}\right) - \theta_c^2 F_5 - \beta_c^2\left(F_3+G_6-\tfrac{F_0}{2}\right) - N_A\Big\{\tfrac{1}{2}(a_1a_3+b_1b_3)(F_7+G_8) + (a_2a_3+b_2b_3)F_8 \\
& + \tfrac{1}{2}(a_1^2+b_1^2)\left(F_3+G_6-\tfrac{F_0}{2}\right) + \tfrac{3}{2}(a_2^2+b_2^2)\left(F_4'-\tfrac{F_0}{6}\right) + \tfrac{1}{2}(a_3^2+b_3^2)F_5 + (a_1a_2+b_1b_2)(F_6+G_4) \\
& + \alpha(a_1b_2-a_2b_1)F_{16} + \tfrac{\alpha}{2}(a_1b_3-a_3b_1)F_{17} + \tfrac{\alpha}{2}(a_2b_1-a_1b_2)F_{18} + \tfrac{\alpha}{2}(a_2b_3-a_3b_2)F_{19} \\
& + \tfrac{\alpha}{2}(a_3b_1-a_1b_3)(F_{20}+G_{21}) + \alpha(a_3b_2-a_2b_3)F_{21} + \tfrac{\alpha^2}{2}(a_1a_2+b_1b_2)F_{34} + \tfrac{\alpha^2}{2}(a_2a_3+b_2b_3)F_{35} \\
& + \tfrac{\alpha^2}{2}(a_2^2+b_2^2)F_{38}\Big\}
\end{aligned}
$$

$RA(7,5) = 0$

$RA(7,6) = 0$

$$
\begin{aligned}
RA(7,7) = {} & -\beta_c^2 G_7 - \phi_c^2 F_8 - N_A\Big\{\tfrac{1}{2}(a_1a_2+b_1b_2)(F_7+G_8) + \tfrac{1}{2}(a_1^2+b_1^2)G_7 + \tfrac{1}{2}(a_2^2+b_2^2)F_8 + (a_2a_3+b_2b_3)F_5 \\
& + (a_1a_3+b_1b_3)G_5 + \tfrac{\alpha}{2}(a_1b_2-a_2b_1)F_{17} + \tfrac{\alpha}{2}(a_2b_1-a_1b_2)G_{19} + \tfrac{\alpha}{2}(a_3b_1-a_1b_3)G_{22} + \tfrac{\alpha}{2}(a_3b_2-a_2b_3)F_{22}\Big\}
\end{aligned}
$$

$RA(7,8) = 0$

$RA(7,9) = 0$

$RA(8,1) = 0$

$$
\begin{aligned}
RA(8,2) = {} & -G_0 - 2\beta_c G_1 - \phi_c F_1 + \theta_c G_2 - \phi_c\theta_c(F_7+G_8) - 2\beta_c\theta_c G_7 - 2\beta_c\phi_c\left(F_3+G_6-\tfrac{F_0}{2}\right) - \phi_c^2(F_6+G_4) \\
& - 3\beta_c^2\left(G_3-\tfrac{G_0}{6}\right) - \theta_c^2 G_5 - N_A\Big\{\tfrac{1}{4}(3a_2a_3+b_2b_3)(F_7+G_8) + \tfrac{1}{2}b_1b_3G_7 + \tfrac{1}{2}b_1b_2\left(F_3+G_6-\tfrac{F_0}{2}\right) \\
& + \tfrac{1}{4}(3a_2^2+b_2^2)(F_6+G_4) + \tfrac{3}{4}(a_1^2+b_1^2)\left(G_3-\tfrac{G_0}{6}\right) + \tfrac{1}{4}(3a_3^2+b_3^2)G_5 + \tfrac{\alpha^2}{2}b_1b_2G_{34} \\
& + \tfrac{\alpha^2}{4}(a_2^2-b_2^2)F_{34} + \tfrac{\alpha^2}{4}(a_1^2+b_1^2)G_{46} + \tfrac{\alpha^2}{4}(a_2a_3+b_2b_3)G_{35} + \tfrac{\alpha^2}{4}(a_2^2+3b_2^2)G_{38} + \tfrac{\alpha^2}{4}(a_3^2+3b_3^2)G_{40} + \tfrac{\alpha^2}{2}b_1b_3G_{42}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
RA(8,3) = {} & \alpha(\phi_c F_{13} + \beta_c G_{13} + \beta_c\phi_c G_{16} + \phi_c^2 F_{16} + \phi_c\theta_c F_{17} + \beta_c\theta_c G_{17} + \beta_c^2 G_{41}) - N_A\Big\{-\tfrac{\alpha}{2}a_1a_2G_{16} \\
& - \tfrac{\alpha}{4}(a_1^2+b_1^2)G_{41} - \tfrac{\alpha}{4}(3a_2^2+b_2^2)F_{16} - \tfrac{3}{4}\alpha a_2a_3F_{17} - \tfrac{\alpha}{2}a_1a_3G_{17} + \tfrac{\alpha}{4}(a_2^2-b_2^2)F_{18} \\
& + \tfrac{\alpha}{2}a_1a_2G_{18} + \tfrac{\alpha}{4}a_2a_3G_{19} + \tfrac{\alpha}{4}(a_3^2-b_3^2)G_{22} + \tfrac{\alpha}{2}a_1a_3G_{20}\Big\}
\end{aligned}
$$

$RA(8,4) = 0$

$$
\begin{aligned}
RA(8,5) = {} & -F_0 - \beta_c F_1 - \theta_c F_2 - \beta_c\theta_c(F_7+G_8) - 2\phi_c\theta_c F_8 - \beta_c^2\left(F_3+G_6-\tfrac{F_0}{2}\right) - 3\phi_c^2\left(F_4'-\tfrac{F_0}{6}\right) - \theta_c^2 F_5 \\
& - 2\beta_c\phi_c(F_6+G_4) - N_A\Big\{\tfrac{1}{4}b_1b_3(F_7+G_8) + \tfrac{1}{2}b_1b_2F_8 + \tfrac{1}{4}(3a_1^2+b_1^2)\left(F_3+G_6-\tfrac{F_0}{2}\right) \\
& + \tfrac{3}{4}(a_2^2+b_2^2)\left(F_4'-\tfrac{F_0}{6}\right) + \tfrac{1}{4}(3a_3^2+b_3^2)F_5 + \tfrac{1}{2}b_1b_2(F_6+G_4) + \tfrac{1}{4}\alpha^2(a_1^2-b_1^2)G_{34} + \tfrac{\alpha^2}{2}b_1b_2F_{34} \\
& + \tfrac{\alpha^2}{2}b_2b_3F_{35} + \tfrac{\alpha^2}{4}(a_2^2+b_2^2)F_{38} - \tfrac{\alpha^2}{4}b_1b_3G_{35} - \tfrac{\alpha^2}{2}b_1b_2G_{38}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
RA(8,6) = {} & \alpha(\beta_c G_{14} + \phi_c F_{14} + \beta_c\phi_c F_{18} + \beta_c^2 G_{18} + \phi_c\theta_c F_{19} + \beta_c\theta_c G_{19}) - N_A\Big\{\tfrac{\alpha}{4}(a_1^2-b_1^2)G_{16} + \tfrac{\alpha}{2}a_1a_2F_{16} \\
& + \tfrac{\alpha}{4}a_1a_3F_{17} - \tfrac{\alpha}{2}a_1a_2F_{18} - \tfrac{\alpha}{4}(3a_1^2+b_1^2)G_{18} - \tfrac{\alpha}{2}a_2a_3F_{19} - \tfrac{3}{4}\alpha a_1a_3G_{19} + \tfrac{\alpha}{2}a_2a_3F_{21} \\
& + \tfrac{\alpha}{4}(a_3^2-b_3^2)F_{22}\Big\}
\end{aligned}
$$

$$RA(8,7) = 0$$

$$RA(8,8) = -\left\{\beta_c G_2 + \phi_c F_2 + \beta_c\phi_c(F_7+G_8) + \beta_c^2 G_7 + \phi_c^2 F_8 + 2\phi_c\theta_c F_5 + 2\beta_c\theta_c G_5\right\} - N_A\left\{\tfrac{1}{4}b_1b_2(F_7+G_8)\right.$$
$$+\tfrac{1}{4}(3a_1^2+b_1^2)G_7 + \tfrac{1}{4}(3a_2^2+b_2^2)F_8 + \tfrac{1}{2}b_2b_3F_5 + \tfrac{1}{2}b_1b_3G_5 + \tfrac{\alpha}{4}(a_1b_2+a_2b_1)(F_{20}+G_{21})$$
$$\left.+\tfrac{\alpha^2}{4}(a_2^2-b_2^2)F_{35} - \tfrac{1}{4}\alpha^2 b_1b_2G_{35} - \tfrac{\alpha^2}{2}b_1b_3G_{40} + \tfrac{\alpha^2}{4}(a_1^2-b_1^2)G_{42}\right\}$$

$$RA(8,9) = \alpha\left\{\beta_c G_{15} + \phi_c F_{15} + \beta_c\phi_c(F_{20}+G_{21}) + \beta_c\theta_c G_{22} + \beta_c^2 G_{20} + \phi_c^2 F_{21} + \phi_c\theta_c F_{22}\right\} - N_A\left\{\tfrac{\alpha}{4}(a_1a_2-b_1b_2)F_{17}\right.$$
$$+\tfrac{\alpha}{4}(a_1^2-b_1^2)G_{17} + \tfrac{\alpha}{4}(a_2^2-b_2^2)F_{19} + \tfrac{\alpha}{4}(a_1a_2-b_1b_2)G_{19} - \tfrac{\alpha}{4}(3a_1a_2+b_1b_2)(F_{20}+G_{21})$$
$$\left.-\tfrac{\alpha}{2}a_1a_3G_{22} - \tfrac{\alpha}{4}(3a_1^2+b_1^2)G_{20} - \tfrac{\alpha}{4}(3a_2^2+b_2^2)F_{21} - \tfrac{\alpha}{2}a_2a_3F_{22}\right\}$$

$$RA(9,1) = 0$$

$$RA(9,2) = -\alpha\left(\phi_c F_{13} + \beta_c G_{13} + \beta_c\phi_c G_{16} + \beta_c^2 G_{41} + \phi_c^2 F_{16} + \phi_c\theta_c F_{17} + \beta_c\theta_c G_{17}\right) - N_A\left\{\tfrac{\alpha}{2}b_1b_2G_{16}\right.$$
$$+\tfrac{\alpha}{4}(a_1^2+b_1^2)G_{41} + \tfrac{\alpha}{4}(a_2^2+3b_2^2)F_{16} + \tfrac{3}{4}\alpha b_2b_3F_{17} + \tfrac{\alpha}{2}b_1b_3G_{17} + \tfrac{\alpha}{4}(a_2^2-b_2^2)F_{18} - \tfrac{\alpha}{2}b_1b_2G_{18}$$
$$\left.-\tfrac{\alpha}{4}b_2b_3G_{19} + \tfrac{\alpha}{4}(a_3^2-b_3^2)G_{22} - \tfrac{\alpha}{2}b_1b_3G_{20}\right\}$$

$$RA(9,3) = -\left\{G_0 + 2\beta_c G_1 + \phi_c F_1 + \theta_c G_2 + \phi_c\theta_c(F_7+G_8) + 2\beta_c\theta_c G_7 + 2\beta_c\phi_c\left(F_3+G_6-\tfrac{F_0}{2}\right) + \phi_c^2(F_6+G_4)\right.$$
$$\left.+3\beta_c^2\left(G_3-\tfrac{G_0}{6}\right) + \theta_c^2 G_5\right\} - N_A\left\{\tfrac{1}{4}(a_2a_3+3b_2b_3)(F_7+G_8) + \tfrac{1}{2}a_1a_3G_7 + \tfrac{1}{2}a_1a_2\left(F_3+G_6-\tfrac{F_0}{2}\right)\right.$$
$$+\tfrac{1}{4}(a_2^2+3b_2^2)(F_6+G_4) + \tfrac{3}{4}(a_1^2+b_1^2)\left(G_3-\tfrac{G_0}{6}\right) + \tfrac{1}{4}(a_3^2+3b_3^2)G_5 + \tfrac{1}{2}\alpha^2a_1a_2G_{34}$$
$$-\tfrac{\alpha^2}{4}(a_2^2-b_2^2)F_{34} + \tfrac{\alpha^2}{4}(a_1^2+b_1^2)G_{46} + \tfrac{\alpha^2}{4}(3a_2a_3+b_2b_3)G_{35} + \tfrac{\alpha^2}{4}(3a_3^2+b_3^2)G_{38} + \tfrac{\alpha^2}{4}(3a_2^2+b_2^2)G_{40}$$
$$\left.+\tfrac{\alpha^2}{2}a_1a_3G_{42}\right\}$$

$$RA(9,4) = 0$$

$$RA(9,5) = -\alpha\left(\beta_c G_{14} + \phi_c F_{14} + \beta_c\phi_c F_{18} + \beta_c^2 G_{18} + \phi_c\theta_c F_{19} + \beta_c\theta_c G_{19}\right) - N_A\left\{\tfrac{\alpha}{4}(a_1^2-b_1^2)G_{16} - \tfrac{\alpha}{2}b_1b_2F_{16}\right.$$
$$-\tfrac{\alpha}{4}b_1b_3F_{17} + \tfrac{\alpha}{2}b_1b_2F_{18} + \tfrac{\alpha}{4}(a_1^2+3b_1^2)G_{18} + \tfrac{\alpha}{2}b_2b_3F_{19} + \tfrac{3}{4}\alpha b_1b_3G_{19} - \tfrac{\alpha}{2}b_2b_3F_{21}$$
$$\left.+\tfrac{\alpha}{4}(a_3^2-b_3^2)F_{22}\right\}$$

$$RA(9,6) = -\left\{F_0 + \beta_c F_1 + \theta_c F_2 + \beta_c\theta_c(F_7+G_8) + 2\phi_c\theta_c F_8 + \beta_c^2\left(F_3+G_6-\tfrac{F_0}{2}\right) + 3\phi_c^2\left(F_4'-\tfrac{F_0}{6}\right) + \theta_c^2F_5 + 2\beta_c\phi_c(F_6+G_4)\right.$$
$$-N_A\left\{\tfrac{1}{4}a_1a_3(F_7+G_8) + \tfrac{1}{2}a_2a_3F_8 + \tfrac{1}{4}(a_1^2+3b_1^2)\left(F_3+G_6-\tfrac{F_0}{2}\right) + \tfrac{3}{4}(a_2^2+b_2^2)\left(F_4'-\tfrac{F_0}{6}\right)\right.$$
$$+\tfrac{1}{4}(a_3^2+3b_3^2)F_5 + \tfrac{1}{2}a_1a_2(F_6+G_4) - \tfrac{\alpha^2}{4}(a_1^2-b_1^2)G_{34} + \tfrac{1}{2}\alpha^2a_1a_2F_{34} + \tfrac{1}{2}\alpha^2a_2a_3F_{35}$$
$$\left.+\tfrac{1}{4}\alpha^2(a_2^2+b_2^2)F_{38} - \tfrac{1}{4}\alpha^2a_1a_3G_{35} - \tfrac{1}{2}\alpha^2a_1a_2G_{38}\right\}$$

$$RA(9,7) = 0$$

$$RA(9,8) = -\alpha\left\{\beta_c G_{15} + \phi_c F_{15} + \beta_c\phi_c(F_{20}+G_{21}) + \beta_c\theta_c G_{22} + \beta_c^2G_{20} + \phi_c^2F_{21} + \phi_c\theta_c F_{22}\right\}$$
$$-N_A\left\{\tfrac{\alpha}{4}(a_1a_2-b_1b_2)F_{17} + \tfrac{\alpha}{4}(a_1^2-b_1^2)G_{17} + \tfrac{\alpha}{4}(a_2^2-b_2^2)F_{19} + \tfrac{\alpha}{4}(a_1a_2-b_1b_2)G_{19}\right.$$
$$\left.+\tfrac{\alpha}{4}(a_1a_2+3b_1b_2)(F_{20}+G_{21}) + \tfrac{\alpha}{2}b_1b_3G_{22} + \tfrac{\alpha}{4}(a_1^2+3b_1^2)G_{20} + \tfrac{\alpha}{4}(a_2^2+3b_2^2)F_{21} + \tfrac{\alpha}{2}b_2b_3F_{22}\right\}$$

$$RA(9,9) = -\left\{ \beta_c G_2 + \phi_c F_2 + \beta_c\phi_c(F_7+G_8) + \beta_c^2 G_7 + \phi_c^2 F_8 + 2\phi_c\theta_c F_5 + 2\beta_c\theta_c G_5 \right\} - N_A\left\{ \tfrac{1}{4}a_1a_2(F_7+G_8) + \tfrac{1}{4}(a_1^2+3b_1^2)G_7 + \tfrac{1}{4}(a_2^2+3b_2^2)F_8 + \tfrac{1}{2}a_2a_3F_5 + \tfrac{1}{2}a_1a_3G_5 - \tfrac{\alpha}{4}(a_1b_2+a_2b_1)(F_{20}+G_{21}) - \tfrac{\alpha^2}{4}(a_2^2-b_2^2)F_{35} - \tfrac{\alpha^2}{4}a_1a_2G_{35} - \tfrac{\alpha^2}{2}a_1a_3G_{40} - \tfrac{\alpha^2}{4}(a_1^2-b_1^2)G_{42} \right\}$$

$$RAO(1) = -\left\{ F_0 + \beta_c^2F_3 + \phi_c^2F_4 + \theta_c^2F_5 + \beta_c\phi_c F_6 + \beta_c\theta_c F_7 + \phi_c\theta_c F_8 \right\} - N_A\left\{ \tfrac{1}{2}(a_1^2+b_1^2)F_3 + \tfrac{1}{2}(a_2^2+b_2^2)F_4 + \tfrac{1}{2}(a_3^2+b_3^2)F_5 + \tfrac{1}{2}(a_1a_2+b_1b_2)F_6 + \tfrac{1}{2}(a_1a_3+b_1b_3)F_7 + \tfrac{1}{2}(a_2a_3+b_2b_3)F_8 + \tfrac{\alpha}{2}(a_1b_2-a_2b_1)F_{16} + \tfrac{\alpha}{2}(a_1b_3-a_3b_1)F_{17} + \tfrac{\alpha}{2}(a_2b_1-a_1b_2)F_{18} + \tfrac{\alpha}{2}(a_2b_3-a_3b_2)F_{19} + \tfrac{\alpha}{2}(a_3b_1-a_1b_3)F_{20} + \tfrac{\alpha}{2}(a_3b_2-a_2b_3)F_{21} + \tfrac{\alpha^2}{2}(a_1a_2+b_1b_2)F_{34} + \tfrac{\alpha^2}{2}(a_2a_3+b_2b_3)F_{35} + \tfrac{\alpha^2}{2}(a_2^2+b_2^2)F_{38} \right\}$$

$RAO(2) = 0$

$RAO(3) = 0$

$$RAO(4) = \left(G_0 + \beta_c^2G_3 + \phi_c^2G_4 + \theta_c^2G_5 + \beta_c\phi_c G_6 + \beta_c\theta_c G_7 + \phi_c\theta_c G_8 \right) + N_A\left\{ \tfrac{1}{2}(a_1^2+b_1^2)G_3 + \tfrac{1}{2}(a_2^2+b_2^2)G_4 + \tfrac{1}{2}(a_3^2+b_3^2)G_5 + \tfrac{1}{2}(a_1a_2+b_1b_2)G_6 + \tfrac{1}{2}(a_1a_3+b_1b_3)G_7 + \tfrac{1}{2}(a_2a_3+b_2b_3)G_8 + \tfrac{\alpha}{2}(a_1b_2-a_2b_1)G_{16} + \tfrac{\alpha}{2}(a_1b_3-a_3b_1)G_{17} + \tfrac{\alpha}{2}(a_2b_1-a_1b_2)G_{18} + \tfrac{\alpha}{2}(a_2b_3-a_3b_2)G_{19} + \tfrac{\alpha}{2}(a_3b_1-a_1b_3)G_{20} + \tfrac{\alpha}{2}(a_3b_2-a_2b_3)G_{21} + \tfrac{\alpha^2}{2}(a_1a_2+b_1b_2)G_{34} + \tfrac{\alpha^2}{2}(a_2a_3+b_2b_3)G_{35} + \tfrac{\alpha^2}{2}(a_2^2+b_2^2)G_{38} + \tfrac{\alpha^2}{2}(a_3^2+b_3^2)G_{40} + \tfrac{\alpha^2}{2}(a_1a_3+b_1b_3)G_{42} + \tfrac{\alpha^2}{2}(a_1^2+b_1^2)G_{46} \right\}$$

$RAO(5) = 0$

$RAO(6) = 0$

$$RAO(7) = -\left(\beta_c^2G_1 + \beta_c\phi_c F_1 + \beta_c\theta_c G_2 + \phi_c\theta_c F_2 \right) - N_A\left\{ \tfrac{1}{2}(a_1^2+b_1^2)G_1 + \tfrac{1}{2}(a_1a_2+b_1b_2)F_1 + \tfrac{1}{2}(a_1a_3+b_1b_3)G_2 + \tfrac{1}{2}(a_2a_3+b_2b_3)F_2 + \tfrac{\alpha}{2}(a_1b_2-a_2b_1)F_{13} + \tfrac{\alpha}{2}(a_2b_1-a_1b_2)G_{14} + \tfrac{\alpha}{2}(a_3b_1-a_1b_3)G_{15} + \tfrac{\alpha}{2}(a_3b_2-a_2b_3)F_{15} \right\}$$

$RAO(8) = 0$

$RAO(9) = 0$

Periodic aerodynamic forces terms for forced oscillations

$$RV(1,1) = 0$$

$$\begin{aligned} RV(1,2) = &-\tfrac{1}{2}\left(R_{13} + \phi_c R_{16} + \theta_c R_{17} + \beta_c \phi_c R_{23} + \beta_c^2 R_{24} + \phi_c^2 R_{25} + \theta_c^2 R_{26}\right) - N_A \tfrac{1}{8}\big\{(a_1 a_2 + 2 b_1 b_2) R_{23} \\ &+ (a_1^2 + b_1^2) R_{24} + (a_2^2 + 3 b_2^2) R_{25} + (a_3^2 + 3 b_3^2) R_{26} - (b_2^2 R_{27} + 2 b_1 b_2 R_{28} + b_2 b_3 R_{31} + b_3^2 R_{32})\big\} \end{aligned}$$

$$\begin{aligned} RV(1,3) = &-\tfrac{1}{2}\left(R_1 + 2\beta_c R_3 + \phi_c R_6 + \theta_c R_7 + \phi_c \theta_c R_9 + 2\beta_c \theta_c R_{10}\right) - N_A \tfrac{1}{8}\big\{(a_2 a_3 + 3 b_2 b_3) R_9 \\ &+ (2 a_1 a_3 + 3 b_1 b_3) R_{10}\big\} \end{aligned}$$

$$RV(1,4) = 0$$

$$\begin{aligned} RV(1,5) = &-\tfrac{1}{2}\left(R_{14} + \beta_c R_{18} + \theta_c R_{19} + \beta_c \phi_c R_{27} + \beta_c^2 R_{28} + \phi_c^2 R_{29} + \theta_c^2 R_{30}\right) - N_A \tfrac{1}{8}\big\{-b_1^2 R_{23} - 2 b_1 b_2 R_{25} \\ &+ (a_1 a_2 + 2 b_1 b_2) R_{27} + (a_1^2 + 3 b_1^2) R_{28} + (a_2^2 + b_2^2) R_{29} + (a_3^2 + 3 b_3^2) R_{30} - b_1 b_3 R_{31} - b_3^2 R_{33}\big\} \end{aligned}$$

$$RV(1,6) = -\tfrac{1}{2}\left(2\phi_c R_4 + \beta_c R_6 + \theta_c R_8 + \beta_c \theta_c R_9 + \phi_c \theta_c R_{11}\right) - N_A \tfrac{1}{8}\left\{a_1 a_3 R_9 + (2 a_2 a_3 + 3 b_2 b_3) R_{11}\right\}$$

$$RV(1,7) = 0$$

$$\begin{aligned} RV(1,8) = &-\tfrac{1}{2}\left(R_{15} + \beta_c R_{20} + \phi_c R_{21} + \theta_c R_{22} + \beta_c \phi_c R_{31} + \beta_c \theta_c R_{32} + \phi_c \theta_c R_{33}\right) - N_A \tfrac{1}{8}\big\{-2 b_1 b_3 R_{26} \\ &- 2 b_2 b_3 R_{30} + (a_1 a_2 + 3 b_1 b_2) R_{31} + (a_1 a_3 + 2 b_1 b_3) R_{32} + (a_2 a_3 + 2 b_2 b_3) R_{33}\big\} \end{aligned}$$

$$\begin{aligned} RV(1,9) = &-\tfrac{1}{2}\left(R_2 + 2\theta_c R_5 + \beta_c R_7 + \phi_c R_8 + \beta_c \phi_c R_9 + \beta_c^2 R_{10} + \phi_c^2 R_{11} + 3\theta_c^2 R_{12}\right) - N_A \tfrac{1}{8}\big\{a_1 a_2 R_9 \\ &+ a_1^2 R_{10} + a_2^2 R_{11} + 3(a_3^2 + b_3^2) R_{12}\big\} \end{aligned}$$

$$\begin{aligned} RV(2,1) = &- N_A \tfrac{1}{4}\big\{(a_2 b_3 + a_3 b_2) R_9 + 2(a_1 b_3 + a_3 b_1) R_{10} + (a_1 a_2 - b_1 b_2)(R_{23} + 2 R_{28}) \\ &+ 2(a_1^2 - b_1^2) R_{24} + (a_2^2 - b_2^2) R_{27} + (a_2 a_3 - b_2 b_3) R_{31} + (a_3^2 - b_3^2) R_{32}\big\} \end{aligned}$$

$$RV(2,2) = 0$$

$$RV(2,3) = 0$$

$$\begin{aligned} RV(2,4) = &- N_A \tfrac{1}{4}\big\{(a_1 b_3 + a_3 b_1) R_9 + 2(a_2 b_3 + a_3 b_2) R_{11} + (a_1^2 - b_1^2) R_{23} + (a_1 a_2 - b_1 b_2)(2 R_{25} + R_{27}) \\ &+ 2(a_2^2 - b_2^2) R_{29} + (a_1 a_3 - b_1 b_3) R_{31} + (a_3^2 - b_3^2) R_{33}\big\} \end{aligned}$$

$$RV(2,5) = 0$$

$$RV(2,6) = 0$$

$$\begin{aligned} RV(2,7) = &- N_A \tfrac{1}{4}\big\{(a_1 b_2 + a_2 b_1) R_9 + 2 a_1 b_1 R_{10} + 2 a_2 b_2 R_{11} + 6 a_3 b_3 R_{12} + 2(a_1 a_3 - b_1 b_3) R_{26} \\ &+ 2(a_2 a_3 - b_2 b_3) R_{30} + (a_1 a_3 - b_1 b_3) R_{32} + (a_2 a_3 - b_2 b_3) R_{33}\big\} \end{aligned}$$

$$RV(2,8) = 0$$

$$RV(2,9) = 0$$

$$RV(3,1) = -\left(R_1 + \phi_c \theta_c R_9 + \beta_c \theta_c R_{10}\right) - N_A \tfrac{1}{4}\Big\{(a_2a_3 + 3b_2b_3)R_9 + 2(a_1a_3 + 3b_1b_3)R_{10} + (3a_1b_2 - a_2b_1)R_{23}$$
$$+ 4a_1b_1R_{24} + 2a_2b_2R_{27} + 2(3a_2b_1 - a_1b_2)R_{28} + (3a_3b_2 - a_2b_3)R_{31} + 2a_3b_3R_{32}\Big\}$$

$$RV(3,2) = 0$$

$$RV(3,3) = 0$$

$$RV(3,4) = -\phi_c\theta_c R_{11} - N_A \tfrac{1}{4}\Big\{(a_1a_3 + 3b_1b_3)R_9 + 2(a_2a_3 + 3b_2b_3)R_{11} + 2a_1b_1R_{23} + 2(3a_1b_2 - a_2b_1)R_{25}$$
$$+ (3a_2b_1 - a_1b_2)R_{27} + 4a_2b_2R_{29} + (3a_3b_1 - a_1b_3)R_{31} + 2a_3b_3R_{33}\Big\}$$

$$RV(3,5) = 0$$

$$RV(3,6) = 0$$

$$RV(3,7) = -(R_2 + \theta_c^2 R_{12}) - N_A \tfrac{1}{4}\Big\{(a_1a_2 + 3b_1b_2)R_9 + (a_1^2 + 3b_1^2)R_{10} + (a_2^2 + 3b_2^2)R_{11}$$
$$+ 3(a_3^2 + 3b_3^2)R_{12} + 2(3a_1b_3 - a_3b_1)R_{26} + 2(3a_2b_3 - a_3b_2)R_{30} + (3a_3b_1 - a_1b_3)R_{32}$$
$$+ (3a_3b_2 - a_2b_3)R_{33}\Big\}$$

$$RV(3,8) = 0$$

$$RV(3,9) = 0$$

$$RV(7,1) = 0$$

$$RV(7,2) = -\tfrac{1}{2}\left(\beta_c S_{13} + \beta_c\phi_c S_{16} + \beta_c\theta_c S_{17} + \phi_c\theta_c R_{17}\right) - N_A \tfrac{1}{8}\Big\{(a_1a_2 + 2b_1b_2)S_{16} + (a_1a_3 + 2b_1b_3)S_{17}$$
$$+ (a_2a_3 + 3b_2b_3)R_{17} - 2b_1b_2S_{18} - b_2b_3(S_{19} + R_{20} + S_{21}) - 2b_1b_3S_{20} - b_3^2S_{22}\Big\}$$

$$RV(7,3) = -\tfrac{1}{2}\Big\{S_0 + 2\beta_c S_1 + \theta_c S_2 + 3\beta_c^2\left(S_3 - \tfrac{S_0}{6}\right) + 2\beta_c\phi_c\left(R_3 - \tfrac{R_0}{2} + S_6\right) + \phi_c^2(S_4 + R_6) + \theta_c^2 S_5$$
$$+ 2\beta_c\theta_c S_7 + \phi_c\theta_c(R_7 + S_8)\Big\} - N_A \tfrac{1}{8}\Big\{3(a_1^2 + b_1^2)\left(S_3 - \tfrac{S_0}{6}\right) + (2a_1a_2 + 3b_1b_2)\left(R_3 + S_6 - \tfrac{R_0}{2}\right)$$
$$+ a_3^2S_5 + (2a_1a_3 + 3b_1b_3)S_7 + (a_2a_3 + 3b_2b_3)(R_7 + S_8) + a_2^2(S_4 + R_6)\Big\}$$

$$RV(7,4) = 0$$

$$RV(7,5) = -\tfrac{1}{2}\left(\beta_c S_{14} + \phi_c R_{14} + \beta_c^2 S_{18} + \beta_c\theta_c S_{19} + \phi_c\theta_c R_{19}\right) - N_A \tfrac{1}{8}\Big\{-b_1^2S_{16} - b_1b_3R_{17} + (a_1^2 + 3b_1^2)S_{18}$$
$$+ (a_1a_3 + 3b_1b_3)S_{19} + (a_2a_3 + 2b_2b_3)R_{19} - b_1b_3(R_{20} + S_{21}) - b_3^2R_{22}\Big\}$$

$$RV(7,6) = -\tfrac{1}{2}\Big\{R_0 + \theta_c R_2 + \beta_c^2\left(R_3 + S_6 - \tfrac{R_0}{2}\right) + 2\beta_c\phi_c(S_4 + R_6) + 3\phi_c^2\left(R_4' - \tfrac{R_0}{6}\right) + \theta_c^2 R_5 + \beta_c\theta_c(R_7 + S_8)\Big\}$$
$$-N_A \tfrac{1}{8}\Big\{a_1^2\left(R_3 - \tfrac{R_0}{2} + S_6\right) + (2a_1a_2 + 3b_1b_2)(S_4 + R_6) + 3(a_2^2 + b_2^2)\left(R_4' - \tfrac{R_0}{6}\right) + a_3^2R_5$$
$$+ a_1a_3(R_7 + S_8)\Big\}$$

$$RV(7,7) = 0$$

$$RV(7,8) = -\tfrac{1}{2}\Big\{\beta_c S_{15} + \beta_c^2 S_{20} + \beta_c\phi_c(R_{20} + S_{21}) + \beta_c\theta_c S_{22} + \phi_c\theta_c R_{22}\Big\} - N_A \tfrac{1}{8}\Big\{-b_1^2S_{17} - b_1b_2(R_{17} + S_{19})$$
$$+ (a_1^2 + 3b_1^2)S_{20} + (a_1a_2 + 3b_1b_2)(R_{20} + S_{21}) + (a_1a_3 + 2b_1b_3)S_{22} + (a_2a_3 + 2b_2b_3)R_{22} - b_2^2R_{19}\Big\}$$

$$RV(7,9) = -\tfrac{1}{2}\Big\{\beta_3 S_2 + \phi_c R_2 + \beta_c\theta_c S_5 + \phi_c\theta_c R_5 + \beta_c^2 S_7 + \beta_c\phi_c(R_7 + S_8)\Big\} - N_A \tfrac{1}{8}\Big\{(2a_1a_3 + 3b_1b_3)S_5$$
$$+ (2a_2a_3 + 3b_2b_3)R_5 + a_1^2S_7 + a_1a_2(R_7 + S_8)\Big\}$$

$$
\begin{aligned}
RV(8,1) = & -N_A\Big\{\tfrac{3}{2}a_1b_1\left(S_3-\tfrac{S_0}{6}\right)+\tfrac{1}{2}a_2b_2(S_4+R_6)+\tfrac{1}{2}a_3b_3S_5+\tfrac{1}{2}(a_1b_2+a_2b_1)\left(R_3-\tfrac{R_0}{2}+S_6\right) \\
& +\tfrac{1}{2}(a_1b_3+a_3b_1)S_7+\tfrac{1}{4}(a_2b_3+a_3b_2)(R_7+S_8)+\tfrac{1}{4}(a_1a_2-b_1b_2)(S_{16}+2S_{18})+(a_3^2-b_3^2)\tfrac{S_{22}}{4} \\
& +\tfrac{1}{4}(a_1a_3-b_1b_3)(S_{17}+2S_{20})+\tfrac{1}{4}(a_2a_3-b_2b_3)(S_{19}+S_{21}+R_{20})\Big\}
\end{aligned}
$$

$$RV(8,2)=0$$

$$RV(8,3)=0$$

$$
\begin{aligned}
RV(8,4) = & -N_A\Big\{\tfrac{1}{2}a_1b_1\left(R_3-\tfrac{R_0}{2}+S_6\right)+\tfrac{3}{2}a_2b_2\left(R_4'-\tfrac{R_0}{6}\right)+\tfrac{a_3b_3}{2}R_5+\tfrac{1}{2}(a_1b_2+a_2b_1)(S_4+R_6) \\
& +\tfrac{1}{4}(a_1b_3+a_3b_1)(R_7+S_8)+\tfrac{1}{4}(a_1^2-b_1^2)S_{16}+\tfrac{1}{4}(a_1a_3+b_1b_3)(R_{17}+R_{20}+S_{21}) \\
& +\tfrac{1}{4}(a_2a_3-b_2b_3)R_{19}+\tfrac{1}{4}(a_3^2-b_3^2)R_{22}\Big\}
\end{aligned}
$$

$$RV(8,5)=0$$

$$RV(8,6)=0$$

$$
\begin{aligned}
RV(8,7) = & -N_A\Big\{\tfrac{1}{2}a_1b_1S_7+\tfrac{1}{2}(a_1b_3+a_3b_1)S_5+\tfrac{1}{2}(a_2b_3+a_3b_2)R_5+\tfrac{1}{4}(a_1b_2+a_2b_1)(R_7+S_8) \\
& +\tfrac{1}{4}(a_1a_2-b_1b_2)(R_{17}+S_{19})+\tfrac{1}{4}(a_2^2-b_2^2)R_{19}+\tfrac{1}{4}(a_1a_3-b_1b_3)S_{22}+\tfrac{1}{4}(a_2a_3-b_2b_3)R_{22} \\
& +\tfrac{1}{4}(a_1^2-b_1^2)S_{17}\Big\}
\end{aligned}
$$

$$RV(8,8)=0$$

$$RV(8,9)=0$$

$$
\begin{aligned}
RV(9,1) = & -\Big\{S_0+\beta_c^2\left(S_3-\tfrac{S_0}{6}\right)+\beta_c\phi_c\left(R_3-\tfrac{R_0}{2}+S_6\right)+\beta_c\theta_c S_7+\phi_c\theta_c(R_7+S_8)\Big\}-N_A\tfrac{1}{4}\Big\{3(a_1^2+3b_1^2)\left(S_3-\tfrac{S_0}{6}\right) \\
& +2(a_1a_2+3b_1b_2)\left(R_3-\tfrac{R_0}{2}+S_6\right)+(a_2^2+3b_2^2)(S_4+R_6)+(a_3^2+3b_3^2)S_5+2(a_1a_3+3b_1b_3)S_7 \\
& +(a_2a_3+3b_2b_3)(R_7+S_8)+(3a_1b_2-a_2b_1)S_{16}+(3a_1b_3-a_3b_1)S_{17}+2(3a_2b_1-a_1b_2)S_{18} \\
& +(3a_2b_3-a_3b_2)S_{19}+2(3a_3b_1-a_1b_3)S_{20}+(3a_3b_2-a_2b_3)(R_{20}+S_{21})+2a_3b_3S_{22}\Big\}
\end{aligned}
$$

$$RV(9,2)=0$$

$$RV(9,3)=0$$

$$
\begin{aligned}
RV(9,4) = & -\Big\{R_0+\beta_c\phi_c(S_4+R_6)+\phi_c^2\left(R_4'-\tfrac{R_0}{6}\right)\Big\}-N_A\tfrac{1}{4}\Big\{(a_1^2+3b_1^2)\left(R_3-\tfrac{R_0}{2}+S_6\right)+2(a_1a_2+3b_1b_2)(S_4+R_6) \\
& +3(a_2^2+3b_2^2)\left(R_4'-\tfrac{R_0}{6}\right)+(a_3^2+3b_3^2)R_5+(a_1a_3+3b_1b_3)(R_7+S_8)+2a_1b_1S_{16}+3(a_1b_3-a_3b_1)R_{17} \\
& +(3a_2b_3-a_3b_2)R_{19}+(3a_3b_1-a_1b_3)(R_{20}+S_{21})+2a_3b_3R_{22}\Big\}
\end{aligned}
$$

$$RV(9,5)=0$$

$$RV(9,6)=0$$

$$
\begin{aligned}
RV(9,7) = & -(\beta_c\theta_cS_5+\phi_c\theta_cR_5)-N_A\cdot\tfrac{1}{4}\Big\{2(a_1a_3+3b_1b_3)S_5+2(a_2a_3+3b_2b_3)R_5+(a_1^2+3b_1^2)S_7 \\
& +(a_1a_2+3b_1b_2)(R_7+S_8)+2a_1b_1S_{17}+(3a_1b_2-a_2b_1)R_{17}+(3a_2b_1-a_1b_2)S_{19}+2a_2b_2R_{19} \\
& +(3a_3b_1-a_1b_3)S_{22}+(3a_3b_2-a_2b_3)R_{22}\Big\}
\end{aligned}
$$

$$RV(9,8)=0$$

$$RV(9,9)=0$$

$$RVO(1) = 0$$

$$RVO(2) = - N_A \frac{1}{4}\{2a_1b_1R_3 + 2a_2b_2R_4 + 2a_3b_3R_5 + (a_1b_2 + a_2b_1)R_6 + (a_1b_3 + a_3b_1)R_7$$
$$+ (a_2b_3 + a_3b_2)R_8 + (a_1a_2 - b_1b_2)(R_{16} + R_{18}) + (a_1a_3 - b_1b_3)(R_{17} + R_{20}) + (a_3^2 - b_3^2)R_{22}$$
$$+ (a_2a_3 - b_2b_3)(R_{19} + R_{21})\}$$

$$RVO(3) = -(R_0 + \beta_c^2R_3 + \phi_c^2R_4 + \theta_c^2R_5 + \beta_c\phi_cR_6 + \beta_c\theta_cR_7 + \phi_c\theta_cR_8) - N_A\frac{1}{4}\{(a_1^2 + 3b_1^2)R_3$$
$$+ (a_2^2 + 3b_2^2)R_4 + (a_3^2 + 3b_3^2)R_5 + (a_1a_2 + 3b_1b_2)R_6 + (a_1a_3 + 3b_1b_3)R_7 + (3a_1b_2 - a_2b_1)R_{16}$$
$$+ (3a_1b_3 - a_3b_1)R_{17} + (3a_2b_1 - a_1b_2)R_{18} + (3a_2b_3 - a_3b_2)R_{19} + (3a_3b_1 - a_1b_3)R_{20} + 2a_3b_3R_{22}$$
$$+ (3a_3b_2 - a_2b_3)R_{21} + (a_2a_3 + 3b_2b_3)R_8\}$$

$$RVO(4) = 0$$

$$RVO(5) = N_A \cdot \frac{1}{4}\{2a_1b_1S_3 + 2a_2b_2S_4 + 2a_3b_3S_5 + (a_1b_2 + a_2b_1)S_6 + (a_1b_3 + a_3b_1)S_7$$
$$+ (a_2b_3 + a_3b_2)S_8 + (a_1a_2 - b_1b_2)(S_{16} + S_{18}) + (a_1a_3 - b_1b_3)(S_{17} + S_{20})$$
$$+ (a_3^2 - b_3^2)S_{22} + (a_2a_3 - b_2b_3)(S_{19} + S_{21})\}$$

$$RVO(6) = S_0 + \beta_c^2S_3 + \phi_c^2S_4 + \theta_c^2S_5 + \beta_c\phi_cS_6 + \beta_c\theta_cS_7 + \phi_c\theta_cS_8 + N_A\frac{1}{4}\{(a_1^2 + 3b_1^2)S_3 + (a_2^2 + 3b_2^2)S_4$$
$$+ (a_3^2 + 3b_3^2)S_5 + (a_1a_2 + 3b_1b_2)S_6 + (a_1a_3 + 3b_1b_3)S_7 + (a_2a_3 + 3b_2b_3)S_8 + (3a_1b_2 - a_2b_1)S_{16} + (3a_1b_3 - a_3b_1)S_{17}$$
$$+ (3a_2b_1 - a_1b_2)S_{18} + (3a_2b_3 - a_3b_2)S_{19} + (3a_3b_1 - a_1b_3)S_{20} + (3a_3b_2 - a_2b_3)S_{21} + 2a_3b_3S_{22}\}$$

$$RVO(7) = 0$$

$$RVO(8) = -N_A\frac{1}{4}\{2a_1b_1S_1 + (a_1b_3 + a_3b_1)S_2 + (a_2b_3 + a_3b_2)R_2 + (a_1^2 - b_1^2)S_{13} + (a_1a_2 - b_1b_2)S_{14}$$
$$+ (a_1a_3 - b_1b_3)S_{15} + (a_2^2 - b_2^2)R_{14}\}$$

$$RVO(9) = -(\beta_c^2S_1 + \beta_c\theta_cS_2 + \phi_c\theta_cR_2) - N_A\frac{1}{4}\{(a_1^2 + 3b_1^2)S_1 + (a_1a_3 + 3b_1b_3)S_2 + (a_2a_3 + 3b_2b_3)R_2$$
$$+ 2a_1b_1S_{13} + (3a_2b_1 - a_1b_2)S_{14} + (3a_3b_1 - a_1b_3)S_{15} + 2a_2b_2R_{14}\}$$

Note:

RV(i,j) for i and j varying from 4 to 6 can be obtained from RV(i-3, j-3) by replacing R_i with $-S_i$ in the expressions

Periodic aerodynamic forces terms for Parametric Resonance

$$RV(1,1) = -N_A\Big\{-\tfrac{1}{4}(a_2a_3-b_2b_3)R_9 - \tfrac{1}{2}(a_1a_3-b_1b_3)R_{10} + \tfrac{1}{8}(a_1b_2+a_2b_1)R_{23} + \tfrac{1}{2}a_1b_1R_{24} + \tfrac{1}{4}a_2b_2R_{27}$$
$$+ \tfrac{1}{4}(a_1b_2+a_2b_1)R_{28} + \tfrac{1}{8}(a_2b_3+a_3b_2)R_{31} + \tfrac{1}{4}a_3b_3R_{32}\Big\}$$

$$RV(1,2) = 0$$

$$RV(1,3) = 0$$

$$RV(1,4) = -N_A\Big\{-\tfrac{1}{4}(a_1a_3-b_1b_3)R_9 - \tfrac{1}{2}(a_2a_3-b_2b_3)R_{11} + \tfrac{1}{4}a_1b_1R_{23} + \tfrac{1}{4}(a_1b_2+a_2b_1)R_{25}$$
$$+ \tfrac{1}{8}(a_1b_2+a_2b_1)R_{27} + \tfrac{1}{2}a_2b_2R_{29} + \tfrac{1}{8}(a_1b_3+a_3b_1)R_{31} + \tfrac{1}{4}a_3b_3R_{33}\Big\}$$

$$RV(1,5) = 0$$

$$RV(1,6) = 0$$

$$RV(1,7) = -N_A\Big\{-\tfrac{1}{4}(a_1a_2-b_1b_2)R_9 - \tfrac{1}{4}(a_1^2-b_1^2)R_{10} - \tfrac{1}{4}(a_2^2-b_2^2)R_{11} - \tfrac{3}{4}(a_3^2-b_3^2)R_{12} + \tfrac{1}{4}(a_1b_3+a_3b_1)R_{26}$$
$$+ \tfrac{1}{4}(a_2b_3+a_3b_2)R_{30} + \tfrac{1}{8}(a_1b_3+a_3b_1)R_{32} + \tfrac{1}{8}(a_2b_3+a_3b_2)R_{33}\Big\}$$

$$RV(1,8) = 0$$

$$RV(1,9) = 0$$

$$RV(2,1) = 0$$

$$RV(2,2) = \tfrac{1}{2}R_1 + \beta_c R_3 + \phi_c R_6 + \tfrac{1}{2}\theta_c R_7 + \tfrac{1}{2}\phi_c\theta_c R_9 + \beta_c\theta_c R_{10} + N_A\Big\{\tfrac{1}{2}a_2a_3R_9\Big\}$$

$$RV(2,3) = -\tfrac{1}{4}(R_{13} + \phi_c R_{16} + \theta_c R_{17} + \beta_c\phi_c R_{23} + \beta_c^2 R_{24} + \phi_c^2 R_{25} + \theta_c^2 R_{26}) - N_A\tfrac{1}{4}\Big\{a_1a_2R_{23}$$
$$+ a_1^2R_{24} + a_2^2R_{25} + a_3^2R_{26}\Big\}$$

$$RV(2,4) = 0$$

$$RV(2,5) = \phi_c R_4 + \tfrac{1}{2}\beta_c R_6 + \tfrac{1}{2}\theta_c R_8 + \tfrac{1}{2}\beta_c\theta_c R_9 + \phi_c\theta_c R_{11}$$

$$RV(2,6) = -\tfrac{1}{4}(R_{14} + \beta_c R_{18} + \theta_c R_{19} + \beta_c\phi_c R_{27} + \beta_c^2 R_{28} + \phi_c^2 R_{29} + \theta_c^2 R_{30}) - N_A\tfrac{1}{4}\Big\{a_1a_2R_{27}$$
$$+ a_1^2R_{28} + a_2^2R_{29} + a_3^2R_{30}\Big\}$$

$$RV(2,7) = 0$$

$$RV(2,8) = \tfrac{1}{2}(R_2 + 2\theta_c R_5 + \phi_c R_8 + \beta_c R_7 + \beta_c\phi_c R_9 + \beta_c^2 R_{10} + \phi_c^2 R_{11} + 3\theta_c^2 R_{12}) + N_A\tfrac{1}{2}\Big\{a_1^2R_{10}$$
$$+ a_2^2R_{11} + a_3^2R_{12}\Big\}$$

$$RV(2,9) = -\tfrac{1}{4}(R_{15} + \beta_c R_{20} + \phi_c R_{21} + \theta_c R_{22} + \beta_c\phi_c R_{31} + \beta_c\theta_c R_{32} + \phi_c\theta_c R_{33}) - N_A\tfrac{1}{4}\Big\{a_1a_2R_{31}$$
$$+ a_1a_3R_{32} + a_2a_3R_{33}\Big\}$$

$$RV(3,1)=0$$

$$\begin{aligned}RV(3,2) = &-\tfrac{1}{4}\left(R_{13}+\phi_c R_{16}+\theta_c R_{17}+\beta_c\phi_c R_{23}+\beta_c^2 R_{24}+\phi_c^2 R_{25}+\theta_c^2 R_{26}\right)-N_A\tfrac{1}{4}\{b_1 b_2 R_{23}\\ &+b_1^2 R_{24}+b_2^2 R_{25}+b_3^2 R_{26}\}\end{aligned}$$

$$RV(3,3) = -\tfrac{1}{2}\left(R_1+2\beta_c R_3+\phi_c R_6+\theta_c R_7+\phi_c\theta_c R_9+\beta_c\theta_c R_{10}\right)-N_A\{\tfrac{1}{2}b_2 b_3 R_9\}$$

$$RV(3,4)=0$$

$$\begin{aligned}RV(3,5) = &-\tfrac{1}{4}\left(R_{14}+\beta_c R_{18}+\theta_c R_{19}+\beta_c\phi_c R_{27}+\beta_c^2 R_{28}+\phi_c^2 R_{29}+\theta_c^2 R_{30}\right)-N_A\tfrac{1}{4}\{b_1 b_2 R_{27}\\ &+b_1^2 R_{28}+b_2^2 R_{29}+b_3^2 R_{30}\}\end{aligned}$$

$$RV(3,6) = -\tfrac{1}{2}\left(2\phi_c R_4+\beta_c R_6+\theta_c R_8+\beta_c\theta_c R_9+2\phi_c\theta_c R_{11}+3\theta_c^2 R_{12}\right)-N_A\{\tfrac{1}{2}b_3^2 R_{12}\}$$

$$RV(3,7)=0$$

$$\begin{aligned}RV(3,8) = &-\tfrac{1}{4}\left(R_{15}+\beta_c R_{20}+\phi_c R_{21}+\theta_c R_{22}+\beta_c\phi_c R_{31}+\beta_c\theta_c R_{32}+\phi_c\theta_c R_{33}\right)-N_A\tfrac{1}{4}\{b_1 b_2 R_{31}\\ &+b_1 b_3 R_{32}+b_2 b_3 R_{33}\}\end{aligned}$$

$$RV(3,9) = -\tfrac{1}{2}\left(R_2+2\theta_c R_5+\beta_c R_7+\phi_c R_8+\beta_c\phi_c R_9+\beta_c^2 R_{10}+\phi_c^2 R_{11}\right)-N_A\{\tfrac{1}{2}(b_1^2 R_{10}+b_2^2 R_{11})\}$$

$$\begin{aligned}RV(7,1) = \; &N_A\tfrac{1}{4}\{3(a_1^2-b_1^2)(S_3-\tfrac{S_0}{6})+2(a_1a_2-b_1b_2)(R_3-\tfrac{R_0}{2}+S_6)+(a_2^2-b_2^2)(S_4+R_6)\\ &+(a_3^2-b_3^2)S_5+2(a_1a_3-b_1b_3)S_7+(a_2a_3-b_2b_3)(R_7+S_8)-\tfrac{1}{2}(a_1b_2+a_2b_1)(S_{16}+2S_{18})\\ &-\tfrac{1}{2}(a_1b_3+a_3b_1)(S_{17}+2S_{20})-\tfrac{1}{2}(a_2b_3+a_3b_2)(S_{19}+R_{20}+S_{21})-a_3b_3S_{22}\}\end{aligned}$$

$$RV(7,2)=0$$

$$RV(7,3)=0$$

$$\begin{aligned}RV(7,4) = \; &N_A\tfrac{1}{4}\{(a_1^2-b_1^2)(R_3-\tfrac{R_0}{2}+S_6)+2(a_1a_2-b_1b_2)(S_4+R_6)+3(a_2^2-b_2^2)(R_4'-\tfrac{R_0}{6})\\ &+(a_3^2-b_3^2)R_5+(a_1a_3-b_1b_3)(R_7+S_8)-a_1b_1S_{16}-\tfrac{1}{2}(a_1b_3+a_3b_1)(R_{17}+R_{20}+S_{21})\\ &-\tfrac{1}{2}(a_2b_3+a_3b_2)R_{19}-a_3b_3R_{22}\}\end{aligned}$$

$$RV(7,5)=0$$

$$RV(7,6)=0$$

$$\begin{aligned}RV(7,7) = \; &N_A\tfrac{1}{4}\{2(a_1a_3-b_1b_3)S_5+2(a_2a_3-b_2b_3)R_5+(a_1^2-b_1^2)S_7+(a_1a_2-b_1b_2)(R_7+S_8)\\ &-a_1b_1S_{17}-\tfrac{1}{2}(a_1b_2+a_2b_1)(R_{17}+S_{19})-a_2b_2R_{19}-\tfrac{1}{2}(a_1b_3+a_3b_1)S_{22}\\ &-\tfrac{1}{2}(a_2b_3+a_3b_2)R_{22}\}\end{aligned}$$

$$RV(7,8)=0$$

$$RV(7,9)=0$$

$$RV(8,1) = 0$$

$$RV(8,2) = \tfrac{1}{2}\left\{S_0 + 2\beta_c S_1 + \theta_c S_2 + 3\beta_c^2\left(S_3 - \tfrac{S_0}{6}\right) + 2\beta_c\phi_c\left(R_3 - \tfrac{R_0}{2} + S_6\right) + \phi_c^2(S_4 + R_6) + \theta_c^2 S_5 + 2\beta_c\theta_c S_7 + \phi_c\theta_c(R_7 + S_8)\right\} + \tfrac{N_A}{2}\left\{a_1^2\left(S_3 - \tfrac{S_0}{6}\right) + a_2^2(S_4 + R_6) + a_3^2 S_5 + a_2 a_3(R_7 + S_8)\right\}$$

$$RV(8,3) = -\tfrac{1}{4}\left\{\beta_c S_{13} + \beta_c\phi_c S_{16} + \beta_c\theta_c S_{17} + \phi_c\theta_c R_{17}\right\} - N_A\tfrac{1}{4}\left\{a_1 a_2 S_{16} + a_1 a_3 S_{17} + a_2 a_3 R_{17}\right\}$$

$$RV(8,4) = 0$$

$$RV(8,5) = \tfrac{1}{2}\left\{R_0 + \theta_c R_2 + \beta_c^2\left(R_3 - \tfrac{R_0}{2} + S_6\right) + 2\beta_c\phi_c(S_4 + R_6) + 3\phi_c^2\left(R_4' - \tfrac{R_0}{6}\right) + \theta_c^2 R_5 + \beta_c\theta_c(R_7 + S_8)\right\} + N_A\tfrac{1}{2}\left\{a_1^2\left(R_3 - \tfrac{R_0}{2} + S_6\right) + a_2^2\left(R_4' - \tfrac{R_0}{6}\right) + a_3^2 R_5\right\}$$

$$RV(8,6) = -\tfrac{1}{4}\left\{\beta_c S_{14} + \phi_c R_{14} + \beta_c^2 S_{18} + \beta_c\theta_c S_{19} + \phi_c\theta_c R_{19}\right\} - N_A\tfrac{1}{4}\left\{a_1^2 S_{18} + a_1 a_3 S_{19} + a_2 a_3 R_{19}\right\}$$

$$RV(8,7) = 0$$

$$RV(8,8) = \tfrac{1}{2}\left\{\beta_c S_2 + \phi_c R_2 + 2\beta_c\theta_c S_5 + 2\phi_c\theta_c R_5 + \beta_c^2 S_7 + \beta_c\phi_c(R_7 + S_8)\right\} + N_A\tfrac{1}{2}\left\{a_1^2 S_7\right\}$$

$$RV(8,9) = -\tfrac{1}{4}\left\{\beta_c S_{15} + \beta_c^2 S_{20} + \beta_c\phi_c(R_{20} + S_{21}) + \beta_c\theta_c S_{22} + \phi_c\theta_c R_{22}\right\} - N_A\tfrac{1}{4}\left\{a_1^2 S_{20} + a_1 a_2(R_{20} + S_{21}) + a_1 a_3 S_{22} + a_2 a_3 R_{22}\right\}$$

$$RV(9,1) = 0$$

$$RV(9,2) = -\tfrac{1}{4}\left\{\beta_c S_{13} + \beta_c\phi_c S_{16} + \beta_c\theta_c S_{17} + \phi_c\theta_c R_{17}\right\} - N_A\tfrac{1}{4}\left\{b_1 b_2 S_{16} + b_1 b_3 S_{17} + b_2 b_3 R_{17}\right\}$$

$$RV(9,3) = -\tfrac{1}{2}\left\{S_0 + 2\beta_c S_1 + \theta_c S_2 + 3\beta_c^2\left(S_3 - \tfrac{S_0}{6}\right) + 2\beta_c\phi_c\left(R_3 - \tfrac{R_0}{2} + S_6\right) + \phi_c^2(S_4 + R_6) + \theta_c^2 S_5 + \beta_c\theta_c S_7 + \phi_c\theta_c(R_7 + S_8)\right\} - N_A\tfrac{1}{2}\left\{b_1^2\left(S_3 - \tfrac{S_0}{6}\right) + b_2^2(S_4 + R_6) + b_3^2(S_5) + b_2 b_3(R_7 + S_8)\right\}$$

$$RV(9,4) = 0$$

$$RV(9,5) = -\tfrac{1}{4}\left(\beta_c S_{14} + \phi_c R_{14} + \beta_c^2 S_{18} + \beta_c\theta_c S_{19} + \phi_c\theta_c R_{19}\right) - N_A\cdot\tfrac{1}{4}\left\{b_1^2 S_{18} + b_1 b_3 S_{19} + b_2 b_3 R_{19}\right\}$$

$$RV(9,6) = -\tfrac{1}{2}\left\{R_0 + \theta_c R_2 + \beta_c^2\left(R_3 - \tfrac{R_0}{2} + S_6\right) + 2\beta_c\phi_c(S_4 + R_6) + 3\phi_c^2\left(R_4' - \tfrac{R_0}{6}\right) + \theta_c^2 R_5 + \beta_c\theta_c(R_7 + S_8)\right\} - N_A\tfrac{1}{2}\left\{b_1^2\left(R_3 - \tfrac{R_0}{2} + S_6\right) + b_2^2\left(R_4' - \tfrac{R_0}{6}\right) + b_3^2 R_5\right\}$$

$$RV(9,7) = 0$$

$$RV(9,8) = -\tfrac{1}{4}\left\{\beta_c S_{15} + \beta_c^2 S_{20} + \beta_c\phi_c(R_{20} + S_{21}) + \beta_c\theta_c S_{22} + \phi_c\theta_c R_{22}\right\} - N_A\tfrac{1}{4}\left\{b_1^2 S_{20} + b_1 b_2(R_{20} + S_{21}) + b_1 b_3 S_{22} + b_2 b_3 R_{22}\right\}$$

$$RV(9,9) = -\tfrac{1}{2}\left\{\beta_c S_2 + \phi_c R_2 + 2\beta_c\theta_c S_5 + 2\phi_c\theta_c R_5 + \beta_c^2 S_7 + \beta_c\phi_c(R_7 + S_8)\right\} - N_A\tfrac{1}{2}\left\{b_1^2 S_7\right\}$$

$$RVO(1) = N_A \frac{1}{4}\{(a_1^2-b_1^2)R_3 + (a_2^2-b_2^2)R_4 + (a_3^2-b_3^2)R_5 + (a_1a_2-b_1b_2)R_6 + (a_1a_3-b_1b_3)R_7$$
$$+(a_2a_3-b_2b_3)R_8 - \frac{1}{2}(a_1b_2+a_2b_1)R_{16} - \frac{1}{2}(a_1b_3+a_3b_1)R_{17} - \frac{1}{2}(a_1b_2+a_2b_1)R_{18}$$
$$-\frac{1}{2}(a_2b_3+a_3b_2)R_{19} - \frac{1}{2}(a_1b_3+a_3b_1)R_{20} - \frac{1}{2}(a_2b_3+a_3b_2)R_{21} - a_3b_3R_{22}\}$$

$$RVO(2) = 0$$

$$RVO(3) = 0$$

$$RVO(4) = -N_A \frac{1}{4}\{(a_1^2-b_1^2)S_3 + (a_2^2-b_2^2)S_4 + (a_3^2-b_3^2)S_5 + (a_1a_2-b_1b_2)S_6 + (a_1a_3-b_1b_3)S_7$$
$$+(a_2a_3-b_2b_3)S_8 - \frac{1}{2}(a_1b_2+a_2b_1)S_{16} - \frac{1}{2}(a_1b_3+a_3b_1)S_{17} - \frac{1}{2}(a_1b_2+a_2b_1)S_{18}$$
$$-\frac{1}{2}(a_2b_3+a_3b_2)S_{19} - \frac{1}{2}(a_1b_3+a_3b_1)S_{20} - \frac{1}{2}(a_2b_3+a_3b_2)S_{21} - a_3b_3S_{22}\}$$

$$RVO(5) = 0$$

$$RVO(6) = 0$$

$$RVO(7) = N_A \frac{1}{4}\{(a_1^2-b_1^2)S_1 + (a_1a_3-b_1b_3)S_2 + (a_2a_3-b_2b_3)R_2 - \frac{1}{2}(a_2b_1+a_1b_2)S_{14}$$
$$-a_1b_1S_{13} - \frac{1}{2}(a_1b_3+a_3b_1)S_{15} - a_2b_2R_{14}\}$$

$$RVO(8) = 0$$

$$RVO(9) = 0$$

<u>Note:</u>

RV(i,j) for i and j varying from 4 to 6 can be obtained from RV(i-3, j-3) by replacing R_i with $-S_i$ in the expressions

APPENDIX VIII

The elements of matrix AA for flapping-lagging-feathering rotor consist of two separate parts

(a) contribution from inertia and stiffness, AI

(b) contribution from aerodynamic forces, AA

Thus

$$A_{ij} = AI(i,j) + \frac{\gamma}{8} AA(i,j)$$

$$A_{i0} = AIO(i) + \frac{\gamma}{8} AAO(i)$$

The various elements are given below:

$$AI(1,1) = 1+\bar{e} - \theta_c^2 - \frac{2}{3}\beta_c^2 - \phi_c^2 - N_I\left\{(a_1^2+b_1^2) + \frac{1}{2}a_2^2 + \frac{1}{2}(1+\alpha^2)(a_3^2+b_3^2) + 2\alpha(a_3 b_1 - a_1 b_3)\right\}$$

$$AI(1,2) = 0$$

$$AI(1,3) = 0$$

$$AI(1,4) = \theta_c - N_I\left\{a_1 a_2 - \alpha a_2 b_3\right\}$$

$$AI(1,5) = \bar{\omega}_\beta^2(\beta_c - \beta_s)$$

$$AI(1,6) = N_I\left\{a_2 b_1\right\}$$

$$AI(1,7) = -N_I\left\{a_1 a_3 + b_1 b_3\right\}$$

$$AI(1,8) = 0$$

$$AI(1,9) = 0$$

$$AI(2,1) = 2\phi_c a_2$$

$$AI(2,2) = (1-\alpha^2)(1-\phi_c^2) - 2\beta_c^2 - \theta_c^2 + \bar{e} - N_I\left\{\frac{1}{2}(a_1^2+b_1^2) + \frac{1}{4}(3-\alpha^2)a_2^2 + \frac{\alpha^2}{4}(a_3^2+3b_3^2) + \frac{1}{4}(3a_3^2+b_3^2)\right\}$$

$$AI(2,3) = -2\alpha\bar{\omega}_\beta\sqrt{\bar{z}}\,\zeta_{1\beta} - N_I\left\{\alpha a_1 a_3\right\}$$

$$AI(2,4) = 0$$

$$AI(2,5) = \bar{\omega}_\beta^2 a_1$$

$$AI(2,6) = 0$$

$$AI(2,7) = a_2$$

$$AI(2,8) = \phi_c(1-\alpha^2) - 2\beta_c\theta_c - N_I\left\{\frac{1}{2}b_1 b_3(1-\alpha^2)\right\}$$

$$AI(2,9) = 2\alpha\beta_c^2 + N_I\left\{\frac{1}{2}\alpha(3a_1^2+b_1^2) + \frac{1}{2}\alpha a_2^2\right\}$$

$AI(3,1) = 2\alpha a_2$

$AI(3,2) = 2\alpha \bar{\omega}_\beta \sqrt{\bar{z}}\, \zeta_\beta + N_I\{\alpha b_1 b_3\}$

$AI(3,3) = (1-\alpha^2)(1-\phi_c^2) - 2\beta_c^2 - \theta_c^2 + \bar{e} - N_I\{\tfrac{1}{2}(a_1^2+b_1^2) + \tfrac{1}{4}(1-3\alpha^2)a_2^2 + \tfrac{1}{4}\alpha^2(3a_3^2+b_3^2)$
$\quad + \tfrac{1}{4}(a_3^2+3b_3^2)\}$

$AI(3,4) = 0$

$AI(3,5) = \bar{\omega}_\beta^2 b_1$

$AI(3,6) = 2a_2 \phi_c \theta_c$

$AI(3,7) = 0$

$AI(3,8) = -2\alpha\beta_c^2 - N_I\{\tfrac{1}{2}\alpha(a_1^2+3b_1^2) - \tfrac{1}{2}\alpha a_2^2\}$

$AI(3,9) = \phi_c(1-\alpha^2) - 2\beta_c\theta_c - N_I\{\tfrac{1}{2}(1-\alpha^2)a_1 a_3\}$

$AI(4,1) = \theta_c - N_I\{a_1 a_2 - \alpha a_2 b_3\}$

$AI(4,2) = 0$

$AI(4,3) = 0$

$AI(4,4) = \bar{e} + \theta_c^2 - \beta_c^2 - N_I\{\tfrac{1}{2}(1-\alpha^2)(a_1^2+b_1^2) - \tfrac{1}{2}(1-\alpha^2)(a_3^2+b_3^2)\}$

$AI(4,5) = \phi_c - \phi_s$

$AI(4,6) = 0$

$AI(4,7) = N_I\{a_2 a_3 + \alpha a_2 b_1\}$

$AI(4,8) = 0$

$AI(4,9) = 0$

$AI(5,1) = 0$

$AI(5,2) = \theta_c - 2\beta_c\phi_c + N_I\{\tfrac{1}{4}a_1 a_2(\alpha^2-3)\}$

$AI(5,3) = 2\alpha(\beta_c + \theta\phi_c) + N_I\{\alpha a_2 a_3 + \tfrac{1}{4}a_2 b_1(3\alpha^2-1)\}$

$AI(5,4) = 0$

$AI(5,5) = a_2$

$AI(5,6) = 0$

$AI(5,7) = 0$

$AI(5,8) = \beta_c(1+\alpha^2) + 2\phi_c\theta_c - N_I\{\tfrac{1}{4}a_2 a_3(\alpha^2-3)\}$

$AI(5,9) = 2\alpha(\theta_c R_m + \beta_c\phi_c) + N_I\{\alpha a_1 a_2 - \tfrac{1}{4}a_2 b_3(3\alpha^2-1)\}$

$AI(6,1) = 0$

$AI(6,2) = -2\alpha(\beta_c + \theta_c\phi_c) - N_I\left\{\frac{1}{2}\alpha a_2 a_3\right\}$

$AI(6,3) = \theta - 2\beta_c\phi_c - N_I\left\{\frac{1}{2}a_1a_2(1+\alpha^2) - \frac{\alpha}{2}a_2b_3\right\}$

$AI(6,4) = 0$

$AI(6,5) = 0$

$AI(6,6) = 2a_2\sqrt{Z}\,\zeta_\phi$

$AI(6,7) = 0$

$AI(6,8) = -2\alpha(R_m\theta_c + \beta_c\phi_c) - N_I\left\{\frac{1}{2}\alpha a_1a_2\right\}$

$AI(6,9) = (1+\alpha^2)\beta_c + 2\phi_c\theta_c + N_I\left\{\frac{1}{2}(1+\alpha^2)a_2a_3 + \frac{1}{2}\alpha a_2b_1\right\}$

$AI(7,1) = -N_I\{a_1a_3 + b_1b_3\}$

$AI(7,2) = 0$

$AI(7,3) = 0$

$AI(7,4) = N_I\{\alpha a_2b_1 + a_2a_3\}$

$AI(7,5) = \bar{\omega}_\theta^2(\theta - \theta_s)R_m$

$AI(7,6) = N_I\{a_2b_3R_m\}$

$AI(7,7) = R_m - \beta_c^2 + \phi_c^2 + N_I\left\{\frac{1}{2}a_2^2 - \frac{1}{2}(a_1^2+b_1^2)\right\}$

$AI(7,8) = 0$

$AI(7,9) = 0$

$AI(8,1) = (1+\alpha^2)a_2$

$AI(8,2) = (1-\alpha^2)\phi_c - 2\beta_c\phi_c + N_I\left\{\frac{1}{2}b_1b_3(\alpha^2-1)\right\}$

$AI(8,3) = -2\alpha\beta_c^2 + N_I\left\{\frac{1}{2}\alpha a_2^2 - \frac{1}{2}\alpha(a_1^2+b_1^2)\right\}$

$AI(8,4) = 0$

$AI(8,5) = \bar{\omega}_\theta^2 R_m a_3$

$AI(8,6) = 0$

$AI(8,7) = 2\phi_c a_2$

$$AI(8,8) = R_m(1-\alpha^2) - (1+\alpha^2)\beta_c^2 + (1-\alpha^2)\phi_c^2 - N_I\left\{\frac{1}{4}(1+\alpha^2)(3a_1^2+b_1^2) + \frac{1}{4}(\alpha^2-3)a_2^2 - \frac{1}{2}\alpha^2(a_1^2-b_1^2)\right\}$$

$AI(8,9) = -2\alpha\bar{\omega}_\theta\sqrt{Z}\,R_m\zeta_\theta$

$$AI(9,1) = 0$$

$$AI(9,2) = 2\alpha\beta_c^2 + N_I\left\{\tfrac{1}{2}\alpha(a_1^2+b_1^2) + \tfrac{1}{2}\alpha a_2^2\right\}$$

$$AI(9,3) = (1-\alpha^2)\phi_c - 2\beta_c\theta_c + N_I\left\{\tfrac{1}{2}(\alpha^2-1)a_1a_3\right\}$$

$$AI(9,4) = 0$$

$$AI(9,5) = \bar{\omega}_\theta^2 R_m b_3$$

$$AI(9,6) = 2\beta_c\phi_c a_2$$

$$AI(9,7) = 2\alpha R_m a_2$$

$$AI(9,8) = 2\alpha\bar{\omega}_\theta\sqrt{Z}\, R_m \zeta_\theta$$

$$AI(9,9) = R_m(1-\alpha^2) - \beta_c^2(1+\alpha^2) + \phi_c^2(1-\alpha^2) + N_I\left\{-\tfrac{1}{4}(a_1^2+3b_1^2)(1+\alpha^2) + \tfrac{1}{4}(1-3\alpha^2)a_2^2 - \tfrac{1}{2}\alpha^2(a_1^2-b_1^2)\right\}$$

$$AIO(1) = N_I\left\{(1+\tfrac{\alpha^2}{2})a_2a_3\right\}$$

$$AIO(2) = 0$$

$$AIO(3) = 0$$

$$AIO(4) = N_I\left\{\tfrac{1}{2}(1-\alpha^2)(a_1a_3+b_1b_3)\right\}$$

$$AIO(5) = (\bar{e} - \alpha^2 - \beta_c^2 + \theta_c^2)a_2$$

$$AIO(6) = 0$$

$$AIO(7) = \beta_c\phi_c + N_I\left\{(\tfrac{1}{2} - R_m\alpha^2)a_1a_2\right\}$$

$$AIO(8) = 0$$

$$AIO(9) = 0$$

Aerodynamic contribution terms

$$AA(1,1) = -(F_1 + \phi_c \theta_c F_9 + \phi_c^2 F_{45}) - N_A \left\{ \tfrac{1}{2} a_2 a_3 F_9 + (a_1 a_3 + b_1 b_3) F_{10} - \tfrac{\alpha}{2} a_2 b_1 F_{23} + \alpha a_2 b_1 F_{28} - \tfrac{\alpha}{2} a_2 b_3 F_{31} + \tfrac{1}{2}\alpha^2 (a_1 a_3 + b_1 b_3) F_{36} + \tfrac{1}{2}\alpha^2 a_2 a_3 F_{44} + \tfrac{1}{2} a_2^2 F_{45} \right\}$$

$$AA(1,2) = 0$$

$$AA(1,3) = 0$$

$$AA(1,4) = -N_A \left\{ \tfrac{1}{4}(a_1 a_3 + b_1 b_3) F_9 + a_2 a_3 F_{11} - \alpha a_2 b_1 F_{25} + \tfrac{1}{2}\alpha a_2 b_1 F_{27} + \tfrac{1}{2}\alpha (a_3 b_1 - a_1 b_3) F_{31} + \tfrac{1}{2}\alpha^2 a_2 a_3 F_{37} - \alpha a_2 b_3 F_{39} + a_1 a_2 F_{45} \right\}$$

$$AA(1,5) = 0$$

$$AA(1,6) = -N_A \left\{ -\tfrac{1}{2} a_2 b_1 F_{16} + \tfrac{1}{2}(a_1 b_3 - a_3 b_1) F_{17} + \tfrac{1}{2} a_2 b_1 F_{18} + \tfrac{1}{2} a_2 b_3 F_{19} + \tfrac{1}{2}(a_3 b_1 - a_1 b_3) F_{20} - \tfrac{1}{2} a_2 b_3 F_{21} \right\}$$

$$AA(1,7) = -(F_2 + \beta_c^2 F_{10} + \phi_c^2 F_{11} + \theta_c^2 F_{12}) - N_A \left\{ \tfrac{1}{2} a_1 a_2 F_9 + \tfrac{1}{2}(a_1^2 + b_1^2) F_{10} + \tfrac{1}{2} a_2^2 F_{11} + \tfrac{3}{2}(a_3^2 + b_3^2) F_{12} + \alpha (a_1 b_3 - a_3 b_1) F_{26} + \alpha a_2 b_3 F_{30} + \tfrac{\alpha}{2}(a_3 b_1 - a_1 b_3) F_{32} - \tfrac{\alpha}{2} a_2 b_3 F_{33} \right\}$$

$$AA(1,8) = 0$$

$$AA(1,9) = 0$$

$$AA(2,1) = -(a_2 F_6 + \theta_c a_2 F_9 + 2\phi_c a_2 F_{45})$$

$$AA(2,2) = -(F_1 + 2\beta_c F_3 + \phi_c F_6 + \theta_c F_7 + \phi_c \theta_c F_9 + 2\beta_c \theta_c F_{10} + \phi_c^2 F_{45}) - N_A \left\{ \tfrac{3}{4} a_2 a_3 F_9 + \tfrac{1}{2} b_1 b_3 F_{10} + \tfrac{1}{2}\alpha^2 b_1 b_3 F_{36} + \tfrac{1}{4}\alpha^2 a_2 a_3 F_{44} + \tfrac{3}{4} a_2^2 F_{45} \right\}$$

$$AA(2,3) = \alpha (F_{13} + \phi_c F_{16} + \theta_c F_{17} + \beta_c \phi_c F_{23} + \beta_c^2 F_{24} + \phi_c^2 F_{25} + \theta_c^2 F_{26}) + N_A \left\{ -\tfrac{1}{4} a_2 b_3 F_9 + \tfrac{\alpha}{2} a_1 a_2 F_{23} + \tfrac{\alpha}{4}(a_1^2 + b_1^2) F_{24} + \tfrac{3}{4}\alpha a_2^2 F_{25} + \tfrac{\alpha}{4}(3a_3^2 + b_3^2) F_{26} - \tfrac{\alpha}{4} a_2^2 F_{27} - \tfrac{\alpha}{2} a_1 a_2 F_{28} + \tfrac{\alpha^2}{4} a_2 b_3 F_{44} - \tfrac{\alpha}{4}(a_3^2 - b_3^2) F_{32} \right\}$$

$$AA(2,4) = -2 a_2 (F_4 + \theta_c F_{11})$$

$$AA(2,5) = 0$$

$$AA(2,6) = 0$$

$$AA(2,7) = a_2 F_8$$

$$AA(2,8) = -(F_2 + 2\theta_c F_5 + \beta_c F_7 + \phi_c F_8 + \beta_c \phi_c F_9 + \beta_c^2 F_{10} + \phi_c^2 F_{11} + 3\theta_c^2 F_{12}) - N_A \tfrac{1}{4} \left\{ (3a_1^2 + b_1^2) F_{10} + 3 a_2^2 F_{11} + 3(a_3^2 + b_3^2) F_{12} + \alpha^2 (a_1^2 - b_1^2) F_{36} + \alpha^2 a_2^2 F_{37} + \alpha a_2 b_1 F_{31} \right\}$$

$$AA(2,9) = \alpha (F_{15} + \beta_c F_{20} + \phi_c F_{21} + \theta_c F_{22} + \beta_c \phi_c F_{31} + \beta_c \theta_c F_{32} + \phi_c \theta_c F_{33} + \phi_c^2 F_{39}) - N_A \left\{ \tfrac{\alpha}{2} a_1 a_3 F_{26} + \tfrac{\alpha}{2} a_2 a_3 F_{30} - \tfrac{3}{4}\alpha a_1 a_2 F_{31} - \tfrac{\alpha}{2} a_1 a_3 F_{32} - \tfrac{\alpha}{2} a_2 a_3 F_{33} - \tfrac{3}{4}\alpha a_2^2 F_{39} \right\}$$

$$AA(3,1) = -\alpha a_2 F_{18}$$

$$AA(3,2) = -\alpha\left(F_{13} + \phi_c F_{16} + \theta_c F_{17} + \beta_c \phi_c F_{23} + \beta_c^2 F_{24} + \phi_c^2 F_{25} + \theta_c^2 F_{26}\right) - N_A \cdot \left\{ \tfrac{\alpha}{4}(a_1^2 + b_1^2) F_{24} + \tfrac{\alpha}{4} a_2^2 F_{25} + \tfrac{\alpha}{4}(a_3^2 + 3b_3^2) F_{26} + \tfrac{\alpha}{4} a_2^2 F_{27} + \tfrac{\alpha}{4}(a_3^2 - b_3^2) F_{32} + \tfrac{\alpha}{4} a_1 a_2 F_{23} + \tfrac{\alpha}{4} a_1 a_2 F_{28} \right\}$$

$$AA(3,3) = -\left(F_1 + 2\beta_c F_3 + \phi_c F_6 + \theta_c F_7 + \phi_c \theta_c F_9 + 2\beta_c \theta_c F_{10} + \phi_c^2 F_{15}\right) - N_A \left\{ \tfrac{1}{4} a_2 a_3 F_9 + \tfrac{1}{2} a_1 a_3 F_{10} - \tfrac{\alpha}{4} a_2 b_1 F_{23} + \tfrac{3}{4}\alpha a_2 b_1 F_{28} + \tfrac{\alpha^2}{2} a_1 a_3 F_{36} + \tfrac{3}{4}\alpha^2 a_2 a_3 F_{44} + \tfrac{a_2^2}{4} F_{45} \right\}$$

$$AA(3,4) = 0$$

$$AA(3,5) = 0$$

$$AA(3,6) = -a_2\left(F_{14} + \beta_c \phi_c F_{27} + \beta_c^2 F_{28} + \theta_c^2 F_{30} + \phi_c^2 F_{29}\right) - N_A \left\{ \tfrac{1}{4} a_2^3 F_9 \right\}$$

$$AA(3,7) = \alpha a_2 F_{19}$$

$$AA(3,8) = -\alpha\left(F_{15} + \beta_c F_{20} + \phi_c F_{21} + \theta_c F_{22} + \beta_c \phi_c F_{31} + \beta_c \theta_c F_{32} + \phi_c \theta_c F_{33} + \phi_c^2 F_{39}\right) - N_A \left\{ -\tfrac{\alpha}{2} b_1 b_3 F_{26} + \tfrac{\alpha}{4} a_2 a_3 F_{30} + \tfrac{\alpha}{4} a_1 a_2 F_{31} + \tfrac{\alpha}{2} b_1 b_3 F_{32} + \tfrac{\alpha}{4} a_2 a_3 F_{33} + \tfrac{\alpha}{4} a_2^2 F_{39} \right\}$$

$$AA(3,9) = -\left(F_2 + 2\theta_c F_5 + \phi_c F_8 + \beta_c F_7 + \beta_c \phi_c F_9 + \beta_c^2 F_{10} + \phi_c^2 F_{11} + 3\theta_c^2 F_{12}\right) - N_A \left\{ \tfrac{1}{4} a_1 a_2 F_9 + \tfrac{1}{4}(a_1^2 + 3b_1^2) F_{10} + \tfrac{1}{4} a_2^2 F_{11} + \tfrac{3}{4}(a_3^2 + b_3^2) F_{12} + \tfrac{3}{4}\alpha a_2 b_3 F_{30} - \tfrac{\alpha}{4} a_2 b_3 F_{33} - \tfrac{\alpha^2}{4}(a_1^2 - b_1^2) F_{36} - \tfrac{\alpha^2}{4} a_1 a_2 F_{44} - \tfrac{\alpha}{4} a_2 b_1 F_{31} - \tfrac{1}{4}\alpha^2 a_2^2 F_{37} \right\}$$

$$AA(4,1) = G_1 + \phi_c \theta_c G_9 + N_A \left\{ \tfrac{1}{2} a_2 a_3 G_9 + (a_1 a_3 + b_1 b_3) G_{10} - \tfrac{\alpha}{2} a_2 b_1 G_{23} + \alpha a_2 b_1 G_{28} - \tfrac{\alpha}{2} a_2 b_3 G_{31} + \tfrac{\alpha^2}{2}(a_1 a_3 + b_1 b_3) G_{36} + \tfrac{1}{2}\alpha^2 a_2 a_3 G_{44} \right\}$$

$$AA(4,2) = 0$$

$$AA(4,3) = 0$$

$$AA(4,4) = N_A \left\{ \tfrac{1}{2}(a_1 a_3 + b_1 b_3) G_9 + a_2 a_3 G_{11} - \alpha a_2 b_1 G_{25} + \tfrac{\alpha}{2} a_2 b_1 G_{27} + \tfrac{\alpha}{2}(a_3 b_1 - a_1 b_3) G_{31} + \tfrac{\alpha^2}{2} a_2 a_3 G_{37} + \tfrac{\alpha^2}{2}(a_1 a_3 + b_1 b_3) G_{43} \right\}$$

$$AA(4,5) = 0$$

$$AA(4,6) = N_A \left\{ -\tfrac{1}{2} a_2 b_1 G_{16} + \tfrac{1}{2}(a_1 b_3 - a_3 b_1) G_{17} + \tfrac{1}{2} a_2 b_1 G_{18} + \tfrac{1}{2} a_2 b_3 G_{19} + \tfrac{1}{2}(a_3 b_1 - a_1 b_3) G_{20} - \tfrac{1}{2} a_2 b_3 G_{21} \right\}$$

$$AA(4,7) = G_2 + \beta_c^2 G_{10} + \phi_c^2 G_{11} + \theta_c^2 G_{12} + N_A \left\{ \tfrac{1}{2}(a_1 a_2) G_9 + \tfrac{1}{2}(a_1^2 + b_1^2) G_{10} + \tfrac{1}{2} a_2^2 G_{11} + \tfrac{3}{2}(a_3^2 + b_3^2) G_{12} + \alpha(a_1 b_3 - a_3 b_1) G_{26} + \alpha a_2 b_3 G_{30} + \tfrac{\alpha}{2}(a_3 b_1 - a_1 b_3) G_{32} - \tfrac{\alpha}{2} a_2 b_3 G_{33} \right\}$$

$$AA(4,8) = 0$$

$$AA(4,9) = 0$$

$$AA(5,1) = a_2 (G_6 + \theta_c G_9)$$

$$AA(5,2) = G_1 + 2\beta_c G_3 + \phi_c G_6 + \theta_c G_7 + \phi_c \theta_c G_9 + 2\beta_c \theta_c G_{10} + N_A \Big\{ \tfrac{3}{4} a_2 a_3 G_9 + \tfrac{1}{2} b_1 b_3 G_{10} + \tfrac{1}{2}\alpha^2 b_1 b_3 G_{36} + \tfrac{1}{4}\alpha^2 a_2 a_3 G_{44} + \tfrac{1}{4}\alpha^2 a_2 a_3 G_{43} \Big\}$$

$$AA(5,3) = -\alpha (G_{13} + \phi_c G_{16} + \theta_c G_{17} + \beta_c \phi_c G_{23} + \beta_c^2 G_{24} + \phi_c^2 G_{25} + \theta_c^2 G_{26} + \beta_c G_{41}) - N_A \Big\{ \tfrac{\alpha}{2} a_1 a_2 G_{23} - \tfrac{1}{4} a_2 b_3 G_9 + \tfrac{\alpha}{4}(a_1^2 + b_1^2) G_{24} + \tfrac{3}{4}\alpha a_2^2 G_{25} + \tfrac{\alpha}{4}(3a_3^2 + b_3^2) G_{26} - \tfrac{\alpha}{4} a_2^2 G_{27} - \tfrac{\alpha}{2} a_1 a_2 G_{28} + \tfrac{\alpha}{4}(a_3^2 - b_3^2) G_{32} + \tfrac{1}{4}\alpha^2 a_2 b_3 G_{44} - \tfrac{3}{4}\alpha^2 a_2 b_3 G_{43} \Big\}$$

$$AA(5,4) = 2a_2 (G_4 + \theta_c G_{11})$$

$$AA(5,5) = 0$$

$$AA(5,6) = 0$$

$$AA(5,7) = a_2 G_8$$

$$AA(5,8) = G_2 + 2\theta_c G_5 + \beta_c G_7 + \phi_c G_8 + \beta_c \phi_c G_9 + \beta_c^2 G_{10} + \phi_c^2 G_{11} + 3\theta_c^2 G_{12} + N_A \Big\{ \tfrac{1}{4}(3a_1^2 + b_1^2) G_{10} + \tfrac{3}{4} a_2^2 G_{11} + \tfrac{3}{4}(a_3^2 + b_3^2) G_{12} + \tfrac{\alpha}{4} a_2 b_1 G_{31} + \tfrac{\alpha^2}{4}(a_1^2 - b_1^2) G_{36} + \tfrac{\alpha^2}{4}(a_2^2) G_{37} \Big\}$$

$$AA(5,9) = -\alpha (G_{15} + \beta_c G_{20} + \phi_c G_{21} + \theta_c G_{22} + \beta_c \phi_c G_{31} + \beta_c \theta_c G_{32} + \phi_c \theta_c G_{33}) + N_A \Big\{ \tfrac{1}{2}\alpha a_1 a_3 G_{26} + \tfrac{1}{2}\alpha a_2 a_3 G_{30} - \tfrac{3}{4}\alpha a_1 a_2 G_{31} - \tfrac{\alpha}{2} a_1 a_3 G_{32} - \tfrac{\alpha}{2} a_2 a_3 G_{33} \Big\}$$

$$AA(6,1) = \alpha a_2 G_{18}$$

$$AA(6,2) = \alpha (G_{13} + \phi_c G_{16} + \theta_c G_{17} + \beta_c \phi_c G_{23} + \beta_c^2 G_{24} + \phi_c^2 G_{25} + \theta_c^2 G_{26} + \beta_c G_{41}) + N_A \Big\{ \tfrac{\alpha}{4} a_1 a_2 G_{23} + \tfrac{\alpha}{4}(a_1^2 + b_1^2) G_{24} + \tfrac{\alpha}{4} a_2^2 G_{25} + \tfrac{\alpha}{4}(a_3^2 + 3b_3^2) G_{26} + \tfrac{\alpha}{4} a_2^2 G_{27} + \tfrac{\alpha}{4} a_1 a_2 G_{28} + \tfrac{\alpha}{4}(a_3^2 - b_3^2) G_{32} \Big\}$$

$$AA(6,3) = G_1 + 2\beta_c G_3 + \phi_c G_6 + \theta_c G_7 + \phi_c \theta_c G_9 + 2\beta_c \theta_c G_{10} + N_A \Big\{ \tfrac{1}{4} a_2 a_3 G_9 + \tfrac{1}{2} a_1 a_3 G_{10} - \tfrac{\alpha}{4} a_2 b_1 G_{23} + \tfrac{3}{4}\alpha a_2 b_1 G_{28} + \tfrac{1}{2}\alpha^2 a_1 a_3 G_{36} + \tfrac{3}{4}\alpha^2 a_2 a_3 G_{44} - \tfrac{\alpha^2}{4} a_2 a_3 G_{43} \Big\}$$

$$AA(6,4) = 0$$

$$AA(6,5) = 0$$

$$AA(6,6) = a_2 (G_{14} + \beta_c \phi_c G_{27} + \beta_c^2 G_{28} + \phi_c^2 G_{29} + \theta_c^2 G_{30}) + N_A \Big\{ \tfrac{1}{4} a_2^3 \Big\} G_9$$

$$AA(6,7) = \alpha a_2 G_{19}$$

$$AA(6,8) = \alpha (G_{15} + \beta_c G_{20} + \phi_c G_{21} + \theta_c G_{22} + \beta_c \phi_c G_{31} + \beta_c \theta_c G_{32} + \phi_c \theta_c G_{33}) + N_A \Big\{ -\tfrac{\alpha}{2} b_1 b_3 G_{26} + \tfrac{\alpha}{4}(G_{30} + G_{33}) a_2 a_3 + \tfrac{\alpha}{4} a_1 a_2 G_{31} + \tfrac{\alpha}{2} b_1 b_3 G_{32} \Big\}$$

$$AA(6,9) = G_2 + 2\theta_c G_5 + \beta_c G_7 + \phi_c G_8 + \beta_c \phi_c G_9 + \beta_c^2 G_{10} + \phi_c^2 G_{11} + 3\theta_c^2 G_{12} + N_A \Big\{ \tfrac{1}{4} a_1 a_2 G_9 + \tfrac{1}{4}(a_1^2 + 3b_1^2) G_{10} + \tfrac{1}{4} a_2^2 G_{11} + \tfrac{3}{4}(a_3^2 + b_3^2) G_{12} - \tfrac{\alpha}{4} a_2 b_1 G_{31} + \tfrac{1}{4}\alpha a_2 b_3 (3G_{30} - G_{33}) - \tfrac{\alpha^2}{4}(a_1^2 - b_1^2) G_{36} - \tfrac{\alpha^2}{4} a_2^2 G_{37} - \tfrac{\alpha^2}{4} a_1 a_2 (G_{43} + G_{44}) \Big\}$$

$$
\begin{aligned}
AA(7,1) = & -G_0 - \phi_c\theta_c(F_7+G_8) - \phi_c^2(F_6+G_4) - \beta_c^2\left(G_3-\tfrac{G_0}{6}\right) - \theta_c^2 G_5 - N_A\Big\{\tfrac{1}{2}a_2a_3(F_7+G_8) \\
& + (a_1a_3+b_1b_3)G_7 + a_1a_2\left(F_3+G_6-\tfrac{F_0}{2}\right) + \tfrac{1}{2}a_2^2(F_6+G_4) + \tfrac{3}{2}(a_1^2+b_1^2)\left(G_3-\tfrac{G_0}{6}\right) \\
& + \tfrac{1}{2}(a_3^2+b_3^2)G_5 - \tfrac{\alpha}{2}a_2b_1G_{16} + \tfrac{\alpha}{2}(a_1b_3-a_3b_1)G_{17} + \alpha a_2b_1G_{18} + \tfrac{\alpha}{2}a_2b_3G_{19} - \tfrac{\alpha}{2}a_2b_3(F_{20}+G_{21}) \\
& + \alpha(a_3b_1-a_1b_3)G_{20} + \tfrac{\alpha^2}{2}(a_1a_2)G_{34} + \tfrac{\alpha^2}{2}a_2a_3G_{35} + \tfrac{\alpha^2}{2}a_2^2G_{38} + \tfrac{\alpha^2}{2}(a_3^2+b_3^2)G_{40} \\
& + \tfrac{\alpha^2}{2}(a_1a_3+b_1b_3)G_{42} + \tfrac{\alpha^2}{2}(a_1^2+b_1^2)G_{46}\Big\}
\end{aligned}
$$

$$AA(7,2) = 0$$

$$AA(7,3) = 0$$

$$
\begin{aligned}
AA(7,4) = & -F_0 - \phi_c^2\left(F_4'-\tfrac{F_0}{6}\right) - \theta_c^2F_5 - \beta_c^2\left(F_3+G_6-\tfrac{F_0}{2}\right) - N_A\Big\{\tfrac{1}{2}(a_1a_3+b_1b_3)(F_7+G_8) + a_2a_3F_8 \\
& + \tfrac{1}{2}(a_1^2+b_1^2)\left(F_3+G_6-\tfrac{F_0}{2}\right) + \tfrac{3}{2}a_2^2\left(F_4'-\tfrac{F_0}{6}\right) + \tfrac{1}{2}(a_3^2+b_3^2)F_5 + a_1a_2(F_6+G_4) \\
& - \alpha a_2b_1F_{16} + \tfrac{1}{2}\alpha(a_1b_3-a_3b_1)F_{17} + \tfrac{1}{2}\alpha a_2b_1F_{18} + \tfrac{1}{2}\alpha a_2b_3F_{19} + \tfrac{1}{2}(a_3b_1-a_1b_3)(F_{20}+G_{21}) \\
& - \alpha a_2b_3F_{21} + \tfrac{\alpha^2}{2}a_1a_2F_{34} + \tfrac{1}{2}\alpha^2a_2a_3F_{35} + \tfrac{1}{2}\alpha^2a_2^2F_{38}\Big\}
\end{aligned}
$$

$$AA(7,5) = 0$$

$$AA(7,6) = N_A\left\{-\tfrac{1}{2}a_2b_1(F_{13}-G_{14}) + \tfrac{1}{2}(a_3b_1-a_1b_3)(G_{15}) - \tfrac{1}{2}a_2b_3F_{15}\right\}$$

$$
\begin{aligned}
AA(7,7) = & -\beta_c^2G_7 - \phi_c^2F_8 - N_A\Big\{\tfrac{1}{2}a_1a_2(F_7+G_8) + \tfrac{1}{2}(a_1^2+b_1^2)G_7 + \tfrac{1}{2}a_2^2F_8 + a_2a_3F_5 \\
& + (a_1a_3+b_1b_3)G_5 + \tfrac{\alpha}{2}(-a_2b_1)F_{17} + \tfrac{\alpha}{2}a_2b_1G_{19} + \tfrac{\alpha}{2}(a_3b_1-a_1b_3)G_{22} - \tfrac{\alpha}{2}a_2b_3F_{22}\Big\}
\end{aligned}
$$

$$AA(7,8) = 0$$

$$AA(7,9) = 0$$

$$AA(8,1) = a_2F_1 + a_2\theta_c(F_7+G_8) + a_2\beta_c\left(F_3+G_6-\tfrac{F_0}{2}\right) + 2a_2\phi_c(F_6+G_4)$$

$$
\begin{aligned}
AA(8,2) = & -G_0 - 2\beta_cG_1 - \phi_cF_1 + \theta_cG_2 - \phi_c\theta_c(F_7+G_8) - 2\beta_c\theta_cG_7 - 2\beta_c\phi_c\left(F_3+G_6-\tfrac{F_0}{2}\right) - \phi_c^2(F_6+G_4) \\
& - 3\beta_c^2\left(G_3-\tfrac{G_0}{6}\right) - \theta_c^2G_5 - N_A\Big\{\tfrac{3}{4}a_2a_3(F_7+G_8) + \tfrac{3}{4}a_1a_2\left(F_3+G_6-\tfrac{F_0}{2}\right) + \tfrac{\alpha^2}{4}a_1a_2G_{34} + \tfrac{1}{2}b_1b_3G_7 \\
& + \tfrac{3}{4}a_2^2(F_6+G_4) + \tfrac{3}{4}(a_1^2+b_1^2)\left(G_3-\tfrac{G_0}{6}\right) + \tfrac{1}{4}(3a_3^2+b_3^2)G_5 + \tfrac{\alpha^2}{4}a_2^2F_{34} + \tfrac{\alpha^2}{4}(a_1^2+b_1^2)G_{46} \\
& + \tfrac{\alpha^2}{4}a_2a_3G_{35} + \tfrac{\alpha^2}{4}a_2^2G_{38} + \tfrac{\alpha^2}{4}(a_3^2+3b_3^2)G_{40} + \tfrac{\alpha^2}{2}b_1b_3G_{42}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
AA(8,3) = & \ \alpha(\phi_cF_{13} + \beta_cG_{13} + \beta_c\phi_cG_{16} + \phi_c^2F_{16} + \phi_c\theta_cF_{17} + \beta_c\theta_cG_{17} + \beta_c^2G_{41}) - N_A\Big\{-\tfrac{1}{2}\alpha a_1a_2G_{16} \\
& - \tfrac{\alpha}{4}(a_1^2+b_1^2)G_{41} + \tfrac{a_2b_3}{4}(F_7+G_8) + \tfrac{1}{4}a_2b_1\left(F_3+G_6-\tfrac{F_0}{2}\right) - \tfrac{3}{4}\alpha a_2^2F_{16} - \tfrac{3}{4}\alpha a_2a_3F_{17} - \tfrac{\alpha^2}{4}a_2b_1G_{34} \\
& - \tfrac{\alpha}{2}a_1a_3G_{17} + \tfrac{\alpha}{4}a_2^2F_{18} + \tfrac{\alpha}{2}a_1a_2G_{18} + \tfrac{\alpha}{4}a_2a_3G_{19} + \tfrac{\alpha}{4}(a_3^2-b_3^2)G_{22} + \tfrac{\alpha}{2}a_1a_3G_{20} - \tfrac{\alpha^2}{4}a_2b_3G_{35}\Big\}
\end{aligned}
$$

$$AA(8,4) = 2a_2\theta_cF_8 + 3a_2\phi_c\left(F_4'-\tfrac{F_0}{6}\right)$$

$$AA(8,5) = 0$$

$$AA(8,6) = 0$$

$$AA(8,7) = a_2F_2 + a_2\theta_cF_5$$

$$
\begin{aligned}
AA(8,8) = & -\left\{\beta_cG_2 + \phi_cF_2 + \beta_c\phi_c(F_7+G_8) + \beta_c^2G_7 + \phi_c^2F_8 + 2\phi_c\theta_cF_5 + 2\beta_c\theta_cG_5\right\} - N_A\Big\{\tfrac{1}{4}(3a_1^2+b_1^2)G_7 + \tfrac{3}{4}a_2^2F_8 \\
& + \tfrac{1}{2}b_1b_3G_5 + \tfrac{\alpha}{4}a_2b_1(F_{20}+G_{21}) + \tfrac{\alpha^2}{4}a_2^2F_{35} - \tfrac{\alpha^2}{2}b_1b_3G_{40} + \tfrac{\alpha^2}{4}(a_1^2-b_1^2)G_{42} + \tfrac{3}{4}a_2a_3F_5\Big\}
\end{aligned}
$$

$$
\begin{aligned}
AA(8,9) = & \ \alpha\left\{\beta_cG_{15} + \phi_cF_{15} + \beta_c\phi_c(F_{20}+G_{21}) + \beta_c\theta_cG_{22} + \beta_c^2G_{20} + \phi_c^2F_{21} + \phi_c\theta_cF_{22}\right\} - N_A\Big\{\tfrac{\alpha}{4}a_1a_2F_{17} + \tfrac{\alpha}{4}(a_1^2-b_1^2)G_{17} \\
& + \tfrac{\alpha}{4}a_2^2F_{19} + \tfrac{\alpha}{4}a_1a_2G_{19} - \tfrac{3}{4}\alpha a_1a_2(F_{20}+G_{21}) - \tfrac{\alpha}{2}a_1a_3G_{22} - \tfrac{\alpha}{4}(3a_1^2+b_1^2)G_{20} - \tfrac{3}{4}\alpha a_2^2F_{21} - \tfrac{\alpha}{2}a_2a_3F_{22} + \tfrac{1}{4}a_2b_3F_5\Big\}
\end{aligned}
$$

$$AA(9,1) = \alpha a_2 G_{14}$$

$$AA(9,2) = -\alpha(\phi_c F_{13} + \beta_c G_{13} + \beta_c\phi_c G_{16} + \beta_c^2 G_{41} + \phi_c^2 F_{16} + \phi_c\theta_c F_{17} + \beta_c\theta_c G_{17}) - N_A\{\tfrac{\alpha}{4}(a_1^2+b_1^2)G_{41} + \tfrac{\alpha}{4}a_2^2 F_{16} + \tfrac{\alpha}{4}a_1a_2G_{16} + \tfrac{\alpha}{2}b_1b_3G_{17} + \tfrac{\alpha}{4}a_1a_2G_{18} + \tfrac{\alpha}{4}a_2^2F_{18} + \tfrac{\alpha}{4}(a_3^2-b_3^2)G_{22} - \tfrac{\alpha}{2}b_1b_3G_{20}\}$$

$$AA(9,3) = -\{G_0 + 2\beta_c G_1 + \phi_c F_1 + \theta_c G_2 + \phi_c\theta_c(F_7+G_8) + 2\beta_c\theta_c G_7 + 2\beta_c\phi_c(F_3+G_6-\tfrac{F_0}{2}) + \phi_c^2(F_6+G_4) + 3\beta_c^2(G_3-\tfrac{G_0}{6}) + \theta_c^2 G_5\} - N_A\{\tfrac{1}{4}a_2a_3(F_7+G_8) + \tfrac{1}{2}a_1a_3G_7 + \tfrac{1}{2}a_1a_2(F_3+G_6-\tfrac{F_0}{2}) + \tfrac{1}{4}a_2^2(F_6+G_4) + \tfrac{3}{4}(a_1^2+b_1^2)(G_3-\tfrac{G_0}{6}) + \tfrac{1}{4}(a_3^2+3b_3^2)G_5 + \tfrac{1}{2}\alpha^2a_1a_2G_{34} - \tfrac{\alpha^2}{4}a_2^2F_{34} + \tfrac{\alpha^2}{4}(a_1^2+b_1^2)G_{46} + \tfrac{3}{4}\alpha^2a_2a_3G_{35} + \tfrac{\alpha^2}{4}(3a_3^2+b_3^2)G_{38} + \tfrac{3}{4}\alpha^2a_2^2G_{40} + \tfrac{\alpha^2}{2}a_1a_3G_{42} - \tfrac{\alpha}{4}b_1a_2G_{16} - \tfrac{\alpha}{4}a_2b_3F_{17} + \tfrac{3}{4}\alpha b_1a_2G_{18} + \tfrac{3}{4}\alpha a_2b_3G_{19}\}$$

$$AA(9,4) = -\alpha a_2 F_{14}$$

$$AA(9,5) = 0$$

$$AA(9,6) = -a_2(\beta_c\phi_c F_{18} + \beta_c^2 G_{18} + \phi_c\theta_c F_{19} + \beta_c\theta_c G_{19})$$

$$AA(9,7) = 0$$

$$AA(9,8) = -\alpha\{\beta_c G_{15} + \phi_c F_{15} + \beta_c\phi_c(F_{20}+G_{21}) + \beta_c\theta_c G_{22} + \beta_c^2 G_{20} + \phi_c^2 F_{21} + \phi_c\theta_c F_{22}\} - N_A\{\tfrac{\alpha}{4}a_1a_2F_{17} + \tfrac{\alpha}{4}(a_1^2-b_1^2)G_{17} + \tfrac{\alpha}{4}a_2^2F_{19} + \tfrac{\alpha}{4}a_1a_2G_{19} + \tfrac{\alpha}{4}a_1a_2(F_{20}+G_{21}) + \tfrac{\alpha}{2}b_1b_3G_{22} + \tfrac{\alpha}{4}(a_1^2+3b_1^2)G_{20} + \tfrac{\alpha}{4}a_2^2F_{21} + \tfrac{\alpha}{4}a_2a_3F_{22}\}$$

$$AA(9,9) = -\{\beta_c G_2 + \phi_c F_2 + \beta_c\phi_c(F_7+G_8) + \beta_c^2 G_7 + \phi_c^2 F_8 + 2\phi_c\theta_c F_5 + 2\beta_c\theta_c G_5\} - N_A\{\tfrac{1}{4}a_1a_2(F_7+G_8) + \tfrac{1}{4}(a_1^2+3b_1^2)G_7 + \tfrac{1}{4}a_2^2F_8 + \tfrac{1}{2}a_2a_3F_5 + \tfrac{1}{2}a_1a_3G_5 - \tfrac{\alpha}{4}a_2b_1(F_{20}+G_{21}) - \tfrac{\alpha^2}{4}a_2^2F_{35} - \tfrac{\alpha^2}{4}a_1a_2G_{35} - \tfrac{\alpha^2}{2}a_1a_3G_{40} - \tfrac{\alpha^2}{4}(a_1^2-b_1^2)G_{42} - \tfrac{\alpha}{4}a_2b_3F_{22}\}$$

$$RAO(1) = -(F_0 + \beta_c^2F_3 + \phi_c^2F_4 + \theta_c^2F_5 + \beta_c\phi_cF_6 + \beta_c\theta_cF_7 + \phi_c\theta_cF_8) - N_A\{\tfrac{1}{2}(a_1^2+b_1^2)F_3 + \tfrac{1}{2}a_2^2F_4 + \tfrac{1}{2}(a_3^2+b_3^2)F_5 + \tfrac{1}{2}a_1a_2F_6 + \tfrac{1}{2}(a_1a_3+b_1b_3)F_7 + \tfrac{1}{2}a_2a_3F_8 + \tfrac{1}{2}\alpha^2a_1a_2F_{34} + \tfrac{1}{2}\alpha^2a_2a_3F_{35} + \tfrac{1}{2}\alpha^2a_2^2F_{38}\}$$

$$RAO(2) = 0$$

$$RAO(3) = 0$$

$$RAO(4) = G_0 + \beta_c^2G_3 + \phi_c^2G_4 + \theta_c^2G_5 + \beta_c\phi_cG_6 + \beta_c\theta_cG_7 + \phi_c\theta_cG_8 + N_A\{\tfrac{1}{2}(a_1^2+b_1^2)G_3 + \tfrac{1}{2}a_2^2G_4 + \tfrac{1}{2}(a_3^2+b_3^2)G_5 + \tfrac{1}{2}a_1a_2G_6 + \tfrac{1}{2}(a_1a_3+b_1b_3)G_7 + \tfrac{1}{2}a_2a_3G_8 + \tfrac{\alpha^2}{2}(a_1a_2G_{34} + a_2a_3G_{35} + a_2^2G_{38}) + \tfrac{\alpha^2}{2}(a_1^2+b_1^2)G_{46} + \tfrac{\alpha^2}{2}(a_3^2+b_3^2)G_{40} + \tfrac{\alpha^2}{2}(a_1a_3+b_1b_3)G_{42}\}$$

$$RAO(5) = 0$$

$$RAO(6) = 0$$

$$RAO(7) = -(\beta_c^2G_1 + \beta_c\phi_cF_1 + \beta_c\theta_cG_2 + \phi_c\theta_cF_2) - N_A\{\tfrac{1}{2}(a_1^2+b_1^2)G_1 + \tfrac{1}{2}a_1a_2F_1 + \tfrac{1}{2}(a_1a_3+b_1b_3)G_2 + \tfrac{1}{2}a_2a_3F_2\}$$

$$RAO(8) = -a_2F_0 - N_A\{\tfrac{3}{4}a_2^3(F_4' - \tfrac{F_0}{6}) + \tfrac{\alpha^2}{4}a_2^3F_{38}\}$$

$$RAO(9) = 0$$

TABLE 1

FORCED SOLUTIONS FOR CASE I

$\frac{\Omega}{\omega_\phi}$	Simple		Linear		Nonlinear	
	a_2	β_c, a_1, b_1 ϕ_c, b_2	a_2	β_c, a_1, b_1 ϕ_c, b_2	a_2	β_c, a_1, b_1 ϕ_c, b_2
.5			-.114	0	-.113	0
.6	-.13	0	-.130	0	-.129	0
.7			-.157	0	-.156	0
.8	-.208	0	-.208	0	-.204	0
.9			-.325	0	-.313	0
1.0	-.880	0	-.880	0	-.713	0
1.025			-1.62	0	-1.00	0
1.05	-11.0	0	-11.3	0	-1.52	0
1.075			2.2	0	-2.35	0
1.1	.989	0	.989	0	.769	0
1.2			.289	0	.288	0
1.3	.169	0	.169	0	.167	0
1.4			.115	0	.115	0

TABLE 2

FORCED SOLUTIONS FOR CASE II

$\frac{\Omega}{\omega_\phi}$	Simple			Linear			Nonlinear								
	β_c	b_1	a_2	β_c	b_1	a_2	β_c	b_1	a_2	β_c	b_1	a_2	β_c	b_1	a_2
.5	.097	0	-.111	.098	-.006	-.113	.098	-.006	-.113	.700	1.159	-.600	.690	-1.200	.388
.6				.084	.001	-.131	.084	.001	-.130	.553	.914	-.622	.541	-.981	.396
.7				.072	.009	-.159	.073	.009	-.158	.434	.709	-.644	.434	-.811	.413
.8	.063	0	-.209	.062	.020	-.213	.065	.020	-.209	.326	.521	-.663	.350	-.661	.437
.9				.053	.042	-.341	.064	.046	-.332	.208	.318	-.659	.275	-.515	.464
.95				.049	.068	-.506	.069	.063	-.397	.172	.254	-.644	.256	-.477	.470
1.0	.047	0	-.900	.043	.141	-1.021	.093	.117	-.530	.111	.150	-.575	.238	-.439	.476
1.025				.038	.271	-2.078	-	-	-	-	-	-	.198	-.358	.484
1.05	.044	0	-8.8	.062	-.717	3.111							.154	-.270	.479
1.1	.041	0	.92	.045	-.144	.792							.106	-.175	.442
1.2				.037	-.051	.277							.050	-.060	.260
1.3	.032	0	.169	.033	-.031	.162							.037	-.033	.159
1.4				.029	-.022	.112							.031	-.023	.111

TABLE 3

FORCED SOLUTIONS FOR CASE III

$\frac{\Omega}{\omega_\phi}$	Simple			Linear			Nonlinear								
	β_c	b_1	a_2	β_c	b_1	a_2	β_c	b_1	a_2	β_c	b_1	a_2	β_c	b_1	a_2
.5	.132	0	-.111	.132	-.002	-.114	.132	-.002	-.113						
.6				.125	0	-.131	.125	0	-.130	1.127	1.012	-2.042	1.132	-1.008	2.071
.7	.118	0	-.165	.118	.004	-.160	.118	.004	-.159	.851	.763	-1.840	.849	-.778	1.809
.8				.110	.010	-.215	.111	.010	-.211	.623	.563	-1.660	.626	-.587	1.605
.9	.104	0	-.336	.103	.024	-.351	.107	.024	-.337	.418	.378	-1.476	.436	-.414	1.419
.95				.099	.043	-.535	.110	.042	-.496	.308	.274	-1.348	.345	-.327	1.313
.99							.141	.096	-.871	.185	.148	-1.085	.271	-.254	1.202
1.0	.096	0	-.900	.095	.106	-1.19	-	-	-	-	-	-	.253	-.235	1.167
1.05	.093	0	-8.8	.098	-.363	3.41							.164	-.140	.912
1.1	.092	0	.92	.091	-.080	.713							.113	-.073	.568
1.2				.084	-.034	.261							.088	-.034	.252
1.3	.077	0	.169	.077	-.022	.155							.079	-.023	.153
1.4				.072	-.017	.107							.073	-.017	.107

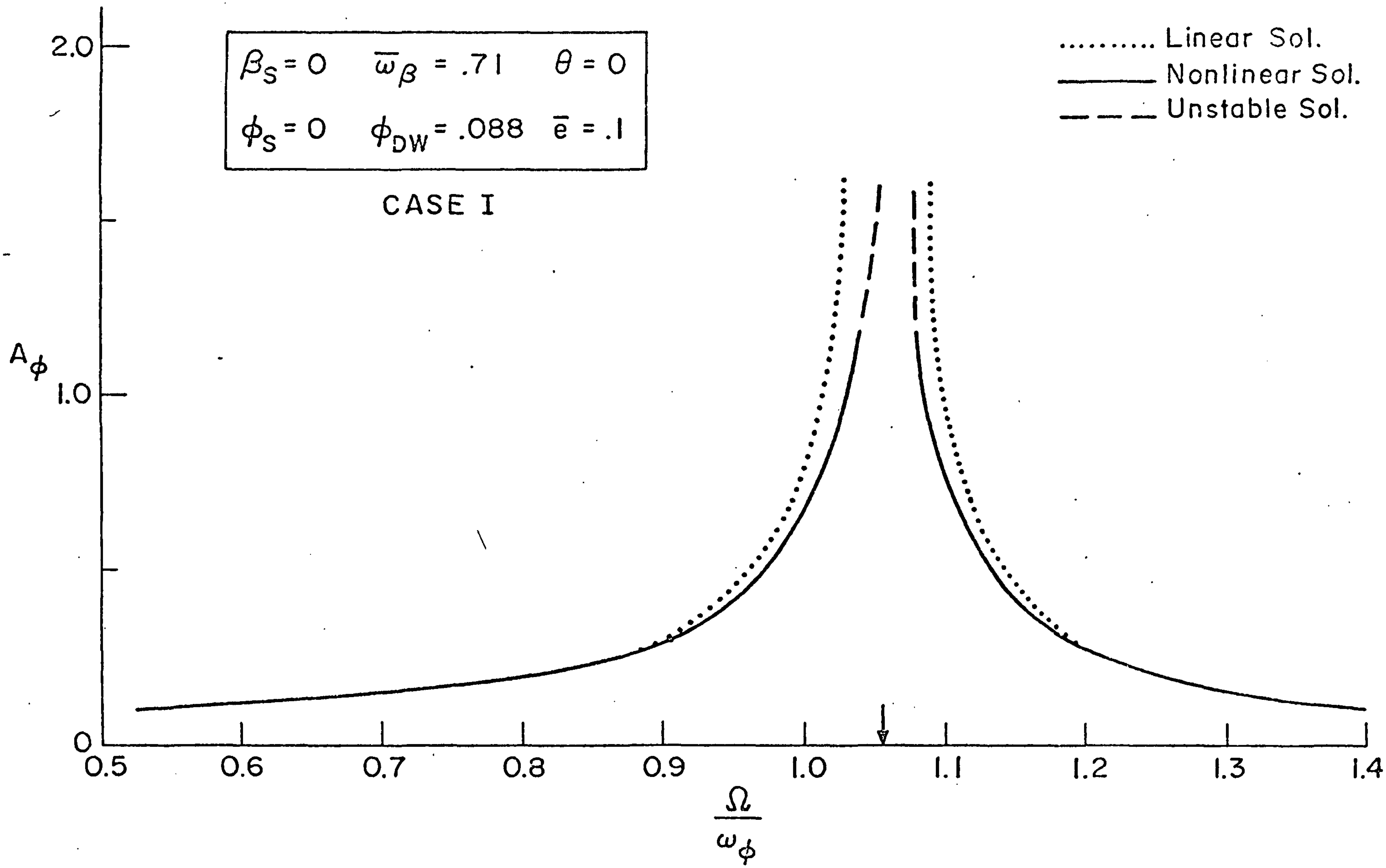

FIG. 1 FORCED OSCILLATIONS CASE I

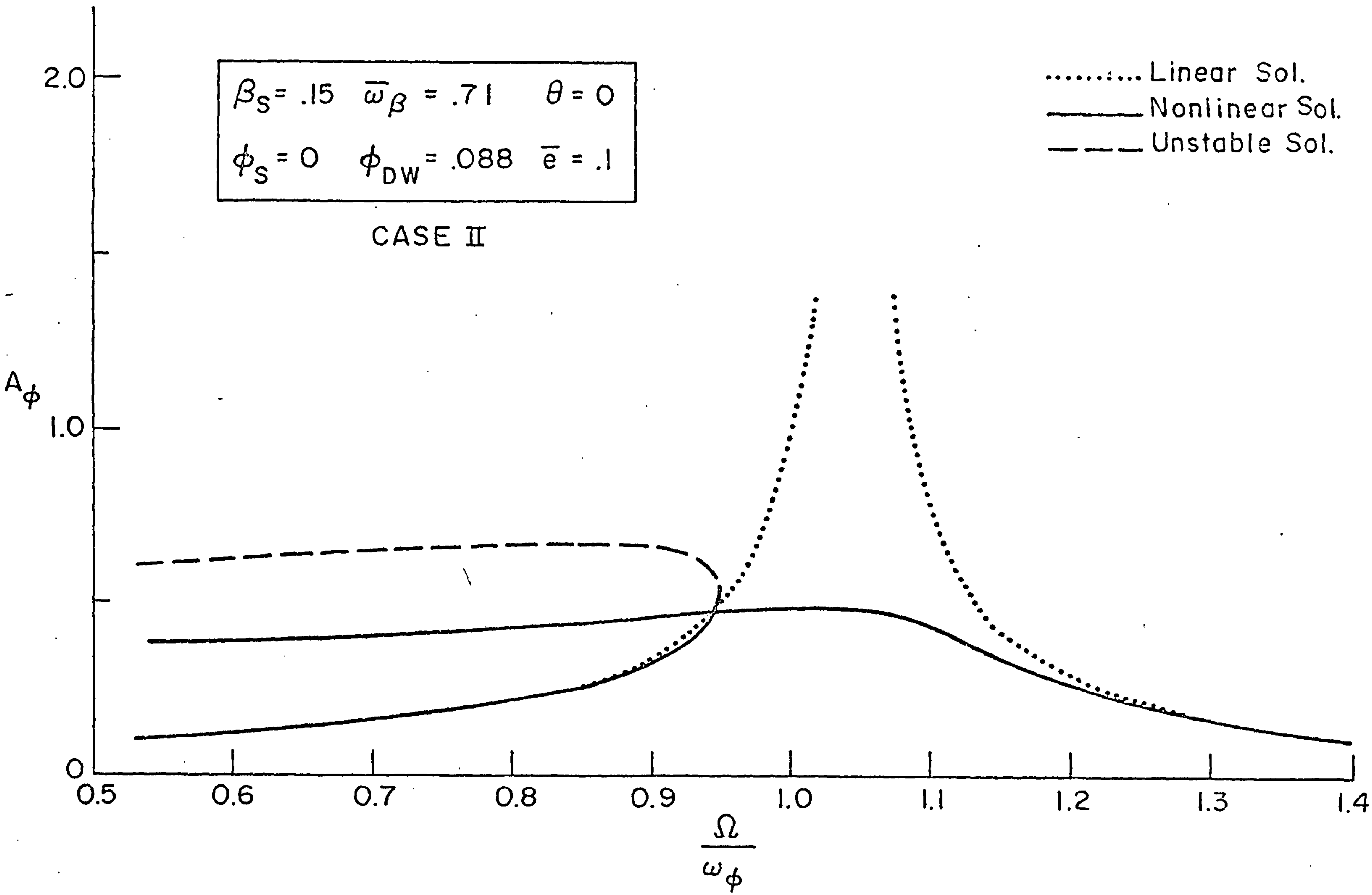

FIG. 2a FORCED OSCILLATIONS, CASE II

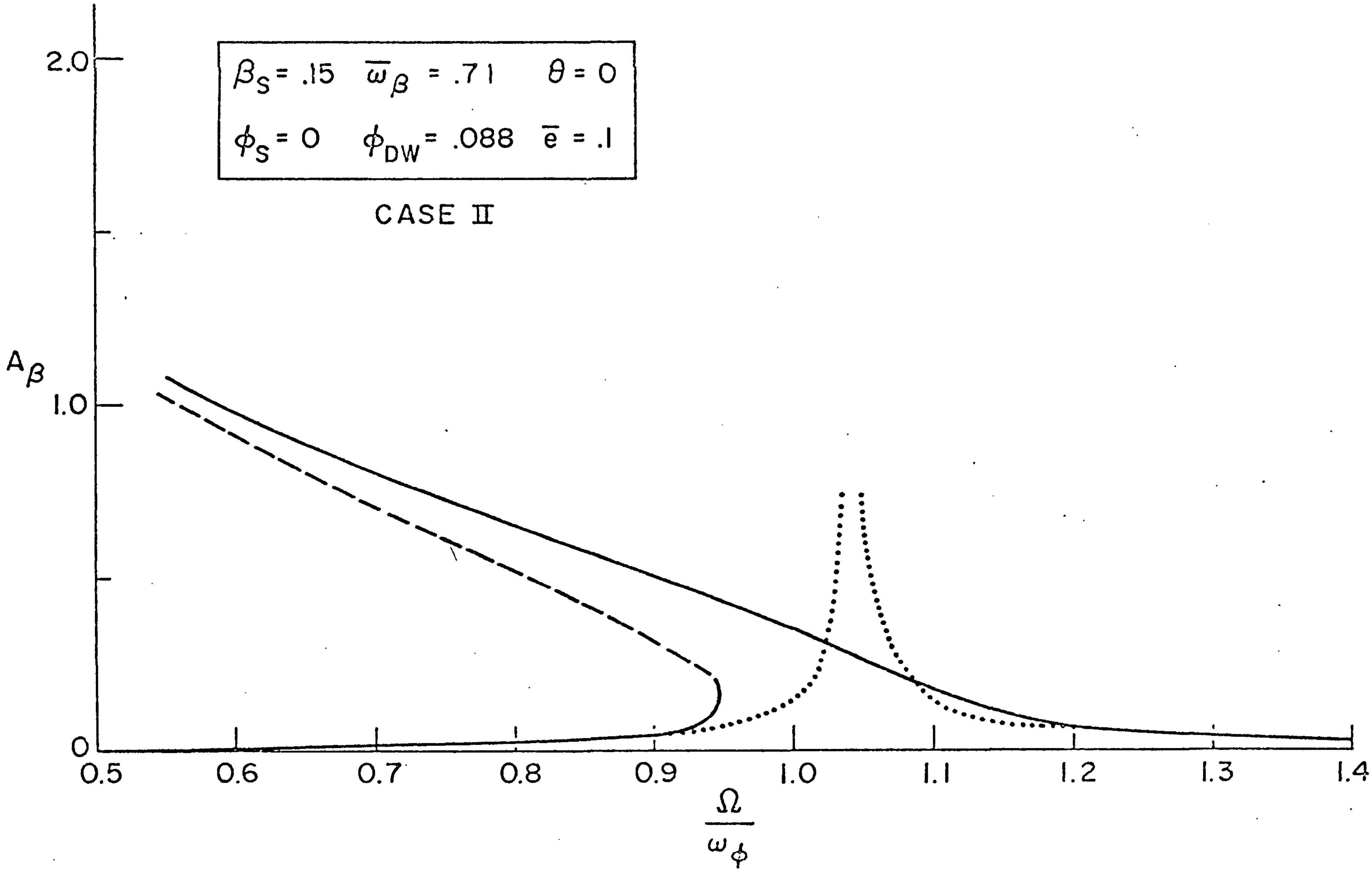

FIG. 2b FORCED OSCILLATIONS, CASE II

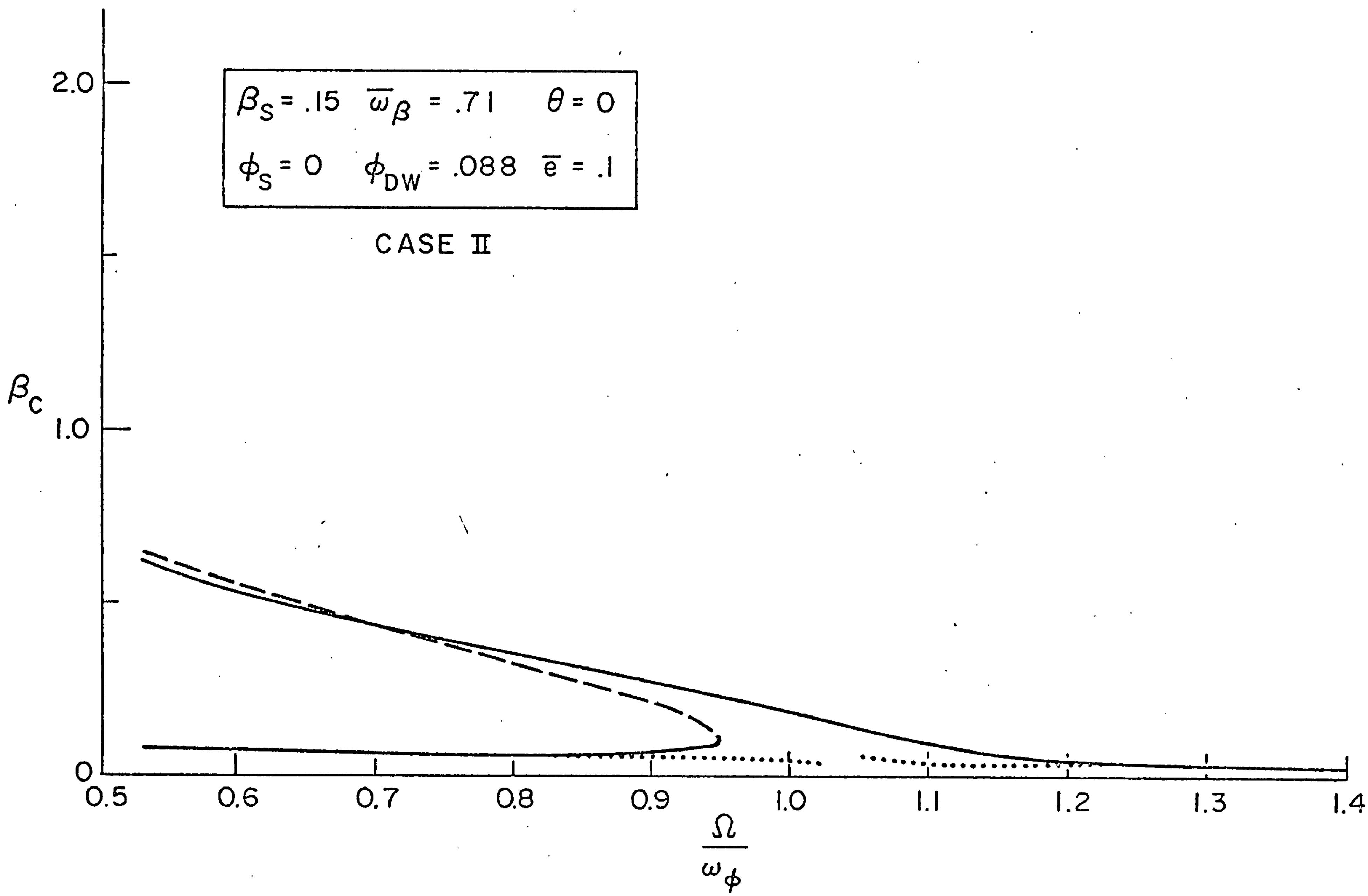

FIG. 2c FORCED OSCILLATIONS, CASE II

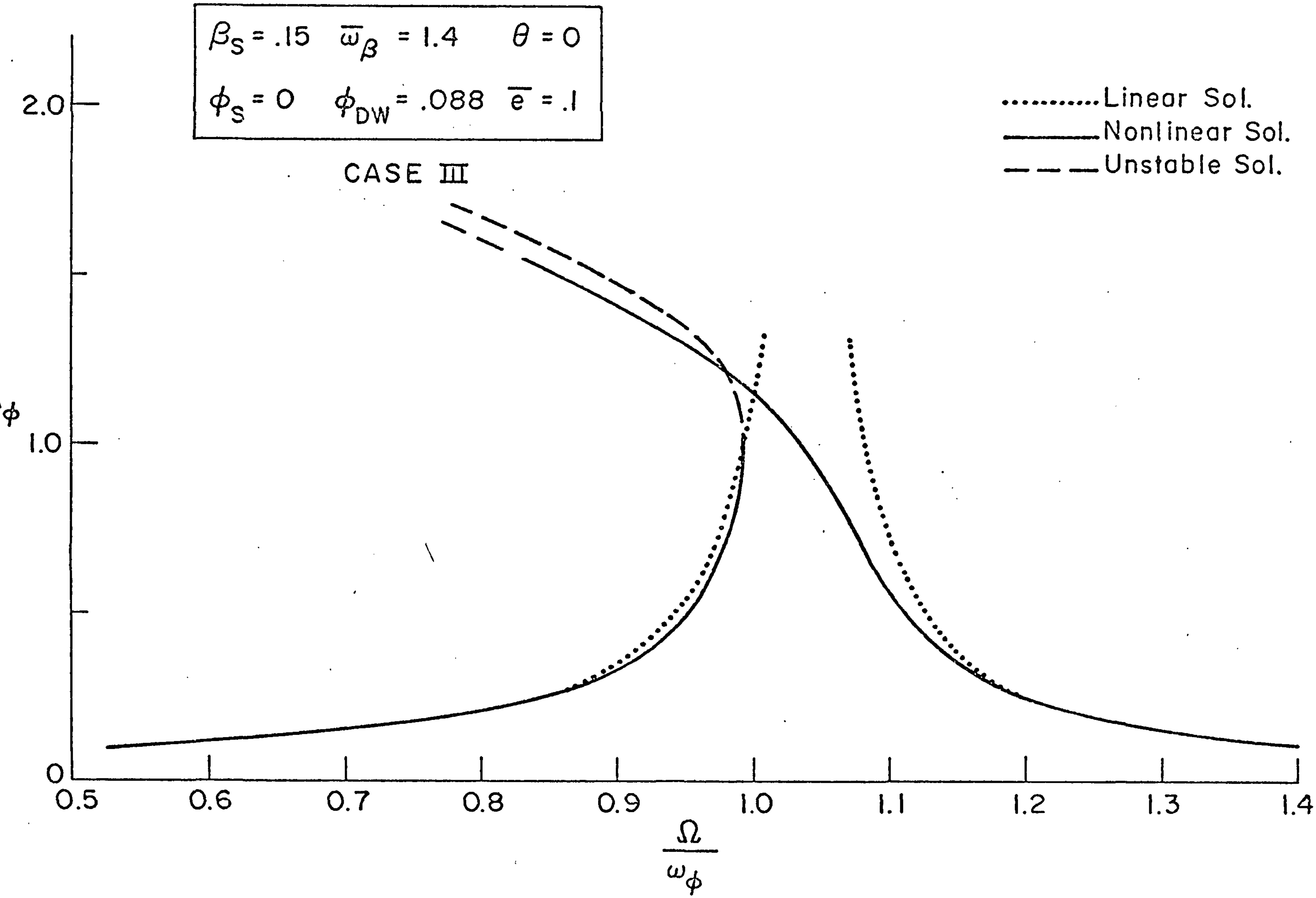

FIG. 3a FORCED OSCILLATIONS, CASE III

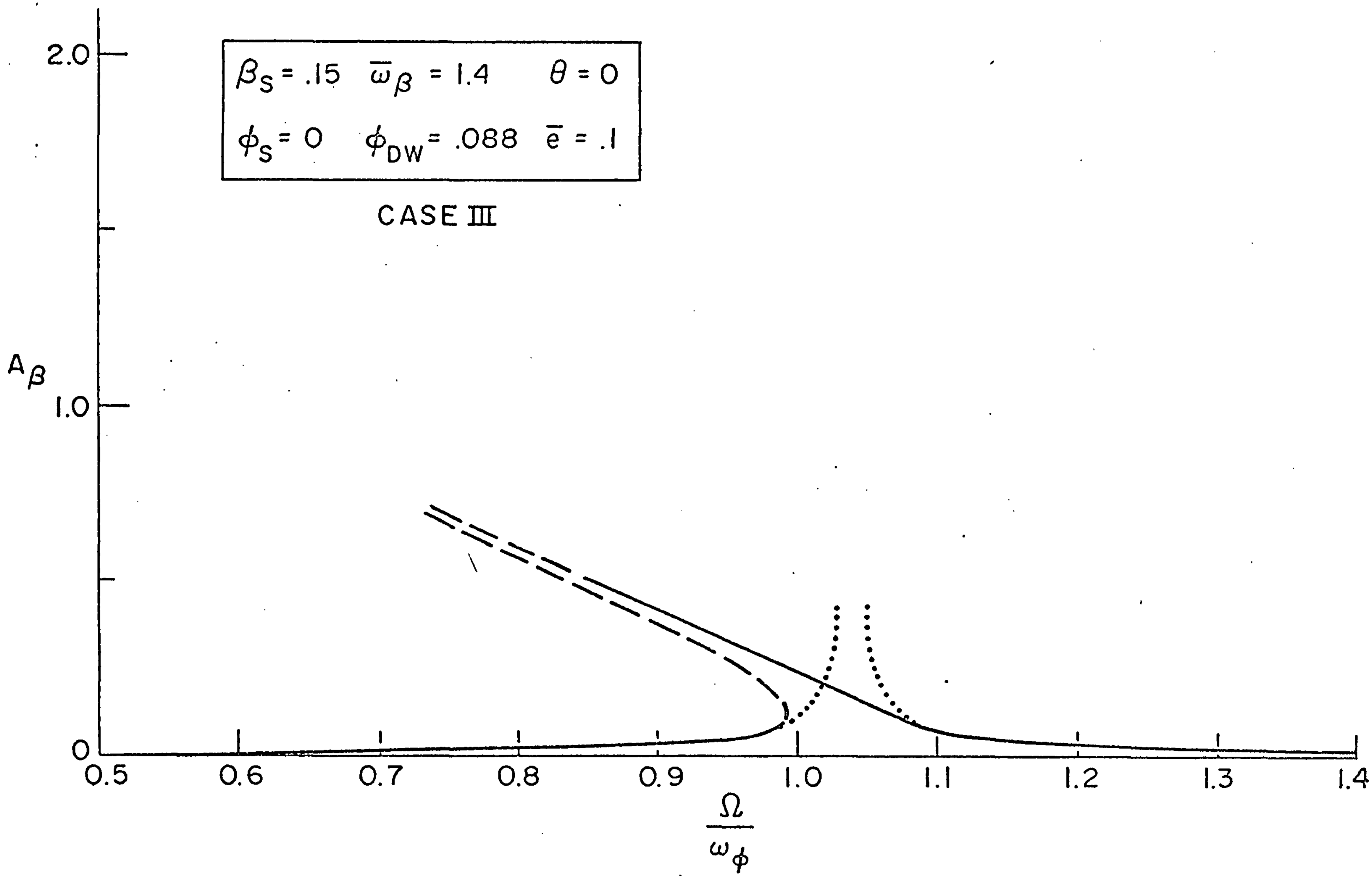

FIG. 3b FORCED OSCILLATIONS, CASE III

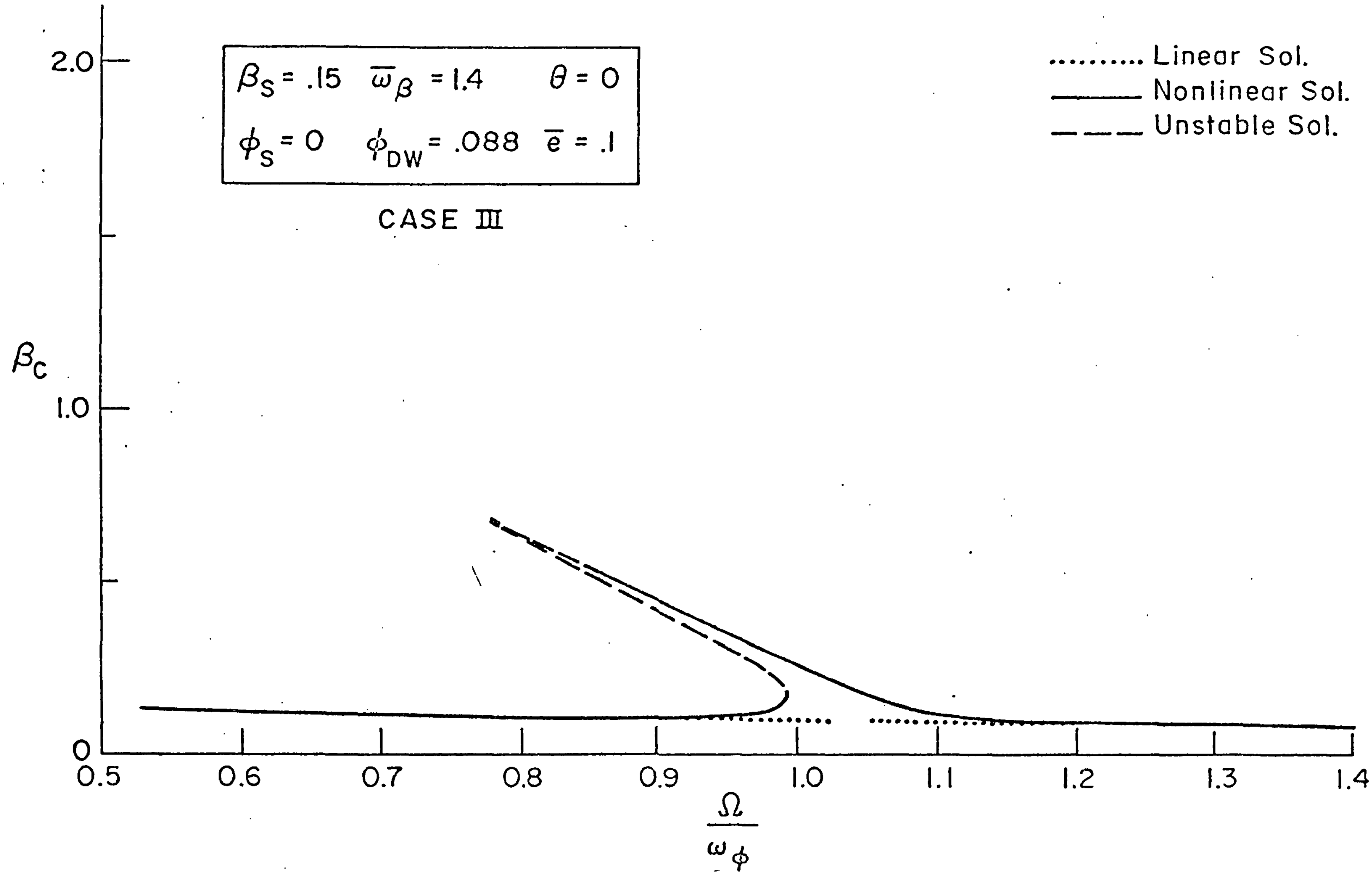

FIG. 3c FORCED OSCILLATIONS, CASE III

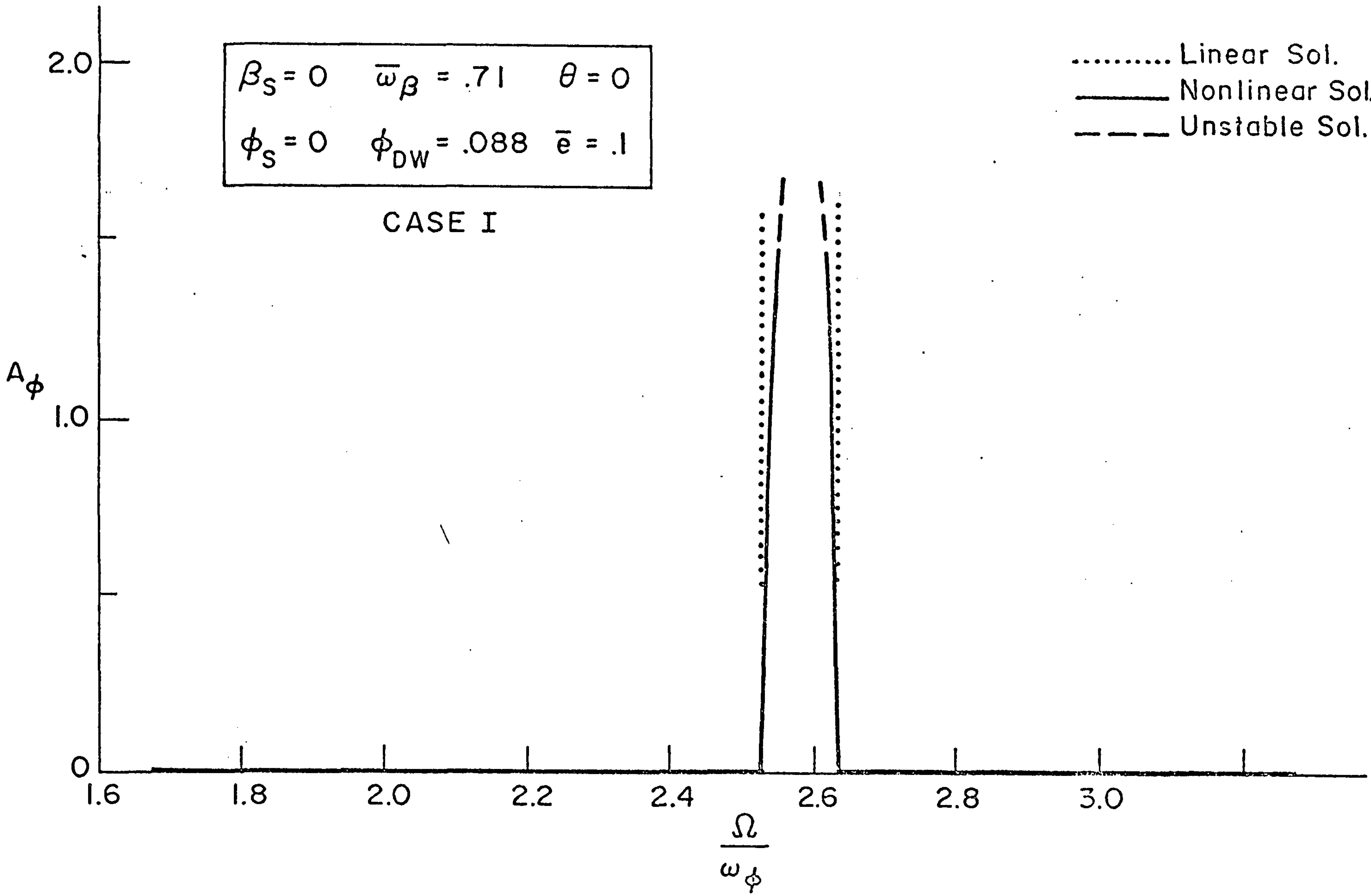

FIG. 4 PARAMETRIC RESONANCE, CASE I

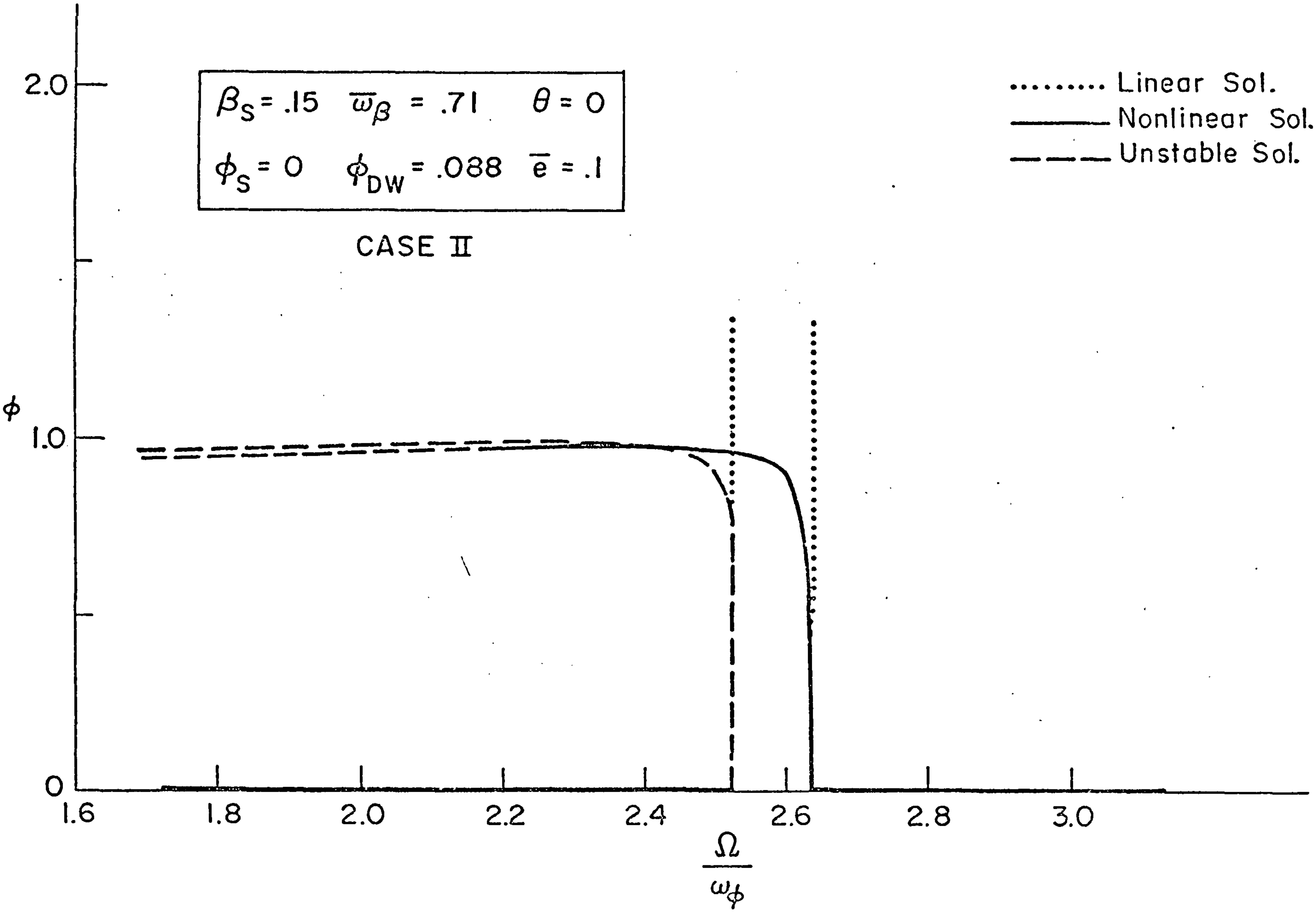

FIG. 5a PARAMETRIC RESONANCE, CASE II

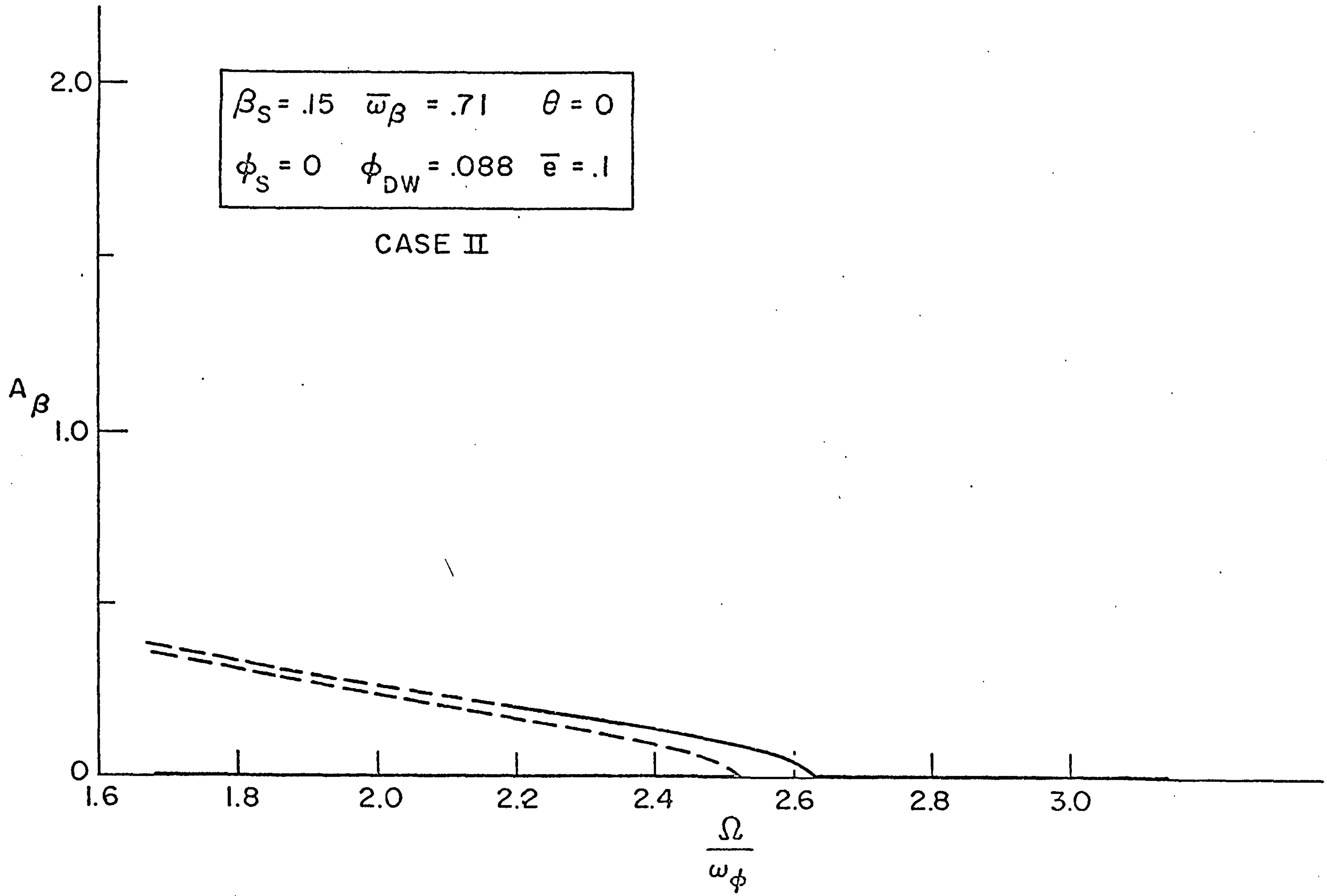

FIG. 5b PARAMETRIC RESONANCE, CASE II

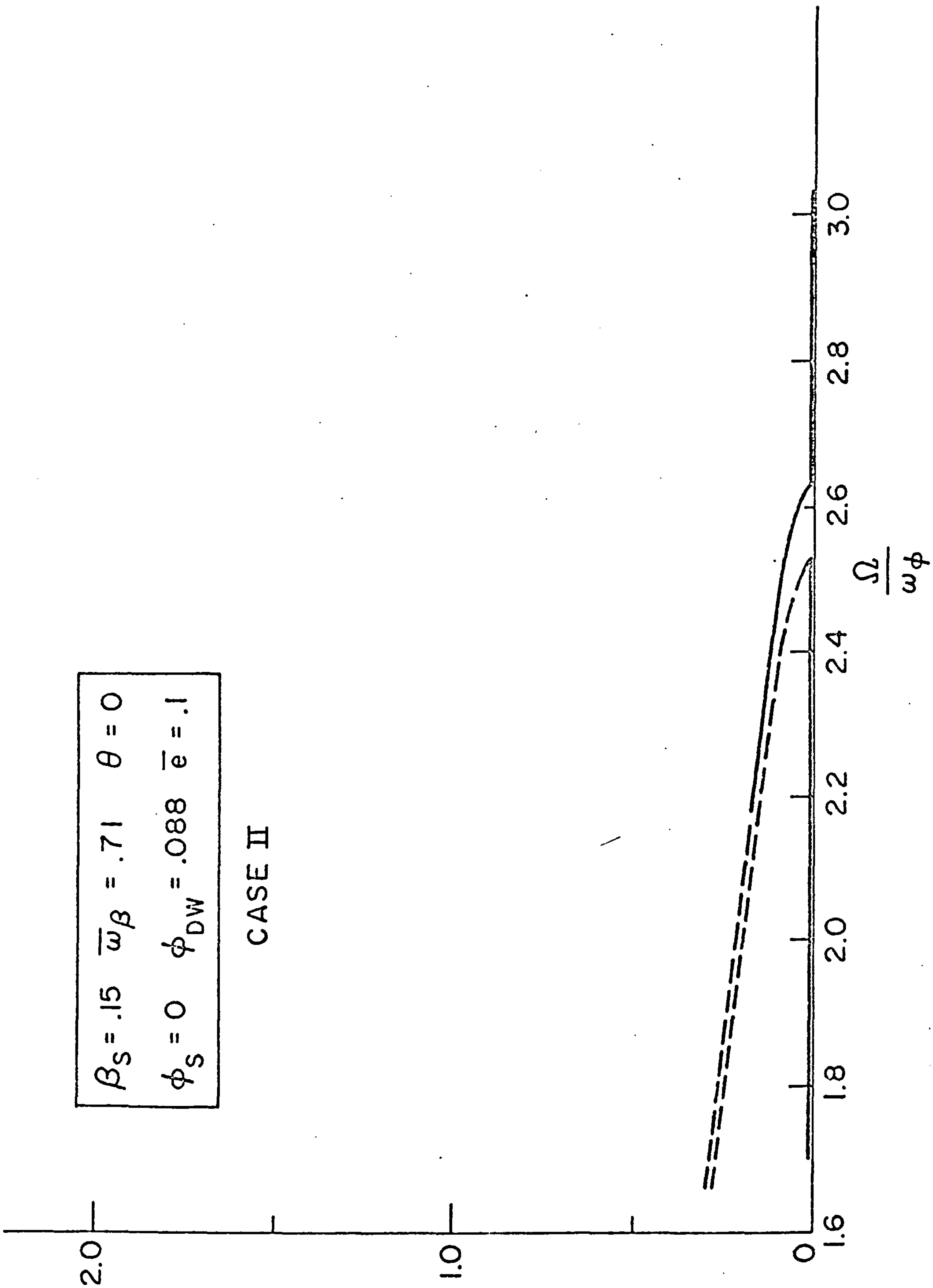

FIG. 5c PARAMETRIC RESONANCE, CASE II

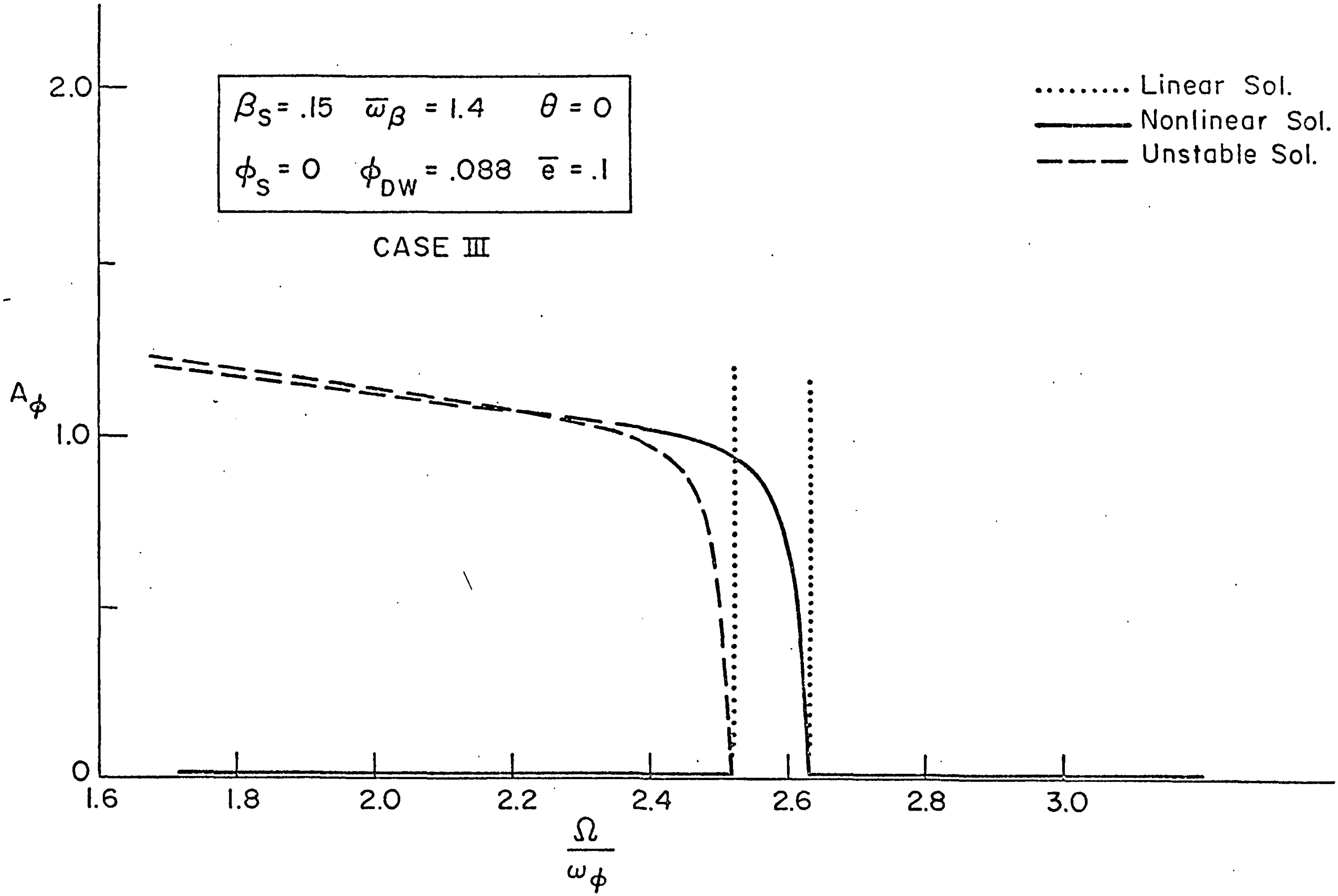

FIG. 6a PARAMETRIC RESONANCE, CASE III

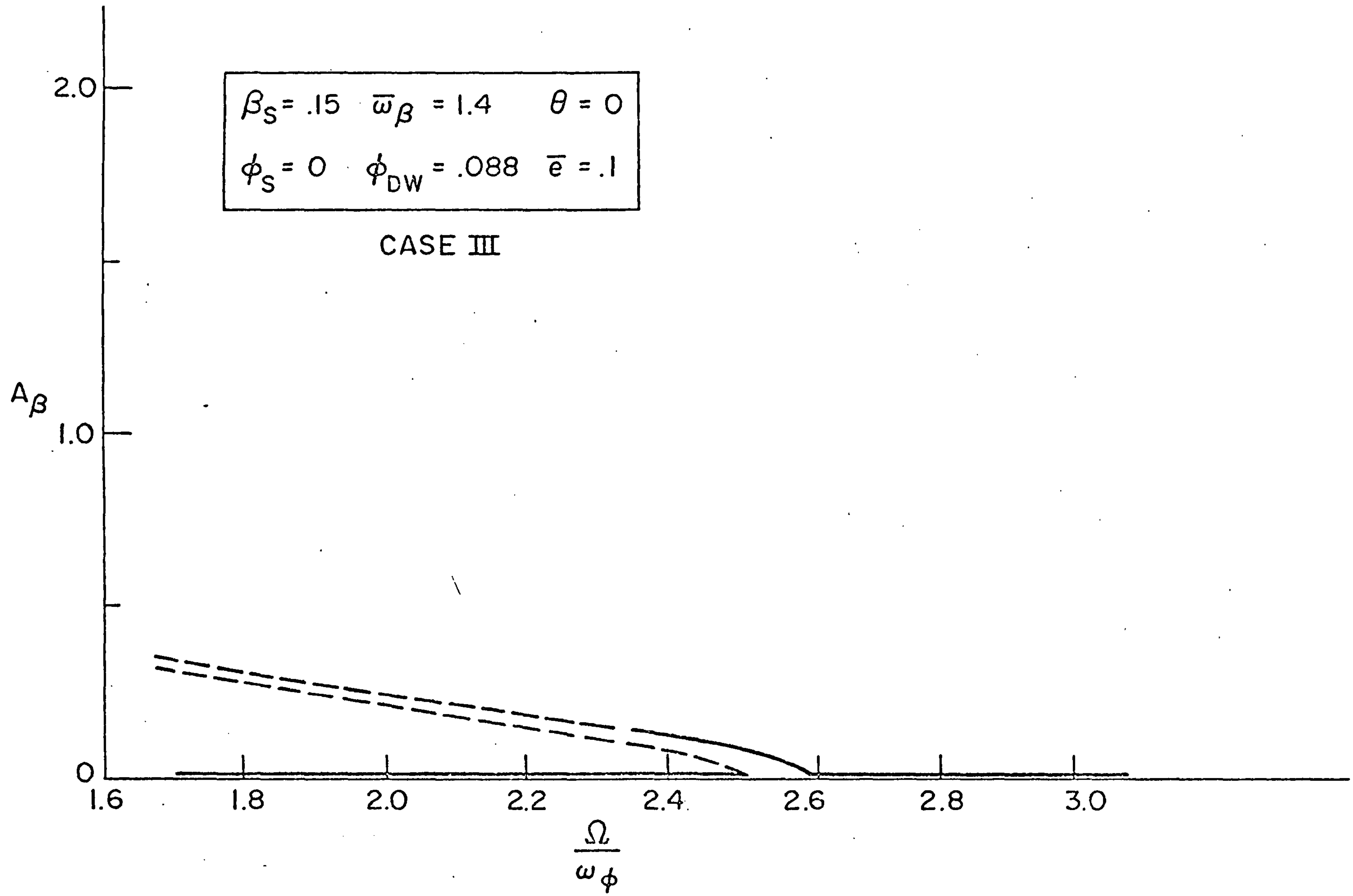

FIG. 6b PARAMETRIC RESONANCE, CASE III

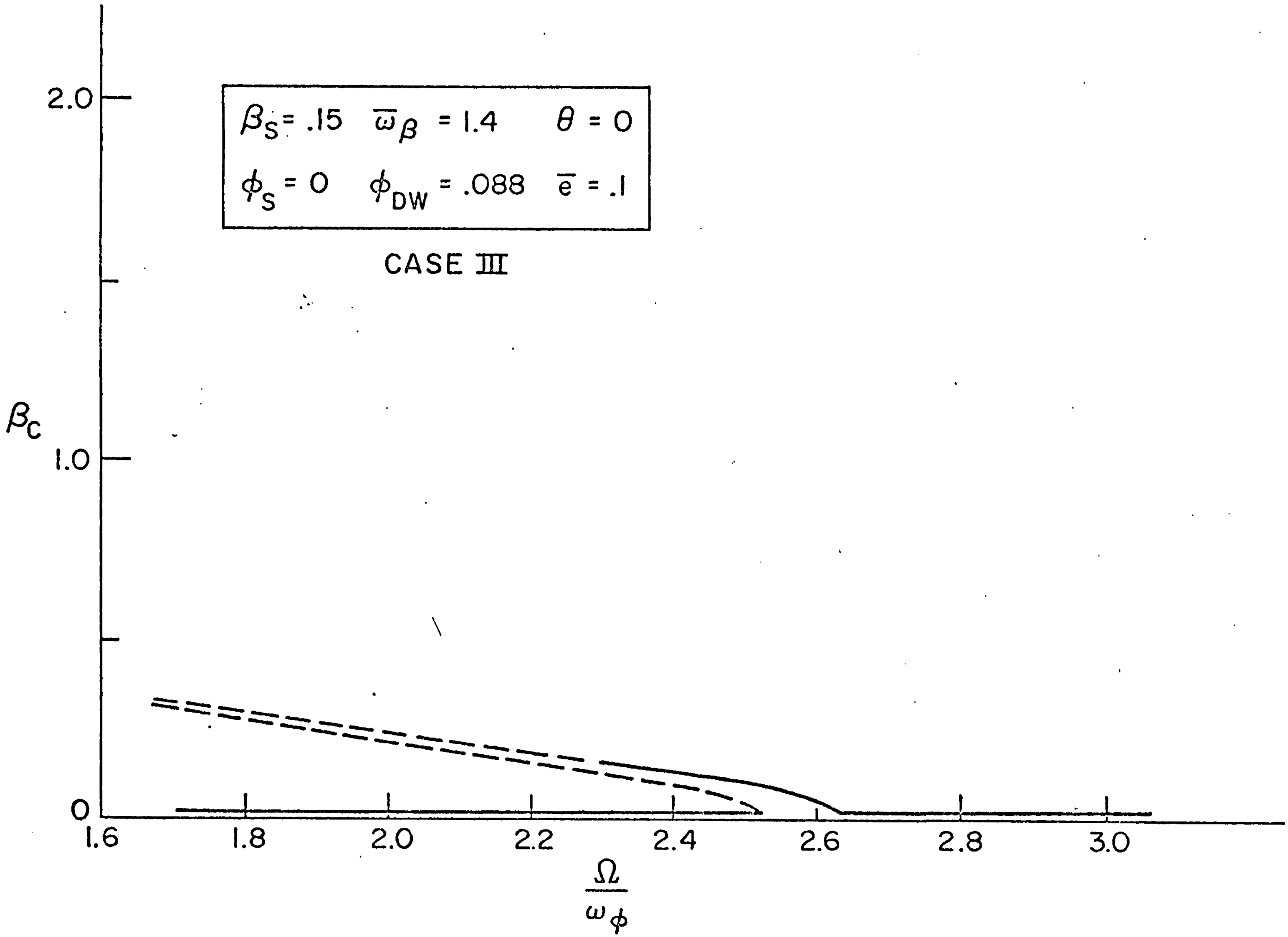

FIG. 6c PARAMETRIC RESONANCE, CASE III

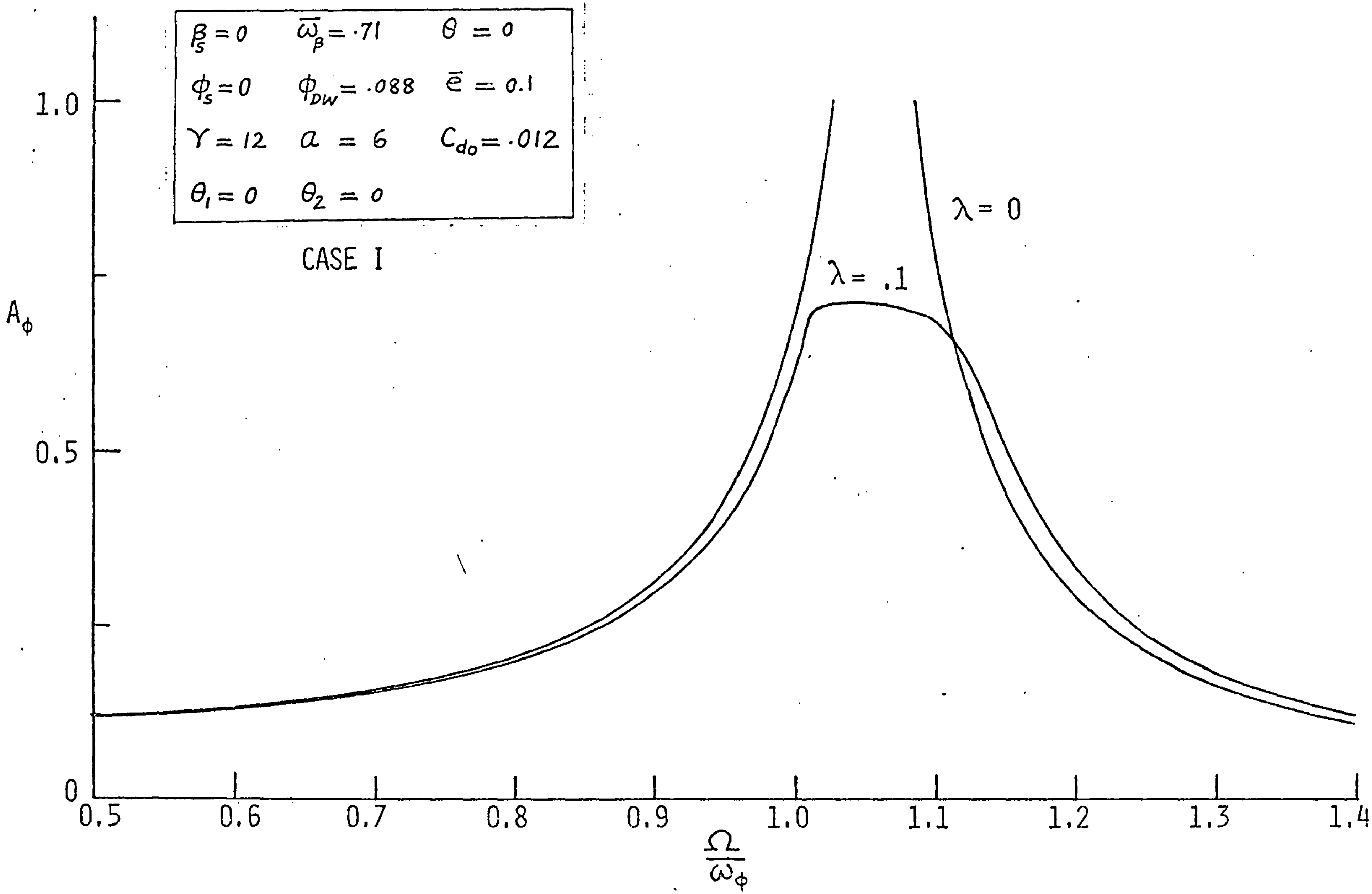

FIG. 7a FORCED OSCILLATIONS WITH AERODYNAMIC FORCES, CASE I

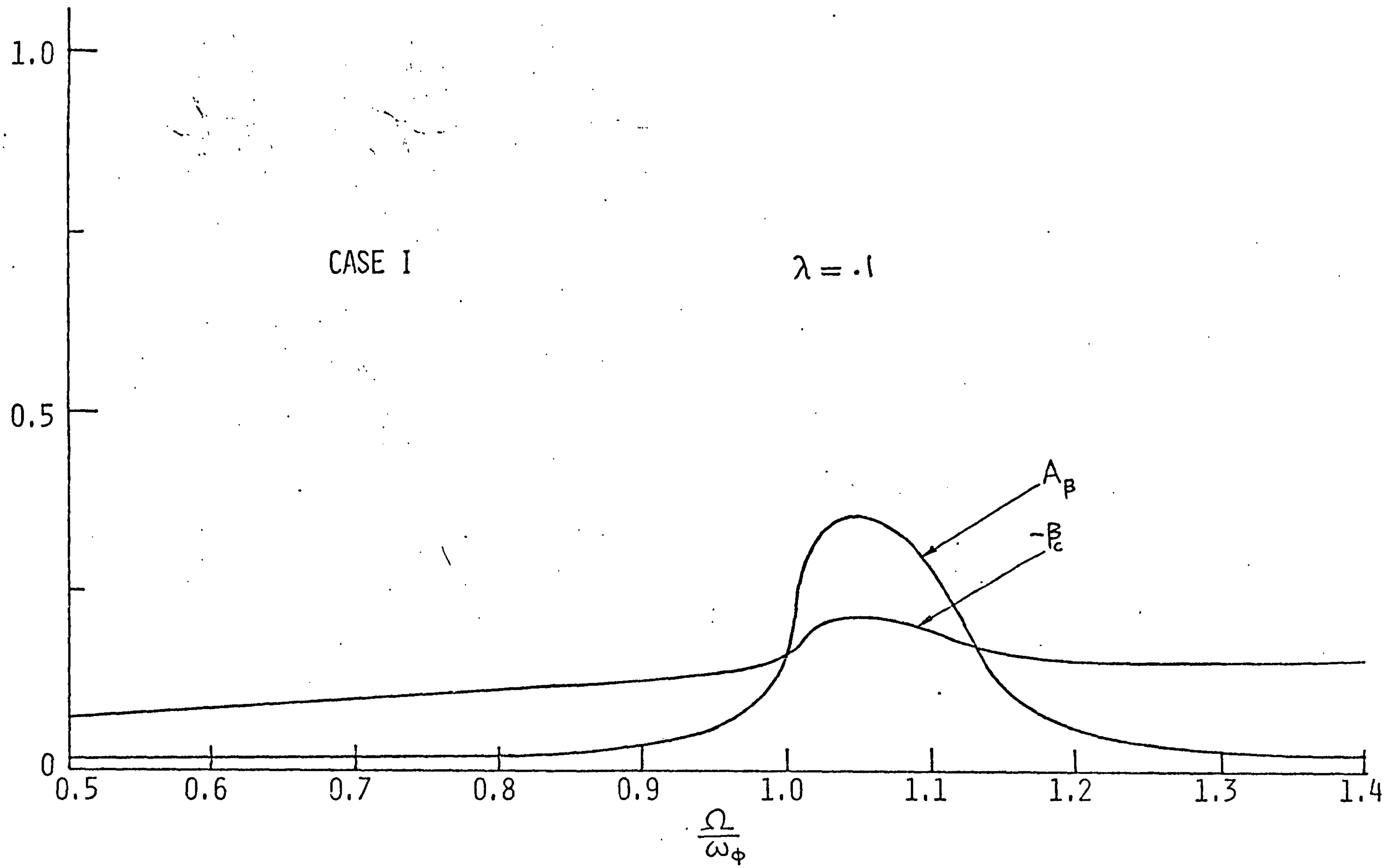

FIG. 7b FORCED OSCILLATIONS WITH AERODYNAMIC FORCES, CASE I

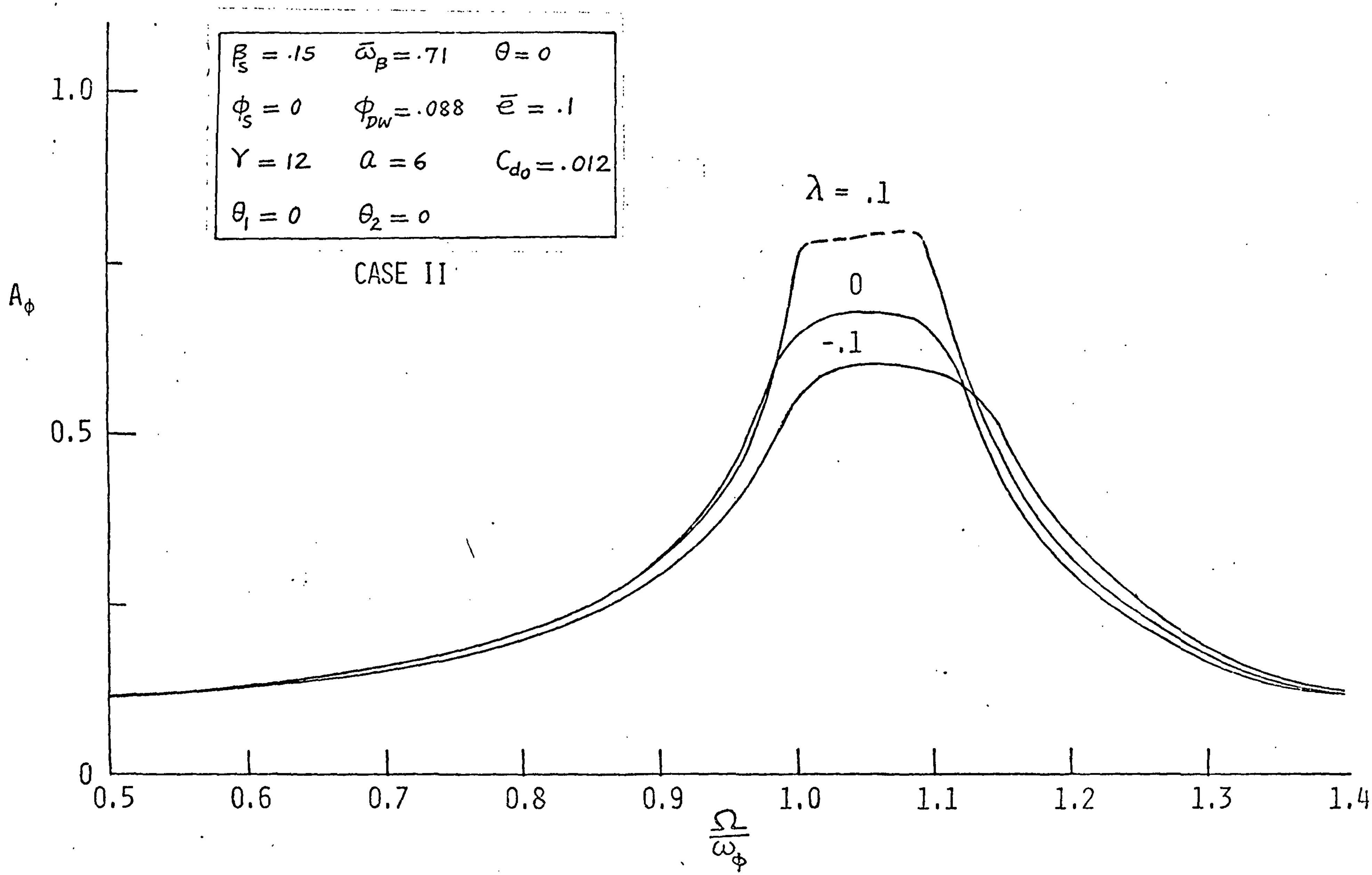

FIG. 8 FORCED OSCILLATIONS WITH AERODYNAMIC FORCES, CASE II

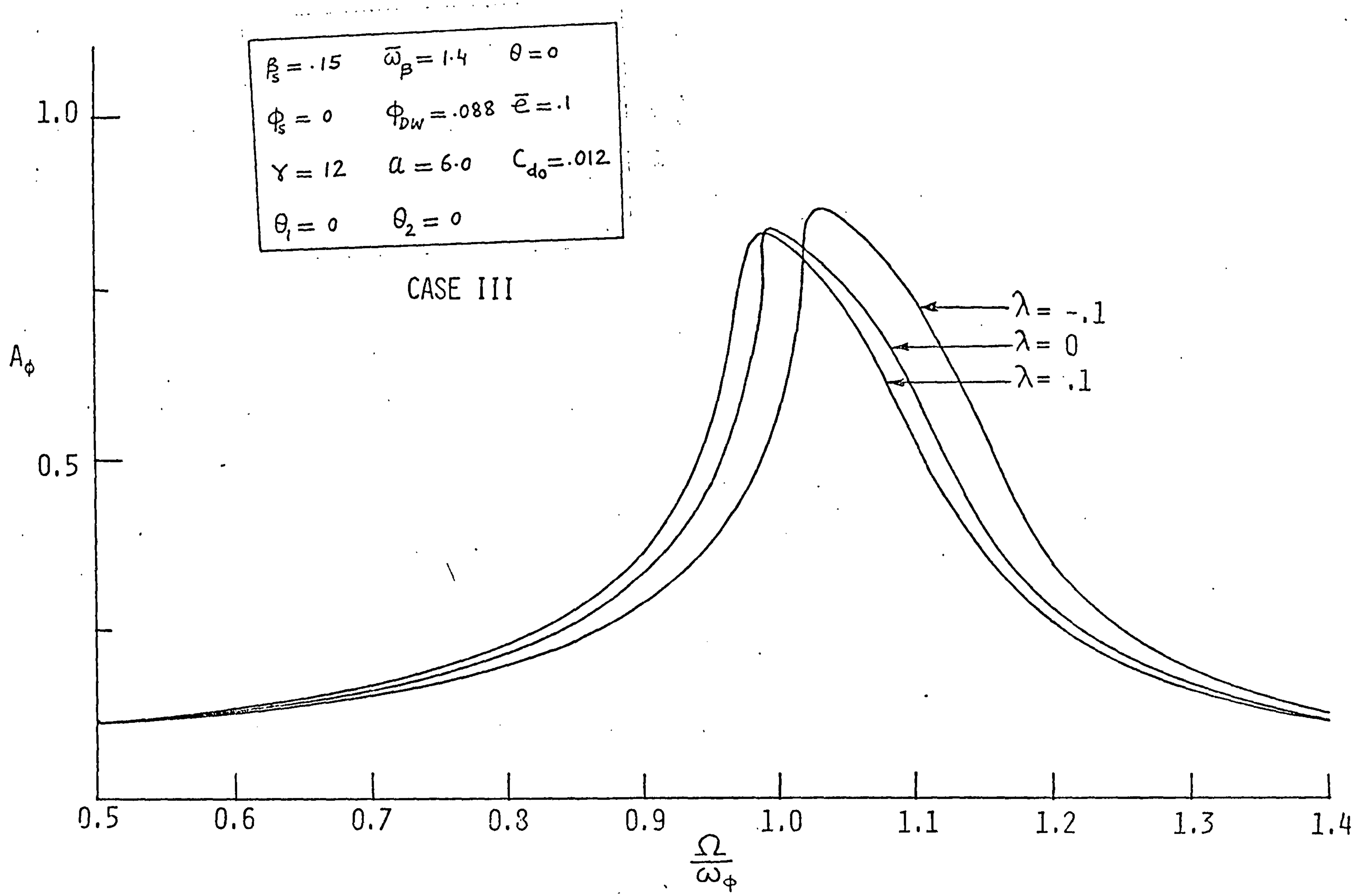

FIG. 9 FORCED OSCILLATIONS WITH AERODYNAMIC FORCES, CASE III

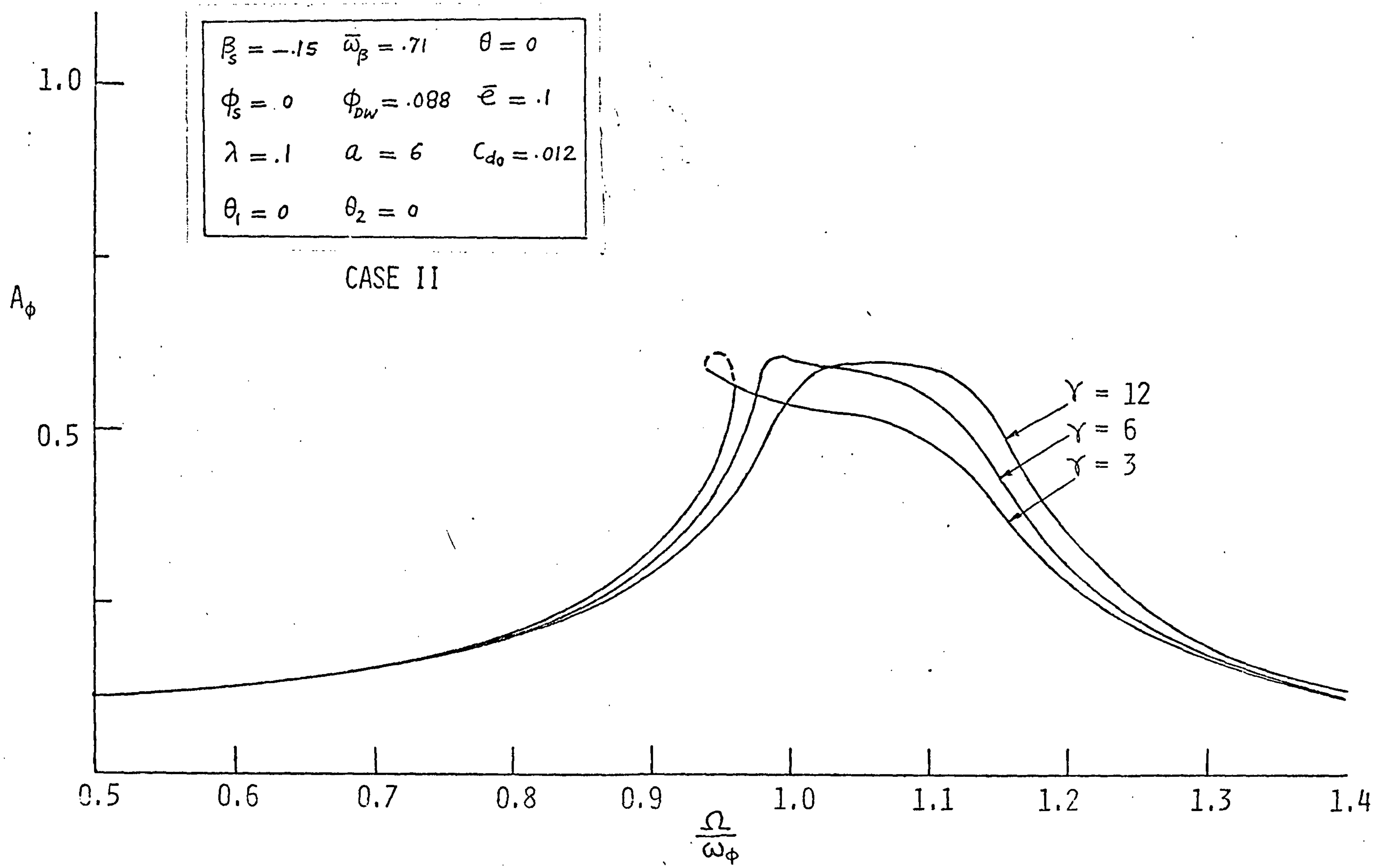

FIG. 10 FORCED OSCILLATIONS WITH AERODYNAMIC FORCES, CASE II (EFFECT OF LOCK NUMBER γ)

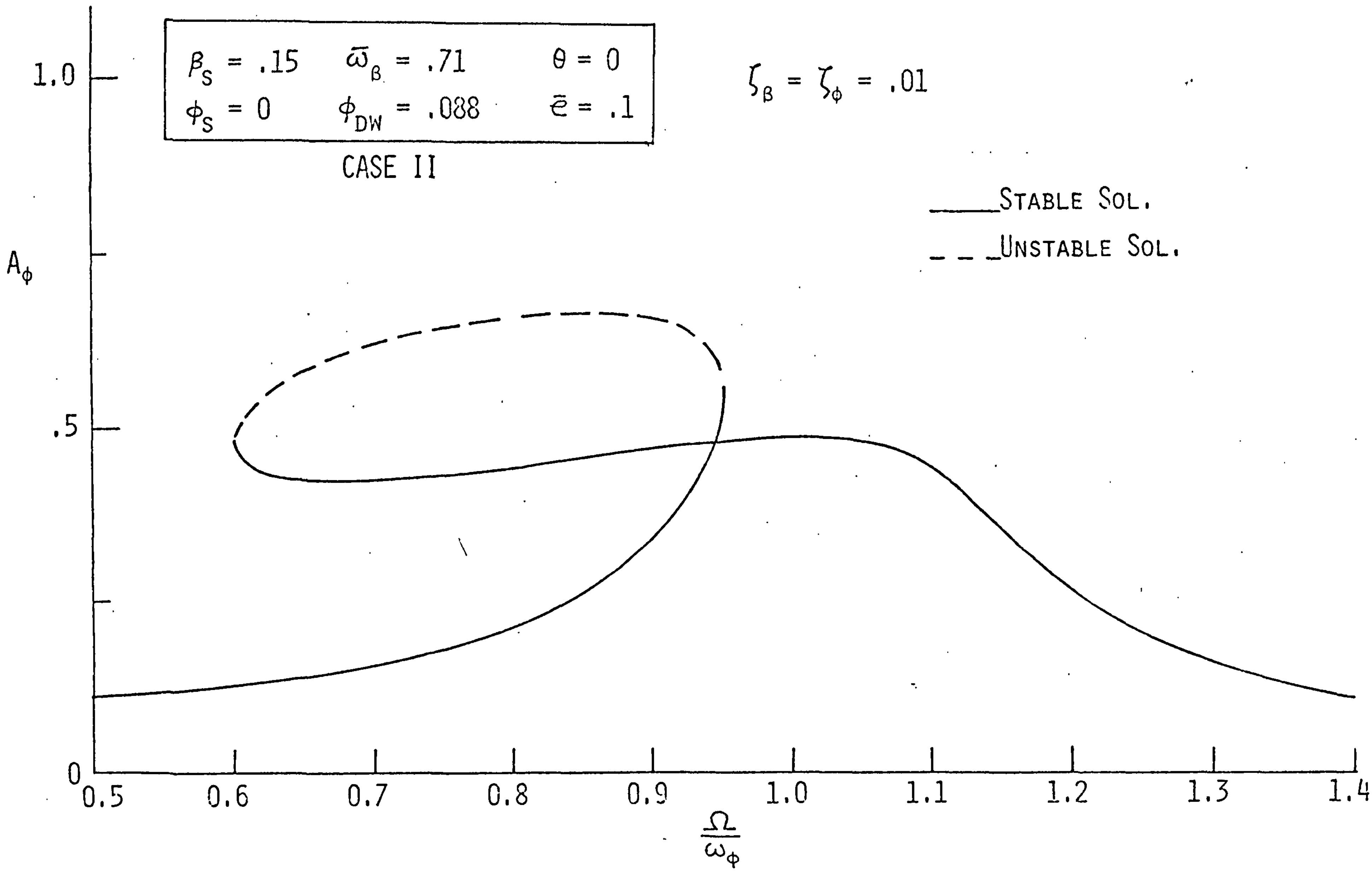

FIG. 11 FORCED OSCILLATIONS WITHOUT AERODYNAMIC FORCES, CASE II (EFFECT OF STRUCTURAL DAMPING)

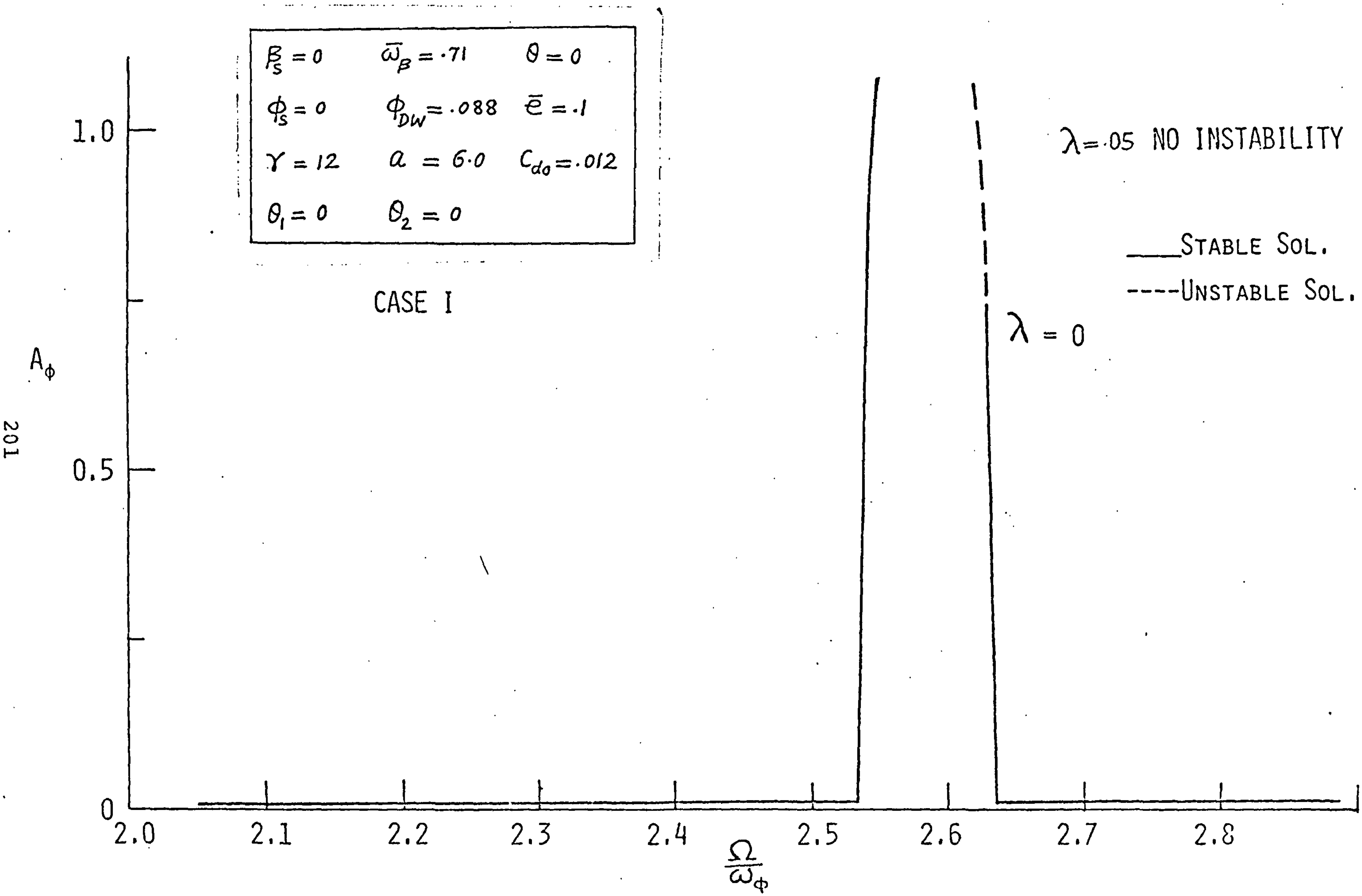

FIG. 12 PARAMETRIC RESONANCE WITH AERODYNAMIC FORCES, CASE I

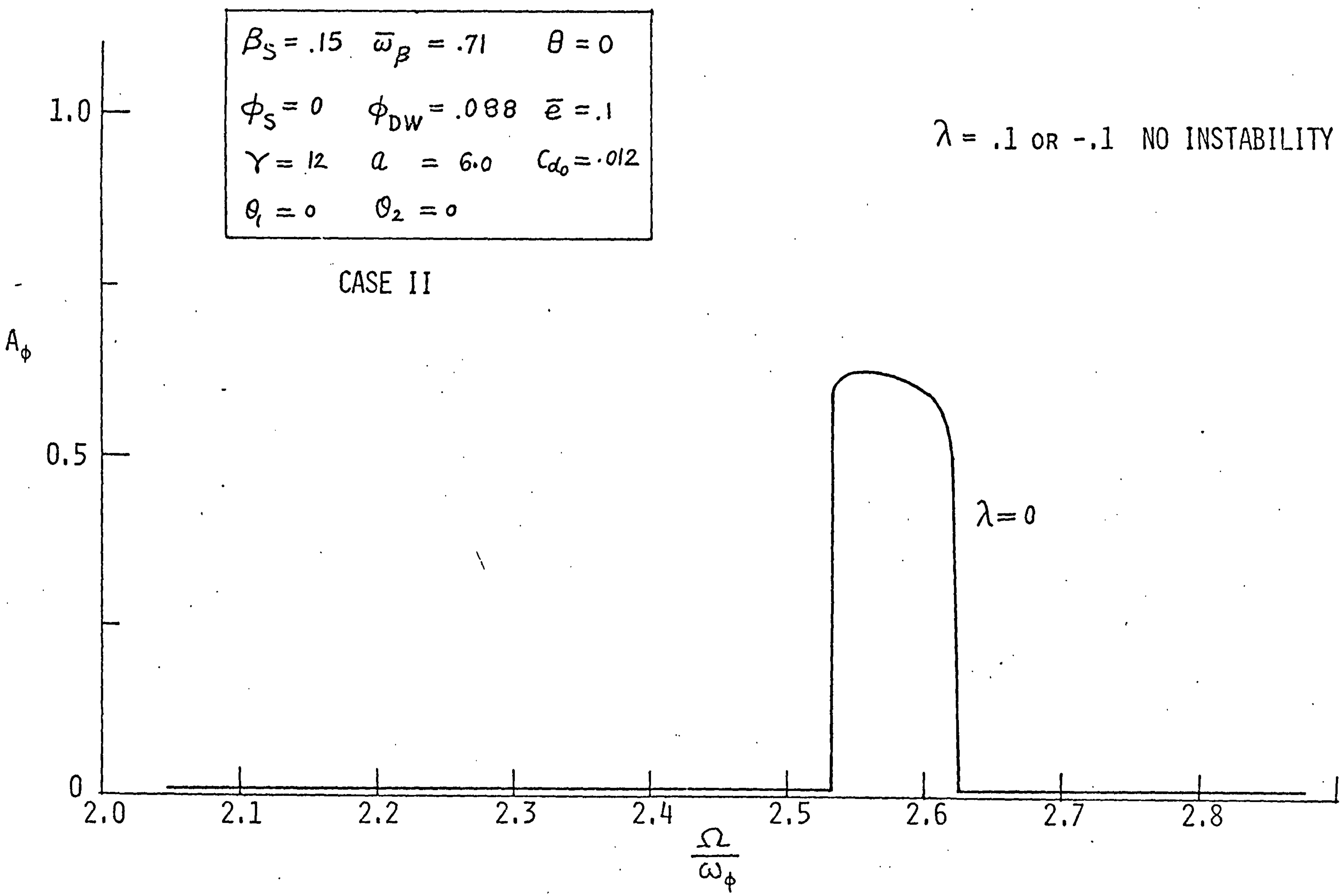

FIG. 13 PARAMETRIC RESONANCE WITH AERODYNAMIC FORCES, CASE II

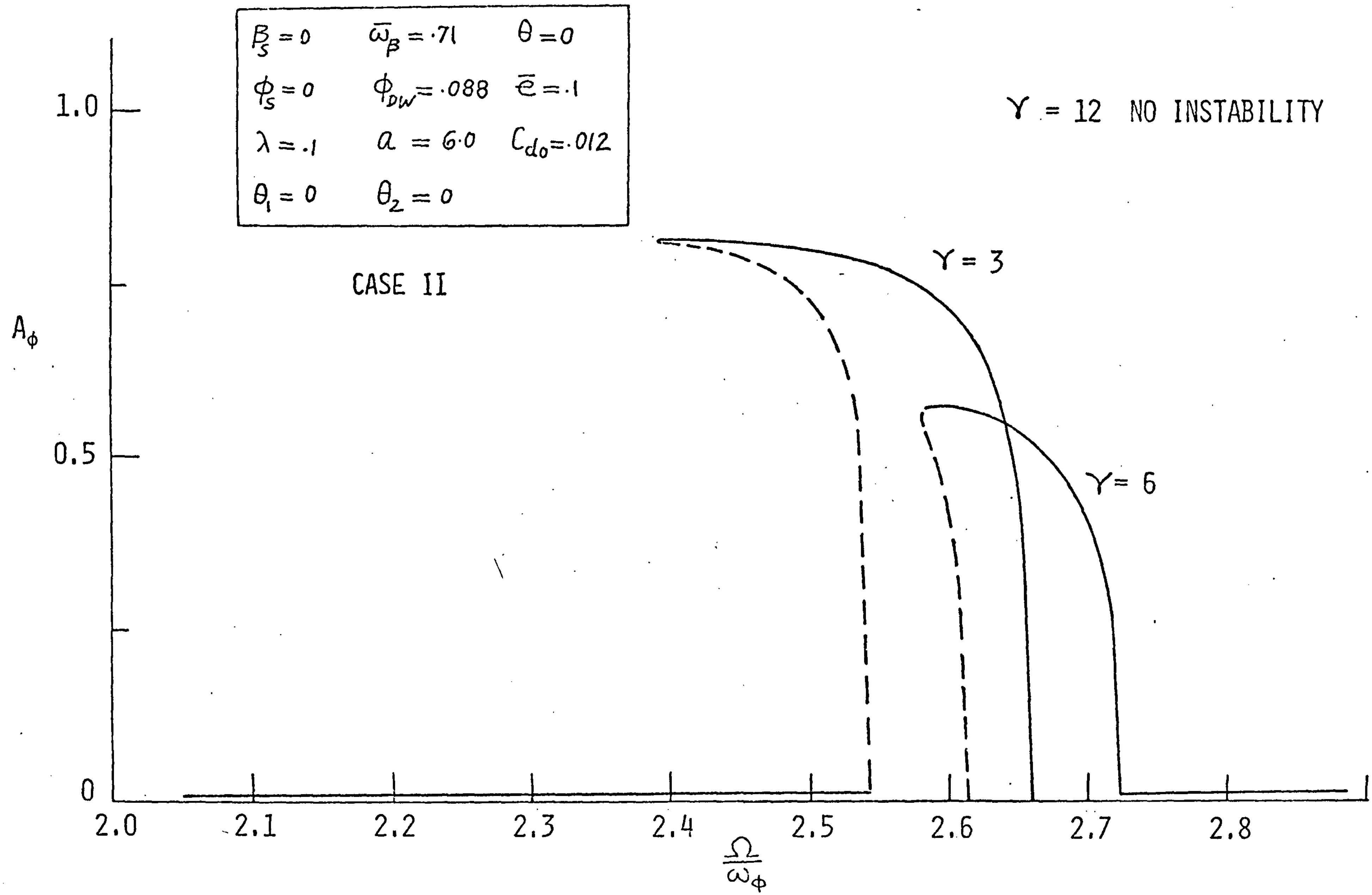

FIG. 14 PARAMETRIC RESONANCE WITH AERODYNAMIC FORCES, CASE II

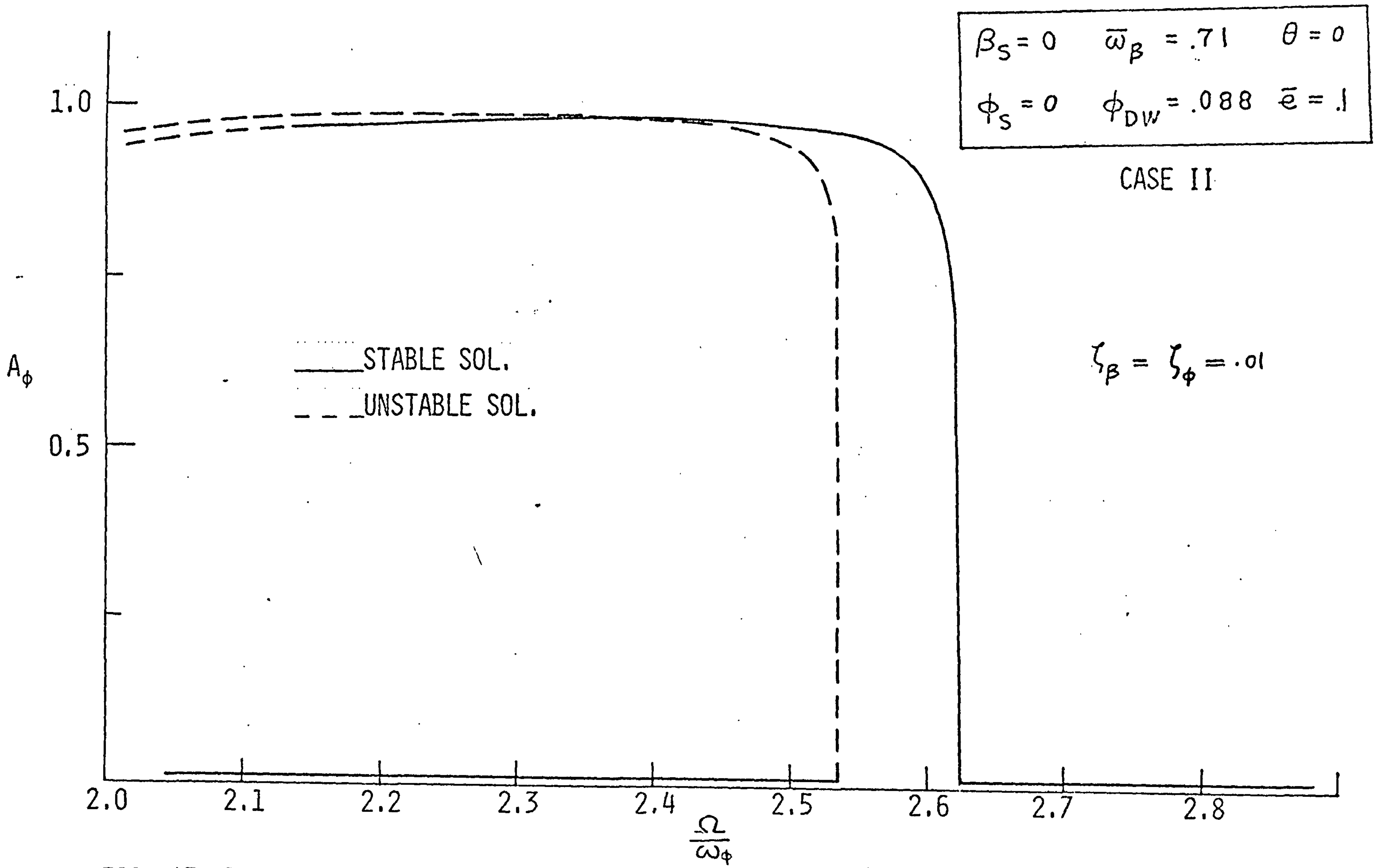

FIG. 15 PARAMETRIC RESONANCE WITHOUT AERODYNAMIC FORCES, CASE II (EFFECT OF STRUCTURAL DAMPING)

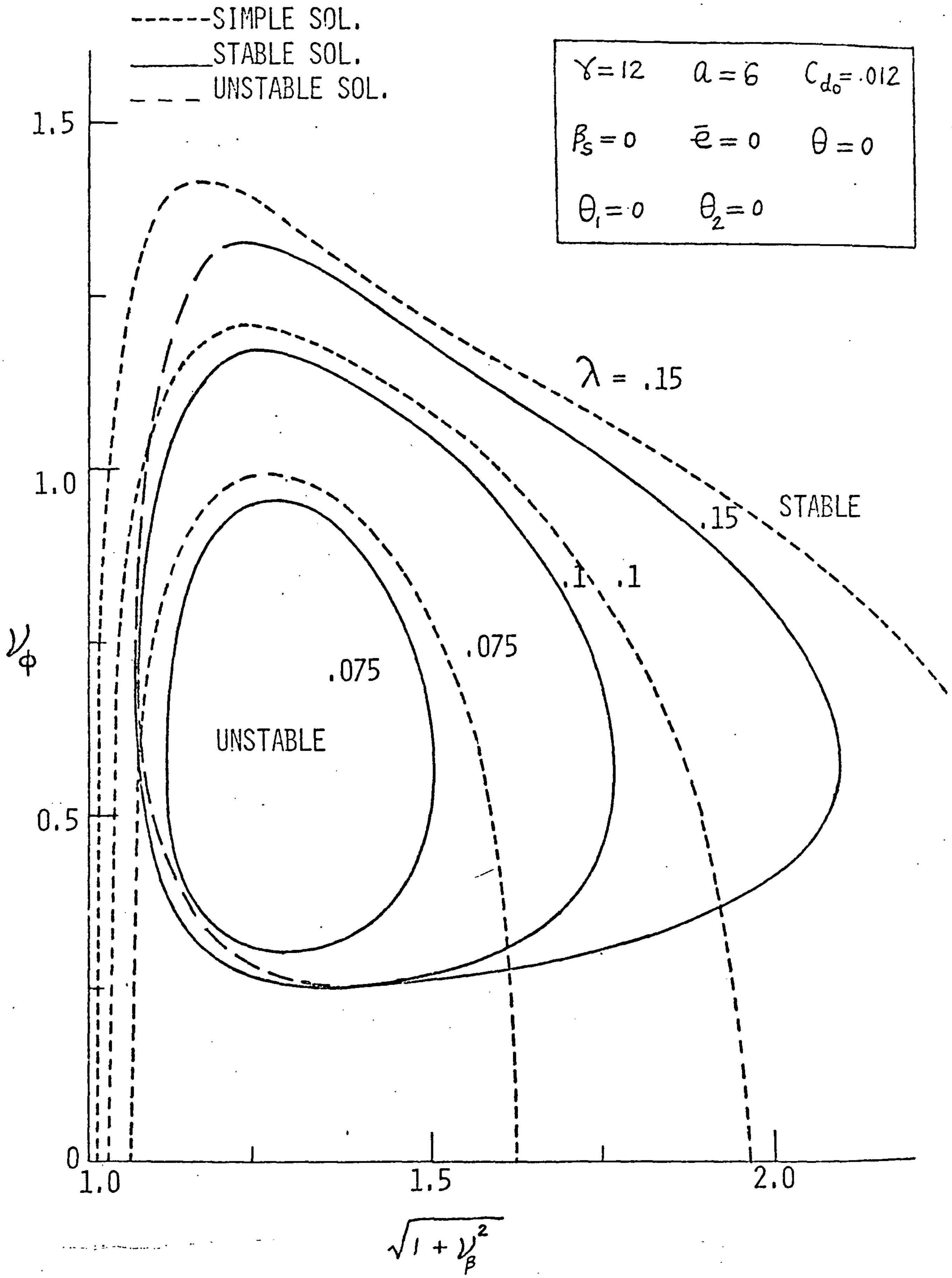

FIG. 16 FLUTTER SOLUTION (EFFECT OF INFLOW λ)

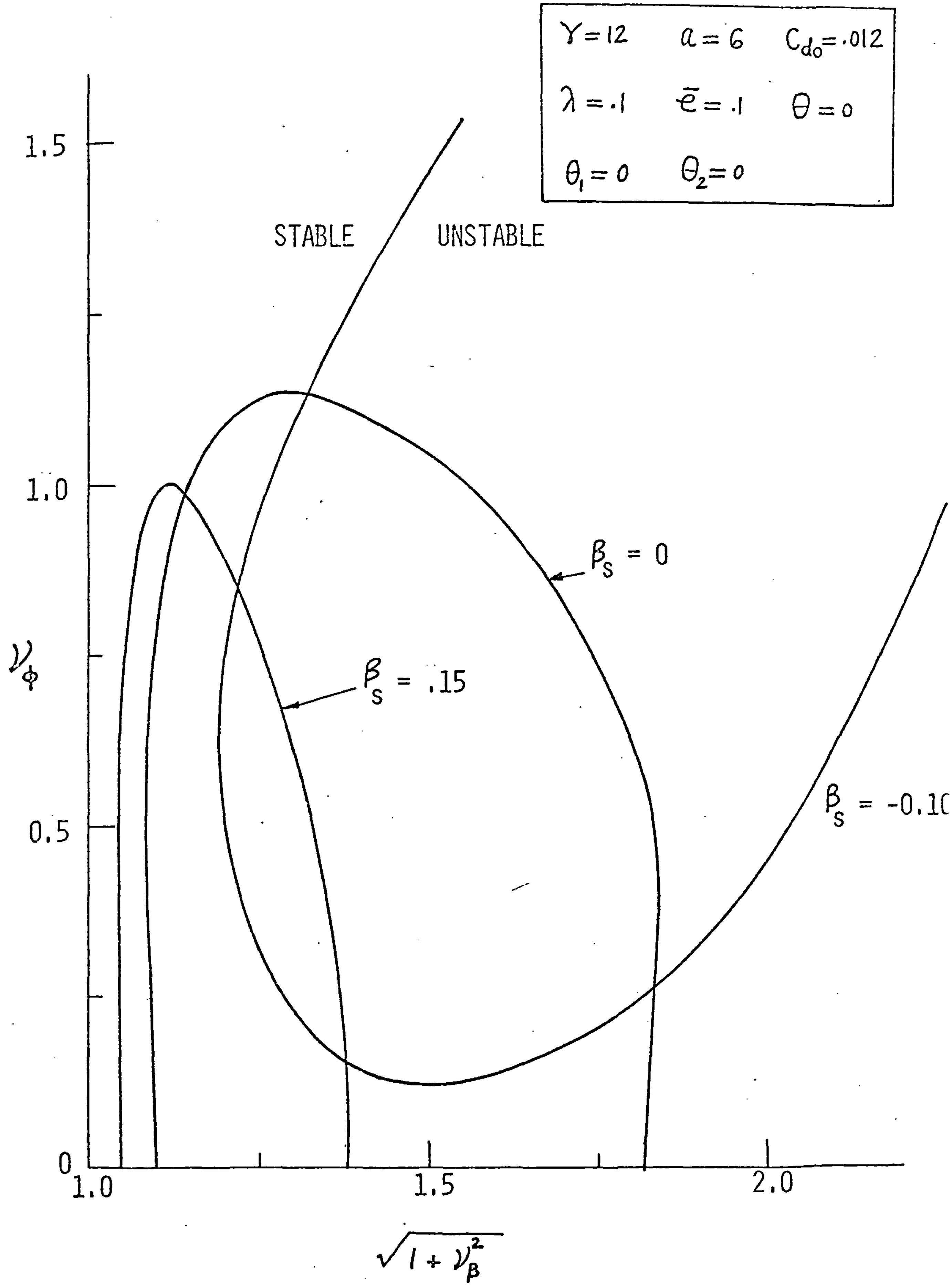

FIG. 17 FLUTTER SOLUTION (EFFECT OF CONING ANGLE)

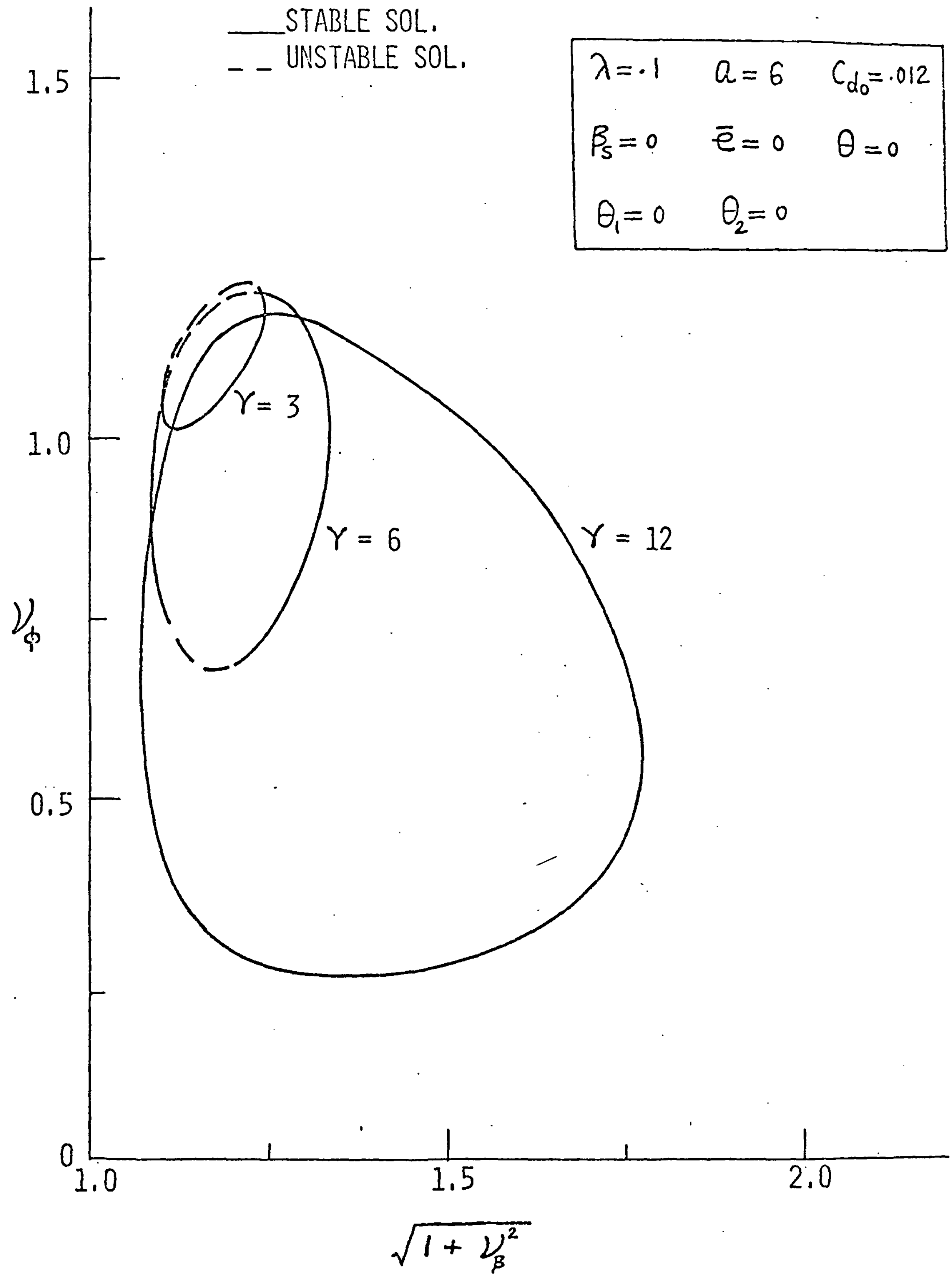

FIG. 18 FLUTTER SOLUTION (EFFECT OF LOCK NUMBER γ)

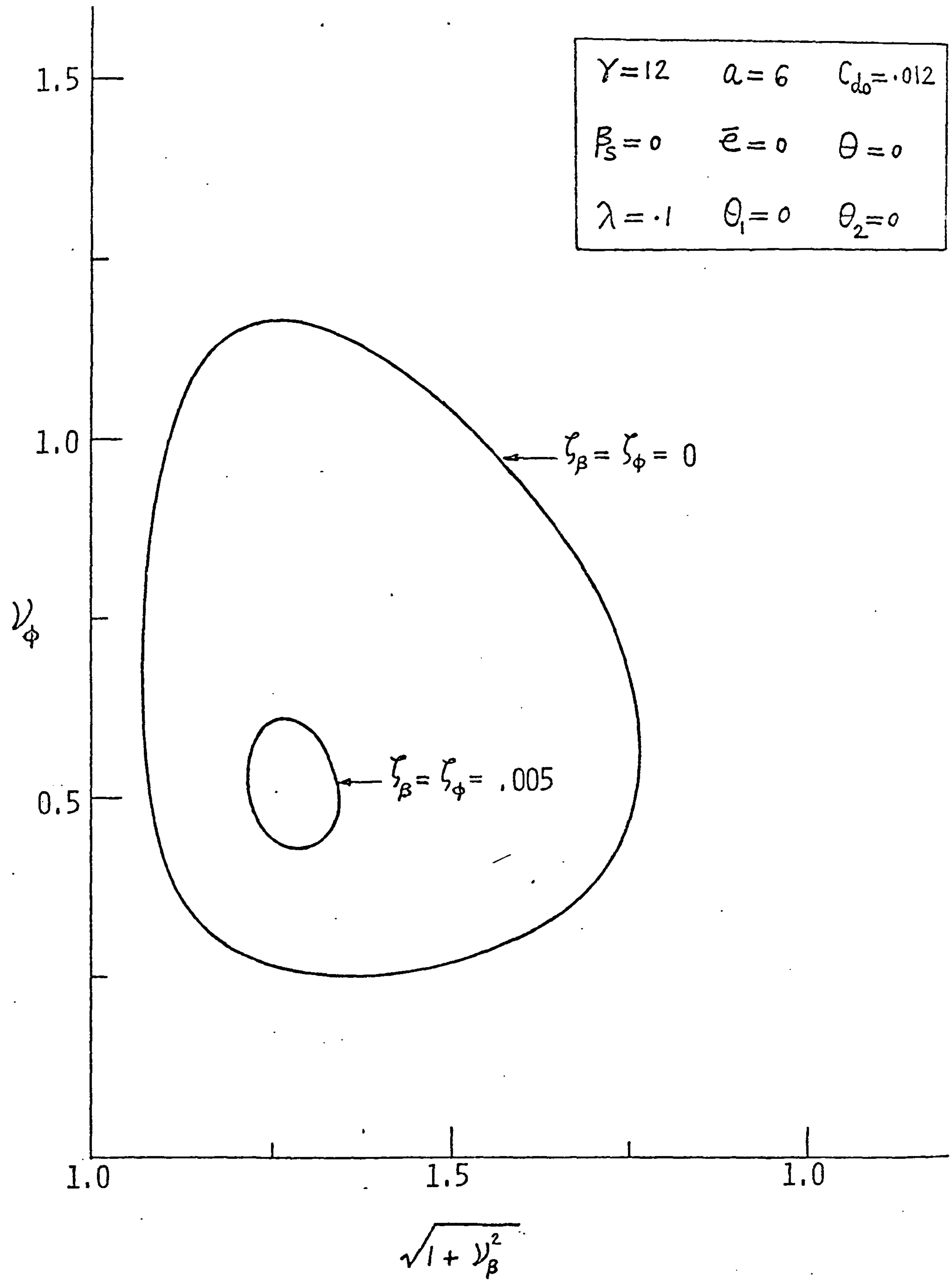

FIG. 19 FLUTTER SOLUTION (EFFECT OF STRUCTURAL DAMPING)

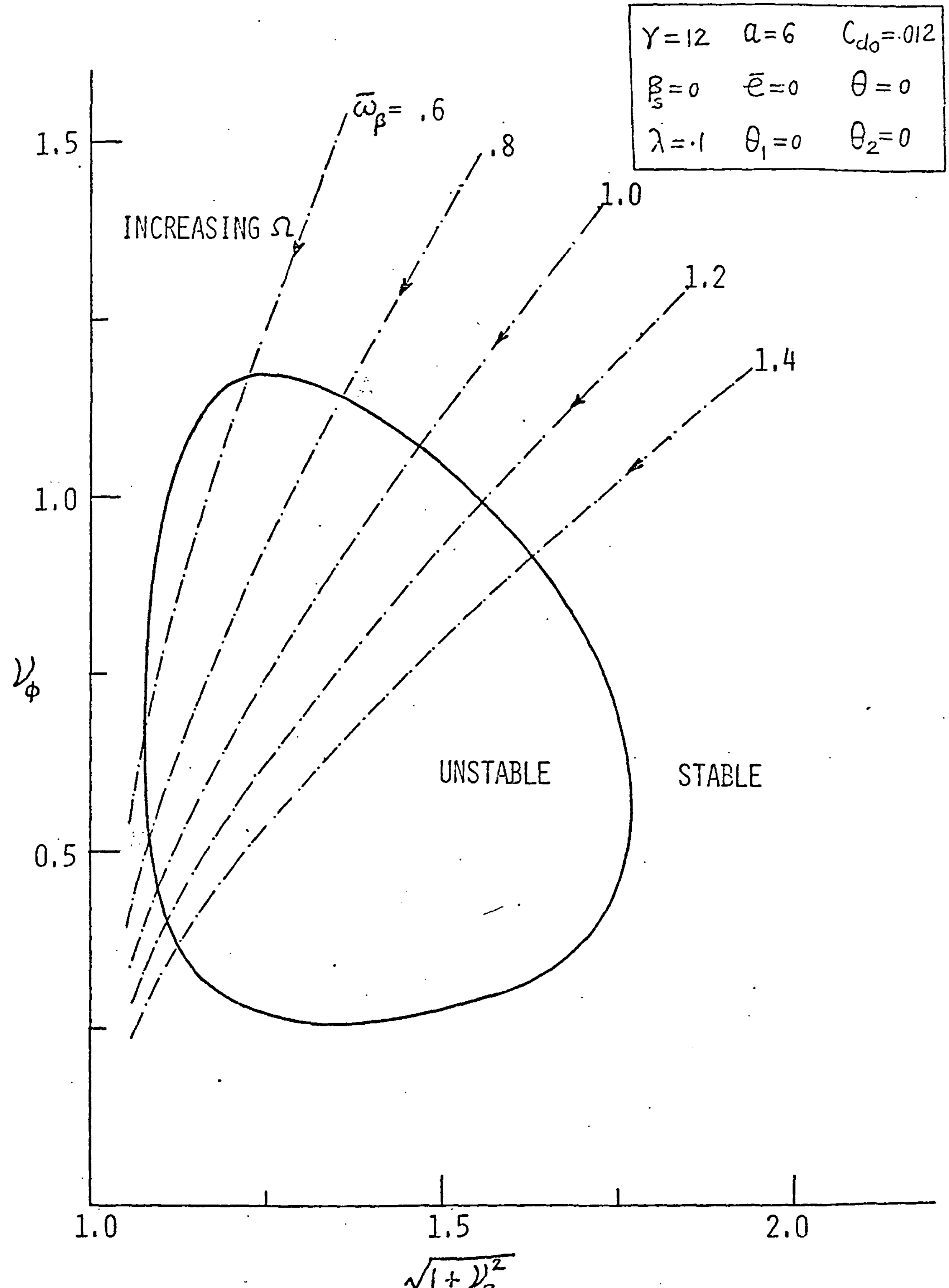

FIG. 20 FLUTTER SOLUTION (PENETRATION OF INSTABILITY BOUNDARIES FOR INCREASING ROTATIONAL SPEED, Ω)

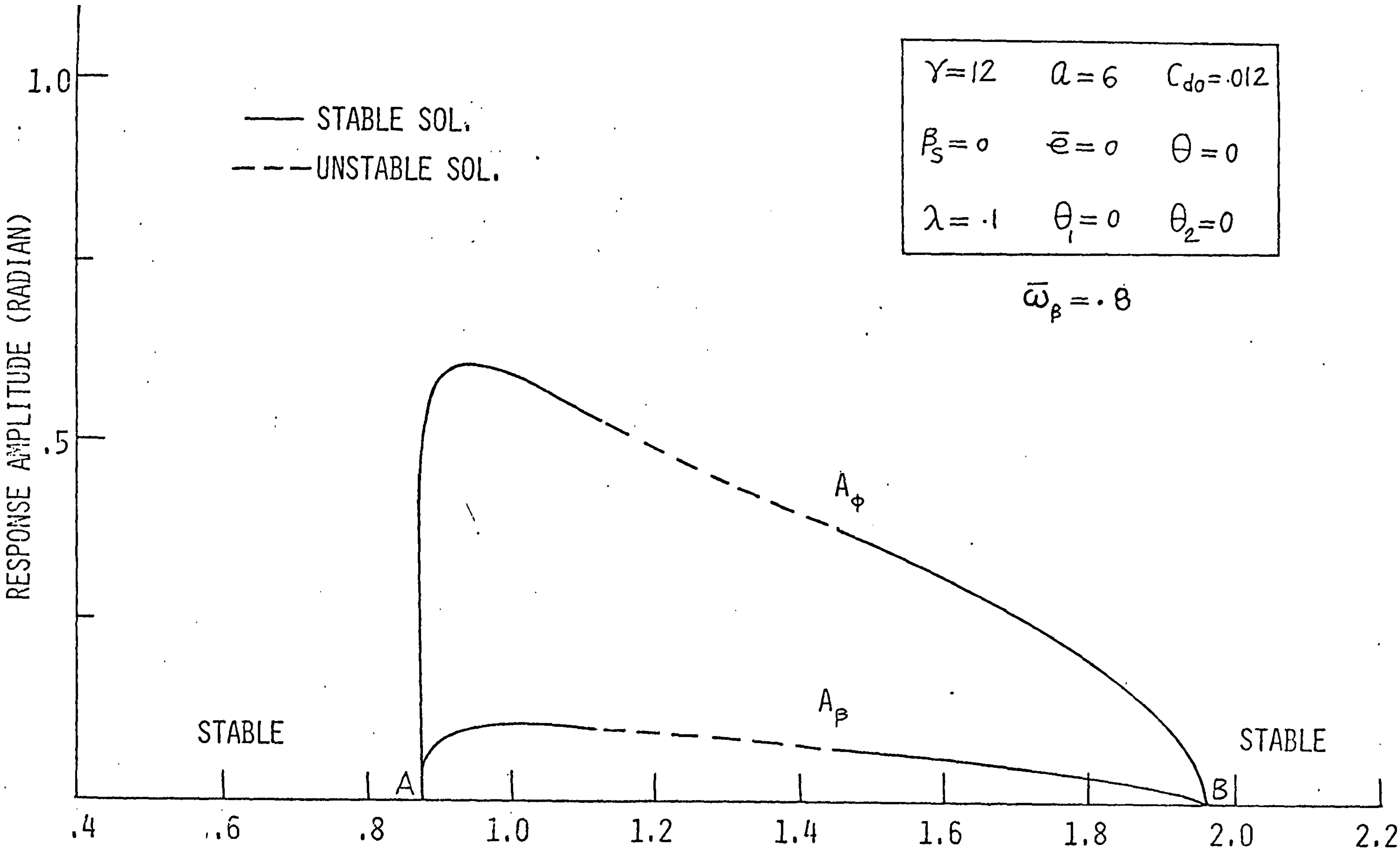

FIG. 21a FLUTTER SOLUTION (NONLINEAR LIMIT CYCLE AMPLITUDES)

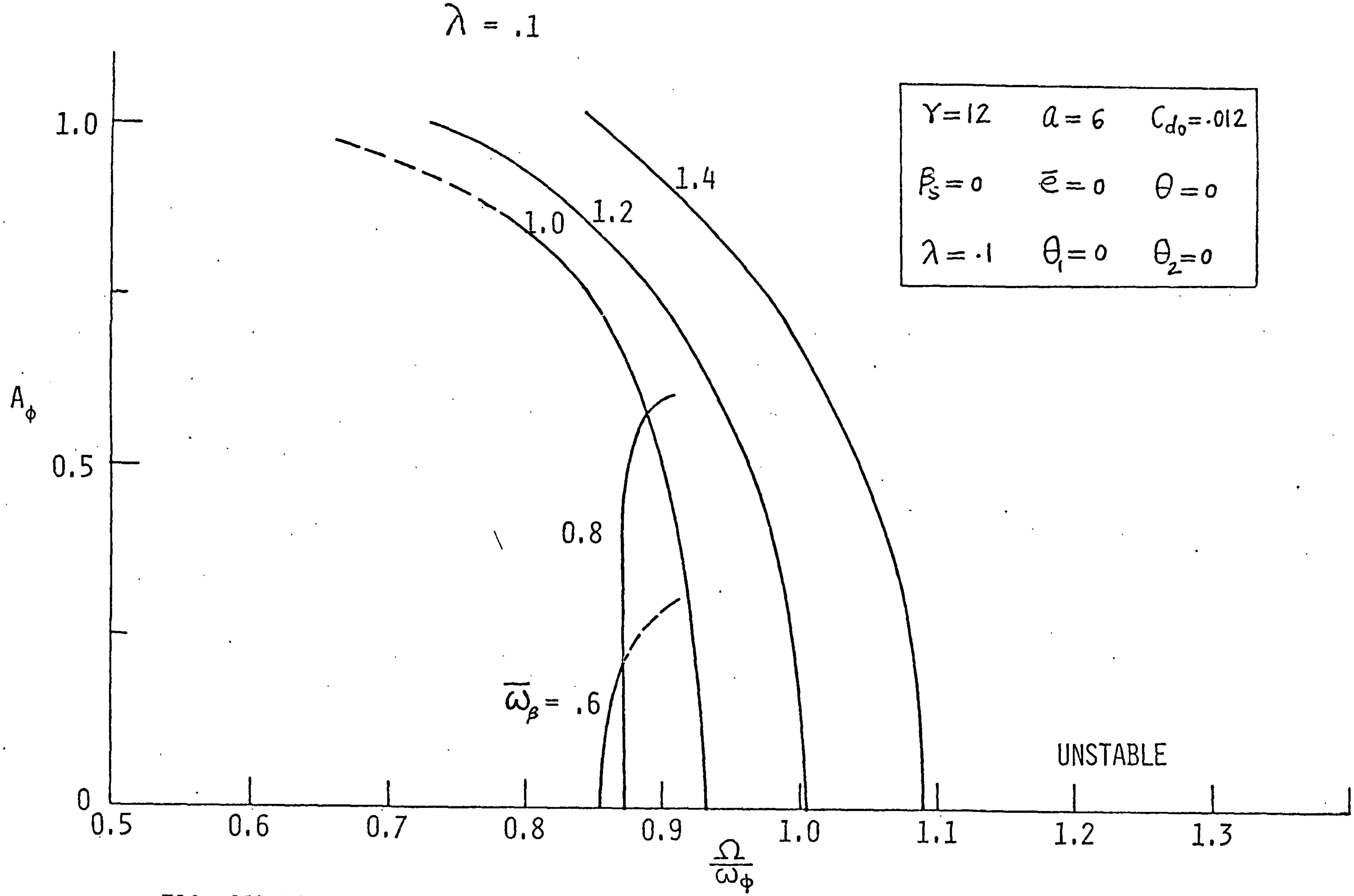

FIG. 21b FLUTTER SOLUTION (NONLINEAR LIMIT CYCLE AMPLITUDES)

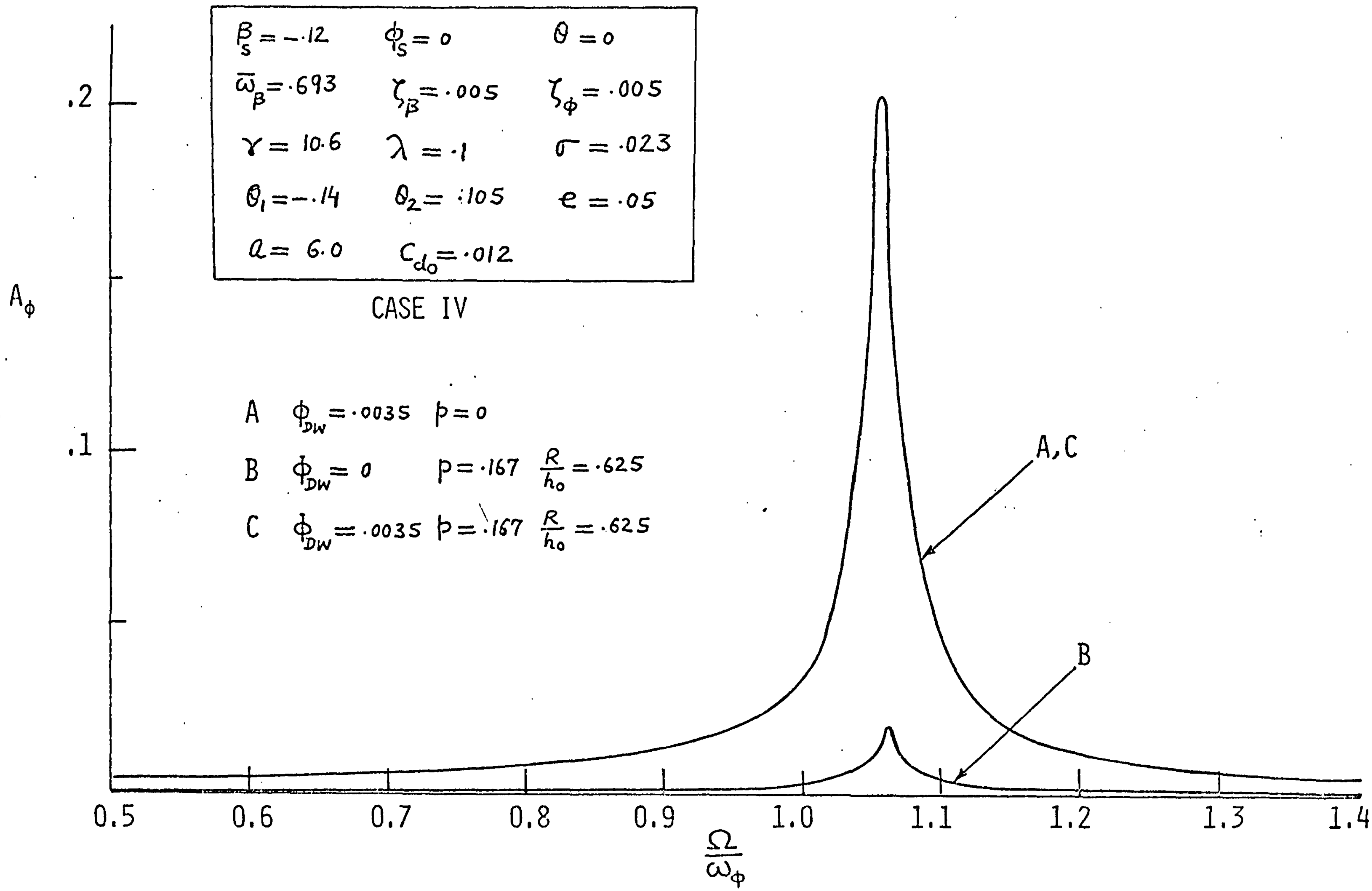

FIG. 22a FORCED OSCILLATIONS, CASE IV

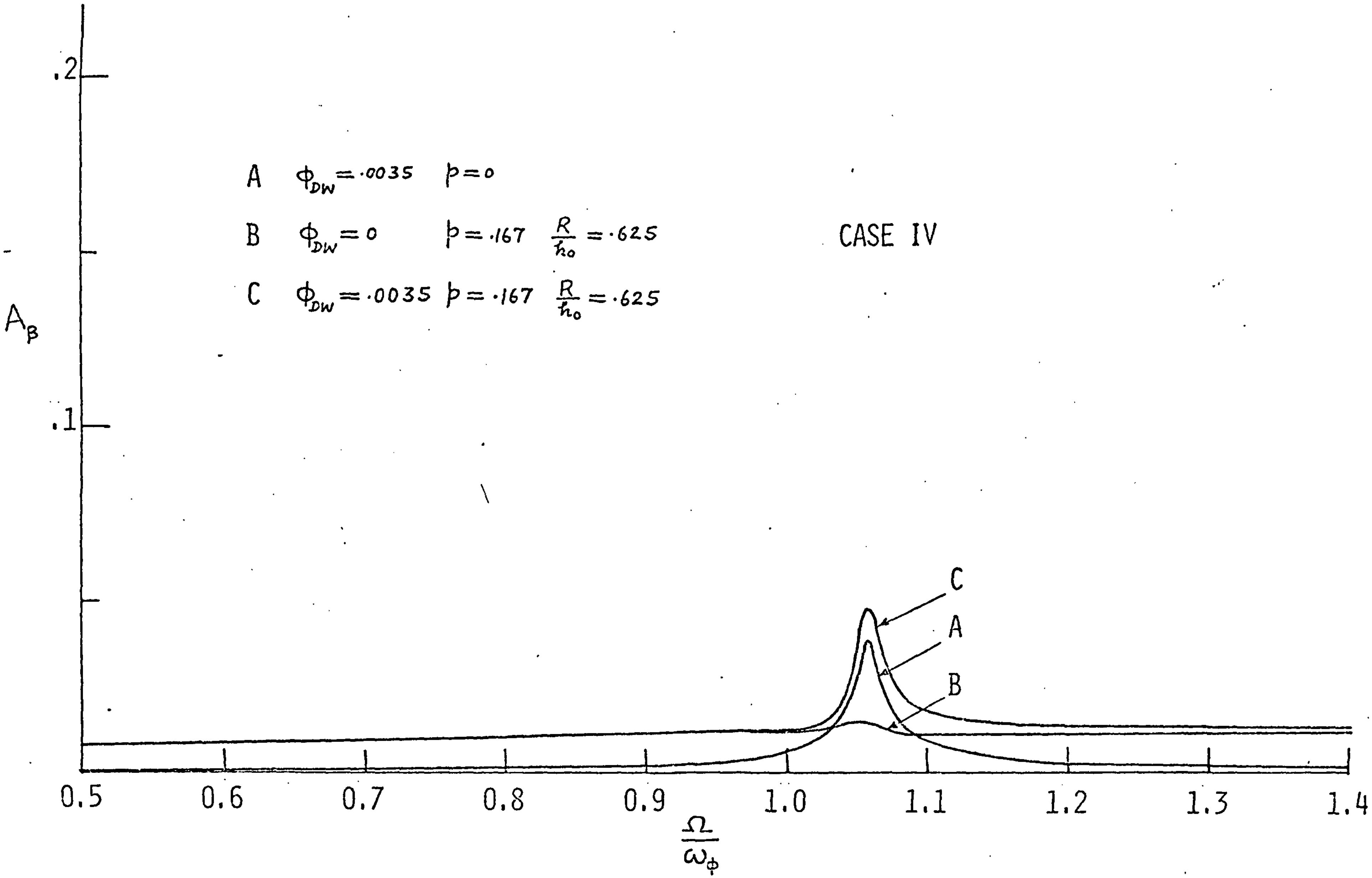

FIG. 22b FORCED OSCILLATIONS, CASE IV

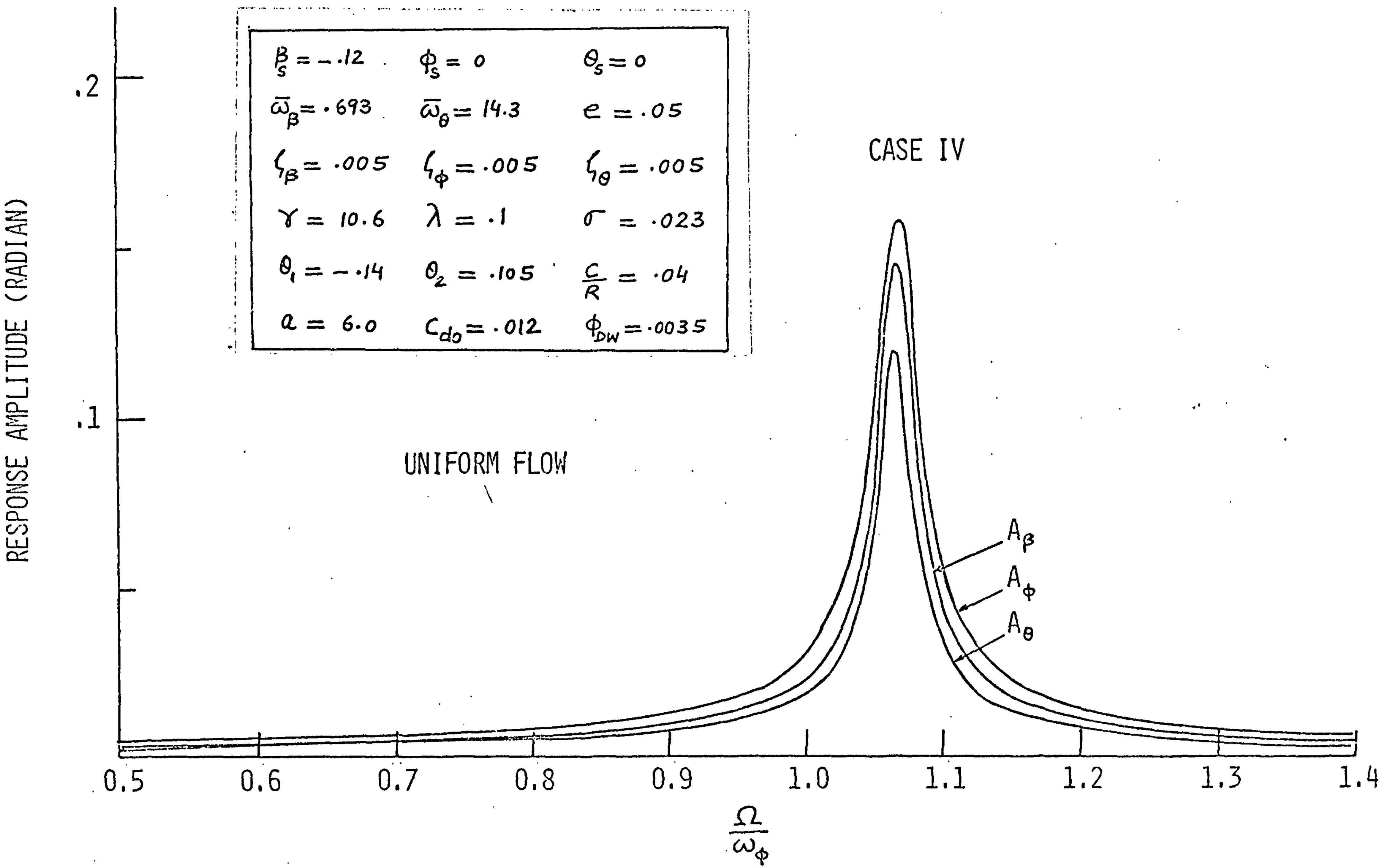

FIG. 23 FORCED OSCILLATIONS OF FLAPPING-LAGGING-FEATHERING ROTOR

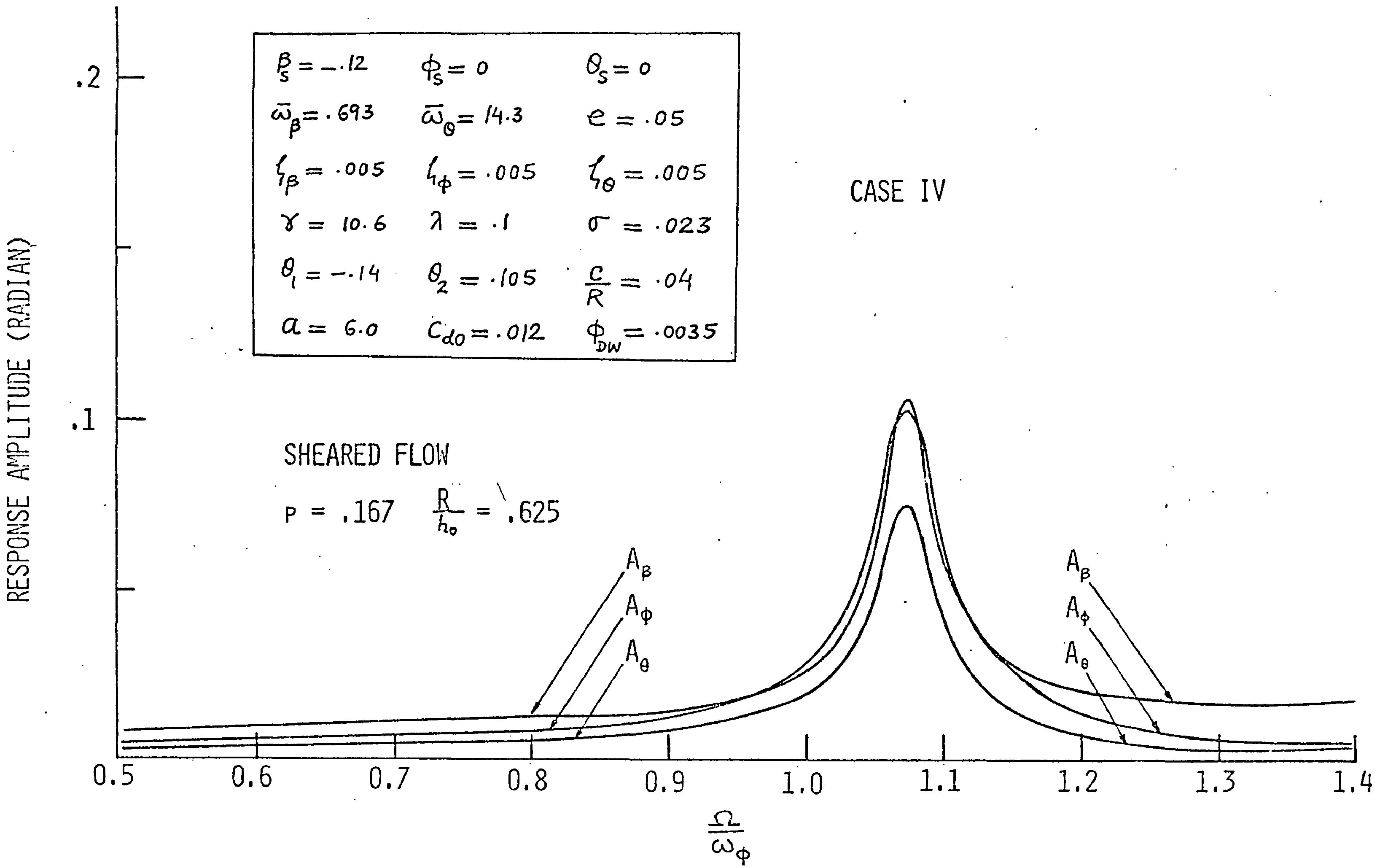

FIG. 24 FORCED OSCILLATIONS OF FLAPPING-LAGGING-FEATHERING ROTOR WITH SHEARED FLOW

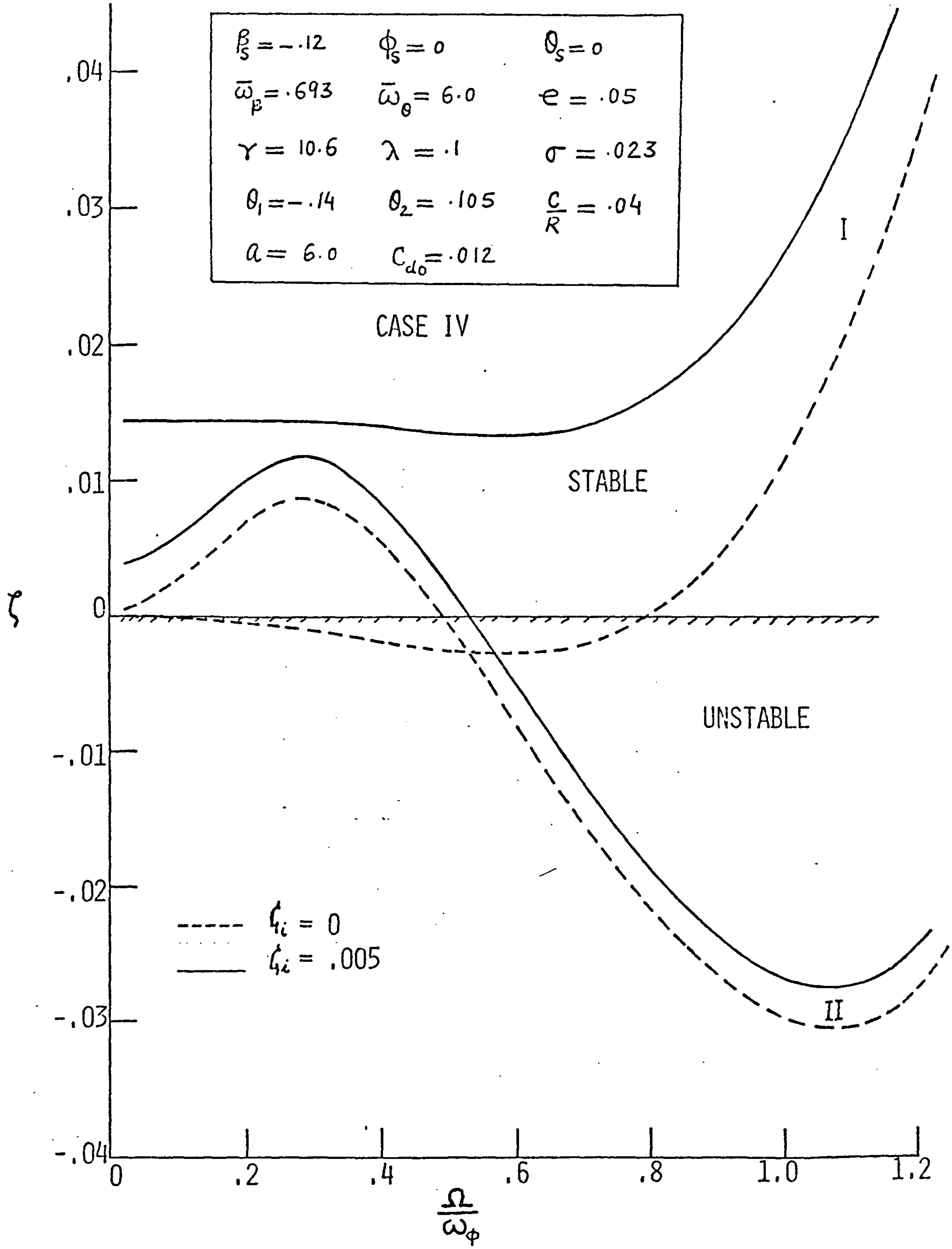

FIG. 25 DAMPING COEFFICIENT FOR FLAPPING-LAGGING-FEATHERING ROTOR ($\bar{\omega}_\theta = 6.0$)

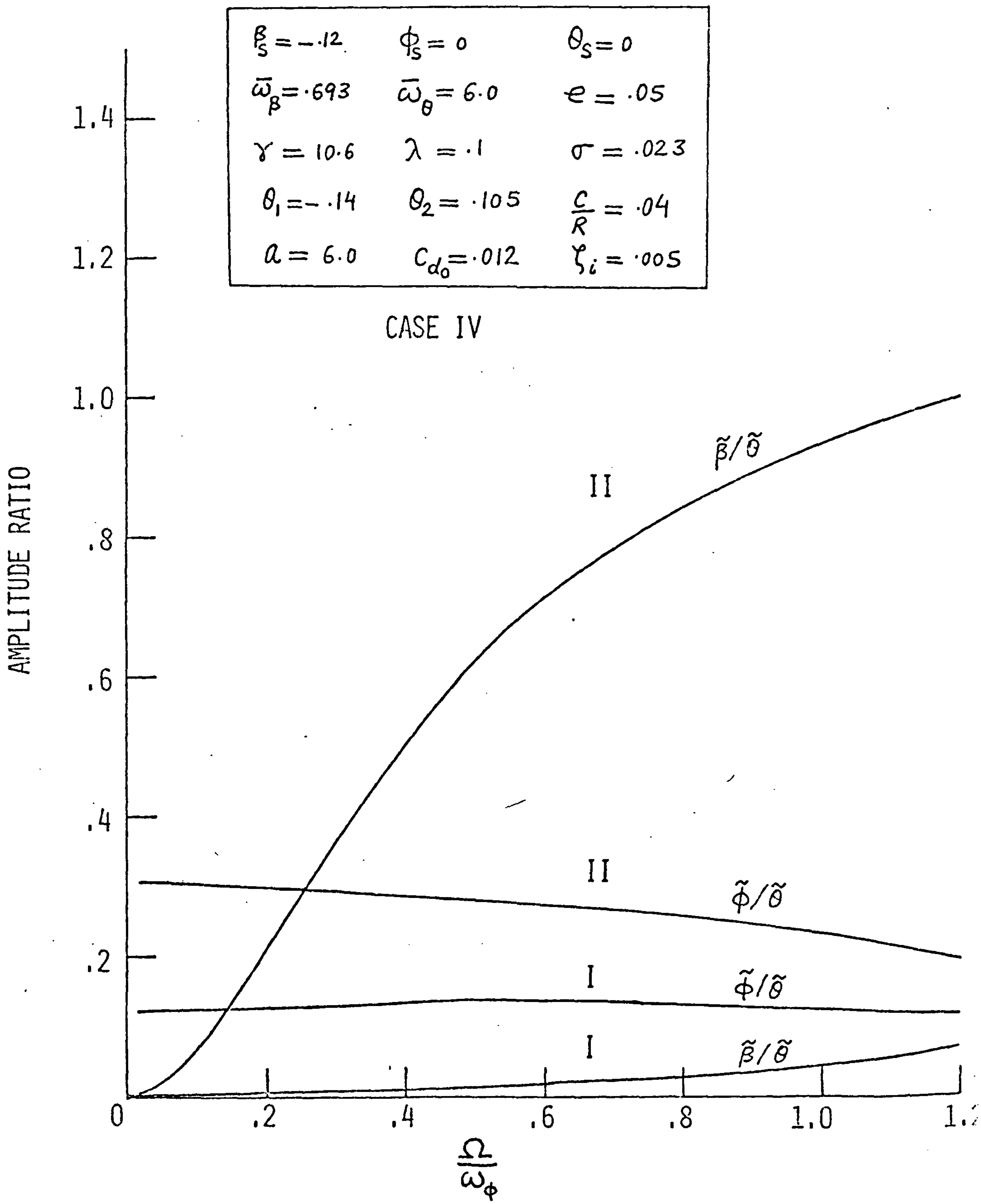

FIG. 26 VIBRATION MODES FOR FLAPPING-LAGGING-FEATHERING ROTOR

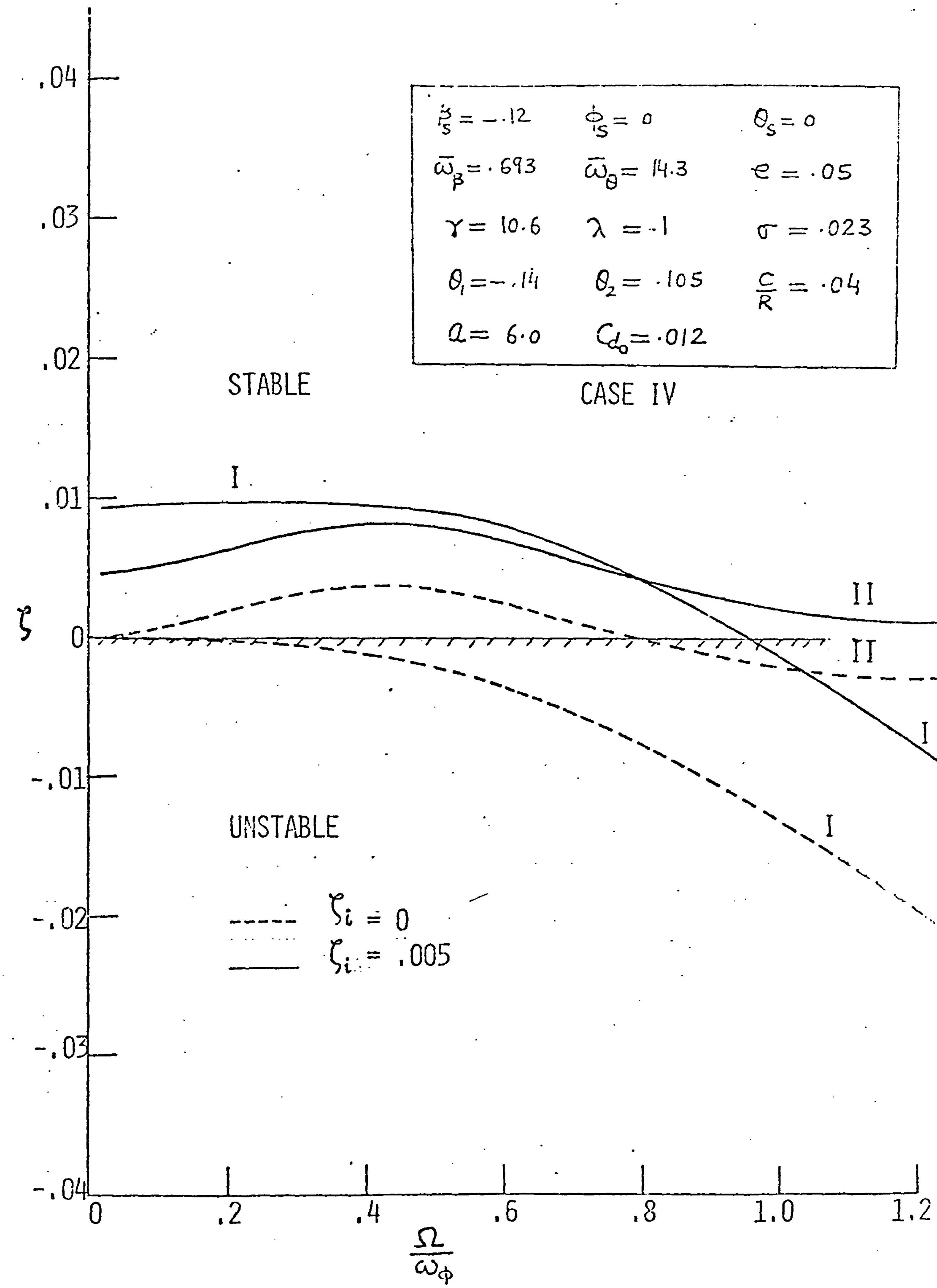

FIG. 27 DAMPING COEFFICIENT FOR FLAPPING-LAGGING-FEATHERING ROTOR ($\overline{\omega}_\theta = 14.3$)

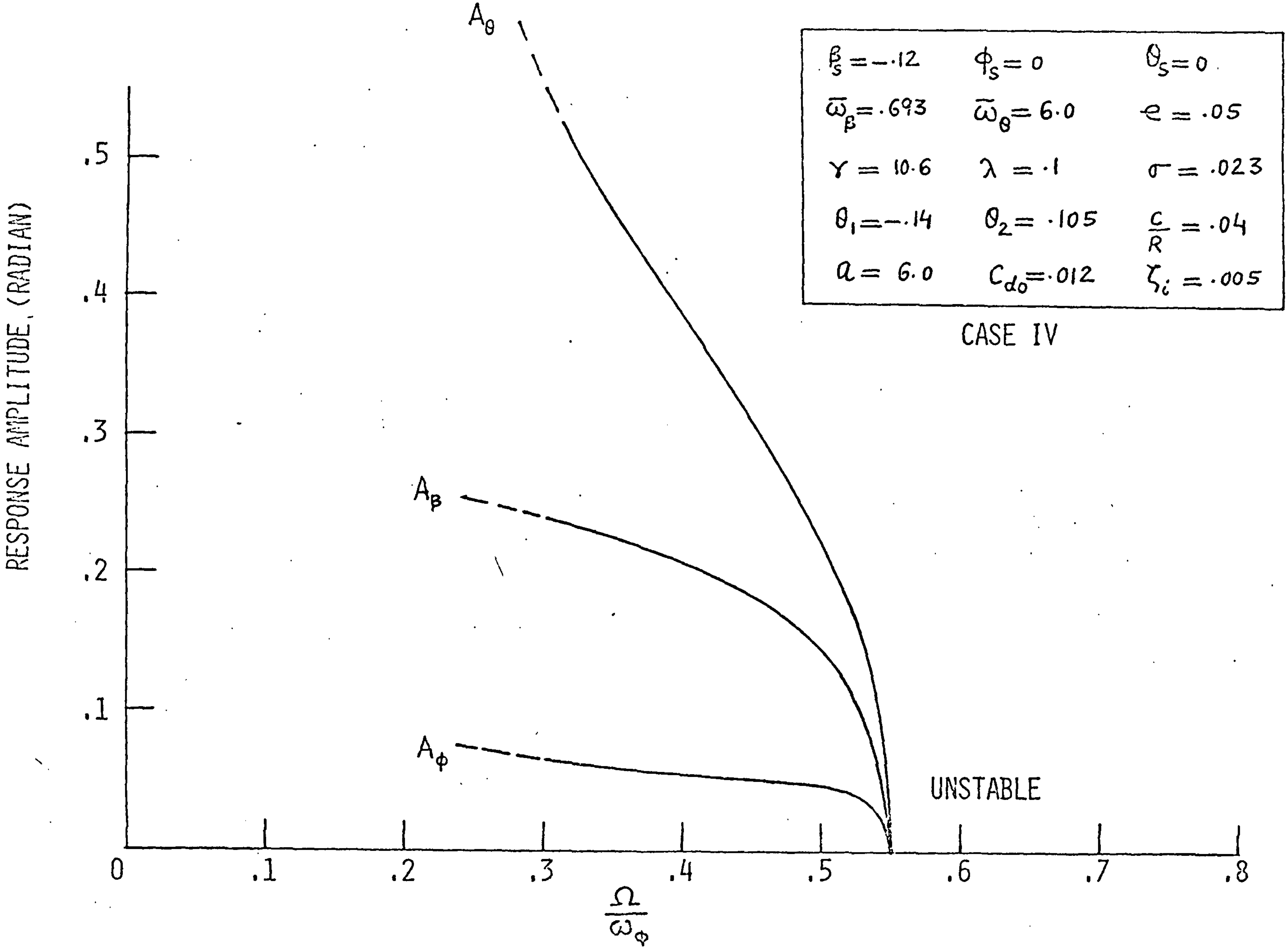

FIG. 28 FLUTTER SOLUTION (NONLINEAR LIMIT CYCLE AMPLITUDES)

BIOGRAPHY

Inderjit Chopra received the degree of Bachelor of Aeronautical Engineering in 1965 from Punjab Engineering College, Chandigarh, and the degree of Master of Engineering with Distinction in Aircraft Structures in 1968 from Indian Institute of Science, Bangalore. From February 1966 until December 1968, he was awarded the Senior Research Fellowship by the Council of Scientific and Industrial Research (India). Since Dec. 1968, he joined as Scientist in Structural Sciences Division of National Aeronautical Laboratory, Bangalore and where he has worked on various problems of aeroelastic analysis of aircraft. In Sept. 1974, he joined MIT as a research assistant in the Aeroelastic and Structures Research Laboratory.

His publications are:

1. Dugundji, J. and Chopra, I., "Further Studies of Stall Flutter and Nonlinear Divergence of Two-Dimensional Wings", NASA CR-144924, August 1975.

2. Chopra, I., "Flutter of a Panel Supported on an Elastic Foundation", AIAA Journal, Vol. 13, No. 5, May 1975, pp. 687-688.

3. Chopra, I., "Vibration of Stepped Thickness Plates", Journal of Mechanical Sciences, Vol. 16, 1974, pp. 337-344.

4. Chopra, I. and Durvasula, S., "Vibration of Simply-Supported Trapezoidal Plates I. Symmetric Trapezoids", Journal of Sound and Vibration, Vol. 19, No. 4, 1971, pp. 379-392.

5. Chopra, I. and Durvasula, S., "Vibration of Simply-Supported Trapezoidal Plates II. Unsymmetric Trapezoids", Journal of Sound and Vibration, Vol. 20, No. 2, 1972, pp. 125-134.

6. Chopra, I. and Durvasula, S., "Natural Frequencies and Modes of Tapered Skew Plates", International Journal of Mechanical Sciences, Vol. 13, 1971, pp. 935-944.

7. Chopra, I., "Flutter Analysis of Two-Dimensional Variable Thickness Plates", Journal of the Aeronautical Society of India, Vol. 24, No. 3, 1972, pp. 368-370.

8. Chopra, I. and Durvasula, S., "Vibration of Tapered Skew Plates", Revue Roumaine des Sciences Techniques - Mecanique Appliquee, Vol. 18, No. 5, 1973, pp. 925-938.

9. Chopra, I., "Perturbation Solution of an Algebraic Eigenvalue Problem of Vibration of Variable Thickness Plates", Journal of the Aeronautical Society of India, Vol. 23, No. 4, 1971, pp. 209-212.