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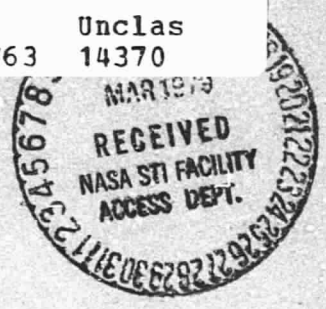

Virginia Polytechnic Institute and State University

Electrical Engineering
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FINAL REPORT

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AIRCRAFT ADAPTIVE LEARNING CONTROL

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1. INTRODUCTION

Optimal control theory of stochastic linear systems has advanced rapidly since the original contributions [1], [2]. For discrete-time linear dynamic systems with additive Gaussian noise, the true optimality is guaranteed by the Separation Theorem [3], [4]. However, nearly all of the results have been obtained under the assumption of the existence of a centralized decision maker which takes all the measurements and generates all control inputs. For systems with multiple decision makers a general computationally feasible theory does not exist at the present time. On the other hand, since most system designs and implementations have been simple and analog in nature, there is a lag in the utilization of digital components and the correspondingly improved performance that could be achieved. This can be partially explained in that in many cases the designers of a control system usually have constraints which make a decentralized computer approach impractical.

The availability of microprocessors, assembled from a small set of LSI logic components, has presented the control system designer with new opportunities for sophisticated control system design. Many designers of control system components have improved their products through the use of microprocessors. This fact is substantiated in reference [5] where a design of a microprocessor-controlled rate gyro is presented. The use of a microprocessor in the rate gyro resulted in an improved performance and a substantial savings in hardware. Incorporation of microcomputers also makes redundant navigation systems available for only a moderate cost and size increase over non-redundant systems. The trend of device manufacturers to use microcomputers as

integral parts of their equipment is expected to continue. The features of a microprocessor--flexibility, modularity, good hardware communications and low cost, have made distributed or decentralized control a suitable answer to many control problems. A distributed-control system offers several potential advantages over a centralized control system, but in the past the high cost of processors has discouraged its use. The advances in microprocessor technology have shifted and are continuing to shift the cost balance in favor of distributed control systems. The following are some of the potential advantages of a distributed-control system.

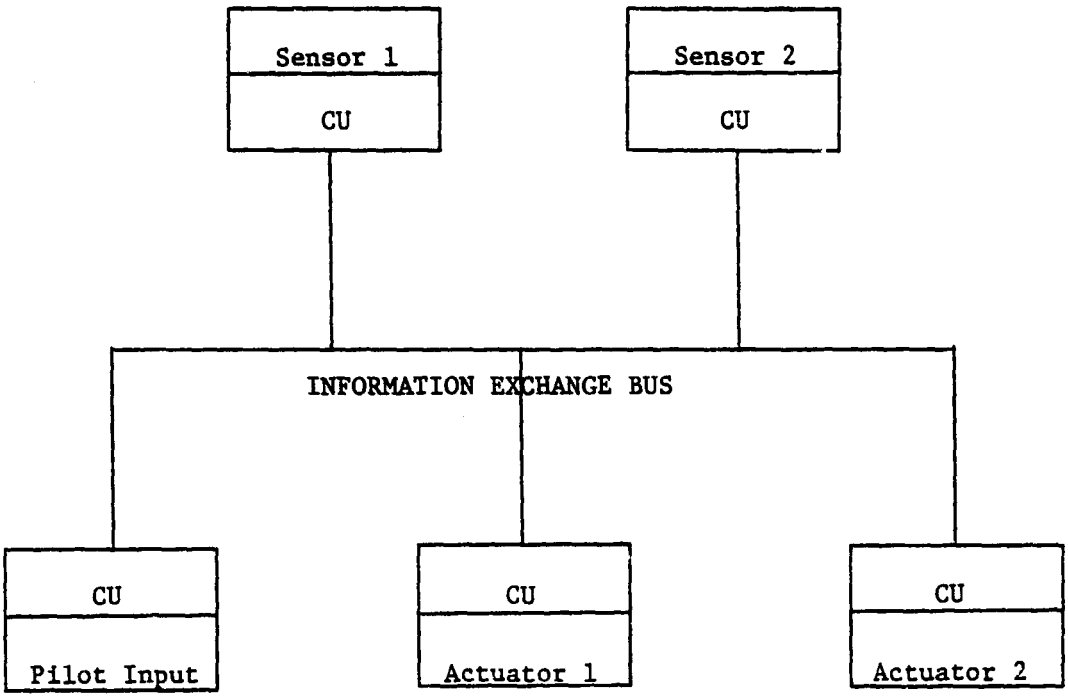
1. The interface between the computer and the process, i.e., the wiring of the process input and output signals to a centralized computer has constituted a considerable portion of the total cost for a centralized control system. In some cases the wiring is susceptible to electromagnetic interferences. Communication difficulties may arise because of non-instantaneous transmission of data between physical locations; by the time the control computer receives the data it may be too late for effective control. Therefore, it is desirable to bring the computers as close as possible to the process.

2. The computation time required to process the data in real time at a centralized computer may be great despite the rapid advance in mainframe computer technology. Intolerable time delays may occur between a portion of the process needing attention and another part which has information for the required action. Parallel computation, with microcomputers each dedicated to a portion of the process, can help to solve this type of problem.

3. Computer down-time has been a nightmare in real-time applications. In most cases it has been so intolerable that a back-up unit is required to ensure reliable operation. But idle facilities can be very hard to justify economically. The structure of a distributed control system is more reliable in the sense that failure in one computer does not usually have catastrophic consequences. For the portion of the process which is critical to the whole operation, a redundant microcomputer can be installed for reliable control at a minimal cost increase.

4. The initial capital investment for centralized control may not be economically feasible because of the high cost of mainframe computers. A distributed control system installed step by step, can be more attractive to the management when there is a constraint on the capital expenditure.

Information exchange is a critical issue in distributed-control systems. In a distributed-control system, it must be assumed that no controller, local or central, possesses a complete description of the system. Figure (1.1) shows an example of a distributed flight control system with minimal module requirements from the flight control point of view. Note that a microcomputer controller is associated with each physical device in the system and that each controller can communicate with other physically distributed elements using an information exchange bus. The controllers, therefore, must exchange information among themselves in order to achieve satisfactory operation. The type of information available to a controller for its decision-making is called its information pattern [6], [7]. Hence, for a distributed processing system we have an information pattern considerably different from that of a



CU = Control Unit

Figure 1.1 A Distributed Flight Control System

centralized control system in which the central controller is assumed to have *complete and instantaneous* information of the system. The following are some practical aspects we must consider in a distributed microprocessor-based control system.

1. Because of the high cost of communication, various techniques for data reduction are needed. For example, in a distributed aircraft control system, an adaptive sampling technique may be employed to reduce the data traffic between sensors and actuators [8],[9],[10]. In an adaptive sampling system, the trade-offs between performance and communication loads must be considered. The current estimate of the state is transmitted to the actuator only as needed to maintain adequate performance of the system. Thus, there will be variable time intervals between information updates to the actuators.

2. As more transmission lines are added to a distributed-control system, it becomes economical to multiplex data on the lines. In some cases, a reduction in weight and size is also a significant improvement. This is especially true in aerospace applications. There are two common methods to merge data over a single transmission line--Frequency Division Multiplexing (FDM) and Time Division Multiplexing (TDM). In FDM, devices share a common transmission line by dividing its frequency spectrum into several subchannels. In TDM, devices queue on one end of the transmission line so that exactly one device is allowed to transmit on it at a time. In the latter case, the data arrival time may be uncertain.

3. For economic as well as system stability reasons, the sampling rate should be selected according to the bandwidth of the loop being

controlled. If loops can be controlled separately, they can usually be sampled at different rates. The necessity for a multivariate system becomes obvious if the bandwidths are more than an order of magnitude apart. The advantages of multirate systems are so great that they have been employed in industry even though there are no complete analytic techniques to design them at the present time [11],[12]. One example is in the Digikon-II program of the space Shuttle flight control system, where multirate results are developed for 2:1 and 4:1 sampling ratios [13].

The above discussion indicates that a distributed control system may have a stochastic information pattern where the data arrives at random times. It is surprising that modern control theory has only a limited amount of available work on this practical aspect of control problems, whereas single rate control and filter problems are well developed [2],[4]. However, it is encouraging that recently, there has been an increasing interest in multirate systems. Primarily, though, it has been assumed that the information pattern is deterministic in both multirate and single rate cases; and control systems with stochastic information patterns have not received enough attention.

In this report, a stochastic model of a sampling process is postulated which accounts for

(1) Variable time intervals in adaptive-sampling systems

(2) Uncertainty of data arrival time in multiplexed communication systems

(3) Multirate-sampling systems

Stochastic modeling of a sampling process was first suggested by A. K. Caglayan and H. F. VanLandingham [14]. In reference [14] an

optimal control policy was obtained for the system with a discrete Markovian sampling process. However, the optimal solution was restricted to a process in which the time interval between each information update was known to the system at the initiation of the interval. In this report an optimal policy is obtained for virtually all Markov type sampling processes. The restriction was relaxed to include the situation where the control unit does not have to know the duration of the information update interval at its initiation, but instead, only has knowledge of the past history of the information update intervals. This assumption is more applicable to a distributed-control system because its control units ordinarily will not know how long they will have to act with the same information. The system with additive plant and measurement noise is, of course, also considered as being a practical constraint of control systems. A separation theory of estimation and control for the randomly-sampled system is presented. In particular, in a randomly-sampled system the stochastic optimal control can be synthesized by cascading an optimal estimator with the optimal control law. This is significant because it considerably simplifies the implementation of the stochastic optimal control of randomly sampled systems. The theory is applied to the control of the longitudinal motion of the F8-DFBW (F-8 Digital Fly-by-Wire) aircraft. Both theoretical and simulation results indicate that, for the application example, the optimal cost obtained using a variable time-increment Markov information update process (where the controllers know only the past information update intervals and the Markov transition mechanism) is almost identical to the cost obtained using a known uniform update interval.

2. OPTIMAL CONTROL OF RANDOMLY-SAMPLED SYSTEMS

2.1 Introduction

The low cost of microprocessors together with increasingly complex control systems have necessitated a re-examination of the information structure in conventional Linear-Quadratic-Gaussian (LQG) controller design. The conventional LQG control methods avoid consideration of any variation in information structure; it always assumes that the system is uniformly sampled, i.e., the information is transferred at fixed times with uniform intervals. In a computer-based control system, especially those with multiple processors, it is not always possible or practical to transfer information uniformly. A more general formulation is needed. A linear sampled-data control system can be classified by its input characteristics as having either random or deterministic inputs; by the measurement model as being noisy, when it is impossible to determine the state of the system exactly, or noise-free, when the state of the system can be known exactly; by the information transfer mechanism as being randomly-sampled, when the information is transferred at random times, or uniformly sampled when the information is transferred at fixed times with uniform intervals. In this chapter it is assumed that the input is deterministic and the measurement process is noise-free, but that the information is transferred at random times. A more general model with additive plant and measurement noise will be discussed in Chapter 4.

A mathematical model describing the information transfer mechanism is first postulated in Section 2.2. Using dynamic programming, a control

law is then derived in Section 2.3 which optimizes the average of an integral of a quadratic form in the state and control variables. In section 2.4 we examine the asymptotic behavior of the optimal control for randomly-sampled systems. Conditions are given for the stability of the closed-loop system using quasi-steady state feedback.

2.2 Stochastic Information Distribution Model

The purpose of this section is to present a model which is useful in describing the stochastic information transfer process in a multiple-processor control system. The next section will present the derivation of the optimal control law that minimizes the average of an integral in the state and control variables.

Consider the continuous dynamical system represented by the linear differential equation:

$$\dot{x} = Ax + Bu \tag{2.2.1}$$

where x is an n-dimensional vector, representing the system states, u is an m-dimensional vector representing the control inputs, A is an n x n matrix describing the dynamics of the system and B is an n x m matrix describing the control effectiveness. It is assumed that the state information is transferred at discrete points in time, σ_k, $k = 0,1,\cdots,N$ where $\sigma_0 < \sigma_1 < \cdots \sigma_1 < \sigma_{k+1} \cdots < \sigma_N$.

The sequence $\{\sigma_k\}$ is defined to be the sum of a stochastic process $\{t_k\}$ in the following sense:

$$\sigma_k = \sigma_0 + \sum_{i=1}^{k} t_i \tag{2.2.2}$$

where t_k is an element of the set $\tau(i)$. Hence, σ_0 denotes the starting time, t_i denotes the $i^{\underline{th}}$ time interval between two consecutive information updates and σ_k is the time that the $k^{\underline{th}}$ information update takes place. Since the time interval between two consecutive information updates cannot be negative, the sample space of the stochastic process $\tau(i)$ for $i = 1, \cdots, N$, is restricted to sets of positive real numbers. There is, however, no restriction on the statistical nature of the process. The set of information update sequences which are admissible for a k-stage process is the Cartesian product of $\tau(i)$, i.e.,

$$T(k) \in T(k) = \tau(1) \times \tau(2) \cdots \times \tau(k) \tag{2.2.3}$$

$$t_k \in \tau(k) \tag{2.2.4}$$

$$T(k) = \{t_1, \cdots, t_k\} \tag{2.2.5}$$

Figure 2.2.1 illustrates the relation between t_k and σ_k. The average of a quadratic integral in the state and control is defined to be the cost functional for the N-interval process to achieve the desired system performance. We assume a cost functional of the form:

$$J = \frac{1}{2} E_{T(N)} \int_{\sigma_0}^{\sigma_N} [x^T(t)Q\, x(t) + u^T(t)R\, u(t)]dt + x^T(\sigma_N)Sx(\sigma_N) \tag{2.2.6}$$

The cost functional can be expressed as the sum of N integrals by dividing the total time into N intervals, viz.

$$J = \frac{1}{2} E_{T(N)} \left\{ \sum_{k=0}^{N-1} \int_{\sigma_k}^{\sigma_{k+1}} [x^T(t)Q\, x(t) + u^T(t)R\, u(t)]dt + x^T(\sigma_N)Sx(\sigma_N) \right. \tag{2.2.7}$$

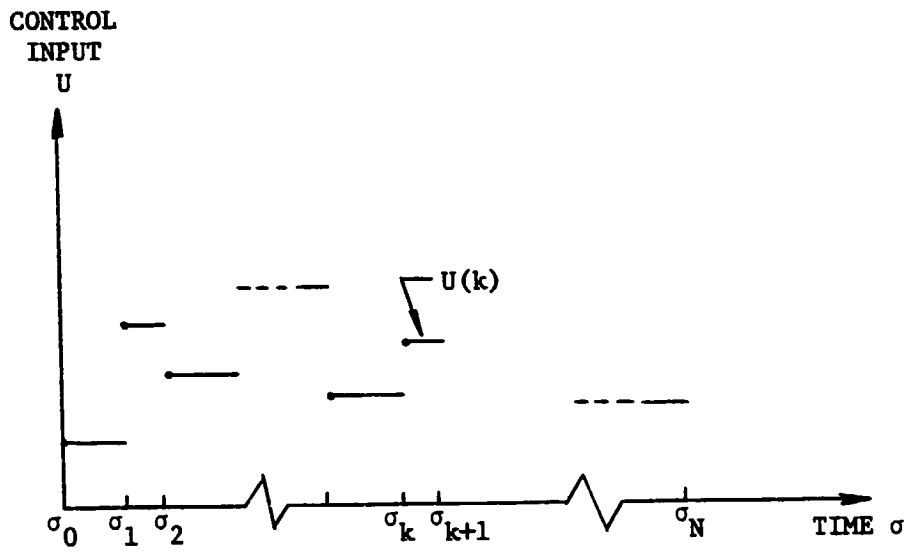


Figure 2.2.1 Stochastic Information Distribution Model

It is interesting to note that the random variable appears in the limits of the integral only. Since the equivalent discrete-time system is obtained by integrating the system differential equation and cost functional over each sampling period, the random variable will appear everywhere in the equivalent discrete-time system! If control inputs are further restricted to piecewise constant functions of time that change only at sampling instants σ_k, i.e.

$$u(t) = u(\sigma_k) \qquad \sigma_k \leq t < \sigma_{k+1} , \tag{2.2.8}$$

it is well known that

$$x(t) = \Phi(t, \sigma_k)\, x(\sigma_k) + \int_{\sigma_k}^{t} \Phi(t,s)\, B(s)ds\, u(\sigma_k) \tag{2.2.9}$$

Therefore, the equivalent discrete-time system is given by

$$x(t_{k+1}) = \Phi(t_{k+1}, t_k)x(t_k) + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)B(s)ds\, u(t_k) \tag{2.2.10}$$

To simplify the notations, the following matrices are defined

$$\Phi(k+1, k) = \Phi(t_{k+1}, t_k) \tag{2.2.11}$$

$$\Gamma(k+1, k) = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)B(s)ds \tag{2.2.12}$$

Using equations (2.2.10) and (2.2.11), the equations describing the process and cost function J can be written in difference equation form as

$$x(k+1) = \Phi(k+1, k)\, x(k) + \Gamma(k+1, k)\, v(k) \tag{2.2.13}$$

$$J = \frac{1}{2} E_{\tau(N)}[x(N)^T Sx(N) + \sum_{k=0}^{N-1} \{x(k)^T Q(k)x(k) + v(k)^T R(k)v(k)\}] \tag{2.2.14}$$

where

$$\Phi(k+1,\ k) = e^{At_{k+1}} - R^{-1}(k)\ W(k)^T \tag{2.2.15}$$

$$\Gamma(k+1\ ,k) = \Gamma(t_{k+1})$$

$$\Gamma(t) = \int_0^t e^{As}B\,ds \tag{2.2.16}$$

$$Q(k) = \int_0^{t_{k+1}} (e^{Ar})^T Q\ e^{Ar}\,dr - W(k)R^{-1}W(k) \tag{2.2.17}$$

$$R(k) = \int_0^{t_{k+1}} [R + \Gamma^T(r,0)Q\Gamma(r,0)]\,dr \tag{2.2.18}$$

and

$$W(k) = \int_0^{t_{k+1}} (e^{Ar})^T Q\ (e^{Ar})\,dr \tag{2.2.19}$$

The above representation was obtained using the techniques of Reference [15] with the transformation

$$v(k) = u(k) + R(k)^{-1}W(k)^T x(k) \tag{2.2.20}$$

employed to eliminate the cross product terms $x^T(k)W(k)u(k)$ that would otherwise appear in the cost function representation.

2.3 The Optimal Solution for Randomly-Sampled Systems

We begin with a formal statement of the optimization problem for randomly-sampled systems:

For the stochastic difference equation describing the system

$$x(k+1) = \Phi(k+1, k)x(k) + \Gamma(k+1, k)v(k) \tag{2.3.1}$$

where $\Phi(k+1, k)$ and $\Gamma(k+1, k)$ are functions of t_k, t_{k+1} and the cost function J given by

$$J = \frac{1}{2} E\{x(N)^T Sx(N) + \sum_{k=0}^{N-1} [x(k)^T Q(k)x(k) + v(k)^T R(k)v(k)]\} \tag{2.3.2}$$

we will select the sequence v(k) to minimize J given the distribution of the stochastic process $\{t_k\}$.

The Principle of Optimality [16],[17] has proven to be a powerful tool for obtaining the solution of optimal control problems. This principle states that whatever the initial state and initial decision may be, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision. Using the principle of optimality, we reduce the N-stage to N one-stage problems. This result is desirable because decisions can then be made sequentially. The optimization is carried out using dynamic programming in a manner similar to that of Reference [18].

First we consider a single stage optimization problem over the last sample interval which starts at σ_{N-1} and terminates at σ_N. Subsequently, we will generalize the result to all N stages by induction.

Single Stage Process

Consider the last stage of the process which starts at σ_{N-1} and terminates at σ_N. The optimal control over the last stage produces a cost

$$J(N-1,N) = \frac{1}{2}\min_{v(N-1)} E\{x^T(n-1)Q(N-1)x(N-1) + v(N-1)^T R(N-1)v(N-1) + x(N)^T P(0)x(N)\} \quad (2.3.3)$$

where $P(0) \triangleq S$.

But $x(N)$ is related to $v(N-1)$ by the state equation.

$$x(N) = \Phi(N, N-1)\, x(N-1) + \Gamma(N, N-1)\, v(N-1)$$

Substituting this equation into the cost function, we have

$$J(N-1, N) = \frac{1}{2}\min_{v(N-1)} E\{x(N-1)^T Q(N-1)x(N-1) + v^T(N-1)R(N-1)v(N-1) + [\Phi(N,N-1) x(N-1) + \Gamma(N, N-1)v(N-1)]^T P(0)[\Phi(N, N-1)x(N-1) + \Gamma(N, N-1)v(N-1)]\} \quad (2.3.4)$$

The control $v(N-1)$ is required to be physically realizable and therefore should be independent of the stochastic process t_N. Using one of the properties of conditional expectation; namely, that $E(x) = E[E(x/y)]$ where the outer expected value on the right-hand side is over y, we have

$$J(N-1, N) = \frac{1}{2}\min_{v(N-1)} E\{E_{t_N/T_{N-1}}\{x(N-1)^T \cdot Q(N-1)x(N-1) + v^T(N-1)R(N-1)v(N-1) + [\Phi(N, N-1)x(N-1) + \Gamma(N, N-1)v(N-1)]^T P(0)[\Phi(N, N-1)x(N-1) + \Gamma(N, N-1)v(N-1)]\}\} \quad (2.3.5)$$

The performance measure can be minimized by minimizing only the inner

expected value in equation (2.3.5) with respect to v(N-1). The minimum is obtained by setting $\frac{\partial J(N-1, N)}{\partial v(N-1)}$ to zero. Thus,

$$E_{\tau_N \, T_{N-1}} \{R(N-1)v(N-1) + \Gamma(N, N-1)^T P(0)(\Phi(N, N-1)x(N-1) + \Gamma(N, N-1)v(N-1)\} = 0 \tag{2.3.6}$$

Since the stochastic process (t_N) is independent of x(N-1), we have

$$v(N-1) = -E_{\tau_N/T_{N-1}}[R(N-1) + \Gamma^T(N, N-1)P(0)\Gamma(N, N-1)]^{-1} \cdot E_{\tau_N/T_{N-1}}[\Gamma^T(N, N-1)P(0)\Phi(N, N-1)]x(N-1) \tag{2.3.7}$$

Introducing K(N-1) to represent the optimal gain at the (N-1)th stage, we have

$$v(N-1) = K(N-1)x(N-1) \tag{2.3.8}$$

where

$$K(N-1) = -E_{\tau_N/T_{N-1}}[R(N-1) + \Gamma^T(N, N-1) \cdot P(0)\Gamma(N, N-1)]^{-1} E_{\tau_N/T_{N-1}}[\Gamma^T(N, N-1)P(0)\Phi(N, N-1)] \tag{2.3.9}$$

For notational simplification we let

$$K(N-1) = K(N-1)/T(N-1) \tag{2.3.10}$$

i.e., the conditioning will be understood in context.

The cost resulting from application of the optimal control can be evaluated as

$$J(N-1, N) = \frac{1}{2} E_{\tau_N/T_{N-1}}\{x^T(N-1)Q(N-1)x(N-1) + [K(N-1)x(N-1)]^T R(N-1)K(N-1)x(N-1) + [\Phi(N, N-1)x(N-1) + \quad \text{(Continued)}$$

$$+ \Gamma(N, N-1)K(N-1)x(N-1)]^T P(0)[\Phi(N, N-1)x(N-1) + \Gamma(N, N-1)$$

$$K(N-1)x(N-1)]\} \qquad (2.3.11)$$

Hence, the cost function takes the form

$$J(N-1, N) = \frac{1}{2}\, x(N-1)^T\, P(1)\, x(N-1) \qquad (2.3.12)$$

where

$$P(1) = E_{\tau_N/T_{N-1}}\{Q(N-1) + K(N-1)^T R(N-1)K(N-1) + [\Phi(N, N-1) + \Gamma(N, N-1)$$

$$K(N-1)]^T P(0) \cdot [\Phi(N, N-1) + \Gamma(N, N-1)K(N-1)]\} \qquad (2.3.13)$$

For notational simplification, let (as in 2.3.10)

$$P(1) = P(1)/T_{N-1} \qquad (2.3.14)$$

The result is analogous to the optimal solution of uniformly-sampled control systems. The optimal cost is still a quadratic function of x(N-1), the initial state for the single-stage problem; however, P(1) is replaced by the mathematical expectation of the same function. This is an important characteristic of randomly-sampled control systems.

We can no longer determine the minimum cost exactly, but instead, only its statistical average before applying the control.

Two Stage Process

The final two intervals of the process have an optimal cost

$$J(N-2, N) = \frac{1}{2} \min_{v(N-1),v(N-2)} E_{\tau_N, T_{N-1}/T_{N-2}}\{x^T(N)S\, x(N) + \sum_{K=N-2}^{N-1} (x(k)^T Q(k)$$

$$x(k) + v(k)^T R(k)v(k))\} \qquad (2.3.15)$$

$$= \frac{1}{2} \min_{v(N-1),v(N-2)} \{E_{\tau_N,\tau_{N-1}/T_{N-2}} \cdot [x^T(N-2)Q(N-2)x(N-2) + v^T(N-2)R(N-2)$$

$$v(N-2)] + E_{\tau_N,\tau_{N-1}/T_{N-2}}[x^T(N-1)Q(N-1)x(N-1)$$

$$+ v^T(N-1)R(N-1)v(N-1) + x^T(N)P(0)x(N)]\} \qquad (2.3.16)$$

$$J(N-2, N) = \frac{1}{2} \min_{v(N-2)} E_{\tau_{N-1}/T_{N-2}} \{x^T(N-2)Q(N-2)x(N-2) + v^T(N-2)R(N-2)v(N-2)$$

$$+ \min_{v(N-1)} E_{\tau_N/T_{N-1}}[x^T(N-1)Q(N-1)x(N-1) + v^T(N-1)R(N-1)v(N-1)$$

$$+ x^T(N)P(0)x(N)]\} \qquad (2.3.17)$$

The last equation follows from the property of conditional distributions, viz.

$$P_{\tau_N,\tau_{N-1}/T_{N-2}}$$

$$= P_{\tau_N/\tau_{N-1},T_{N-2}} P_{\tau_{N-1}/T_{N-2}}$$

$$= P_{\tau_N/T_{N-1}} P_{\tau_{N-1}/T_{N-2}} \qquad (2.3.18)$$

From this property a similar identity can be derived for the mathematical expectation operator. That is

$$E_{\tau_N,\tau_{N-1}/T_{N-2}}(\cdot) = E_{\tau_N/T_{N-2}}[E_{\tau_N/T_{N-1}}(\cdot)] \qquad (2.3.19)$$

From the Principle of Optimality, the cost functional for the two-stage problem can be written as

$$J(N-2, N) = \frac{1}{2} \min_{v(N-2)} E_{\tau_{N-1}/T_{N-2}} \{x^T(N-2)Q(N-2)x(N-2) + v^T(N-2)R(N-2)v(N-2)$$

$$+ J^*(N-1, N)\} \qquad (2.3.20)$$

$$J(N-2,\ N) = \frac{1}{2} \min E_{\tau_{N-1}/T_{N-2}} [x^T(N-2)Q(N-2)x(N-2) + v^T(N-2)R(N-2)v(n-2)$$

$$+ x^T(N-1)P(1)x(N-1)] \qquad (2.3.21)$$

The cost functional for the two-stage process is analogous to that of the single stage process; $v(N-2)$ can therefore be determined easily by repeating the steps which led to the expression for $v(N-1)$. For the two-stage process, we conclude immediately that

$$J(N-2,\ N) = x^T(N-2)P(2)x(N-2) \qquad (2.3.22)$$

$$P(2) = E_{\tau_{N-1}/T_{N-2}} \{Q(N-2) + K^T(N-2)R(N-2)K(N-2) + [\Phi(N-1,\ N-2) + \Gamma(N-1,$$

$$N-2)K(N-2)]^T P(1) \cdot [\Phi(N-1,\ N-2) + \Gamma(N-1,\ N-2)K(N-2)]\} \qquad (2.3.23)$$

where

$$K(N-2) = -E_{\tau_{N-1}/T_{N-2}} [R(N-2) + \Gamma^T(N-1,\ N-2)P(1)\Gamma(N-1,\ N-2)]^{-1}$$

$$\cdot E_{\tau_{N-1}/T_{N-2}} [\Gamma^T(N-1,\ N-2)P(1)\Phi(N-1,\ N-2)] \qquad (2.3.24)$$

where for notational simplification

$$P(2) = P(2)/T_{N-2}$$

and

$$K(N-2) = K(N-2)/T_{N-2} \qquad (2.3.25)$$

Utilizing the principle of induction, we can prove the theorem for the N-stage problem by carrying out the above derivation for k-stage, problem, assuming the form for the (k-1) stage problem, i.e. we will first assume that equations (2.3.22), (2.3.23) and (2.3.24) hold for the (k-1)-stage problem and then we will show it also holds for the k-stage problem.

Assumed Relations for the (k-1)-Stage Problem

The optimal control at time N-k+1 for the (k-1)-stage process is characterized by the following equations.

$$J(N-k+1,\ N) = \frac{1}{2}\, x^T(N-k+1)P(k-1)x(N-k+1) \tag{2.3.26}$$

$$P(k-1) = E_{\tau_{N-k+2}/T_{N-K+1}}\{Q(N-k+1) + K^T(N-k+1)R(N-k+1)K(N-k+1)$$
$$+ [\Phi(N-k+2,\ N-k+1) + \Gamma(N-k+2,\ N-k+1)K(N-k+1)]^T \cdot P(1)$$
$$[\Phi(N-k+2,\ N-k+1) + \Gamma(N-k+2,\ N-k+1)K(N-k+1)]\} \tag{2.3.27}$$

and

$$K(N-k+1) = -E_{\tau_{N-k+2}/T_{N-k+1}}[R(N-k+1) + \Gamma^T(N-k+2,\ N-k+1)P(k-2)\Gamma(N-k+2,$$
$$N-k+1)]^{-1} \cdot E_{\tau_{N-k+2}/T_{N-k+1}}[\Gamma^T(N-k+2,\ N-k+1)P(k-2)$$
$$\cdot \Phi(N-k+2,\ N-k+1)] \tag{2.3.28}$$

Using the Principle of Optimality the cost function for the k-stage problem can be written as

$$J(N-k,\ N) = \min_{v(N-k)} E_{\tau_{N-k+1}/T_{N-k}}\{\frac{1}{2}\, x^T(N-k) \cdot Q(N-k)x(N-k) + \frac{1}{2}\, v(N-k)^T$$
$$R(N-k)v(N-k) + J(N-k+1,\ N)\} \tag{2.3.29}$$

Using equations (2.3.26) and (2.3.1), we have

$$J(N-k,\ N) = \min_{v(N-k)} E_{\tau_{n-k+1}/T_{N-k}}\{\frac{1}{2}\, x^T(N-k) \cdot Q(N-k)x(N-k) + \frac{1}{2}\, v(N-k)^T$$
$$R(N-k)v(N-k) + [\Phi(N-k+1,\ N-k)x(N-k)$$
$$+ \Gamma(N-k+1,\ N-k)v(N-k)]^T \cdot P(k-1)[\Phi(N-k+1,\ N-k)x(N-k)$$

(Continued)

$$+ \Gamma(N-k+1,\ N)v(N-k) \tag{2.3.30}$$

The minimum is obtained by setting

$$\frac{\partial J(N-k,\ N)}{\partial v(N-k)} = 0 \tag{2.3.31}$$

Repeating the steps similarly used for the single-stage problem, the following recursive relations are obtained for the k-stage process

$$J(N-k,\ N) = \frac{1}{2}\, x(N-k)^T P(k)\ x(N-k) \tag{2.3.32}$$

$$P(k) = E_{\tau_{N-k+1}/T_{N-k}} \{Q(N-k) + K(N-k)^T \cdot R(N-k)K(N-k) + [\Phi(N-k+1,\ N-k) + \Gamma(N-k+1,\ N-k) \cdot K(N-k)]^T P(k-1)[\Phi(N-k+1,\ N-k) + \Gamma(N-k+1,N-k)K(N-k)]\} \tag{2.3.33}$$

and

$$K(N-k) = -E_{\tau_{N-k+1}/T_{N-k}} [R(N-k) + \Gamma^T(N-k+1,\ N-k)P(k-1)\Gamma(N-k+1,\ N-k)]^{-1} \cdot E_{\tau_{N-k+1}/T_{N-k}} [\Gamma^T(N-k+1,\ N-k)P(k-1)\Phi(N-k+1,\ N-k)] \tag{2.3.34}$$

Markov Process Assumption

The statistical property characterizing a Markov process is analogous to the property of Newtonian dynamics. In a dynamical system the equation of motion describing the future trajectory can be determined if the position and velocity are given at any time t; the past trajectory before time t is irrelevant. In a Markov process, the corresponding property is that the present state of the system contains all relevant statistics pertaining to the future. Mathematically, a process $x(1), x(2), \cdots$ is called a Markov process if [19], [20]

$$P(x(N+1)/x(N), \cdots, x(1)) = P(x(N+1)/x(N)).$$

If $T_k = \{t_o \cdots t_k\}$ with T_k its sample space and if τ_k is the sample space of t_k, then assuming a Markov relationship,

$$P_{\tau_k/T_{k-1}}(t_k/T_{k-1}) = P_{\tau_k/\tau_{k-1}}(t_k/t_{k-1}). \tag{2.3.35}$$

This equation states that the present sampling interval determines the probability of the next sampling interval in the future. For such a process a transition mechanism can be defined and the joint density $P_{T_k}(T_k)$ can be written as

$$\begin{aligned} P_{T_k}(T_k) &= P_{\tau_k, T_{k-1}}(t_k, T_{k-1}) \\ &= P_{\tau_k/T_{k-1}}(t_k/T_{k-1}) P_{T_{k-1}}(T_{k-1}) \\ &= \prod_{i=1}^{k} P_{\tau_i/T_{i-1}}(t_k/T_{i-1}) P_{\tau_o}(t_o) \\ &= \prod_{i=1}^{k} P_{\tau_i/\tau_{i-1}}(t_i/t_{i-1}) P_{\tau_o}(t_o) \end{aligned} \tag{2.3.36}$$

Hence, for a Markov process, specification of the prior density function $P_{\tau_o}(t_o)$ together with the transition probabilities $P_{\tau_k/\tau_{k-1}}(t_k/t_{k-1})$ completely determine the distribution of the process. As a result, equations (2.3.32-34) can be modified to replace conditioning on T_{N-k} by conditioning on τ_{N-k}. Hence, the probabilistic Riccati Equations are

$$J(N-k, N) = x^T(N-k)\, P(k) x(N-k) \tag{2.3.37}$$

$$P(k) = E_{\tau_{N-k+1}/\tau_{N-k}} \{Q(N-k) + K^T(N-k) \cdot R(N-k)K(N-k) + [\Phi(N-k+1, N-k)$$

$$+ \Gamma(N-k+1, N-k)K(N-k)]^T P(k-1)[\Phi(N-k-1, N-k) + \Gamma(N-k+1, N-K)K(N-k)]\} \tag{2.3.38}$$

$$K(N-k) = -E_{\tau_{N-k+1}/\tau_{N-k}} [R(N-k) + \Gamma(N-k+1, N-k)P(k-1)\Gamma(N-k+1, N-k)]^T \cdot$$

$$E_{\tau_{N-k+1}/\tau_{N-k}} [\Gamma(N-k+1, N-k)P(k-1) \cdot \Phi(N-k+1, N-k)] \tag{2.3.39}$$

where

$$P(k) = P(k)/\tau_{N-k}$$

and

$$K(N-k) = K(N-k)/\tau_{N-k}$$

The computation for the above probabilistic Riccati Equations is simpler than the previous ones because the set that the expectation operators are conditioned on is more restricted. However, if the expectation operator is time-dependent and if the sample space of $\{t_k\}$ is continuous, integration still has to be performed at each stage of the recursive Riccati equation. The computational time for this case can be exhorbitant. On the other hand, if the expectation operator is stationary (independent of time) and the sample space of $\{t_k\}$ is continuous, integration for the expectation operator has to be performed only once before solving the probabilistic Riccati equations. A simpler case in which the expectation operator is stationary and the sample space is discrete will be discussed in the next section.

Discrete Stationary Markov Processes

If the sample space τ_k is discrete and finite and if the conditional probability is stationary, then the transition mechanism can be represented by the matrix $T = T_{t_k, t_{k-1}} = \{T_{ij}\} = \{P/t_k = S_j/t_{k-1} = S_i)\}$ for a given sample space $\tau_k = \{s_1, s_2, \cdots, s_{NS}\}$ for $k = 1, \cdots, N$. In that case the optimal solution consists of NS gain and sensitivity matrices at each stage so that

$$P(k)/\tau_{N-k} \in \{P(k)/s_1, P(k)/s_2 \cdots P(k)/s_{NS}\} \tag{2.3.40}$$

and

$$K(N-k)/\tau_{N-k} \in \{K(N-k)/s_1, K(N-k)/s_2, \cdots K(N-k)/s_{NS}\} \tag{2.3.41}$$

The equation operator for a discrete sample space can be replaced by a summation. That is

$$E_{\tau_{N-k+1}/\tau_{N-k}}([f]/t_{N-k} = S_i) = \sum_{j=1}^{NS} f(S_j) \cdot P(t_{N-k+1} = S_j/t_{N-k} = S_i)$$
$$= \sum_{j=1}^{NS} T_{ij}\, f(S_j) \tag{2.3.42}$$

The following notations will be used for simplification.

$$Q(N-k, i) = Q(t_{N-k} = S_i) = Q_i \tag{2.3.43}$$

$$R(N-k, i) = R(t_{N-k} = S_i) = R_i \tag{2.3.44}$$

$$\Phi(N-k, i) = \Phi(t_{N-k} = S_i) = \Phi_i \tag{2.3.45}$$

$$\Gamma(N-k, i) = \Gamma(t_{N-k} = S_i) = \Gamma_i \tag{2.3.46}$$

where Q_i, R_i, Φ_i, Γ_i are defined in equation (3). The probabilistic equations can now be written in difference equation form.

$$J(N-k,\ N) = x^{T}(N-k)P(k/i)x(N-k) \tag{2.3.47}$$

$$P(k/i) = \sum_{j=1}^{NS} T_{ij}\{Q(N-k,\ j) + K^{T}(N-k/j)R(N-k,\ j)K(N-k\ /\ J) + [\Phi(N-k,\ j) + \Gamma(N-k,\ j)K(N-k/j)]^{T}P(k-1/j)[\Phi(N-k,\ j) + \Gamma(N-k,\ j) \cdot K(N-k/j)] \tag{2.3.48}$$

$$K(N-k/i) = -\{\sum_{j=1}^{NS} T_{ij}[R(N-k,\ j) + \Gamma^{T}(N-k,\ j) \cdot P(k-1/j)\Gamma(N-k,\ j)]^{-1}\} \{\sum_{m=1}^{NS} T_{im}[\Gamma^{T}(N-k,\ m) \cdot P(K-1/m)\Phi(N-k,\ m)]\} \tag{2.3.49}$$

The recursive computations required to generate the optimal control for the probabilistic Riccati equation above is similar to that for a deterministic Riccati equation. However, the probabilistic Riccati equation also requires the prior distribution function for t_o, viz. $P_{\tau_o}(t_o)$, to initiate the computation. Furthermore, at each stage of the recursive computation NS gain matrices and NS sensitivity matrices must be calculated. The computations required to generate the optimal control sequences for the randomly-sampled system is equivalent to solving NS Riccati equations coupled by transition probabilities. If the transition probability matrix is an identity matrix then the corresponding Riccati equations are decoupled. Each decoupled probabilistic Riccati equation reduces to a deterministic Riccati equation with uniform sampling. The solution with N and k infinite is called the quasi-steady-state solution. The quasi-steady solution is obtained as the NS gain matrices approach constant values. In general, the NS gain matrices have different quasi-steady values. For the purpose of computing quasi-steady-state gains, the prior distribution function can

be chosen to be

$$P_{\tau_o}(t_o = S_i) = 1 \qquad (2.3.50)$$

and

$$P_{\tau_o}(t_o \neq S_i) = 0$$

This is equivalent to selecting an initial sampling interval $t_o = S_i$ to start the process. Hence, there are generally NS simulations of interest in the class of distributions considered, one for each element of the discrete sample space τ_o. The optimal cost is not generally asymptotically stationary. That is, the cost will depend on the initial sampling interval selected. One trivial example is the system with identity transition probability. The optimal cost depends on the initial sampling interval chosen in this case because all subsequent sampling intervals are the same as the initial sampling interval. Each simulation requires that the initial conditions x(o) and the initial sample interval t_o contained in $\tau = \{s_1 \cdots s_{NS}\}$ be given. If the quasi-steady state gains are used and if $t_o = S_i$, then the control u(o) applied over the interval t_1 is given by $u(o) = K_i x(o)$, where $K_i = K(N-k/i)$ for N and k infinite with i = 1, 2, $\cdots$ NS. Subsequent control actions are computed according to $u(k) = K_i x(k)$ if $t_{k-1} = S_i$. Fig. (2.3.1) shows the closed loop control system using quasi-steady-state feedback.

2.4 The Stability of the Closed-Loop System

In this section we will study the stability of the randomly-sampled system in which the sampling interval varies randomly over a fixed,

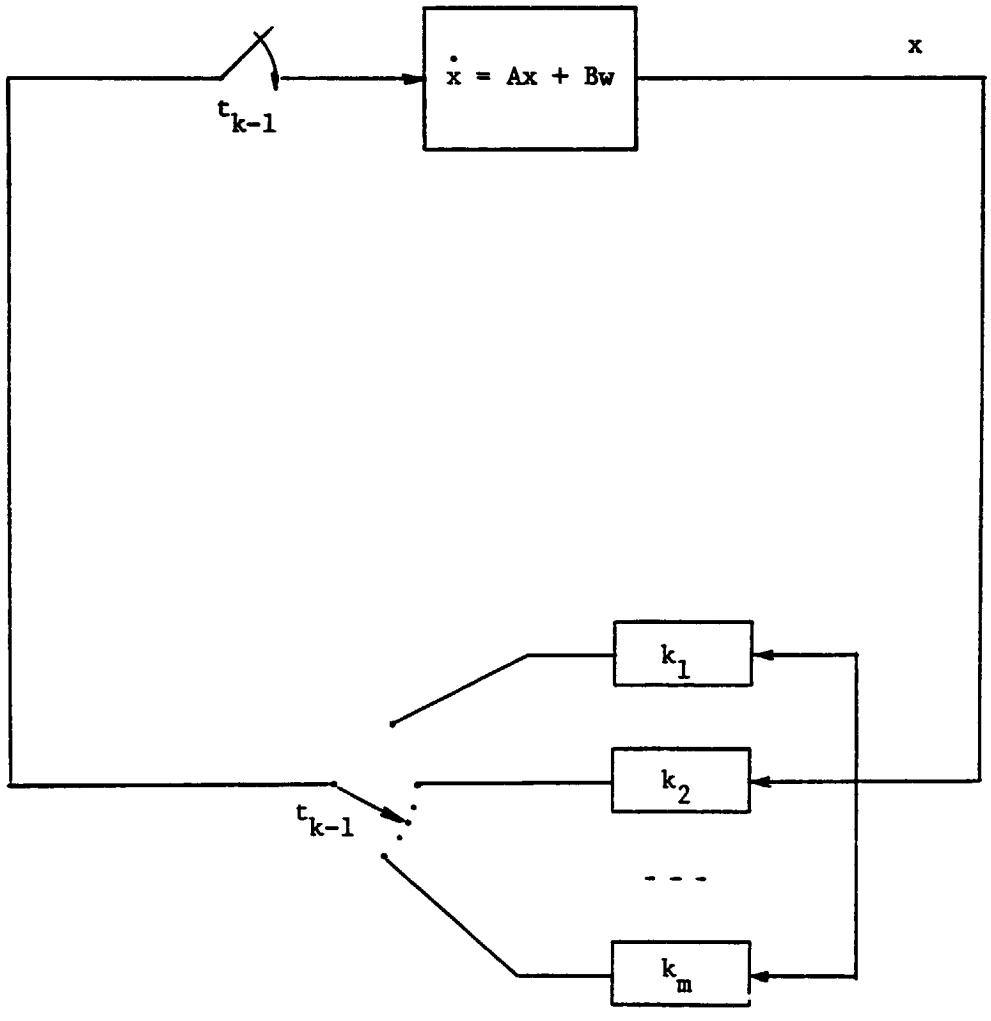


Figure 2.3.1. The Closed-Loop System Using Quasi-Steady-State Feedback.

finite set of positive numbers. The stability of the randomly sampled system can be analyzed by re-examining the discretization procedure for a continuous system. The optimal gains calculated in section (2.3) are used to stabilize a linear system of the form

$$\dot{x} = Ax + Bu \tag{2.4.1}$$

If a finite set of positive real number $\{s_1 \cdots s_N\}$ is selected for possible sampling intervals and a constant u(k) is applied to produce the state x(k+1), s_i time units later, the state at the next sampling instant is given by

$$x(k+1) = e^{s_i A} x(k) + \int_0^{s_i} e^{At} dt\, Bu(k) \tag{2.4.2}$$

For the case in which the controller knows the duration of the sampling interval at its initiation, the constant u(k) is the product of the state and the gain matrix, which depends only on s_i(the duration of the sampling interval), i.e.

$$u(k) = F_i x(k) \tag{2.4.3}$$

For the case in which the controller does not know the duration of the sampling interval at its initiation, there are possible combinations of gain and sampling intervals, but the basic approach will be the same. For simplicity we will stay with the first case.

Introducing (2.4.3) into (2.4.2),

$$x(k+1) = (e^{s_i A} + \int_0^{s_i} e^{At} B dt\, F_i) x(k) \tag{2.4.4}$$

We define

$$\Phi_i = e^{s_i A}$$

and

$$\Gamma_i = \int_0^{s_i} e^{At} B dt \quad \text{for} \quad i = 1, \cdots, N. \tag{2.4.5}$$

Equation (2.4.4) can now be written as

$$x(k+1) = (\Phi_i + \Gamma_i F_i)x(k) \tag{2.4.6}$$

Defining

$$H_i = (\Phi_i + \Gamma_i F_i), \tag{2.4.7}$$

$$x(k+1) = H_i x(k) \tag{2.4.8}$$

Since the sampling interval varies in time, we introduce the index i_k to denote $i^{\underline{th}}$ configuration at the $k^{\underline{th}}$ sampling instant. Hence,

$$x(k+1) = (\Phi_{i_k} + \Gamma_{i_k} F_{i_k})x(k) \tag{2.4.9}$$

where $1 \leq i_k \leq N$.

Let

$$H_{i_k} = \Phi_{i_k} + \Gamma_{i_k} F_{i_k} \tag{2.4.10}$$

Using Equation (2.4.9), the state at the $k^{\underline{th}}$ sampling instant can be expressed as

$$x(k) = \prod_{j=o}^{k-1} H_{i_j} \; x(o)$$

with $1 \leq i_j \leq N$.

The stability of the system (2.4.6) is, therefore, guaranteed if the following sequence converges

$$\prod_{j=o}^{k-1} H_{i_j}$$

Before discussing the convergence of a set of matrices, let us define some definitions on vector and matrix norms, [29], [30].

Vector norms must satisfy the following relations:

$$||x|| \geq 0 \quad \text{and} \quad ||x|| = 0 \text{ iff } x = 0$$

$$||\alpha x|| = |\alpha| \; ||x|| \quad \text{for any scalar } \alpha$$

$$||x+y|| \leq ||x|| + ||y|| \tag{2.4.11}$$

Similarly, matrix norms have the following properties:

$$||A|| \geq 0 \text{ and } ||A|| = 0 \text{ iff } A = 0$$

$$||\alpha A|| = |\alpha| \; ||A|| \text{ for any scalar } \alpha$$

$$||A+B|| \leq ||A|| + ||B||$$

$$||AB|| \leq ||A|| \; ||B|| \tag{2.4.12}$$

Corresponding to each vector norm, the induced matrix norm is defined as

$$||A|| = \max_{x \neq 0} \frac{||Ax||}{||x||} \tag{2.4.13}$$

It can be proved that $||A|| = \max_{||x||=1} ||Ax||$

Three most commonly used vector norms are

$$||x||_P = (|x_1|^P + |x_2|^P + \cdots |x_N|^P)^{1/P}$$

for $P = 1, 2, \infty$. (2.4.14)

The norm $||x||_2$ is called the Euclidean norm and $||x||_\infty$ can be interpreted as $\max_i |x_i|$. It is conceivable that a measure of magnitude for matrices could be based on the magnitude of the eigenvalues. If

$u_1 \cdots u_n$ are the eigenvalues of A, then $u_A = \max_j |u_j|$ $(1 \leq j \leq n)$ is called the spectral radius of A. The following theorem on spectral radius of A^*A will be used to establish the stability conditions of the closed-loop system.

Theorem 2.4.1 If $||\cdot||_s$ denote the matrix norm (known as spectral norm) induced by the Euclidean norm, then $||A||_s = \lambda_A^{1/2}$, where λ_A is the spectral radius of $A^* \cdot A$.

Proof: Since the matrix A^*A is Hermitian and positive-definite, it is well known that the eigenvalues of A^*A are real and non-negative. The spectral radius λ_A is, therefore, an eigenvalue of A^*A.

Let $\{x_1, x_2, \cdots, x_n\}$ be a set of orthonormal right eigenvectors of A^*A with associated eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$. For any x with $||x||_2 = 1$ write $x = \sum_{j=1}^{n} \xi_j x_j$, then

$$A^*Ax = \sum_{j=1}^{n} \xi_j \lambda_j x_j$$

and

$$(||Ax||_2)^2 = (Ax)^*Ax = x^*(A^*Ax)$$

$$= (\Sigma_j \xi_j x_j)^* (\Sigma_k \xi_k \lambda_k x_k)$$

$$= \Sigma_j |\xi_j|^2 \lambda_j$$

which uses the fact that $x^*_j x_k = \delta_{jk}$. Therefore,

$$||Ax||_2 = \{\Sigma_j |\xi_j|^2 \lambda_j\}^{1/2}$$

But

$$\sum_j |\xi_j|^2 = 1$$

and

$$\lambda_j \geq 0 \quad \text{for} \quad j = 1, \cdots, n.$$

It follows immediately that

$$||A||_s = \max ||Ax||_2 = \lambda_A^{1/2}$$

Since H_{i_j} is chosen from the finite set $\{H_1, \cdots H_N\}$. We will state the definition of convergence of a set of matrices.

Definition 2.4.1: A set $\{H_1, H_2 \cdots \}$ of m x m matrices is convergent if every sequence

$$\{x, H_{i_o} x, H_{i_1} H_{i_o} x, \cdots, (\prod_{j=o}^{k} H_{i_j})x, \cdots\}$$

(for $1 \leq i_j \leq N$) converges to zero for every x in R^n.

The convergence can be written as

$$\lim_{k\to\infty} (\prod_{j=o}^{k} H_{i_j})x = 0$$

for all $1 \leq i_j \leq N$.

It is well known that the single element set $\{H_1\}$ is convergent if and only if the spectral radius of H_1 is less than one--i.e., H_1 satisfies the relation $||H_1 x||_2 < ||x||_2$ for all x in R^n, where $||\cdot||_2$ denotes the Euclidean norm. One is tempted to conjecture that a set $\{H_1 \cdots H_N\}$ of m x m matrices is convergent if and only if every matrix H_i, $1 \leq i \leq N$ has a spectral radius less than one. Unfortunately, this is not true. The following is a counterexample.

Example 2.4.1 Let the set of matrices be $\{H_1, H_2\}$ where

$$H_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

It is clear that both H_1 and H_2 have spectral radii which equal zero. However, the following sequence will not contract the vector $[0\ 1]^T$

$$H_1H_2H_1H_2 \cdots H_1H_2 \cdots$$

Therefore, the set of matrices $\{H_1, H_2\}$ is not convergent.

At present a necessary and sufficient condition for the convergence of a set of matrices is unknown. We have, however, established a sufficient condition for the convergence of a set of matrices.

Theorem 2.4.2 A set $\{H_1, H_2, \cdots, H_N\}$ of m x m matrices is convergent if for every matrix H_i, $1 \leq i \leq N$, $H_i^*H_i$ has a spectral radius less than one.

Proof: from the definition of the spectral norm (the matrix norm induced by the Euclidean norm) we have

$$||H_{i_k} H_{i_{k-1}} \cdots H_{i_o} x_o|| \leq ||H_{i_k} H_{i_{k-1}} \cdots H_{i_o}|| \ ||x_o||$$

where

$$1 \leq i_j \leq N \quad \text{for} \quad j=0, \cdots, k$$

But

$$||H_{i_k} H_{i_{k-1}} \cdots H_{i_o}|| \cdot ||x_o|| \leq ||H_{i_k}|| \cdots ||H_{i_o}|| \ ||x_o||$$

$$= \lambda_{i_k} \lambda_{i_{k-1}} \cdots \lambda_{i_o} ||x_o||_2$$

from Theorem 2.4.1, where λ_{i_j} is the spectral radius of $H_{i_j}^{*} H_{i_j}$.

Since every matrix $H_i^{*} H_i$ has a spectral radius less than one for $1 \leq i \leq N$ and $1 \leq i_j \leq N$, therefore $|\lambda_{i_j}| < 1$ for $j = 0, \cdots, k$. Hence, each number in the sequence $\lambda_{i_k} \lambda_{i_{k-1}} \cdots \lambda_{i_o}$ has an absolute value less than one, and

$$\lim_{k \to \infty} \prod_{j=o}^{k} \lambda_{i_j} = 0$$

Therefore, the sequence of matrices will also converge.

To verify Theorem 2.4.2, let us return to Example 2.4.1. It is obvious that both $H_1^{*} H_1$ and $H_2^{*} H_2$ have a spectral radius equal to 1. Therefore, the set of matrices $\{H_1, H_2\}$ is not necessarily convergent under Theorem 2.4.2.

From the above discussion, we can see that the convergence of the set of matrices $\{(\Phi_i + \Gamma_i F_i), i=1, \cdots, N\}$ is a sufficient condition for the stability of the randomly-sampled system. Although the condition looks very restrictive, it is required for reliable operation of the control system. Namely, if the set of matrices $\{(\Phi_i + \Gamma_i F_i), i=1, \cdots, N\}$ is convergent, the system will remain stable regardless of what the actual transition probability between adjacent sampling intervals is, since it is conceivable that the actual transition probability could change due to software or hardware failure in a digital system. The convergence is, however, not a necessary condition for finding a stable sequence. We will state the example from Reference [20] to show why it is not a necessary condition. The example is interesting because it shows that for any given vector it is possible to find a stable sequence of matrices from a set of matrices in which each

element is unstable by itself.

Example 2.4.2 Let the set of matrices be

$$\{H_1, H_2, H_3, H_4\}$$

where

$$H_1 = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \quad H_2 = \begin{bmatrix} 1.25 & -.75 \\ -.75 & 1.25 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix} \quad H_4 = \begin{bmatrix} 1.25 & .75 \\ .75 & 1.25 \end{bmatrix}$$

It can be easily verified that each matrix has a spectral radius of 2. The set of matrices is not convergent by Theorem 2.4.2 and Definition 2.4.1. But for every vector in R^2 it is always possible to find a sequence of matrices from the set which will contract the vector to zero for the following reasons: H_1 contracts every vector in the closed cone R_1, co-axial with the x axis, having 45° vertex angle at the origin. Similarly H_2, H_3 and H_4 contract vectors in the cones R_2, R_3 and R_4 which are counter-clockwise rotations of R_1 through 45°, 90° and 135°, respectively. Apparently, R_1, R_2, R_3 and R_4 cover the whole region of interest. Hence, we can contract every vector in R^2 using the set $\{H_1, H_2, H_3, H_4\}$. The above example shows that while the convergence of the set of matrices guarantees the stability of the closed-loop system, it is not a necessary condition to find a stabilizing sequence of matrices.

To summarize, we state the following theorem for the stability of our closed-loop system.

Theorem 2.4.4 If each spectral radius of the matrices in the set $\{(\Phi_i + \Gamma_i F_i)^*(\Phi_i + \Gamma_i F_i), i=1, \cdots, N\}$ is less than unity, the closed-loop optimal control system with random-sampling is stable regardless of what the actual set of transition probabilities between the adjacent sampling intervals is.

2.5 The Existence of Quasi-Steady-State Solutions

Optimal control problems with an infinite optimization interval are always of special interest. The resulting control strategy, if it exists, is usually simple to implement, and with suitable assumptions yields a stable closed-loop system. The single sample-rate discrete-time system is said to be stabilizable if there exists a matrix F such that all the eigenvalues of $(\Phi + \Gamma F)$ are inside the unit circle [2], [28]. It is obvious that if a system is stablizable, there always exists a control input which makes the performance index with infinite optimization interval finite. For the randomly-sampled system we state the following theorem on stablizability.

Theorem 2.5.1 A randomly-sampled system with NS possible configurations $\{(\Phi_i, \Gamma_i), i=1, \cdots, NS\}$ is stabilizable if for each pair (Φ_i, Γ_i) there is an F_i such that the set of matrices $\{(\Phi_i + \Gamma_i F_i)\ i=1, \cdots, NS\}$ is convergent.

From Theorem 2.4.4 the following theorem is immediate.

Theorem 2.5.2 If for each pair (Φ_i, Γ_i) in the set $\{(\Phi_i, \Gamma_i), i=1, \cdots, NS\}$ of randomly-sampled system there exists an F_i such that all the eigenvalues of $[\Phi_i + \Gamma_i F_i]^T[\Phi_i + \Gamma_i F_i]$ are inside the unit circle,

the system is stabilizable.

In order to establish the existence of an optimal solution as the time index $k \to \infty$, we must prove certain properties of the matrices $P(k/i)$, see (2.3.48), $i=1, \cdots, NS$. Specifically, we will show that the sequence is bounded above and is non-decreasing. It is clear that $P(k/i)$ is bounded if the system is stabilizable. The inequality $P(k+1/i) > P(k/i)$ can be established if we have $x^T P(k+1/i)x > x^T P(k/i)x$. We will prove $x^T P(k+1/i)x > x^T P(k/i)x$ as follows:

Since each term in the cost function

$$J = \frac{1}{2} E\{x(N)^T S x(N) + \sum_{k=0}^{N-1} [x(k)^T Q(k) x(k) + V^T(k) R(k) V(k)]\}$$

is nonnegative, we have

$$J^*(N-k, N+1) \geq J^*(N-k, N).$$

From equation (2.3.48), we have

$$J^*(N-k, N) = x^T(N-k) P(k/i) x(N-k).$$

It should be noted that the value of $P(k/i)$ in equation (2.3.48) depends only on the number of iterations. The time invariance of the statistics of the system allows a shift from the interval $(N-(k+1), N)$ to $(N-k, N+1)$. Thus, we have

$$J(N - (k+1), N) = J(N-k, N+1)$$

Therefore,

$$P(k+1/i) \geq P(k/i)$$

***This completes the proof. The quasi-steady state can, of course, be calculated by substituting $P(k)$ for $P(k-1)$ in equation (2.3.48) and

solving the resulting algebraic Riccati equations. To summarize, we state the following theorem:

Theorem 2.5.3 If a linear system with random sampling is stabilizable, the quasi-steady state solution for the probabilistic Riccati Equations (2.3.48) and (2.3.49) exist and can be determined from the following algebraic Riccati equations.

$$P(k/i) = \sum_{j=1}^{NS} T_{ij}\{Q(N-k, j) + K^T(N-k/j)R(N-k, j) \cdot K(N-k, j) + [\Phi(N-k, j) + \Gamma(N-k, j)K(N-k/j)]^T \cdot P(k/j)[\Phi(N-k, j) + \Gamma(N-k, j) \cdot K(N-k/j)] \tag{2.5.1}$$

$$K(N-k/i) = -\{\sum_{j=1}^{NS} T_{ij}[R(N-k, j) + \Gamma^T(N-k, j) \cdot P(k-1/j)\Gamma(N-k, j)]^{-1}\}\{\sum_{m=1}^{NS} T_{im}[\Gamma^T(N-k, m) \cdot P(k-1/m)\Phi(N-k, m)]\} \tag{2.5.2}$$

Considerable computational time may be saved by using equations (2.5.1-2.5.2) instead of equations (2.3.48-2.3.49). If Theorem 2.5.3 fails to indicate the stabilizability of the system, equations (2.3.48-2.3.49) can be iterated recursively, backward in time until either they converge or pass some test of non-convergence. Since the P(k/i) matrix is non-decreasing, it is not possible to have limit cycles.

3. APPLICATION TO AIRCRAFT CONTROL

3.1 Introduction

In this chapter we illustrate the application of the optimal solution of Chapter 2 to the longitudinal control of the F8-DFBW (Digital-Fly-By-Wire) aircraft. The longitudinal dynamics of the F8-DFBW is modeled as a linear process and the information distribution process is modeled as a variable time-increment process where, at the time that information is supplied to the control effectors, the control effector knows the time of the next information update only in a stochastic sense. Section (3.2) describes the longitudinal dynamics of the F8-DFBW aircraft. The unaugmented response of the aircraft to initial conditions indicates the need for control. In section (3.3) the aircraft control problem is modeled as an optimal control problem with random sampling. The optimal control problem for the aircraft is solved in section (3.4). Theoretical and simulation results indicate that the optimal cost obtained using a variable time-increment Markov information update process, i.e. where the control effectors know only the past information update intervals and the Markov transition mechanism, is almost identical to that obtained using a known uniform information update interval.

3.2 The Equations of Motion

The longitudinal equations of motion for a rigid aircraft are:

$$m\dot{v} = \Sigma F_v = T \cos \alpha - mg \sin (\theta - \alpha) - D \qquad (3.2.1)$$

$$mv\dot{\alpha} = mvq - T \sin \alpha + mg \cos (\theta - \alpha) - L \qquad (3.2.2)$$

$$I_y\dot{q} = \Sigma M_y = M_a + M_t \qquad (3.2.3)$$

$$\dot{\theta} = q \qquad (3.2.4)$$

where

v = forward velocity

α = angle of attack

q = pitch rate

θ = pitch angle

T = thrust, D = drag, L = lift

M_a = pitching moment due to aerodynamic forces

$M_t = 0$ = pitching moment due to thrust

m = mass, g = gravity, I_y = moment of inertia about the y-axis.

The state variables of the aircraft longitudinal system are forward velocity, angle of attack α, pitch rate q and pitch angle θ. The aerodynamic forces and moments are the lift and drag forces and the pitching moment. These forces and moments can be expressed in terms of aerodynamic data as follows:

$$D = \frac{1}{2} \rho v^2 S C_D (\mu, \alpha, \dot{\alpha}, q, \delta e, \sigma f) \qquad (3.2.5)$$

$$L = \frac{1}{2} \rho v^2 S C_L (\mu, \alpha, \dot{\alpha}, q, \delta e, \delta f) \qquad (3.2.6)$$

$$M = \frac{1}{2} \rho v^2 S \bar{c} C_M (\mu, \alpha, \dot{\alpha}, q, \delta e, \delta f) \qquad (3.2.7)$$

where

S = the effective aerodynamic surface area

ρ = atmosphere density

$\bar{c}$ = the mean aerodynamic wing chord

C_D, C_L, C_M = nonlinear drag, lift, and moment coefficients.

μ, δe, δf = perturbation in velocity and elevator and flap surface deflections

For a nonlinear dynamic system

$$\dot{\bar{x}} = f(\bar{x}, \bar{\mu}) \tag{3.2.8}$$

The small perturbations around the nominal value $(\bar{x}_o, \bar{\mu}_o)$ can be expressed as

$$\bar{x} = \bar{x}_o + \bar{\delta x} \tag{3.2.9}$$

$$\bar{\mu} = \bar{\mu}_o + \bar{\delta \mu}_o \tag{3.2.10}$$

Substituting equations (3.2.9) and (3.2.10) into (3.2.8). we have

$$\dot{\bar{x}}_o + \dot{\bar{\delta x}} = f(\bar{x}_o + \bar{\delta x}, \bar{\mu}_o + \bar{\delta \mu}) \tag{3.2.11}$$

Taking a Taylor series expansion, we have

$$\dot{\bar{x}}_o + \dot{\bar{\delta x}} = f(\bar{x}_o, \bar{\mu}_o) + \frac{\partial f}{\partial x}(\bar{x}_o, \bar{\mu}_o)\ \bar{\delta x}$$

$$+ \frac{\partial f}{\partial \mu}(\bar{x}_o, \bar{\mu}_o)\ \delta\mu + \text{Higher order terms} \tag{3.2.12}$$

Neglecting higher order terms, we have the following linearized equation.

$$\delta\dot{\bar{x}} = A\delta\bar{x} + B\delta\bar{u} \tag{3.2.13}$$

where

$$A = \frac{\partial f}{\partial \bar{x}} (\bar{x}_o, \bar{u}_o) \tag{3.2.14}$$

$$B = \frac{\partial f}{\partial \bar{u}} (\bar{x}_o, \bar{u}_o) \tag{3.2.15}$$

For the F8-DFBW aircraft the linearized equations of motion at a selected trimmed flight condition have the matrix form

$$\begin{bmatrix} \dot{V} \\ \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} P_1 & P_4 & 0 & -g \\ P_2 & P_5 & 1 & 0 \\ P_3 & P_6 & P_{12} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V \\ \alpha \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} P_7 & 0 \\ P_8 & P_{10} \\ P_9 & P_{11} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta e \\ \delta f \end{bmatrix} \tag{3.2.16}$$

where

$$P_1 = -\frac{\bar{q}S}{mV_o} C_{D\mu}$$

$$P_2 = -\frac{\bar{q}S}{mV_o^2} C_{L\mu}$$

$$P_3 = \frac{\bar{q}S\bar{c}}{V_o I_y} \left(C_{m\mu} - \frac{\bar{q}S\bar{c}}{2mV_o^2} C_{m\dot{\alpha}} C_{L\mu}\right)$$

$$P_4 = -\frac{57.3\bar{q}S}{m} C_{D\alpha}$$

$$P_5 = -\frac{57.3\bar{q}S}{mV_o} C_{L\alpha}$$

$$P_6 = \frac{57.3\bar{q}Sc}{Iy} \left(C_{m\alpha} - \frac{\bar{q}Sc}{2mV_o} C_{m\dot{\alpha}} C_{L\dot{\alpha}}\right)$$

$$P_7 = \frac{57.3\bar{q}Sc}{Iy} \left(C_{m\delta e} - \frac{\bar{q}Sc}{2\ V_o^2} C_{m\dot{\alpha}} C_{L\delta e}\right)$$

$$P_8 = \frac{57.3\bar{q}S}{m} C_{D\delta e}$$

$$P_9 = \frac{57.3\bar{q}S}{mV_o} C_{L\delta e}$$

$$P_{10} = \frac{57.3\bar{q}S}{m} C_{D\delta f}$$

$$P_{11} = \frac{57.3\bar{q}S}{mV_o} C_{L\delta f}$$

$$P_{12} = \frac{\bar{q}Sc}{2V_o I_y} \left(C_{mq} + C_{m\dot{\alpha}}\right)$$

$$\bar{q} = \frac{1}{2} \rho V_o^2.$$

For the equilibrium flight condition at an altitude of 20,000 feet and a Mach number of 0.67, the linearized longitudinal dynamics take the following numerical values

$$A = \begin{bmatrix} -9.529\ E-3 & -1.283\ E+1 & 0 & -3.217\ E+1 \\ -1.175\ E-4 & -9.782\ E-1 & 1.0 & 0 \\ 3.324\ E-7 & -4.723\ E+0 & -4.729\ E-1 & 0 \\ 0 & 0 & 1.0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -6.554\ E+0 & 0 \\ -2.253\ E-1 & -1.513\ E-1 \\ -1.539\ E+0 & -1.333\ E+0 \\ 0 & 0 \end{bmatrix}$$

3.3 Formulation of a Randomly-Sampled Aircraft Control Problem

For a "staticly stable aircraft" all eigenvalues of the A matrix must have negative real parts. A staticly stable aircraft will return to its equilibrium position after a small disturbance has occurred. Although all aircraft are designed to be staticly stable, certain disturbances may result in unsatisfactory responses. Figure (3.3.1) shows the unaugmented response of the aircraft to initial conditions. The motion is characterized by two oscillatory modes, one of short period and one of long period (phugoid). The predominant oscillation, seen in Figure (3.3.1), is the short-period oscillation which has a period of 3 seconds. The large variations of the angle of attack and the pitch angle are not acceptable. In addition, the short-period mode causes the aircraft to have a prolonged oscillation. The main control objective is to damp the short-period mode. The weighting matrices, Q and R, of Equation (2.2.6), were selected so that only elevator deflections are used to control the aircraft and so that only the short-period mode is regulated. This requires high weights on the α- and θ-error terms and a high weight on the flap cost term. The v and θ weighting terms are small so as not to over-control the long-period mode.

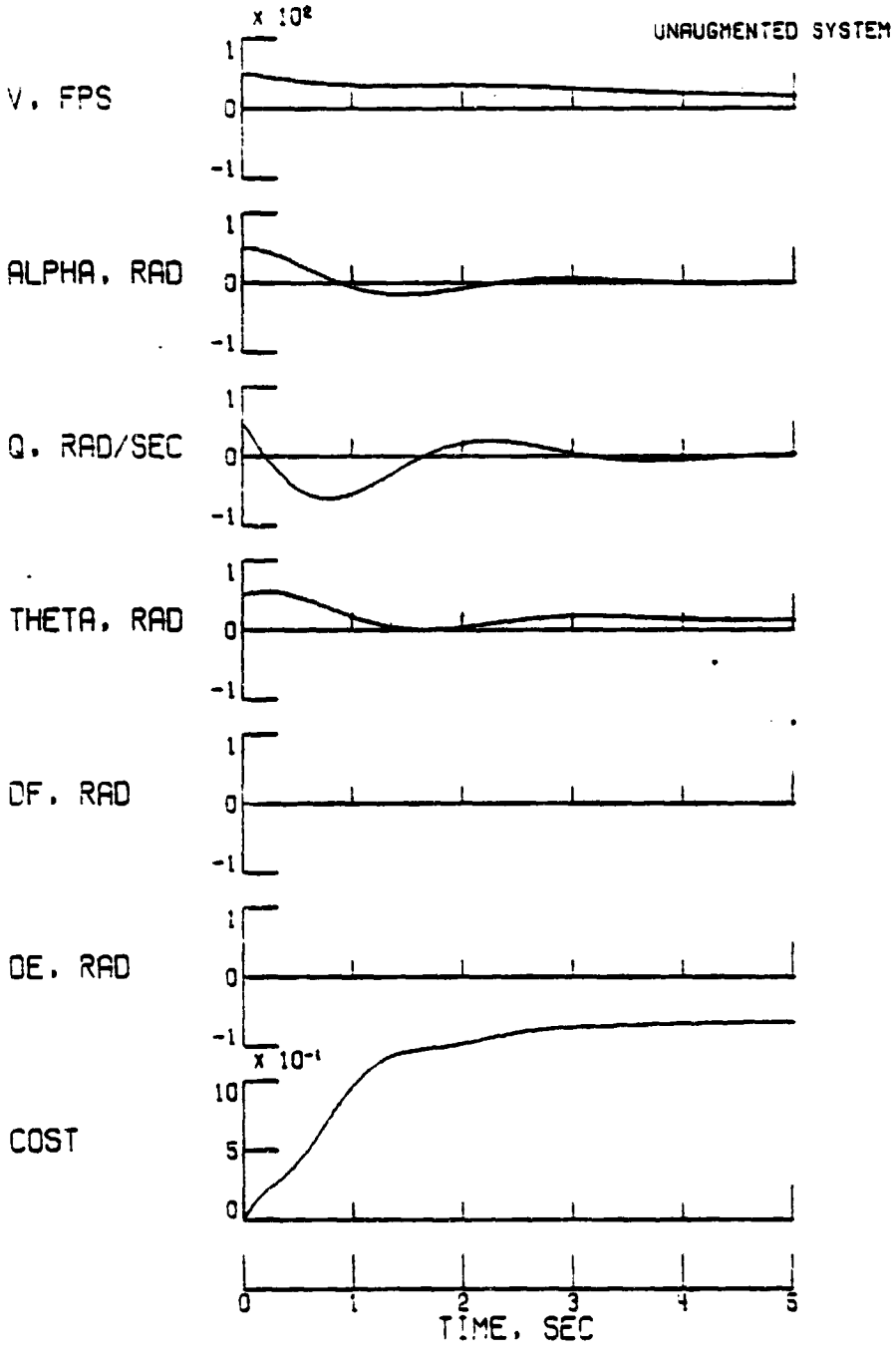


Figure 3.3.1. Response of Unaugmented Aircraft

Errors in the state variables are weighted relative to control effort by adjusting the Q and R matrices in the cost functional. The specified values for the Q and R matrices are

$$Q = \begin{bmatrix} 2.5\ E-5 & 0 & 0 & 0 \\ 0 & 3.0\ E+0 & 0 & 0 \\ 0 & 0 & 3.0\ E+0 & 0 \\ 0 & 0 & 0 & 1.0\ E-2 \end{bmatrix},$$

$$R = \begin{bmatrix} 4.0\ E+0 & 0 \\ 0 & 4.0\ E-1 \end{bmatrix}$$

The sample space is chosen to be $\tau = \{0.02, 0.03, 0.1\}$ and the transition probability is chosen to be

$$T = \begin{bmatrix} .95 & .05 & 0 \\ 0 & .8 & .2 \\ .7 & 0 & .3 \end{bmatrix}$$

Hence, if the sampling interval at t_{k-1} is S_1 the probability that $t_k = S_1$ is 0.95, the probability that $t_k = S_2$ is 0.05, and the probability that $t_k = S_3$ is zero. Figure (3.3.2) is a graphical representation of transition mechanism. A characteristic of this transition mechanism is that it leads to a process which is cyclic. That is, it tends to change gradually from the fastest sampling interval to the slowest and back to the fastest. Table (3.3.1)

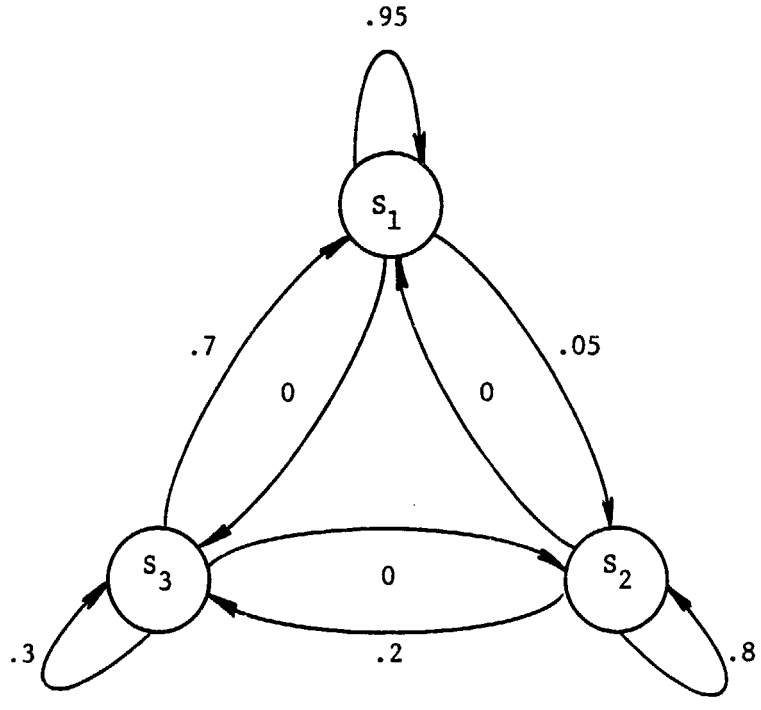


Figure 3.3.2. Probabilistic Transition Diagram

Table 3.3.1 Sequences generated by Random Number Generator

Sequence 1 = $\{S_1, S_1, S_1, S_1, S_2, S_2, S_2, S_2, S_2, S_3, S_1, S_1, S_1\}$
Sequence 2 = $\{S_1, S_1, S_1, S_1, S_1, S_1, S_2, S_3, S_1, S_1, S_1, S_2, S_2\}$
Sequence 3 = $\{S_1, S_1, S_1, S_1, S_1, S_1, S_1, S_1, S_1, S_1, S_2, S_2, S_2\}$

shows several sampling sequences generated by a random number generator using the indicated transition probability.

3.4 Numerical Results and Conclusions

In this section some numerical results on the F8-DFBW aircraft longitudinal system with random sampling will be presented. For the chosen sample space $\tau = \{0.02, 0.03, 0.1\}$, the equivalent discrete model is obtained for each sampling interval using equations (2.2.13-2.2.18) and the software in Reference [19]. The results are shown in Tables (3.4.1), (3.4.2), and (3.4.3). For the purpose of comparison, the optimal control system with uniform sampling is solved for each sampling interval. The optimal gains for the uniformly-sampled system were first obtained by solving the stochastic Riccati equations (2.3.47 and 2.3.48) iteratively with the transition probability matrix $T = I$. In this approach three steady-state gains for the system with uniform sampling intervals S_1, S_2 and S_3 were obtained simultaneously. Each gain matrix was verified by solving the regular Riccati equation individually using the software in Reference [19]. The results were identical. The steady-state solutions of the uniformly-sampled system are shown on Tables (3.4.4-3.4.6). The uniformly-sampled system was simulated for each sampling interval using the corresponding optimal gain matrix. The optimal responses for each system are shown in Figures (3.4.1-3.4.3). It should be noted that the response improves as the time interval becomes smaller. It should be emphasized once more that the uniformly-sampled systems

are not realizable by our assumption and the systems are simulated here only for the purpose of comparison. The optimal gains for the randomly-sampled cases were also obtained by solving the stochastic Riccati equations (2.3.47 and 2.3.48) iteratively with the transition probability T as given in Figure (3.3.2). In this case the optimal control law consists of three quasi-steady state gain matrices $K_i = K(\infty/i)$ with $i = 1, 2, 3$. If the sampling interval at the $(k-1)$-th interval t_{k-1} is S_i, the control $u(k)$ applied over the k-th time interval t_k is given by $u(k) = K_i x(k)$. The quasi-steady-state gain matrices and sensitivity matrices are shown in Tables (3.4.8) and (3.4.7), respectively. It is interesting to see that the norms of the sensitivity matrices are very close to each other. Therefore, the expected cost, which is equal to $x^T(0)\ P(\infty/i)\ x(0)$, is almost the same regardless which initial sampling interval t_o is assumed. The quasi-steady-state gain matrices, on the other hand, are considerably different from each other. Hence, it is important to use the correct gain for each sampling interval.

A randomly-sampled system is simulated with the transition probability between the adjacent sampling intervals equal to the matrix shown in Figure (3.2.2). The quasi-steady gain from the optimal stochastic regulators is first applied to the randomly-sampled system. The response of the system is probabilistic, i.e. each run is different from the other even if the initial state is the same. Figure (3.4.4) shows a typical response of the randomly-sampled system using optimal control. Comparing these results with the

deterministic optimal solution with known uniform sampling intervals in Figure (3.4.1), we can see that the trajectories differ only insignificantly. The expected value of the cost of the process can also be evaluated using the sensitivity matrices according to the formular $x^T(0)\ P(\infty/i)\ x(0)$. That value is 0.5096 for uniform sampling(.02 sec/sample) and 0.5159 for stochastic sampling. This indicates that little is lost in the stochastic information update process provided that the stochastic nature of the process is accounted for in the control logic. The closeness of the stochastic and deterministic trajectories are remarkable since the deterministic optimal control requires absolute knowledge of the whole sampling sequence, which, of course, is not realizable for the problem we are considering. Non-optimal gains are applied to the randomly-sampled system to study the sensitivity of the feedback gains. Figures (3.4.5), (3.4.6) and (3.4.7) illustrate typical responses of the randomly-sampled systems using "optimal gains" calculated for a uniformly-sampled system with sampling interval equal to 0.02 sec., 0.03 sec. and 0.1 sec., respectively. To study the average cost, 50 simulations were performed for the optimal feedback system and each non-optimal feedback system. The results are summarized in Table (3.4.9). They are consistent with our theory, i.e. in the randomly-sampled system the optimal stochastic gains give the lowest average cost (0.5174) over all other gains, including those calculated for the uniformly-sampled system.

Table 3.4.1. Discrete Model for Sampling Interval = .02 Sec.

Phi Matrix	4 Rows	4 Columns	
9.9980E-01	-2.5395E-01	-8.9546E-03	-6.4333E-01
-2.3265E-06	9.7969E-01	1.9705E-02	7.5103E-07
1.1653E-07	-9.3074E-02	9.8965E-01	-2.5765E-08
8.0091E-10	-9.3540E-04	1.9899E-02	1.0000E+00
Gam Matrix	**4 Rows**	**2 Columns**	
-1.3040E-01	1.1827E-03		
-4.7669E-03	-5.6363E-03		
-3.0430E-02	-2.6517E-01		
-3.0553E-04	-2.6568E-03		
Q Matrix	**4 Rows**	**4 Columns**	
4.9990E-07	-1.3299E-07	-3.2405E-11	-1.6081E-07
-1.3299E-07	5.8979E-02	-2.2011E-03	-2.0356E-08
-3.2405E-11	-2.2011E-03	5.9406E-02	1.9939E-06
-1.6081E-07	-2.0356E-08	1.9939E-06	2.0006E-04
W Matrix	**4 Rows**	**2 Columns**	
-6.4985E-08	-1.5411E-10		
-1.6300E-04	7.1016E-04		
-1.8245E-03	-1.5837E-02		
-1.2901E-08	-3.5469E-07		
R Matrix	**2 Rows**	**2 Columns**	
8.0019E-02	1.6258E-04		
1.6258E-04	9.4109E-03		

Table 3.4.2. Discrete Model for Sampling Interval = .03 Sec.

Phi Matrix	4 Rows	4 Columns	
9.9971E-01	-3.7854E-01	-2.0093E-02	-9.6496E-01
-3.4712E-06	9.6900E-01	2.9333E-02	1.6839E-06
2.5597E-07	-1.3854E-01	9.8382E-01	-8.4251E-08
2.6189E-09	-2.0941E-03	2.9767E-02	1.0000E+00
Gam Matrix	**4 Rows**	**2 Columns**	
-1.9500E-01	3.5489E-03		
-7.3405E-03	-1.0381E-02		
-4.5363E-02	-3.9656E-01		
-6.8466E-04	-5.9662E-03		
Q Matrix	**4 Rows**	**4 Columns**	
7.4978E-07	-2.9745E-07	-3.5255E-10	-3.6180E-07
-2.9745E-07	8.7869E-02	-4.9137E-03	-6.8748E-08
-3.5255E-10	-4.9137E-03	8.8637E-02	4.4800E-06
3.6180E-07	-6.8748E-08	4.4800E-06	3.0023E-04
W Matrix	**4 Rows**	**2 Columns**	
-1.4569E-07	-1.3404E-09		
-2.4918E-04	2.5790E-03		
-4.0778E-03	-3.5439E-02		
-4.3452E-06	-1.1955E-06		
R Matrix	**2 Rows**	**2 Columns**	
1.2006E-01	5.4536E-04		
5.4536E-04	1.6741E-02		

Table 3.4.3. Discrete Model for Sampling Interval = .10 Sec.

Phi Matrix	4 Rows	4 Columns	
9.9905E-01	-1.1885E+00	-2.1859E-01	-3.2154E+00
-1.1101E-05	8.8512E-01	9.2281E-02	1.8223E-03
2.6658E-06	-4.3586E-01	9.3174E-01	-2.9155E-06
9.0630E-08	-2.2417E-02	9.6923E-02	9.9999E-01
Gam Matrix	**4 Rows**	**2 Columns**	
-6.3000E-01	1.0718E-01		
-2.8607E-02	-7.7585E-02		
-1.4419E-01	-1.2888E+00		
-7.3792E-03	-6.5259E-02		
Q Matrix	**4 Rows**	**4 Columns**	
2.4976E-06	-3.1953E-06	-3.0913E-08	-4.0174E-06
-3.1953E-06	2.8821E-01	-5.1246E-02	-2.5491E-06
-3.0913E-08	-5.1246E-02	2.8283E-01	4.9393E-05
-4.0174E-06	-2.5491E-06	4.9393E-05	1.0086E-03
W Matrix	**4 Rows**	**2 Columns**	
-1.5976E-06	-2.4818E-07		
5.6752E-03	1.0009E-01		
-4.2932E-02	-3.7628E-01		
-1.5797E-06	-4.3838E-05		
R Matrix	**2 Rows**	**2 Columns**	
4.0222E-01	1.9270E-02		
1.9270E-02	2.0976E-01		

Table 3.4.4. Optimal Solution for Uniformly Sampled System with Sampling Interval = .02 Sec.

Gain for Tau (1) = .20000E-01

G(1) Matrix	2 Rows	4 Columns	
1.8535E-04	7.8963E-02	2.2683E-02	-2.5638E-02
-5.2238E-03	3.1285E-01	1.9756E+00	8.8219E-01

Sensitivity Matrix for Tau (3) = .10000E+00

P Matrix	4 Rows	4 Columns	
1.1994E-04	1.0675E-03	-3.3902E-04	-9.4343E-03
1.0675E-03	1.4353E+00	1.3888E-02	-3.9375E-01
-3.3902E-04	1.3888E-02	1.1996E-01	5.8485E-02
-9.4343E-03	-3.9375E-01	5.8485E-02	1.7723E+00

Table 3.4.5. Optimal Solution for Uniformly Sampled System with Sampling Interval = .03 Sec.

Gain for Tau (2) = .30000E-01

G(1) Matrix	2 Rows	4 Columns	
1.9892E-04	7.7499E-02	1.9770E-02	-2.7282E-02
-4.6552E-03	2.2307E-01	1.7167E+00	7.7317E-01

Sensitivity Matrix for Tau (2) = .30000E-01

P Matrix	4 Rows	4 Columns	
1.1922E-04	1.0879E-03	-2.4621E-04	-9.3346E-03
1.0879E-03	1.4311E+00	2.5550E-03	-3.9702E-01
-2.4621E-04	2.5550E-03	8.6224E-02	4.3228E-02
-9.3346E-03	-3.9702E-01	4.3228E-02	1.7566E+00

Table 3.4.6. Optimal Solution for Uniformly Sampled System with Sampling Interval = .10 Sec.

Gain for Tau (3) = .10000E+00			
G(1) Matrix	2 Rows	4 Columns	
2.2852E-04	7.1368E-02	1.0267E-02	-3.2493E-02
-2.2871E-03	-7.7732E-02	8.3336E-01	3.8282E-01

Sensitivity Matrix for Tau (1) = .20000E-01			
P Matrix	4 Rows	4 Columns	
1.1354E-04	1.1378E-03	-2.3509E-04	-9.1732E-03
1.1378E-03	1.4297E+00	1.7943E-03	-3.9664E-01
-2.3509E-04	1.7943E-03	8.4000E-02	4.1950E-02
-9.1732E-03	-3.9664E-01	4.1950E-02	1.7453E+00

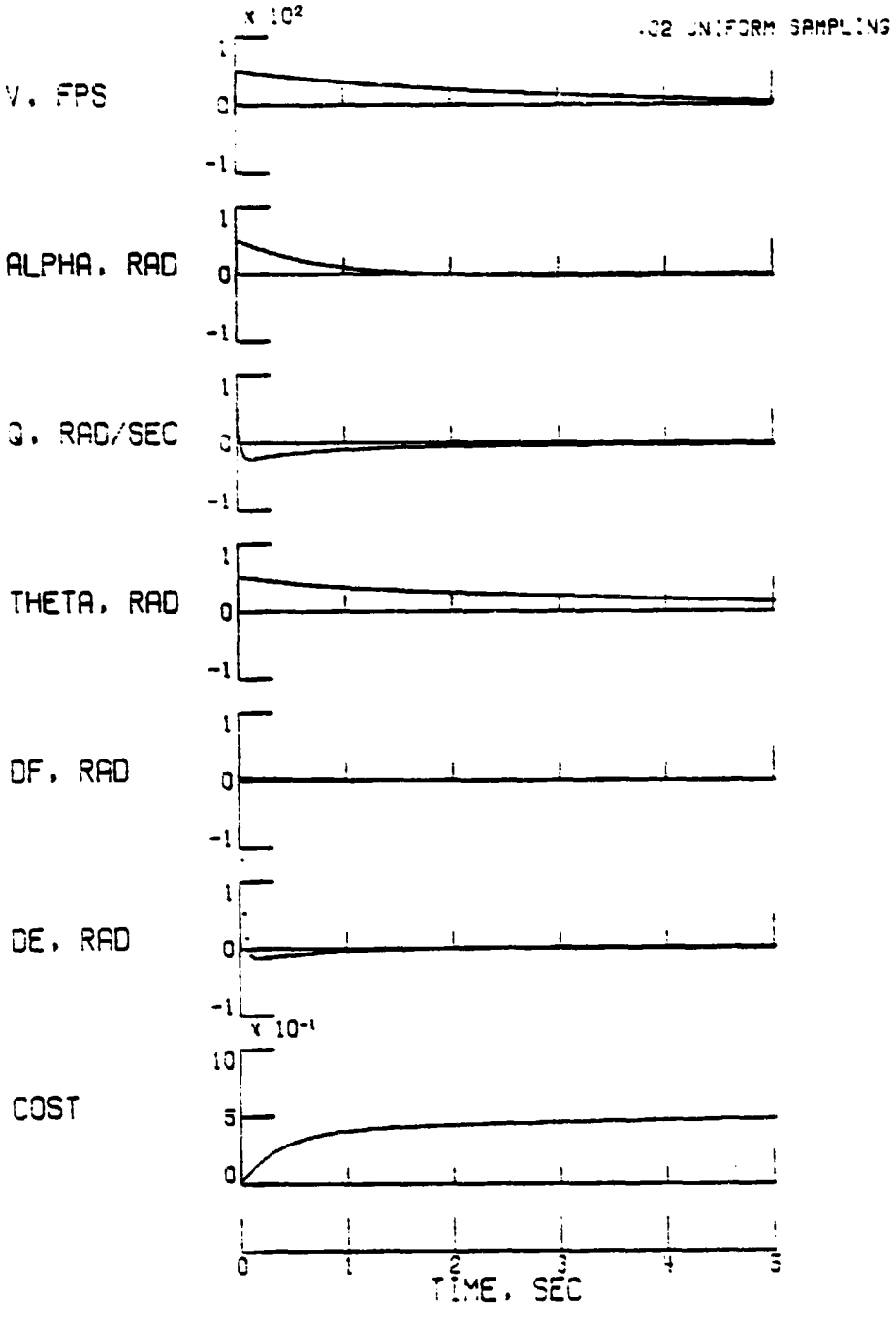


Figure 3.4.1. Response of the Aircraft with .02 Uniform Sampling

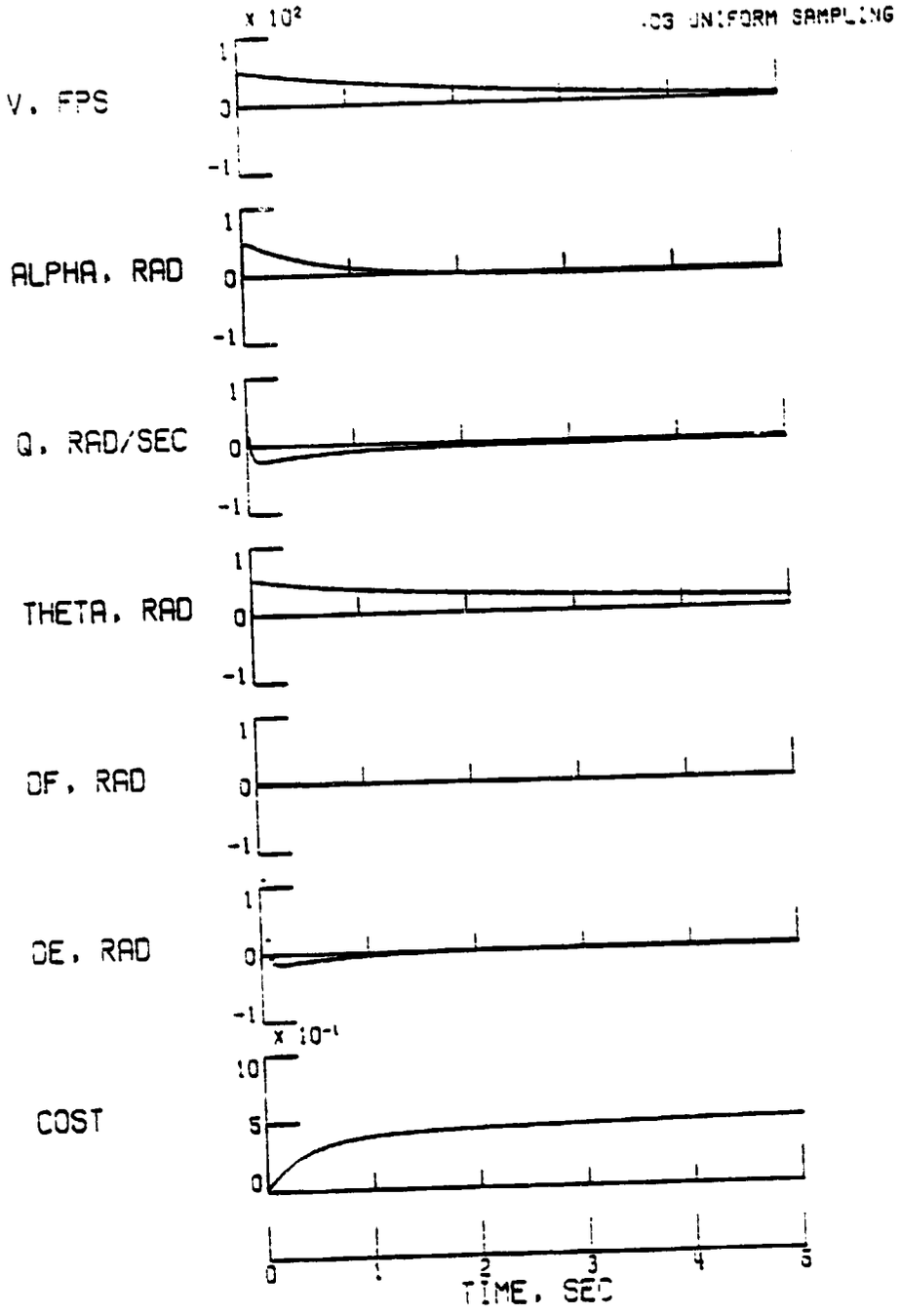


Figure 3.4.2. Response of the Aircraft with .03 Uniform Sampling

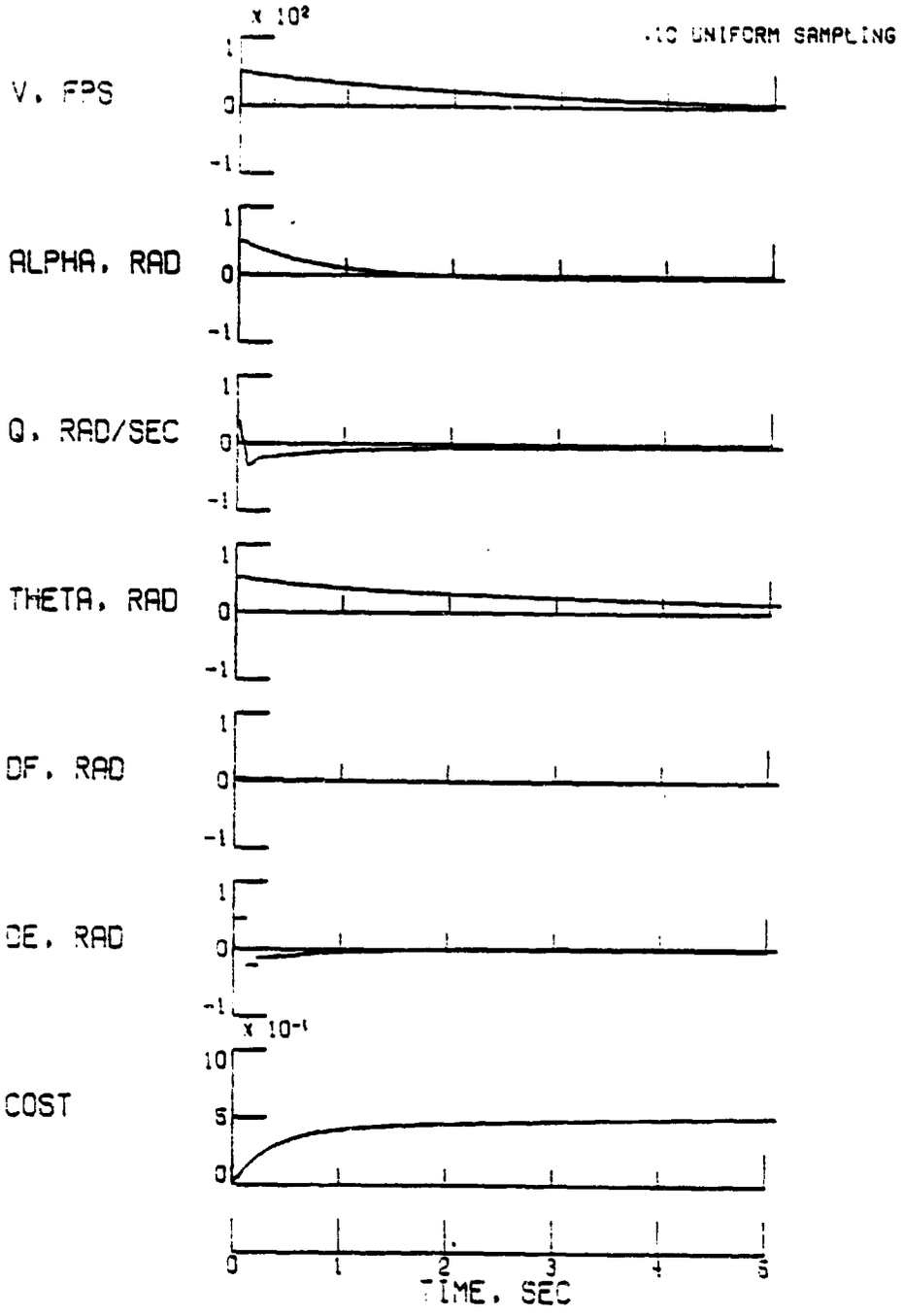


Figure 3.4.3. Response of the Aircraft with .10 Uniform Sampling

Table 3.4.7. Quasi-Steady State Sensitivity Matrix of Stochastic Ricatti Equations.

Sensitivity Matrix for Tau (1) = .20000E-01

P Matrix	4 Rows	4 Columns	
1.1978E-04	1.1080E-03	-2.4544E-04	-9.4068E-03
1.1080E-03	1.4327E+00	1.8046E-03	-4.0116E-01
-2.4544E-04	1.8046E-03	8.5011E-02	4.3119E-02
-9.4068E-03	-4.0116E-01	4.3119E-02	1.7696E+00

Sensitivity Matrix for Tau (2) = .30000E-01

P Matrix	4 Rows	4 Columns	
1.1996E-04	1.0959E-03	-3.0399E-04	-9.4375E-03
1.0959E-03	1.4343E+00	6.2613E-03	-3.9918E-01
-3.0399E-04	6.2613E-03	1.0405E-01	5.2773E-02
-9.4375E-03	-3.9918E-01	5.2773E-02	1.7747E+00

Sensitivity Matrix for Tau (3) = .10000E+00

P Matrix	4 Rows	4 Columns	
1.1985E-04	1.0977E-03	-2.7469E-04	-9.4191E-03
1.0977E-03	1.4338E+00	5.4788E-03	-3.994E-01
-2.7469E-04	5.4788E-03	9.6078E-02	4.7938E-02
-9.4191E-03	-3.9947E-01	4.7938E-02	1.7717E+00

Table 3.4.8. Quasi-Steady State Gain Matrix of Stochastic Riccati Equations.

Gain for Tau (2) = .30000E-01			
G(1) Matrix	2 Rows	4 Columns	
2.1973E-04	7.3006E-02	1.4131E-02	-3.0984E-02
-2.9917E-03	1.2004E-01	1.2040E+00	4.9917E-01
Gain for Tau (1) = .20000E-01			
G(1) Matrix	2 Rows	4 Columns	
1.9223E-04	7.8889E-02	2.2274E-02	-2.6143E-02
-5.3798E-03	3.0465E-01	1.9414E+00	8.9281E-01
Gain for Tau (3) = .10000E+00			
G(1) Matrix	2 Rows	4 Columns	
2.2576E-04	7.4435E-02	1.1344E-02	-3.1862E-02
-2.5799E-03	-5.9389E-02	9.4410E-01	4.3141E-01

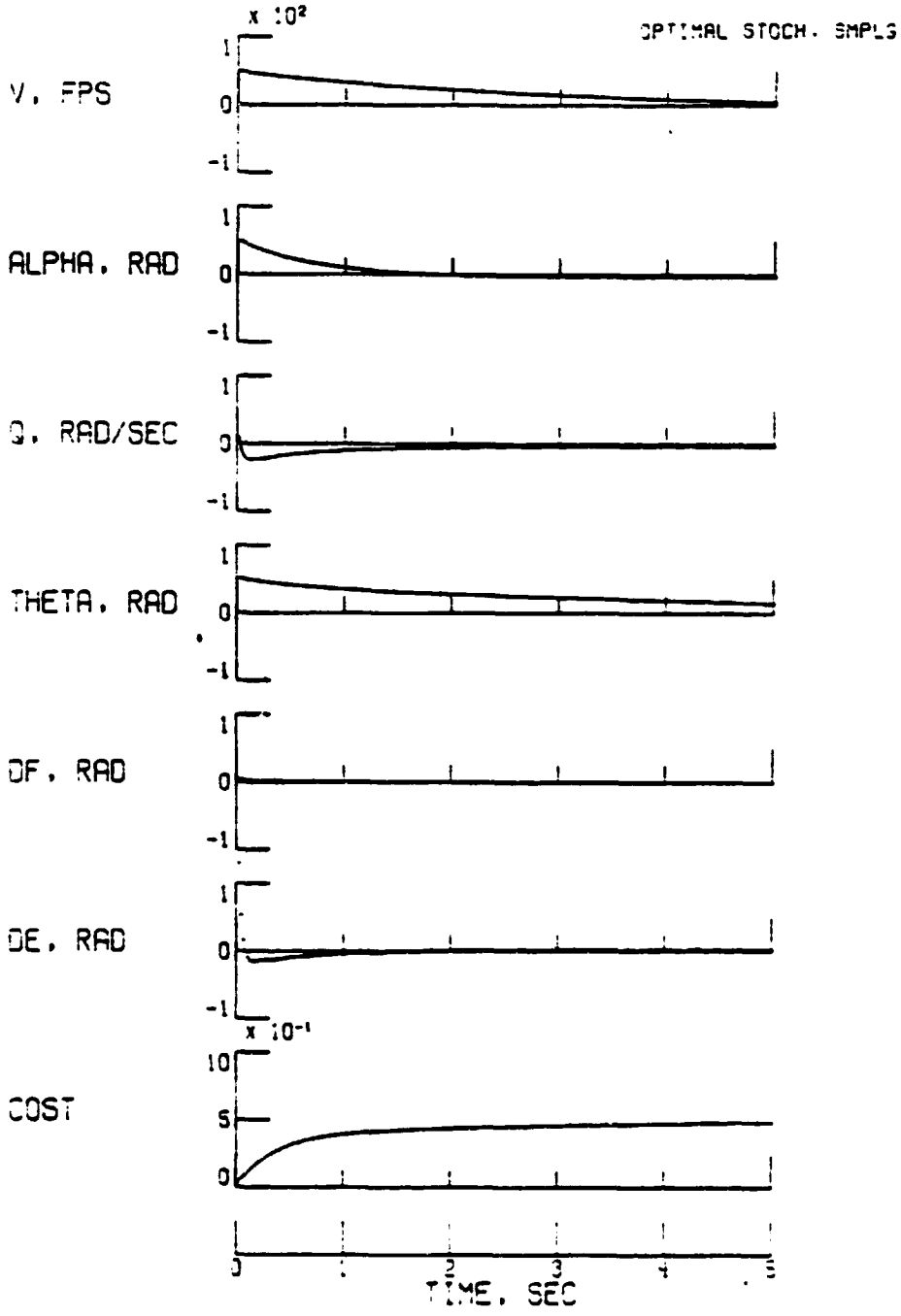


Figure 3.4.4. Response of Randomly-Sampled System Using Optimal Stochastic Gains

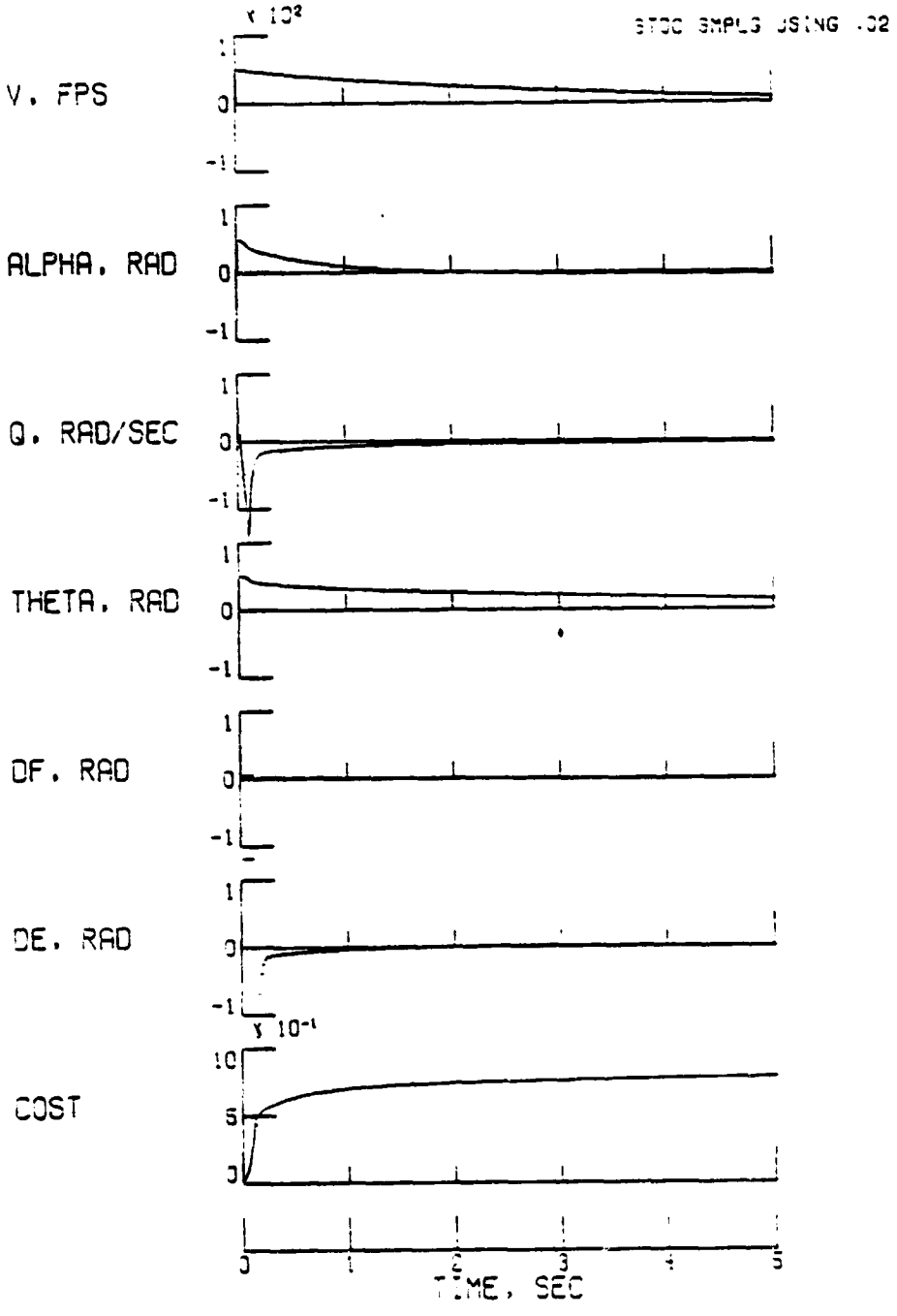


Figure 3.4.5. Response of Randomly-Sampled System Using Optimal Gain Calculated for Uniformly-Sampled System with Sampling Interval = .02 Sec.

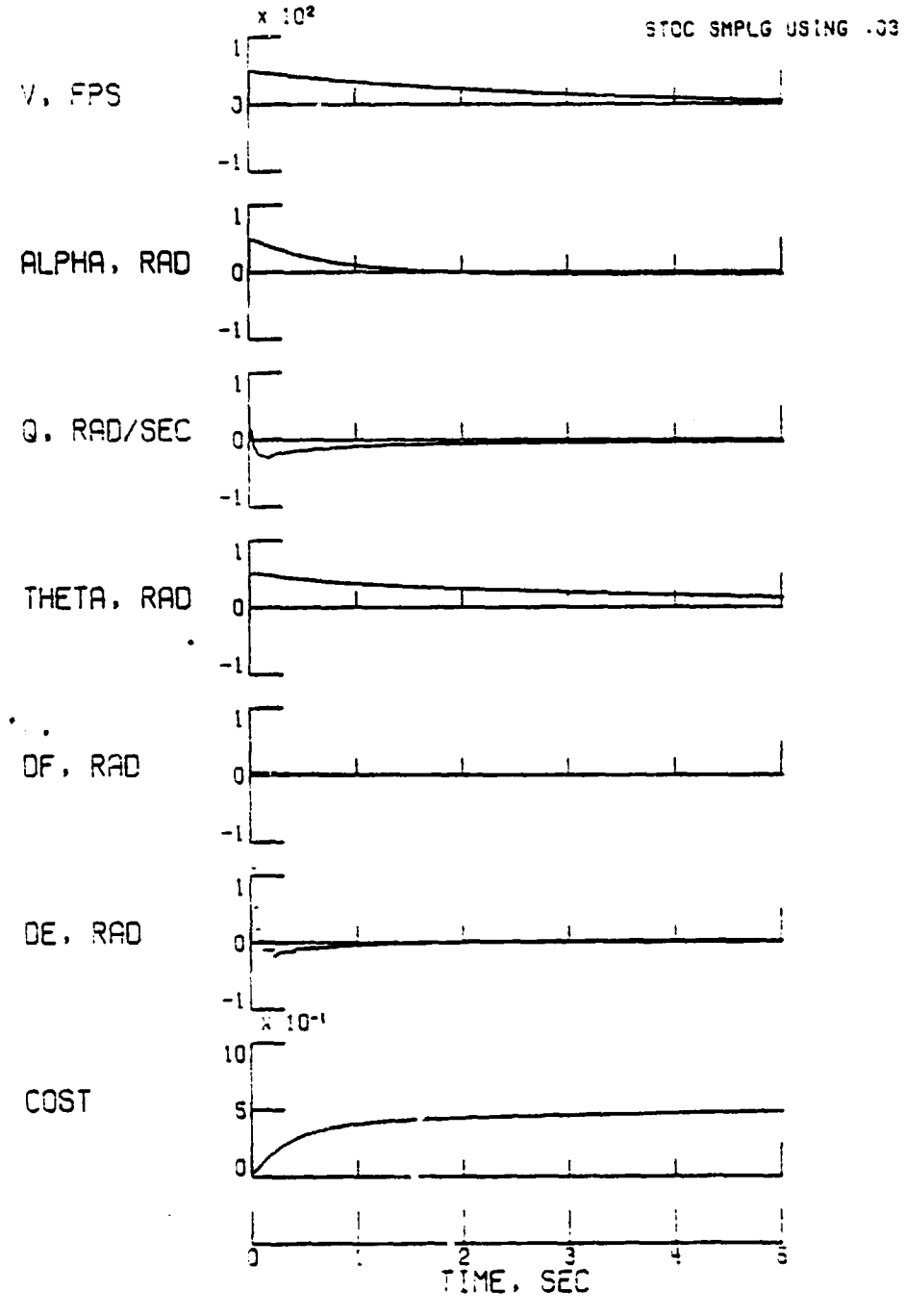


Figure 3.4.6. Response of Randomly-Sampled System Using Optimal Gains Calculated for Uniformly-Sampled System with Sampling Interval = .03 Sec.

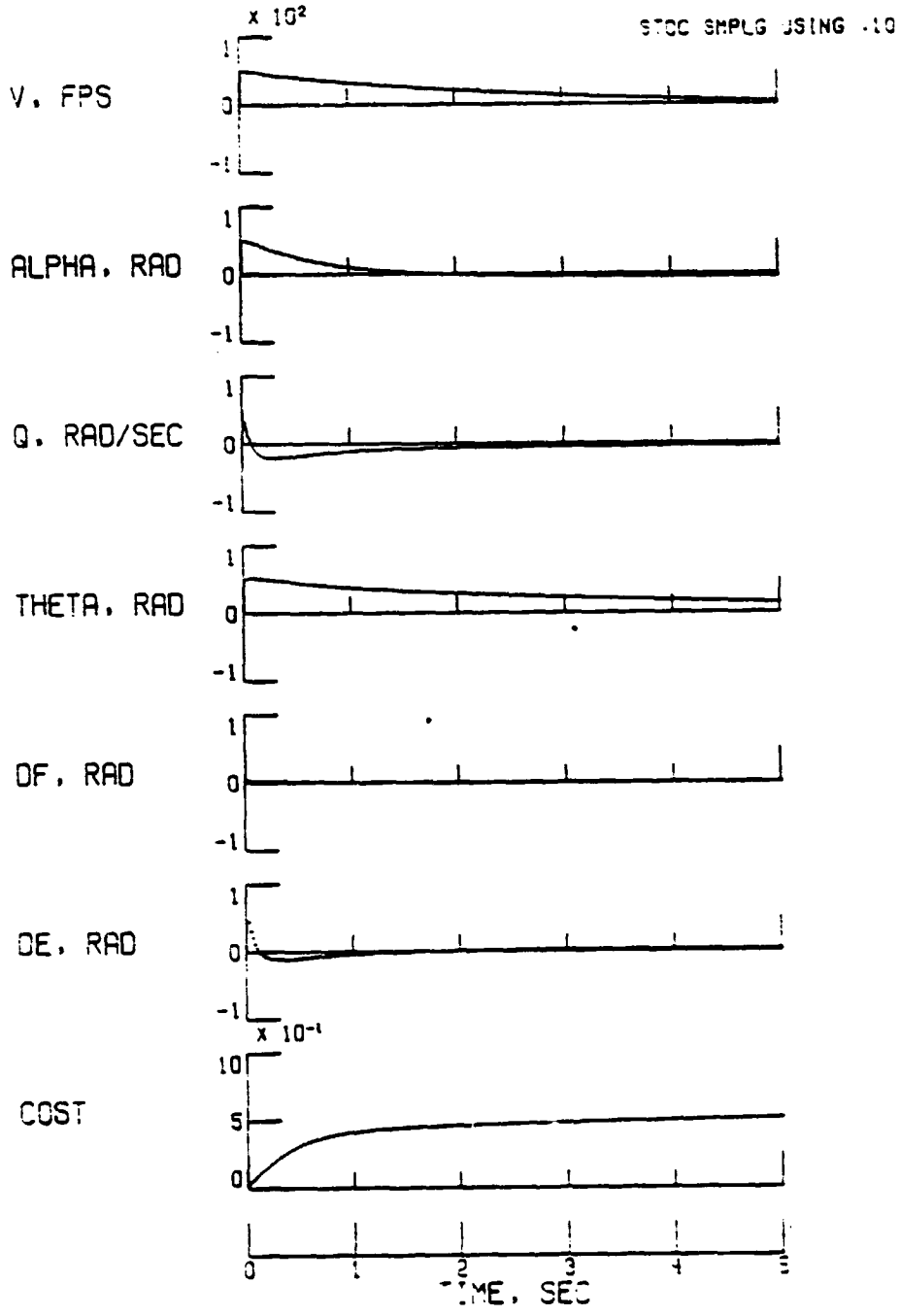


Figure 3.4.7. Response of Randomly-Sampled System Using Optimal Gains Calculated for Uniformly-Sampled System with Sampling Interval = .10 Sec.

Table 3.4.9. Results of Simulation Study: Averages of 50 Random Runs of 10-Second Durations

Gains Used	Optimal Gain for a 0.02 sec. Uniformly-Sampled System	Optimal Gain for a 0.03 sec. Uniformly-Sampled System	Optimal Gain for a 0.10 sec. Uniformly-Sampled System	Unaugmented System	System with Optimal Stochastic Gains
Average Cost	0.6158	0.5526	0.5372	1.427	0.5174

4. RANDOMLY-SAMPLED STOCHASTIC SYSTEM CONTROL

4.1 Introduction

In Chapter (2) optimal control of linear systems with stochastic sampling was discussed. A more practical concern is with a more general stochastic control problem. Specifically, we focus our attention on a randomly-sampled linear process model in which additive process and measurement noise are present. The control effectors in the process have variable sampling intervals between information updates. The information update intervals are modeled as an independent stochastic process with known transition probabilities relating the duration of adjacent intervals. We will minimize the mathematical expectation of a quadratic form in the state variables and control variables. As in the optimal control of stochastic systems with uniform sampling, the determination of the optimal strategy involves two problems, i.e. the problem of optimal control and the problem of optimal estimation. On the standard deterministic linear-quadratic-Gaussian (LQG) problem the separation theorem for the control and estimation holds [3], [4]. In this case the stochastic optimal feedback controller can be synthesized by cascading an optimal estimation (Kalman-Bucy filter) with the deterministic optimal control [21], [22]. Unfortunately, in most stochastic control systems with multiplicative noise, the optimal solution does not separate in the sense that the filter gains are not independent of the control computation [23], [24], [25], [26].

In other words, the optimal control has to perform simultaneously the estimation and control of the state in order to minimize the expected cost of some real-valued cost function.

For the stochastic control system with random sampling, the optimal estimator is equivalent to the time-varying Kalman filter because the estimator has perfect recall of the previous sampling intervals. Furthermore, the separation theorem still holds. The separation of estimation and control is important because it considerably simplifies the implementation of the stochastic optimal controller for randomly-sampled systems [4], [27]. In section (4.2) we discussed the formulation of the stochastic control system with random sampling. Section (4.3) solves the optimal estimation and one-stage prediction problems for randomly-sampled stochastic systems. Section (4.4) presents the Separation Theorem, and section (4.5) provides some simulation results.

4.2 The Formulation of the Stochastic Control Problem

In a manner similar to our treatment of the deterministic system with random sampling, the control effectors are assumed to receive state information at time σ_k for $k = 1, 2, 3, \ldots$ and the control actions are subject to the constraint $u(\sigma) = u(\sigma_k)$ for $\sigma_k \leq \sigma \leq \sigma_{k+1}$, $k = 1, 2, 3, \ldots$ The sequence $\{\sigma_k\}$ is random and can be described by the incremental process $\{t_k\}$ with an initial value σ_o, so that

$$\sigma_k = \sigma_o + \sum_{j=1}^{k} t_j \qquad k = 1, 2, 3, \ldots \tag{4.2.1}$$

with $\{t_k\}$ a stochastic process. The systems we are considering are governed by the following stochastic difference equations

$$x(k+1) = \Phi(k+1, k)\, x(k) + \psi(k+1, k)\, w(k) + \Gamma(k+1, k)\, u(k) \tag{4.2.2}$$

with the measurement equations

$$y(k+1) = Hx(k+1) + \theta(k+1) \tag{4.2.3}$$

where $w(k)$ and $\theta(k+1)$ are mutually independent, zero-mean Gaussian white noises with known statistics given by

$$E\{w(k), w(k)\} = \alpha(k) \tag{4.2.4}$$

$$E\{\theta(k), w(k)\} = 0 \tag{4.2.5}$$

$$E\{\theta(k), \theta(k)\} = \beta(k) \tag{4.2.6}$$

It should be pointed out that $\{w(k)\}$, $\{\theta(k)\}$ and $\{t_k\}$ are mutually independent processes, but that the matrices $\Phi(k+1, k)$, $\Gamma(k+1, k)$ and $\psi(k+1, k)$ depend on t_k. It is also assumed that $\{w(k)\}$, $\{\theta(k)\}$ and $\{t_k\}$ are independent of $x(N)$ and the initial state $x(0)$, which has a normal distribution with

$$E\{x(0)\} = m \tag{4.2.7}$$

$$\text{Cov}\{x(0), x(0)\} = \gamma(0) \tag{4.2.8}$$

The optimal stochastic control problem is to determine a closed-loop control strategy based on the past and current

measurements, past control and past sampling history to minimize the mathematical expectation of a quadratic function of the state and control variables.

$$V(N) = \underset{u(0), \ldots u(N-1)}{\text{Min}} \; E\{ \sum_{i=1}^{N} x^T(i)\, Q(i)\, x(i) + u^T(i-1)\, R(i-1)\, u(i-1)\} \tag{4.2.9}$$

subject to the dynamics of Equation (4.2.2) and the measurements of Equation (4.2.3). The weighting matrices Q(i) are assumed to be positive semi-definite and the matrices R(i), positive definite. Similar to the deterministic control problem in Chapter (2), we denote the entire sampling history by T(k)

$$T(k) = (t_0, t_1, \ldots, t_k) \tag{4.2.10}$$

where $T(k) \in \mathcal{T}(k)$.

In analogy to T(k), the vector Y(k) is introduced here to describe the entire measurement history

$$Y(k) = \{y(o), \ldots, y(k)\} \tag{4.2.11}$$

where $Y(k) \in \mathcal{Y}(k)$.

4.3 Optimal Estimation for Stochastic Systems with Random Sampling

The optimal estimation problem is to obtain a best estimate $\hat{x}(k)$ for x(k) given the measurement history Y(k) and sampling history T(k). For the system we have been considering if T(k) is given, all system matrices $\Phi(t)$, $\psi(t)$ and $\Gamma(t)$ are known for $t = 1, \ldots, k$.

The optimal estimator is, therefore, same as the standard Kalman filter up to t = k, which satisfies the recursive relations stated in the following theorem:

Theorem 4.3.1 For the stochastic system with random sampling, the optimally filtered estimate $\hat{x}(k + 1/k + 1)$ is given by the recursive relation

$$\hat{x}(k + 1/k + 1) = \Phi(k + 1, k)\, \hat{x}(k/k) + \Gamma(k + 1, k)\, u(k) + K(k + 1)[y(k + 1) - H\Phi(k + 1, k)\, \hat{x}(k/k)] \tag{4.3.1}$$

where the gain matrix is determined by the following recursive relations

$$K(k) = \Phi(k + 1, k)\, P(k)\, H^T[HP(k)\, H^T + \alpha(k)]^{-1} \tag{4.3.2}$$

$$P(k + 1) = [\Phi(k + 1, k) - K(k)\, H]\, P(k)\, [\Phi(k + 1, k) - K(k)\, H]^T + \beta(k) + K(k)\, \alpha(k)\, K(k)^T$$

$$\text{for } k = 1, 2, \ldots \tag{4.3.3}$$

where P(k) is the covariance of the vector estimation error. The initial conditions are

$$P(0) = \gamma(0)$$

$$\hat{x}(0) = m$$

The filter error $\{\tilde{x}(k + 1/k + 1)\}$ defined as

$$\tilde{x}(k + 1/k + 1) = x(k + 1) - \hat{x}(k + 1/k + 1)$$

for k = 0, 1, . . . is a zero-mean Gauss-Markov sequence whose covariance is given by (4.3.3).

The Kalman filter (4.3.1-4.3.3) for the stochastic system with random sampling is, however, time-variant because the system matrices switch randomly among M possible configurations. In general, the gain matrix K(k + 1) is not expected to reach any type of steady-state. But there is no need to store any measurement data since the measurements can be processed on line as they occur. $\hat{x}(k/k)$ is the only estimate of state needed to be stored at time k. Since we will not have a steady-state gain, all possible configurations must be stored in order to calculate the optimal gain which is usually time-variant. Figure 4.3.1 gives the block diagram for the stochastic system with random sampling showing the information flow inside the filter. It should be emphasized that the duration of the last interval along with the new measurement must be provided to the filter in order to proceed one step in time.

The prediction problem for a stochastic system with random sampling is considerably complicated in that the predictor does not have exact knowledge of future sampling intervals and instead knows only their probability distributions. For simplicity, let us assume the sampling rate switches randomly among m possible discrete values. The $i^{\underline{th}}$ of m linearized models is represented by Φ_i, Ψ_i and Γ_i, the set of which is denoted by S_i. Since the discrete-time system constantly switches among m models, $S_i(k)$ represents the ith model (configuration) at the kth sampling instant. Since the stochastic

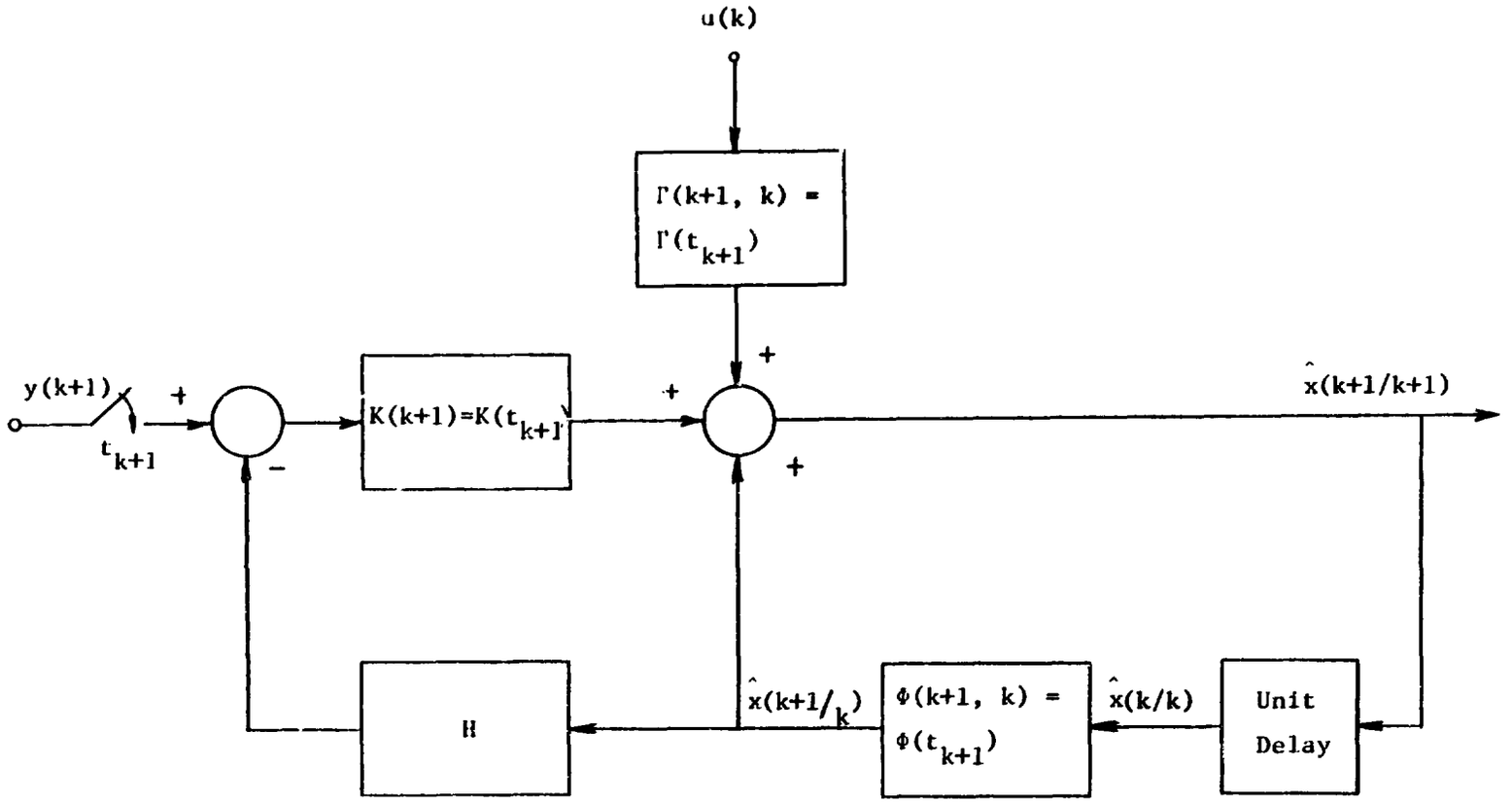


Figure 4.3.1. Block Diagram of a Kalman Filter for a Randomly-Sampled System

sampling process is modeled as a semi-Markov process in which the adjacent sampling intervals switch according to the transition probability matrix, the probability of ith model at time k + 1 is conditioned on the last sampling interval value and represented by

$$\Pi_i(k+1) = \text{Prob. } \{S_i(k+1)/t_k\} \tag{4.3.4}$$

For simplicity we will write the system model (4.2.2) in the following form

$$x(k+1) = \Phi(k)\, x(k) + \Gamma(k)\, u(k) + \psi(k)\, w(k) \tag{4.3.5}$$

$$y(k) = Hx(k) + \Theta(k) \tag{4.3.6}$$

where

$$\Phi(k) \in \{\Phi_1, \ldots, \Phi_m\}$$

$$\psi(k) \in \{\psi_1, \ldots, \psi_m\}$$

$$\Gamma(k) \in \{\Gamma_1, \ldots, \Gamma_m\}$$

for k = 1, ..., n

$$S_i \triangleq \{\Phi_i, \psi_i, \Gamma_i\}$$

The optimal predition of the state x(k + 1) at time k + 1 is

$$\bar{x}(k+1) = E[x(k+1)/\hat{x}(k)] \tag{4.3.7}$$

$$\bar{x}(k+1) = \int xp\ \{x(k+1) = x/\hat{x}(k)\}\, dx \tag{4.3.8}$$

Using $S_i(k)$ to represent the ith configuration at time σ_k, the conditional probability in Equation (4.3.6) can be written as

$$P(x(k+1) = x/\hat{x}(k))$$

$$= \sum_{i=1}^{M} P(x(k+1) = x/\hat{x}(k),\ S_i(k+1))\ \Pi_i(k+1) \qquad (4.3.9)$$

Combining equations (4.3.9) and (4.3.4), we have

$$\bar{x}(k+1) = \int x \sum_{i=1}^{m} P(x(k+1) = x/\hat{x}(k),\ S_i(k+1))\ dx\ \Pi_i(k+1)$$

$$\bar{x}(k+1) = \sum_{i=1}^{m} \left[\int xP(x(k+1) = x/\hat{x}(k),\ S_i(k+1))\ dx\right] \Pi_i(k+1)$$

$$\bar{x}(k+1) = \sum_{i=1}^{m} \bar{x}_i(k+1)\ \Pi_i(k+1)\ . \qquad (4.3.10)$$

Since $\bar{x}_i(k+1)$ can be obtained from the standard Kalman filter algorithm for predition,

$$\bar{x}_i(k+1) = \Phi_i\ \hat{x}(k) + \Gamma_i\ u(k) \qquad (4.3.11)$$

Hence, Equation (4.3.10) becomes

$$\bar{x}(k+1) = \sum_{i=1}^{m} (\Phi_i\ \hat{x}(k) + \psi_i\ u(k))\ \Pi_i(k+1) \qquad (4.3.12)$$

$$\bar{x}(k+1) = \bar{\Phi}(k)\ \hat{x}(k) + \bar{\psi}(k)\ u(k) \qquad (4.3.13)$$

where $\bar{\Phi}(k)$ and $\bar{\Gamma}(k)$ denote the expected values of the corresponding matrices conditioned on knowledge of previous sampling intervals. Equation (4.3.13) is used for one-stage prediction.

4.4 A Separation Theorem for Randomly-Sampled Systems

In a manner analogous to our treatment of the randomly-sampled system without noise in Chapter 2, we will prove the Separation Property for randomly-sampled systems by dynamic programming. First we consider a single-stage optimization problem over the last interval which starts at σ_{N-1} and terminates at σ_N, and then we generalize the result to all N stages by mathematical induction.

Single-Stage Problem

The optimal terminal cost for the single-stage process which starts at time N - 1 and ends at time N is equal to

$$V(1) = \min_{u(n-1)} E\{x^T(N)\, Q(N)\, x(N) + u^T(N-1) \cdot R(N-1)\, u(N-1)\} \tag{4.4.1}$$

From the equation of our system model, we have

$$x(N) = \Phi(N, N-1)\, x(N-1) + \psi(N, N-1) w(N-1) + \Gamma(N, N-1)\, u(N-1) \tag{4.4.2}$$

Introducing equation (4.4.2) into (4.4.1), we have

$$V(1) = \min_{u(N-1)} E\{(\Phi x + \psi w + \Gamma u)^T Q (\Phi x + \psi w + \Gamma u) + u^T R u\} \tag{4.4.3}$$

where $x = x(N-1)$, $w = w(N-1)$, $u = u(N-1)$,

$$Q = Q(N),\ R = R(N-1),\ \Phi = \Phi(N, N-1),$$

$$\Gamma = \Gamma(N, N-1) \text{ and } \psi = \psi(N, N-1) .$$

After some manipulation, we have

$$V(1) = \min_u E\{x^T\Phi^T Q\Phi X^T + 2x^T\Phi^T Q\psi w + 2x^T\Phi^T Q\Gamma u + 2w^T\psi^T Q\Gamma u + w^T\psi^T Q\psi w + u^T(\Gamma^T Q\Gamma + R)\ u\} \qquad (4.4.4)$$

According to our assumptions, x(i), w(i) and S(i + 1) are statistically independent of each other and w(i) has zero mean. Therefore, the expected value of the second, third and fourth terms in Equation (4.4.4) are zero. The optimal cost becomes

$$V(1) = \min_u E\{x^T\Phi^T Q\Phi x + 2x^T\Phi^T Q\Gamma u + w^T\psi^T Q\psi w + u^T(\Gamma^T Q\Gamma + R)\ u\} .$$

Using a property of conditional expectation; namely, that $E\{x\} = E\{E(x/y)\}$, where the outer expectation is over y, we have

$$V(1) = \min_u E\{E[x^T\Phi^T Q\Phi x + 2x^T\Phi^T Q\Gamma u + w^T\psi^T Q\psi w + u^T(\Gamma^T Q\Gamma + R)\ u/y(N-1), t_{N-1}]\} .$$

We will evaluate the second term of the above expression to show how the conditional expectation should be evaluated

$$E\{2x^T\phi^T Q\Gamma u/y(N-1),\ t_{N-1}\}$$

$$= E\{2x^T\phi^T Q\Gamma/y(N-1),\ t_{N-1}\}\ u \tag{4.4.5}$$

$$= \sum_{i=1}^{m} [E\{2x^T\phi^T Q\Gamma/y(N-1),\ Si(N)\}]\cdot \Pi_i(N)\cdot u \tag{4.4.6}$$

where

$$\Pi_i(k) = \text{Prob}\{S_i(k)/t_{k-1}\}$$

The expected value can be evaluated as the following

$$E\{2x^T\phi^T Q\Gamma/y(N-1),\ S_i(N)\} = 2\hat{x}\phi^T{}_i Q_i \Gamma_i \tag{4.4.7}$$

where the index i denotes the ith configuration Taking the summation over i, we have

$$E\{2x^T\phi^T Q\Gamma u/y(N-1),\ t_{N-1}\} = 2\hat{x}\ \overline{\phi^T Q\Gamma}\cdot u \tag{4.4.8}$$

where $\overline{\phi^T Q\Gamma}$ denotes the expected value of $\phi^T Q\Gamma$ given the value of the last sampling interval. Taking the gradient of V(1) with respect to u and setting the result equal to zero, we have

$$u = -(\overline{\Gamma^T Q\Gamma} + \bar{R})^{-1}\ \overline{\phi^T Q\Gamma}\ \hat{x} \tag{4.4.9}$$

It should be emphasized that $\hat{x}$ is the optimal estimate of x(N - 1) from the Kalman filter algorithm. Substituting the equation for u into the equation for V(1),

$$V(1) = E\{x^T \overline{\Phi^T Q \Phi} x - 2x^T \overline{\Phi^T Q \Gamma}\ \overline{(\Gamma^T Q \Gamma + \bar{R})^{-1}}\ \overline{\Gamma^T Q \Phi}\ \hat{x}$$

$$+ w^T \overline{\Psi^T Q \Psi} w + \hat{x}^T\ \overline{\Phi^T Q \Gamma}\ \overline{(\Gamma^T Q \Gamma + \bar{R})^{-1}}\ \overline{\Gamma^T Q \Phi}\ \hat{x}\} \qquad (4.4.10)$$

After some manipulation,

$$V(1) = E\{x^T [\overline{\Phi^T Q \Phi} - \overline{\Phi^T Q \Gamma}\ \overline{(\Gamma^T Q \Gamma + \bar{R})^{-1}}\ \overline{\Gamma^T Q \Phi}]\ x$$

$$+ E\{\tilde{x}^T [\overline{\Phi^T Q \Gamma}\ \overline{(\Gamma^T Q \Gamma + \bar{R})^{-1}}\ \overline{\Gamma^T Q \Phi}]\ \tilde{x} + w^T \overline{\Psi^T Q \Psi} w\} \qquad (4.4.11)$$

$$\text{where } \tilde{x} = x - \hat{x} \qquad (4.4.12)$$

Summarizing the results for the single-stage problem where $P(N) = Q(N)$, we have

$$u(N-1) = S(N-1)\ \hat{x}(N-1/y(N-1),\ t_{N-1}) \qquad (4.4.13)$$

where

$$S(N-1) = -[\overline{\Gamma^T(N, N-1)\ P(N)\ \Gamma(N, N-1)}$$

$$+ \overline{R(N-1)}]^{-1}\ \overline{\Gamma^T(N, N-1)\ P(N)\ \Phi(N, N-1)}$$

$$V(1) = E\{x^T(N-1)\ M(N-1)\ x(N-1) + \alpha(N-1)\} \qquad (4.4.14)$$

where

$$M(N-1) = \overline{\Phi^T(N, N-1)\ P(N)\ \Phi(N, N-1)} - \overline{\Phi^T(N, N-1)}$$

$$\cdot\ \overline{P(N)\ \Gamma(N, N-1)}\ \overline{(\Gamma^T(N, N-1)\ P(N)\ \Gamma^T(N, N-1)}$$

$$+ \overline{R(N-1)})^{-1}\ \overline{\Gamma^T(N, N-1)\ P(N)\ \Phi(N, N-1)} \qquad (4.4.15)$$

$$u(N-1) = E\{\hat{x}(N-1)[\overline{\Gamma^T(N, N-1)\, R(N)\, \Gamma(N, N-1)} \cdot$$

$$\overline{(\Gamma^T(N, N-1)\, P(N)\, \Gamma(N, N-1)} + \overline{R(N-1)}^{-1} \cdot$$

$$\overline{\Gamma^T(N, N-1)\, P(N)\, \Gamma(N, N-1)}]\, \hat{x}(N-1)$$

$$+ w(N-1)^T\, \overline{\psi(N, N-1)^T\, P(N)\, \psi(N, N-1)}\, w(N-1)\} \qquad (4.4.16)$$

Double-Stage Problem

The optimal cost for the double-stage problem is

$$V(2) = \min_{u(N-2)} \min_{u(N-1)} E\{[x^T(N-1)\, P(N-1)\, x(N-1)$$

$$+ u^T(N-2)\, R(N-2)\, u(N-2)] + [x^T(N)\, P(N)\, x(N)$$

$$+ u^T(N-2)\, R(N-1)\, u(N-1)]\} \qquad (4.4.17)$$

From the Principle of Optimality and the expression for V(1)

$$V(2) = \min_{u(N-2)} E\{x^T(N-1)\, Q(N-1)\, x(N-1)$$

$$+ u^T(N-2)\, R(N-2)\, u(N-2) + V(1)\} \qquad (4.4.18)$$

From Equation (4.4.14)

$$V(2) = \min_{u(N-2)} E\{x^T(N-1)\, Q(N-1)\, x(N-1)$$

$$+ u^T(N-2)\, R(N-2)\, u(N-2) + E[x^T(N-1)\, M(N-1)$$

$$x(N-1)] + \alpha(N-1)\} \qquad (4.4.19)$$

Hence, Equation (4.4.18) can be simplified as

$$V(2) = \min_{u(N-2)} E \{[x^T(N-1)(Q(N-1) + M(N-1))\, x(N-1) + u^T(N-2)\, R(N-2)\, u(N-2)] + \alpha(N-1)\} \quad (4.4.20)$$

Since the choice of $u(N-2)$ will not affect $\alpha(N-1)$, only the first two terms of (4.4.20) will be involved in optimization.

Defining $P(N-1) = M(N-1) + Q(N-1)$,

$$V(2) = \min_{u} E \{x^T(N-1)\, P(N-1)\, x(N-1) + u^T(N-2)\, R(N-2)\, u(N-2) + \alpha(N-1)\} \quad (4.4.21)$$

From the equation of our system model,

$$x(N-1) = \Phi(N-1, N-2)\, x(N-2) + \psi(N-1, N-2)\, w(N-2) + \Gamma(N-1, N-2)\, u(N-2) \quad (4.4.22)$$

Comparing Equations (4.4.21) and (4.4.22) with Equations (4.4.1) and (4.4.2), the equations for the double-stage problem are of the same form as the equations for the single-stage problem with the exception of the additive term $\alpha(N-1)$ which is unrelated to the optimization. The optimization can, therefore, be carried out in the same manner as in the single-stage problem.

The results are

$$u(N-2) = S(N-2)\, \hat{x}(N-2/y(N-2), t_{N-2}) \quad (4.4.23)$$

$$S(N-2) = -\overline{[\Gamma^T(N-1,N-2)\,P(N-1)\,\Gamma(N-1,N-2) + R(N-2)]}^{-1}$$
$$\cdot\,\overline{\Gamma^T(N-1,N-2)\,P(N-1)\,\Phi(N-1,N-2)} \qquad (4.4.24)$$

$$V(2) = E\{x^T(N-2)\,M(N-2)\,x(N-2) + \alpha(N-2)\} \qquad (4.4.25)$$

$$P(N-1) = M(N-1) + Q(N-1) \qquad (4.4.26)$$

where

$$M(N-2) = \overline{\Phi^T(N-1,N-2)\,P(N-1)\,\Phi(N-1,N-2)}$$
$$-\,\overline{\Phi^T(N-1,N-2)\,P(N-1)\,\Gamma(N-1,N-2)}\cdot$$
$$\overline{[\Gamma^T(N-1,N-2)\,P(N-1)\,\Gamma(N-1,N-2) +}$$
$$\overline{R(N-2)]}^{-1}\,\overline{\Gamma(N-1,N-2)\,P(N-1)\,\Phi(N-1,N-2)} \qquad (4.4.27)$$

and

$$\alpha(N-2) = E\{-\bar{x}^T(N-2)[\Phi^T(N-1,N-2)\,P(N-1)\,\Gamma(N-1,N-2)$$
$$\cdot\,S(N-2)]\,\bar{x}(N-2) + w^T(N-2)\,\psi^T(N-1,N-2)$$
$$\cdot\,P(N-1)\,\psi(N-1,N-2)\,w(N-2)\} + \alpha(N-1) \qquad (4.4.28)$$

Utilizing the Principle of Induction, we can prove the Separation Theorem for the general N-stage problem by carrying out the derivation for the (k-1)-stage and k-stage problem. We will first assume the recursive relations (4.4.24), (4.4.26), (4.4.27) and (4.4.28)

hold for the (k-1)-stage problem and then prove the Separation Theorem by showing that they also hold for k-stage problem.

(k-1)-Stages

The optimal strategy at time N - k + 1 for the (k-1)-stage process is characterized by the following equations

$$u(N \cdot k + 1) = S(N - k + 1)\, \hat{x}(N - k + 1/y(N - k + 1),\ t_{N-k+1}) \tag{4.4.29}$$

$$S(N-k+1) = -\overline{[\Gamma^T(N-k+2,\ N-k+1)\ P(N-k+2)\ \Gamma(N-k+2,\ N-k+1) + \overline{R(N-k+1)}]^{-1}} \cdot \overline{\Gamma^T(N-k+2,\ N-k+1)\ P(N-k+2)\ \Phi(N-k+2,\ N-k+1)}$$

$$P(N - k + 2) = M(N - k + 2) + Q(N - k + 2) \tag{4.4.30}$$

$$V(k-1) = E\ \{x^T(N-k+1)\ M(N-k+1)\ x(N-k+1) + \alpha(N-k+1)\} \tag{4.4.31}$$

where

$$M(N-k+2) = \overline{\Phi^T(N-k+2,\ N-k+1)\ P(N-k+2)\ \Phi(N-k+2,\ N-k+1)} \cdot \overline{[\Gamma^T(N-k+2,\ N-k+1)\ P(N-k+2)\ \Gamma(N-k+2,\ N-k+1) + \overline{R(N-k+1]}^{-1}} \cdot \overline{\Gamma(N-k+2,\ N-k+1)\ P(N-k+2)\ \Phi(N-k+2,\ N-k+1)} \tag{4.4.32}$$

and

$$\alpha(N-k+1) = E\ \{-\tilde{x}^T(N-k+1)[\Phi^T(N-k+2,\ N-k+1)\ P(N-k+2) \cdot \Gamma(N-k+2,\ N-k+1)\ S(N-k+1)]\ \tilde{x}(N-k+1) +$$ (continued)

$$w^T(N-k+1)\ \psi^T(N-k+2,\ N-k+1)\ P(N-k+2,\ N-k+1)\ w(N-k+1)\}$$

$$+\ \alpha(N-k+2) \qquad (4.4.33)$$

k Stages

Using the Principle of Optimality the cost function for the k-stage problem can be written as

$$V(k) = \min_{u(N-j)} E\ \{x^T(N-j+1)\ Q(N-j+1)\ x(N-j+1)$$

$$+\ u^T(N-j)\ R(N-j)\ u(N-j) + V(K-1)] \qquad (4.4.34)$$

Combining (4.4.34) and (4.4.31),

$$V(k) = \min_{u(N-j)} E\ \{x^T(N-j+1)\ P(N-j+1)\ x(N-j+1)$$

$$+\ u^T(N-j)\ R(N-j)\ u(N-j) + \alpha(N-j+1)\} \qquad (4.4.35)$$

where

$$P(N-j+1) = M(N-j+1) + Q(N-j+1)$$

From the equations of system dynamics

$$x(N-j+1) = \Phi(N-j+1,\ N-j)\ x(N-j)$$

$$+\ \psi(N-j+1,\ N-j)\ w(N-j)$$

$$+\ \Gamma(N-j+1,\ N-j)\ u(N-j) \qquad (4.4.36)$$

With the exceptions of the difference in time arguments and the additive term $\alpha(N-j+1)$ which is unrelated to the minimization,

the equations (4.4.35) and (4.4.36) are in the same form as the equations (4.4.21) and (4.4.22). The optimization can be carried out by repeating the similar steps. We will state the results in the following theorem.

Theorem 4.4.1 (The Separation Theorem)

The optimal strategy for the stochastic linear system with random sampling can be implemented by cascading the optimal Kalman filter with the optimal feedback gain matrix of the probabilistic linear regulator. The parameters for the two parts of the control system can be determined separately. The optimal Kalman filter is governed by equations (4.3.1-4.3.3). The probabilistic linear regulator satisfies the following recursive relations:

$$P(k+1) = M(k+1) + Q(k+1) \tag{4.4.37}$$

$$S(k) = -[\overline{\Gamma^T(k+1, k)\ P(k+1)\ \Gamma(k+1, k)} + \overline{R(k)}]^{-1}$$

$$\overline{\Gamma^T(k+1, k)\ P(k+1)\ \Phi(k+1, k)} \tag{4.4.38}$$

$$M(k) = \overline{\Phi^T(k+1, k)\ P(k+1)\ \Phi(k+1, k)} +$$

$$\overline{\Phi^T(k+1, k)\ P(k+1)\ \Gamma(k+1, k)\ S(k)} \tag{4.4.39}$$

for $k = N - 1, N - 2, \ldots, 0$.

If we are only interested in generating the control, the term $M(k + 1)$ can be eliminated by substituting $M(k + 1) = P(k + 1) - Q(k + 1)$ into Equation (4.4.39). This leads us to the corollary:

Corollary 4.4.1. The optimal control for the stochastic linear system with random sampling can be generated using the following recursive relations:

$$u(k) = S(k)\, x(k) \tag{4.4.40}$$

$$S(k) = -\overline{[\Gamma^T(k+1,\, k)\, P(k+1)\, \Gamma(k+1,\, k) + R(k)]}^{-1}$$

$$\overline{\Gamma^T(k+1,\, k)\, P(k+1)\, \Phi(k+1,\, k)} \tag{4.4.41}$$

$$P(k) = \overline{\Phi^T(k+1,\, k)\, P(k+1)\, \Phi(k+1,\, k)}$$

$$+ \overline{\Phi^T(k+1,\, k)\, P(k+1)\, \Gamma(k+1,\, k)\, S(k)} + \overline{Q(k)} \tag{4.4.42}$$

$k = N - 1, N - 2, \ldots, 0$.

With the exception of notational differences, Equations (4.4.40-4.4.42) are the same as Equations (2.3.32-2.3.34). This result is significant because the optimization problem can be reduced to two separate optimizations whose solutions are in closed form. The most important feature of the optimal strategy for the stochastic system with random sampling is that the feedback gain matrix is independent of the statistics of the additive noise but dependent on the statistics of the sampling process, wh reas the optimal filter is independent of the weighting matrices in the performance measure. The block diagram for the Control system is shown in Figure (4.4.1).

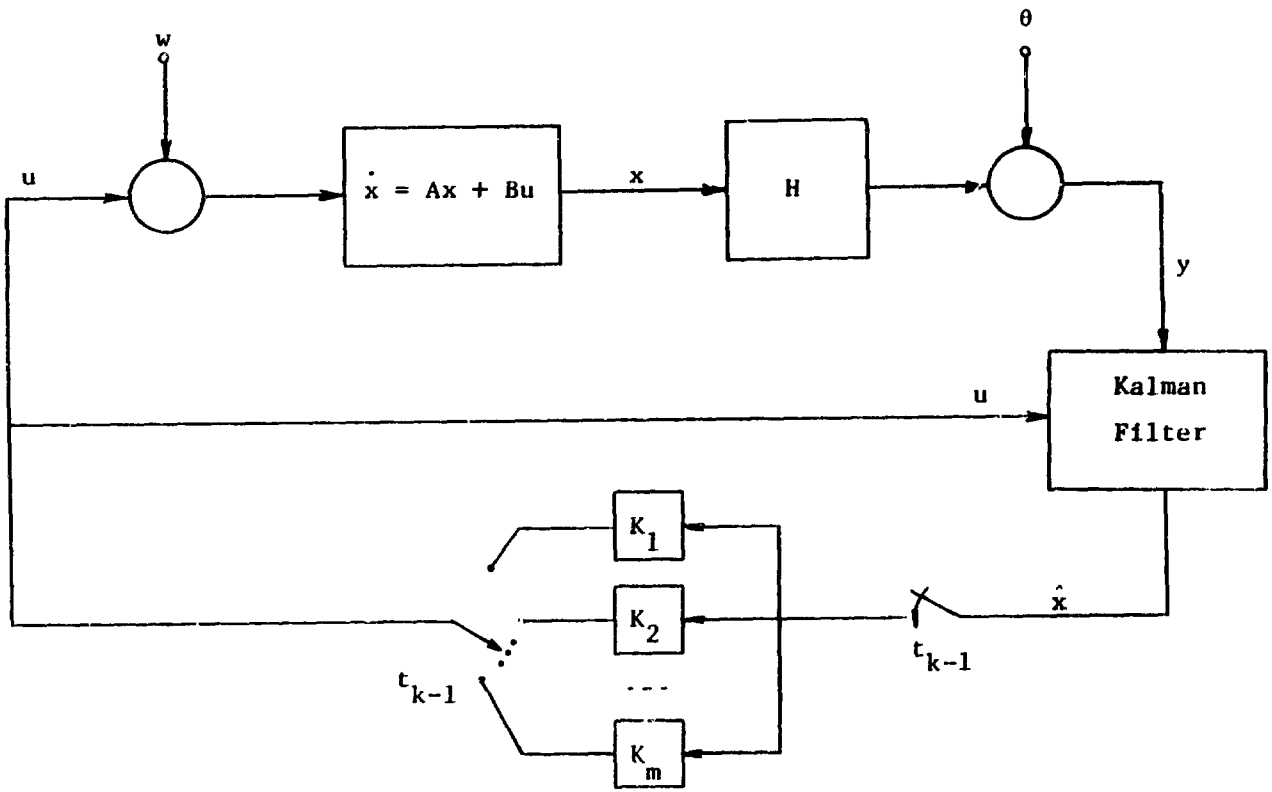


Figure 4.4.1. Stochastic Optimal Control with Random Sampling

4.5 Numerical Results

In this section we present some numerical results of optimal control and estimation applied to the F8-DFBW aircraft in a noisy environment. The aircraft in the noisy environment is modeled as

$$\dot{x} = Ax + Bu + w \tag{4.5.1}$$

$$y = Hx + \Theta \tag{4.5.2}$$

where A is (4 x 4) matrix

B is (4 x 2) matrix

w and Θ are (4 x 1) random vectors

H is the (4 x 4) identity matrix

With the exception of additive Gaussian noise terms, the aircraft is assumed to be in the same flight configuration as that used in chapter 3. We also use the same random sampling process. The noise processes w and Θ are independent random vectors with zero means. The covariance of w is equal to diag. [.04, .04, .04, .04]. The covariance of Θ is equal to diag. [.01, .01, .01, .01].

The equivalent discrete model for the system (4.5.1) and (4.5.2) is given by

$$x(t_{k+1}) = \Phi(t_{k+1}, t_k)\, x(t_k) + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)\, Bu(\tau)\, d\tau + \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)\, w(\tau)\, d\tau \tag{4.5.3}$$

C-2

$$y(t'_k) = H\Phi(t'_k, t_k)\, x(t_k) + H \int_{t_k}^{t'_k} \Phi(t'_k, \tau)\, Bu(\tau)\, d\tau$$

$$+ H \int_{t_k}^{t'_k} \Phi(t'_k, \tau)\, w(\tau)\, d\tau + \Theta(t'_k) \qquad (4.5.4)$$

But as $t'_k \to t_k$ the second and third terms approach zero and $\Phi(t'_k, t_k)$ approaches unity. We have the following equivalent model after dropping the t notation for simplification.

$$x(k+1) = \Phi(k+1, k)\, x(k) + \Gamma(k+1, k)\, u(k) + v(k) \qquad (4.5.5)$$

$$y(k) = Hx(k) + \Theta(k) \qquad (4.5.6)$$

where

$$E\{v(k)\, v(k)^T\} = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, \tau)\, R(\tau)\, \Phi^T(t_{k+1}, \tau)\, d\tau \qquad (4.5.7)$$

$$R(t)\, \delta(t-\tau) = E\{w(t)\, w^T(\tau)\} \qquad (4.5.8)$$

For the purpose of comparison both the unaugmented system and each uniformly-sampled system were simulated. The results are shown in Figures (4.5.1-4.5.4). A randomly-sampled system is simulated with transition probabilities as shown on Figure (3.3.2). Typical responses of the randomly-sampled system using optimal gains calculated for the uniformly-sampled system are shown in Figures (4.5.5-4.5.7). A typical response of the randomly-sampled system with optimal gains is shown in Figure (4.4.8).

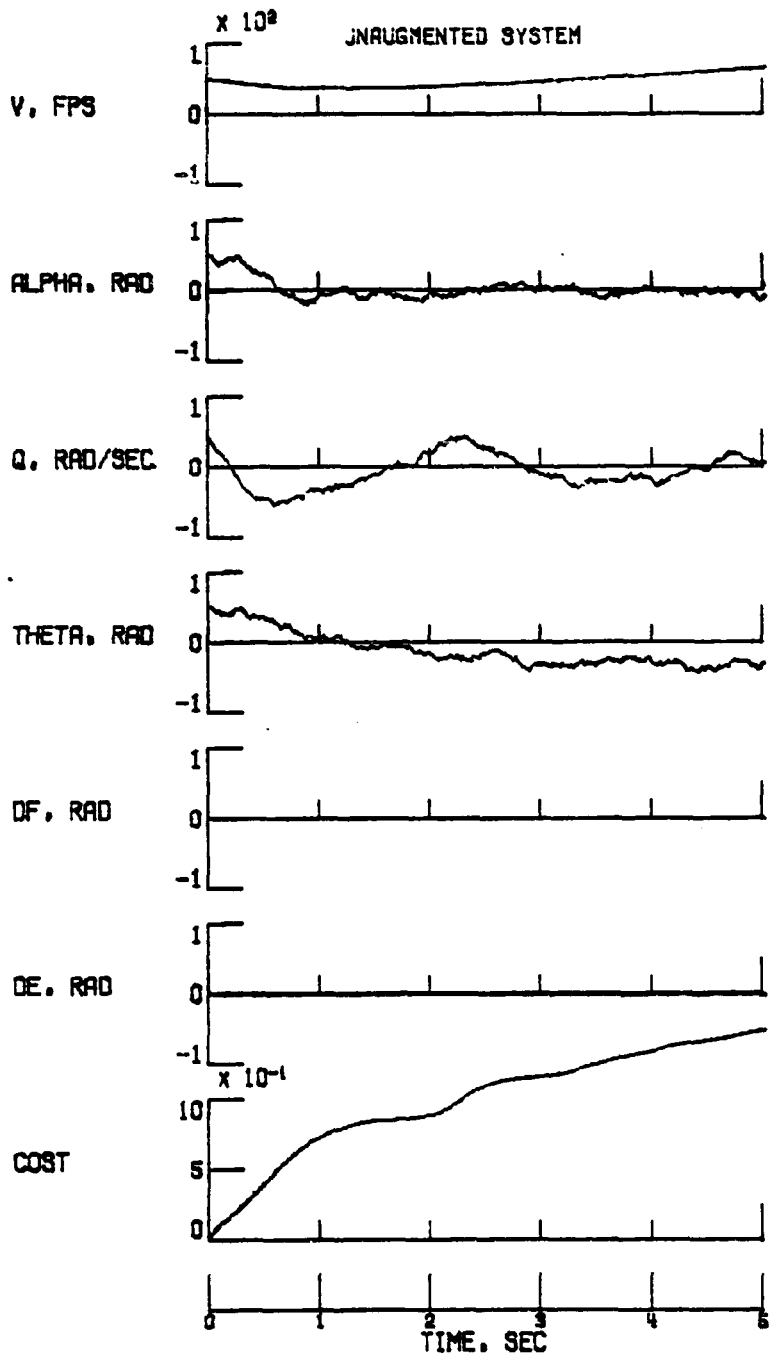


Figure 4.5.1. Response of the Unaugmented Aircraft

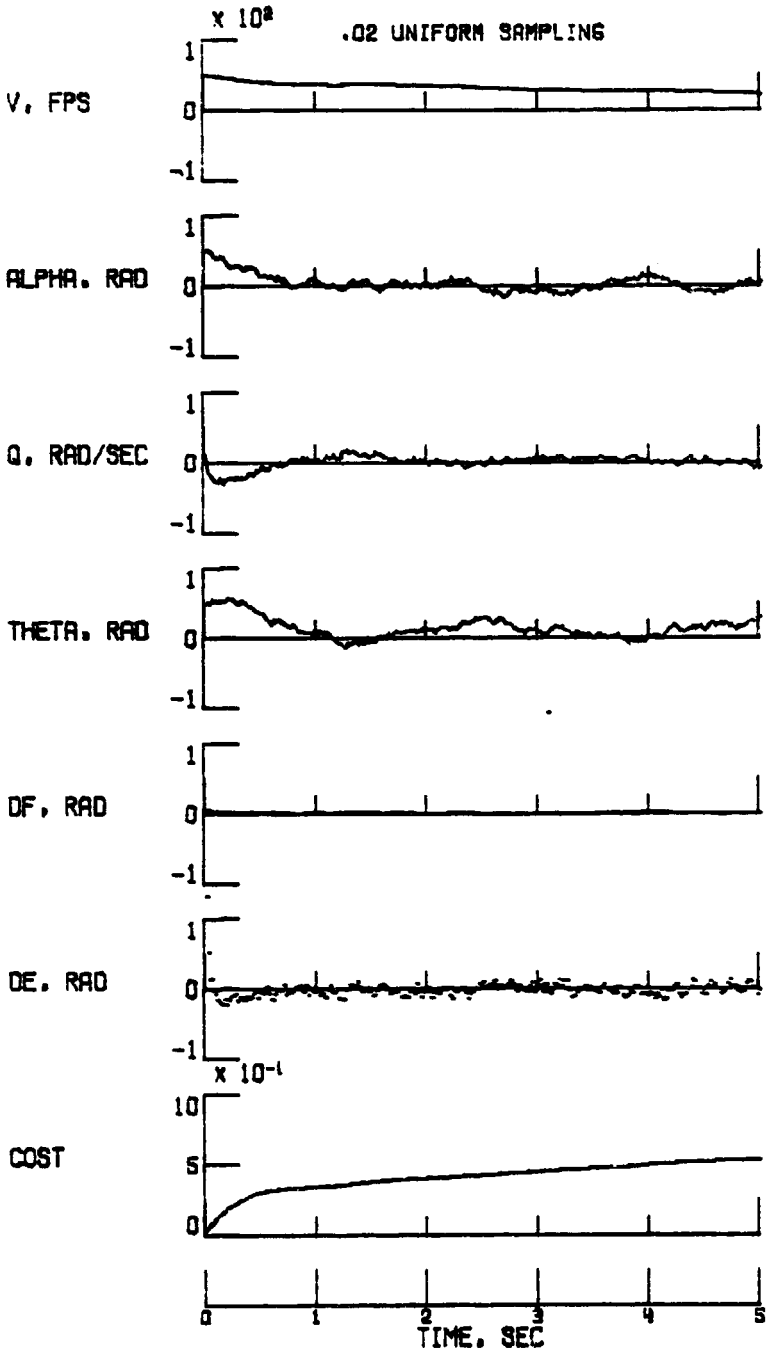


Figure 4.5.2. Response of the Aircraft with 0.02 Sec. Uniform Sampling

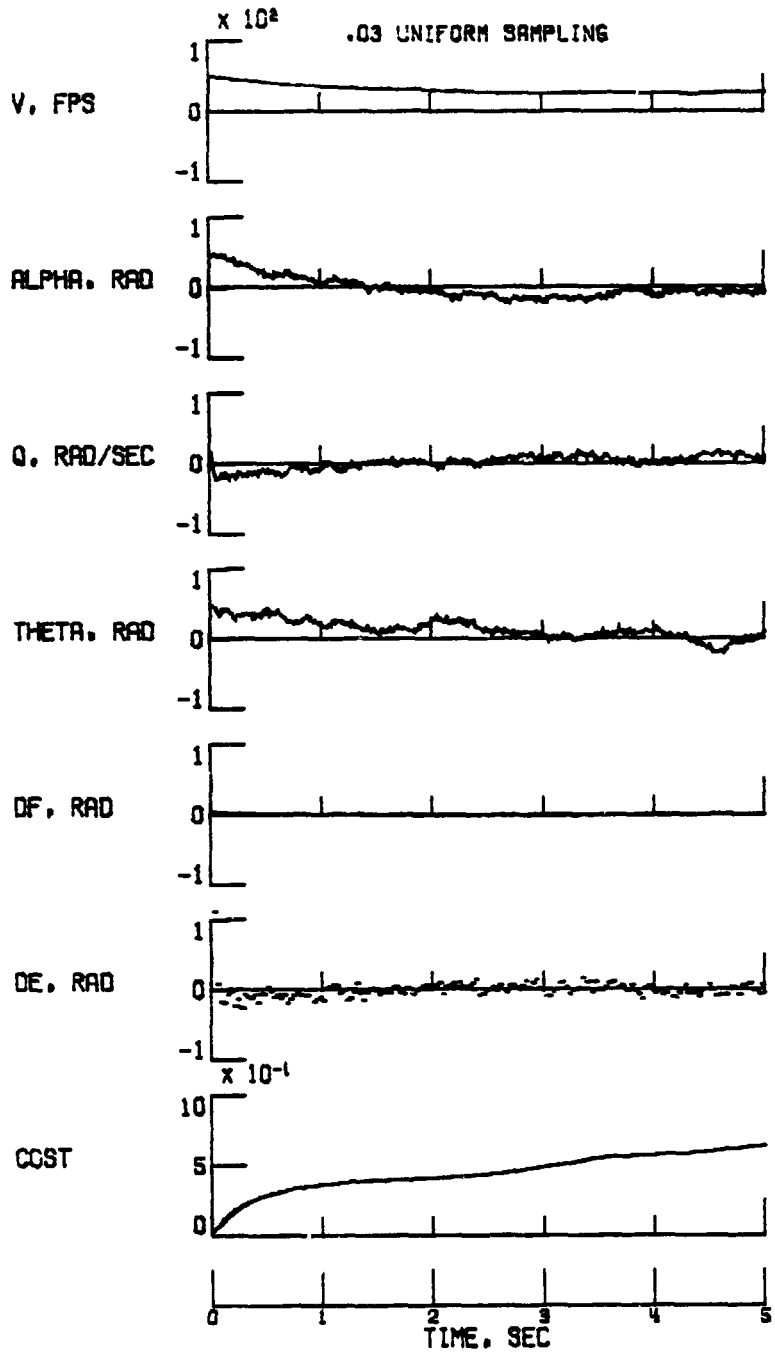


Figure 4.5.3. Response of the Aircraft with 0.03 Sec. Uniform Sampling

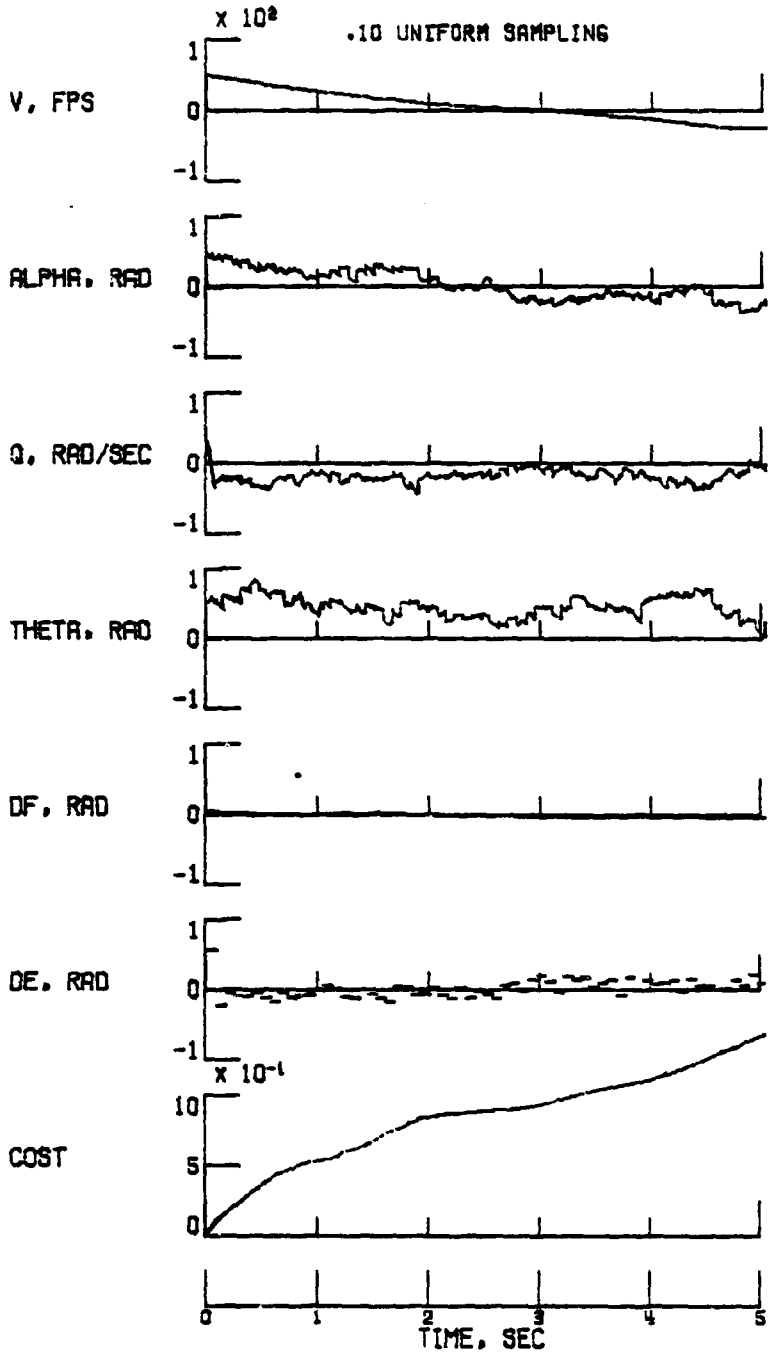


Figure 4.5.4. Response of the Aircraft with 0.10 Sec. Uniform Sampling

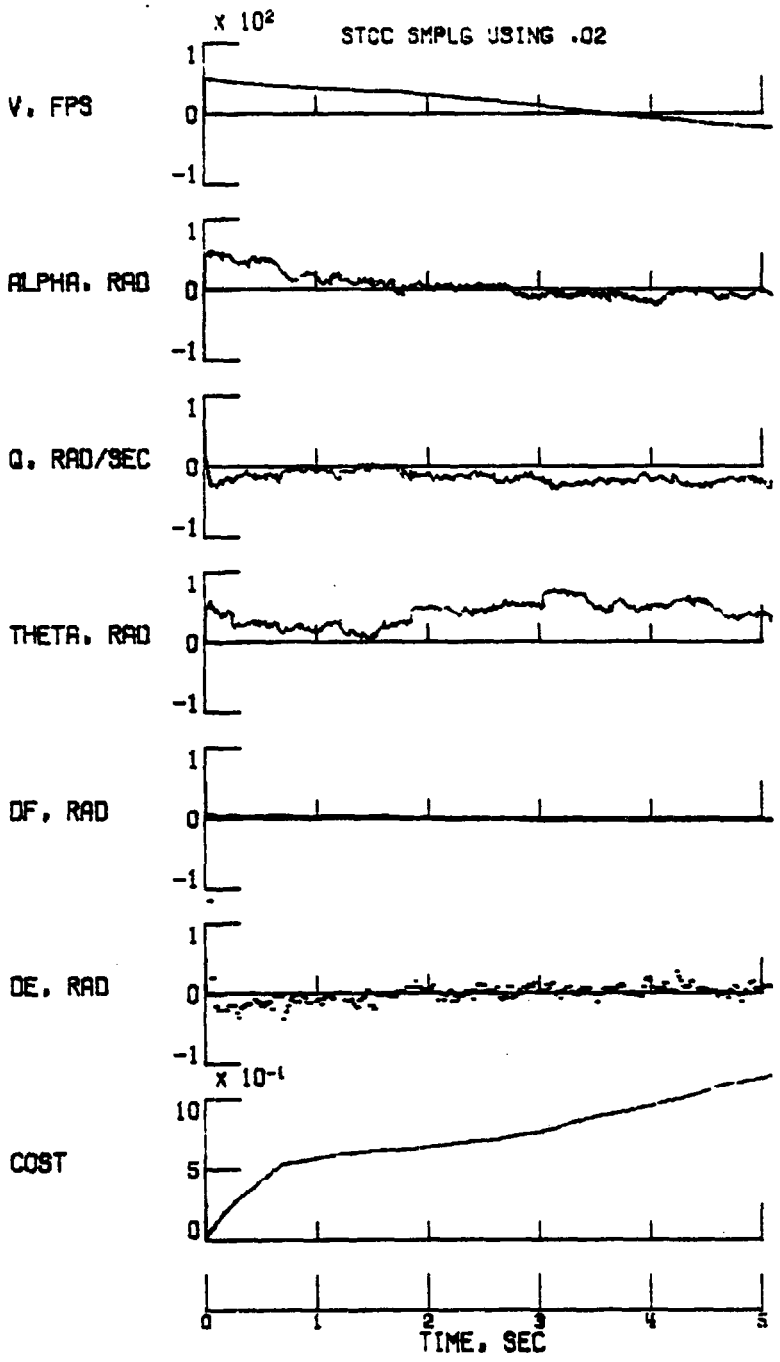


Figure 4.5.5. Response of the Randomly-Sampled System Using Optimal Gains Calculated for the Uniformly-Sampled System with Sampling Intervals = 0.02 Sec.

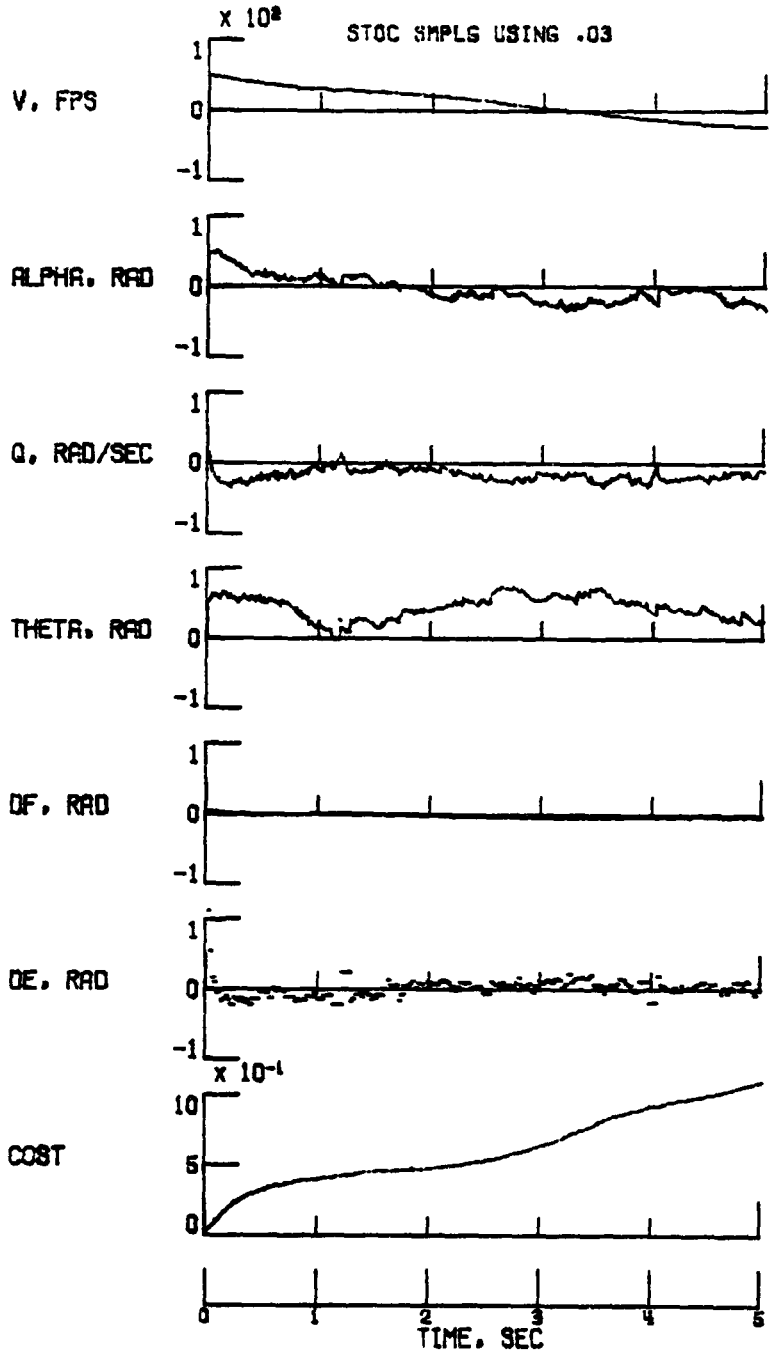


Figure 4.5.6. Response of the Randomly-Sampled System Using Optimal Gains Calculated for the Uniformly-Sampled System with Sampling Intervals = 0.03 Sec.

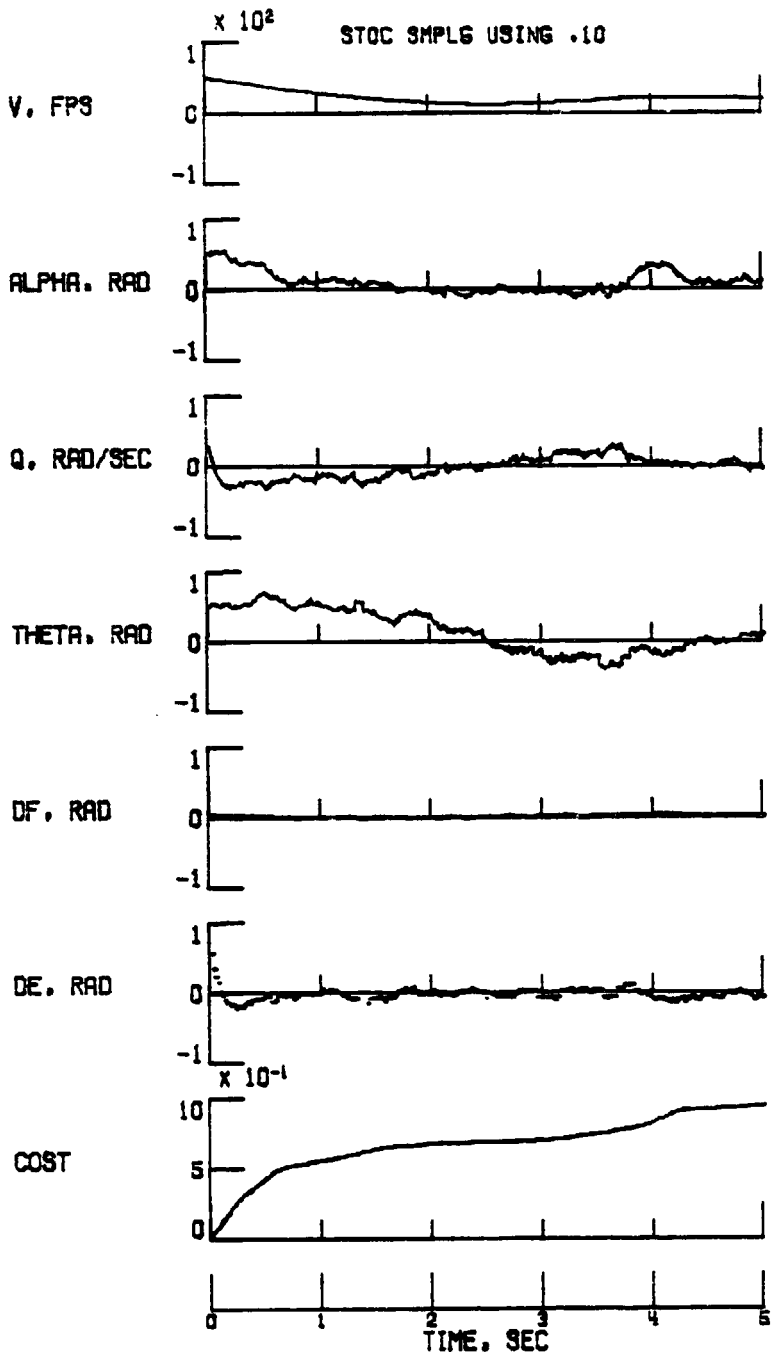


Figure 4.5.7. Response of the Randomly-Sampled System Using Optimal Gains Calculated for the Uniformly-Sampled System with Sampling Interval = 0.10 Sec.

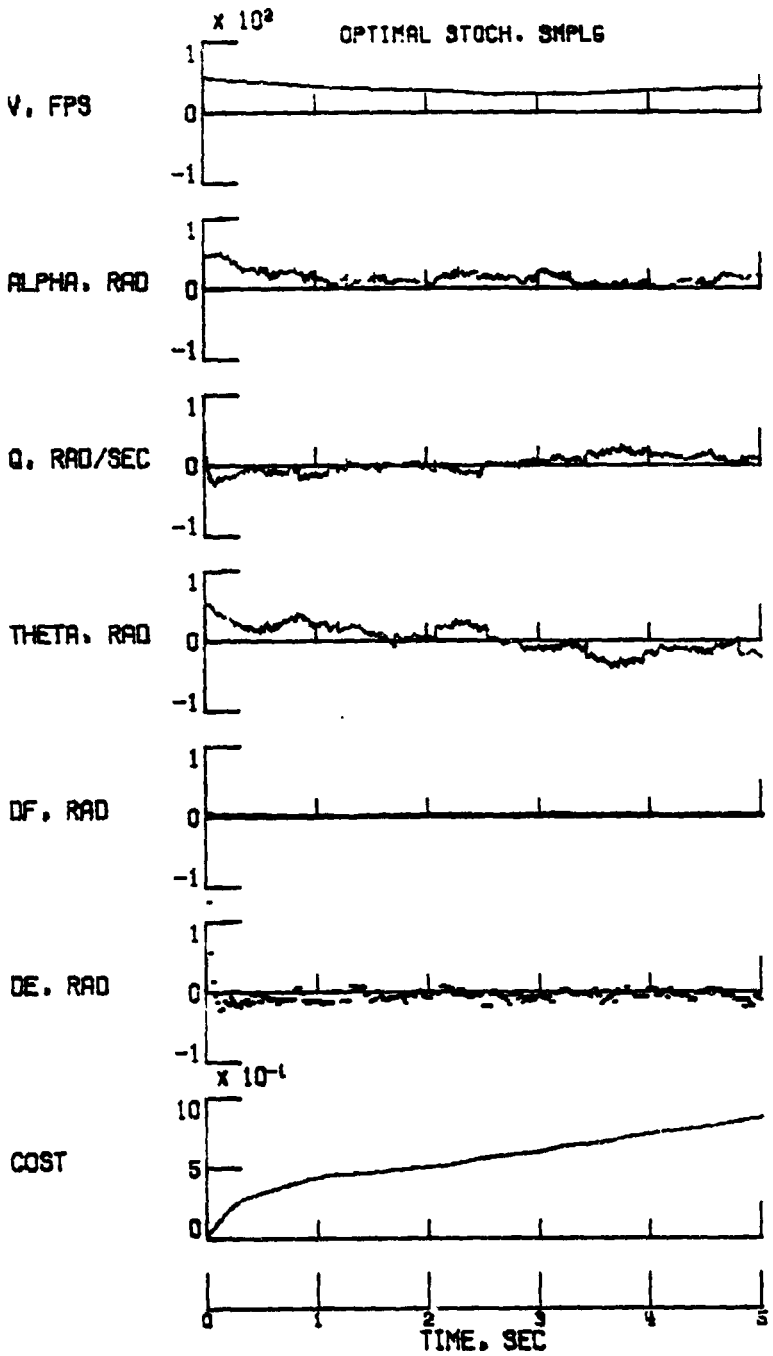


Figure 4.5.8. Response of the Randomly-Sampled System Using Optimal Stochastic Gains.

Table 4.5.1. Results of Simulation Study on Stochastic System with Random Sampling: 20 Random Runs of 10-Second Duration.

GAIN USED	Optimal Gain for 0.02 sec. Uniformly-Sampled System	Optimal Gain for 0.03 sec. Uniformly-Sampled System	Optimal Gain for 0.10 sec. Uniformly-Sampled System	Unaugmented System	Optimal Stochastic Gains
Average Cost	0.982	1.045	1.029	1.490	1.901

5. EXTENSIONS AND RELATED TOPICS

5.1 Introduction

In this chapter the major results will generalized to cover a wider range of applications of probabilistic Riccati equations. The problem can be re-formulated so that the probabilistic Riccati equations (2.3.47-2.3.48) form the optimal solution of the system which switches randomly among NS different configurations. There are a variety of causes for the change of configurations in a dynamic system. In a large-scale system the failure of an individual sensor or actuator may change the system dynamics. The occurrence of a failure is more frequent in this case due to the size and complexity of large systems. In a nonlinear system the dynamics can be thought to change as the system moves from one linearized region to another. In an adaptive-sampling system, each sampling rate results in a unique equivalent discrete model determined by Equations (2.2.15-2.2.19). The generalized probabilistic Riccati equations will be presented in Section 5.2. We will discuss failure detection and accommodation of large-scale system in Section 5.3. In Section 5.4 we will show that the solution of the probabilistic Riccati equation is the optimal solution for nonlinear systems in which the non-linearity is formulated as a semi-Markov switched-linear process. The application of stochastic sampling theory to an adaptive-sampling system is presented in Section 5.5.

5.2 Generalization of the Probabilistic Riccati Equations

In this section, we will consider a discrete-time system which switches randomly among m possible configurations $\{(\Phi_i, \Gamma_i, R_i, Q_i), i = 1, \ldots, m\}$. The cost function for the discrete-time system is defined as

$$J = \frac{1}{2} E \{x(N)^T Sx(N) + \sum_{k=0}^{N-1} [x(k)^T Q(k) x(k) + u^T(k) R(k) u(k)]\} \tag{5.2.1}$$

where

$$Q(k) \in \{Q_i, i = 1, \ldots, m_q\}$$

$$R(k) \in \{R_i, i = 1, \ldots, m_r\}$$

$$\text{for } k = 1, 2, 3, \ldots$$

The dynamics of the system can be described as

$$x(k+1) = \Phi(k+1, k) x(k) + \Gamma(k+1, k) u(k) \tag{5.2.2}$$

where

$$\Phi(k+1, k) \in \{\Phi_i, i = 1, \ldots, m_\Phi\}$$

$$\Gamma(k+1, k) \in \{\Gamma_i, i = 1, \ldots, m_\gamma\}$$

For notational simplicity we will represent the ith configuration by S_i and the configuration of the system at time k by S(k); i.e.

$$S_i = (\Phi_i, \Gamma_i, R_i, Q_i) \text{ for } i = 1, \ldots, m$$

and

$$S(k) = (\Phi(k+1, k), \Gamma(k+1, k), R(k), Q(k))$$

$$\text{for } k = 1, \ldots, N .$$

If the transition mechanism among m possible configurations can be described as a stationary Markov process, i.e. the transition mechanism can be represented by the matrix T defined by

$$T = \{P(S(k) = S_i / S(k-1) = S_j)\}$$

then the optimal control is governed by the following recursive relations

$$J(N-k, N) = x^T(N-k)\, P(k/i)\, x(N-k) \tag{5.2.3}$$

$$P(k/i) = \sum_{j=1}^{m} T_{ij} \{Q_j + K^T(N-k/j)\, R_j\, K(N-k/j) + [\Phi_j + \Gamma_j\, K(N-k/j)]^T\, P(k-1/j)\, [\Phi_j + \Gamma_j\, K(N-k/j)]\} \tag{5.2.4}$$

$$K(N-k/i) = -\{\sum_{j=1}^{m} T_{ij}\, [R_j + \Gamma_j^T\, P(k-1/j)\, \Gamma_j]^{-1}\} \cdot \{\sum_{j=1}^{m} T_{ij}\, [\Gamma_j^T\, P(k-1/j)\, \Phi_j]\} \tag{5.2.5}$$

$$u(N-k/i) = K(N-k/i)\, x(N-k) \tag{5.2.6}$$

The proof is omitted here because it is similar to that in Section 2.3.

The optimal solution (5.2.3-5.2.6) was obtained under the assumption that the configuration of the system can be identified perfectly with at most a one-step delay. If the structure of the system cannot be observed perfectly with at most a one-step delay, the solution is no longer optimal. This is because with imperfect observation of the system, the control must perform dual functions; one is to control the state, and the other is to identify the structure of the system [31], [32]. The phenomenon that the control influences the estimation of the state or the structure of the system is called the dual effect. In the randomly-sampled system there are no dual effects since the control strategy does not affect the estimation of the state or the determination of the system structure. This is also the basic reason for the separation of estimation and control in randomly-sampled systems. Unfortunately, these results can not be extended optimally to systems in which the structure is uncertain. On the other hand, the optimal control for a system with dual effects requires unrealistically large amounts of computer resources. Therefore, the use of Equations (5.2.3-5.2.6) as a suboptimal strategy should still be considered.

5.3 Failure Accommodation in Large Scale Systems

With ever increasing complexity of digital control systems, there is a need for design techniques for the control system that will respond rapidly and maintain overall system performance when a failure occurs. In designing a reliable control system there are two considerations: 1. The control system must respond rapidly

when a failure occurs; and 2. There must not be a significant degradation in performance during normal operation. These two considerations are usually conflicting. In the probabilistic Riccati Equations (5.2.3-5.2.6) the trade-off between these considerations is assessed by the Markov transition probability matrix T. Therefore, in a failure-tolerant control system, the Markov transition probability must reflect the probability of failure, maintenance time and the probability of recovery. From the discussion in Section 5.2 we know that the solution given by Equations (5.2.3-5.2.6) is optimal only if the system configuration is identified with at most a one-step delay. It is, therefore, important to select detection filters which can rapidly identify any change in the system configuration. The detection filters developed by Beard [33] and Jones [34] are very promising. Their work has led to a systematic design procedure for the detection of a wide variety of abrupt failures in linear time-invariant systems. The failure modes that Beard and Jones have considered include actuator and sensor shifts and shifts in the matrices A and B. Let us consider using these detection filters in a decentralized control system. A decentralized control system can be represented as [35]

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i u_i(t) \qquad (5.3.1)$$

$$y_j(t) = C_j x(t) \quad j = 1, \ldots, m \qquad (5.3.2)$$

where $B = (B_1, \ldots, B_m)$

$C = (C_1, \ldots, C_m)^T$

Suppose we want to detect a failure in the jth actuator, which can be modeled as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} B_i u_i(t) + B_j v(t) \qquad (5.3.3)$$

where v(t) is an arbitrary time function.

The detection filter is governed by the following dynamics in the absence of failure and is illustrated in Figure (5.3.1)

$$\dot{\hat{x}}(t) = A\hat{x}(t) + D_j(y_i(t) - C_j\hat{x}(t)) + \sum_{i=1}^{m} B_i u_i \qquad (5.3.4)$$

The objective of the detection filter is to choose the gain matrix D_j so that the effects of jth actuator failure are accentuated in the detection filter residual

$$r(t) = x(t) - \hat{x}(t) \qquad (5.3.5)$$

From Equations (5.3.3), (5.3.4) and (5.3.5), we know that the detection filter residual has the following dynamics in the absence of failure

$$\dot{r} = (A - D_j C_j)\, r(t) \qquad (5.3.6)$$

The output residual is given by

$$r'(t) = C_j\, r(t) \qquad (5.3.7)$$

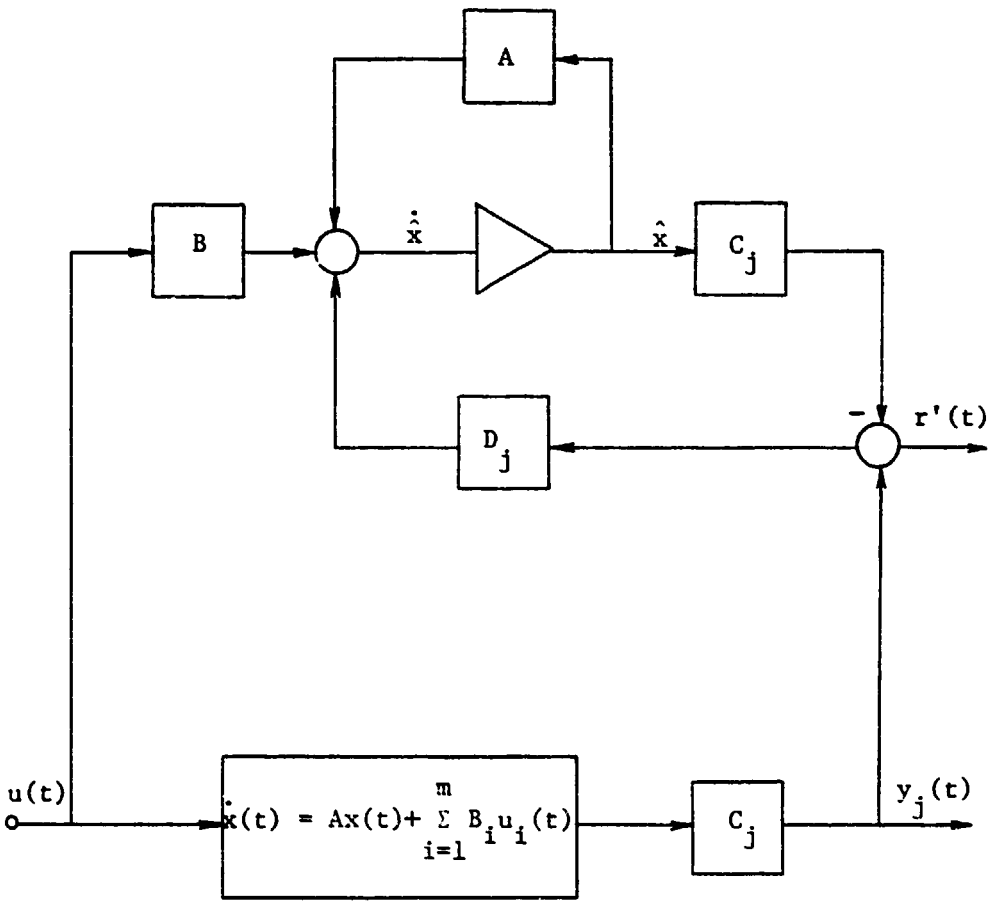


Figure 5.3.1. Failure Detection in a Large-Scale Control System Using the Beard-Jones Failure Detection Filter.

The residual dynamics in the presence of the jth actuator failure is governed by

$$r(t) = e^{(A-D_jC_j)(t-t_o)}t(t_o) + \int_{t_o}^{t} e^{(A-D_jC_j)(t-\tau)}B_j v(\tau)\, d\tau \tag{5.3.8}$$

The detection filter residual is contained in the controllable subspace W_j of B_j, which is equal to

$$W_j = [B_j,\ (A-D_jC_i)\ B_j,\ \ldots\ (A-D_jC_j)^{n-1}\ B_j] \tag{5.3.9}$$

The output residual is contained in the subspace spanned by C_jW_j. The failure of jth actuator is, therefore, detectable if D_j can be chosen so that rank of C_jW_j is one and the eigenvalues of $(A-D_jC_j)$ are stable. The choice of D_j will influence the eigenvalues of $(A-D_jC_j)$ and hence the speed of failure detection. This can be illustrated if we judiciously chose

$$(A-D_jC_j) = -\sigma I \tag{5.3.10}$$

Furthermore, if we assume that $v(t) = V$, which is a constant, then we have

$$r(t) = e^{-\sigma(t-t_o)}\ r(t_o) + V\ \left(\frac{1 - e^{-\sigma t}}{\sigma}\right)\ B_j \tag{5.3.11}$$

It is apparent now that the greater the detection filter gain σ, the faster the residual will approach its steady state which is equal to $\frac{V}{\sigma} B_j$ in the direction of B_j. The output residual is equal to $\frac{V}{\sigma} C_jB_j$ which is in the direction C_jB_j. Hence, in a noise-free system, the speed of failure detection can be made as fast as

desired by increasing the detection filter gain. This is, however, not possible if noise is present because the magnitude of the output residual is inversely proportional to the filter gain.

From the above discussion, we conclude that the optimal control strategy determined by Equations (5.2.3-5.2.6) together with the Beard-Jones failure detection filter is an interesting and logical approach, especially in systems where the noise level is relatively low.

5.4 Applications to Nonlinear Systems

A general nonlinear system can be represented by

$$\dot{x} = f(x, u) \tag{5.4.1}$$

where x is a vector of system states and u is the vector of inputs. The entire state space of interest can be partitioned into a number of regions, viz. $\{S_i, i = 1, \ldots, m\}$. The dynamics in each region can be approximated by a linear model which is derived by using a Taylor series expansion in the neighborhood of the "center" (x_i, u_i) of the region. Using a first-order approximation as in Section (3.2), we have the following linearized model for each region,

$$\delta\dot{x} = A_i \delta x + B_i \delta u \tag{5.4.2}$$

where

$$A_i = \frac{\partial f}{\partial x}(x_i, u_i) \tag{5.4.3}$$

$$B_i = \frac{\partial f}{\partial u}(x_i, u_i) \tag{5.4.4}$$

for $i = 1, \ldots, m$

For each linear model an equivalent linear discrete-time model can be calculated of the form

$$x(k+1) = \Phi_i(k+1, k)\, x(k) + \Gamma_i(k+1, k)\, u(k)$$

$$\text{for } i = 1, \ldots, m \tag{5.4.5}$$

As the state of the nonlinear system moves from one region to the other, our linear approximation model should change accordingly. A nonlinear system can, therefore, be thought of as a linear system with time-varying parameters. Several methods have been developed to identify these parameters. One well known technique is called "Partitioned Adaptive Estimation," first introduced by Magill and later refined by Lainiotis [36], [37]. In this approach the optimal estimate is a weighted sum of estimates derived from a bank of Kalman filters. The weighting coefficients are functions of the measurement residuals. The weighting coefficient corresponding to the actual plant model will approach unity while the other coefficients approach zero given sufficient time for the identification. The partitioned adaptive control known as "Multiple Model Adaptive Control" was developed in the Electronic Systems Laboratory of Massachusetts Institute of Technology.

The method was successful in controlling the nonlinear dynamics of the F-8 aircraft in its various flight configurations [38]. These methods, however, failed to track nonlinear plants undergoing relatively rapid maneuvers as reported in reference [39]. Moose, Wang and Zwicke developed a better estimator by assuming that the random configuration changes could be modeled as a semi-Markov process [40], [41], [42]. This formulation is exactly the same as that presented in Section 5.2. The optimal control, assuming that the plant configuration can be identified with at most a one-step delay, is given by Equations (5.23-5.2.6). One feature of Equations (5.2.3-5.2.6) is that the optimal control has accounted for a possible change of configuration in the future since the feedback gain matrix is generated from a recursive probabilistic Riccati equation. Unfortunately, a nonlinear plant can seldom be identified within one iteration. On the other hand, since dual optimal control requires immensely large amounts of computer resources, the suboptimal approach by assuming, valid or not, that the plant can be identified with no more than a one-step delay is still attractive.

5.5 Minimization of Information Flow Using Adaptive Sampling

The exchange of information between microcomputers is an item of concern because its cost is high relative to the cost of computation within a given micro-computer. This section explains an analytical procedure that assesses the trade-off between the

stability and control objectives of the control system and its information exchange requirements. Figure (5.5.1) shows a typical trade-off between the objectives of the control system and information exchange requirements. The optimal trade-off involves determining the information pattern of the system (i.e. variables exchanged between the distributed microcomputer subsystems and the frequency of the exchange). Past research has considered control system designs using quadratic optimization for a fixed information pattern P_i. For that process, the variable exchanged and the frequency of the exchange are fixed and the control system gains and filter constants are selected as in classical regulator theory to minimize an objective function of the form

$$J_o = \int_o^\infty (x^T(t)\ Q(t)\ x(t) + u^T(t)\ R(t)\ u(t))\ dt \qquad (5.5.1)$$

Since the information pattern P_i specifies the variables available for information and the rate of information exchange, the result of the minimization of J_o will depend implicitly on P_i. The variation of the optimal control objective function with P_i is recognized and accounted for in the adaptive sampling. This is done by considering a total objective function that not only involves performance of the objectives of the control system as in classical regulator theory but also penalizes the use of information flow. The total objective function to be minimized is of the form

$$J = J_o + C(P_i)$$

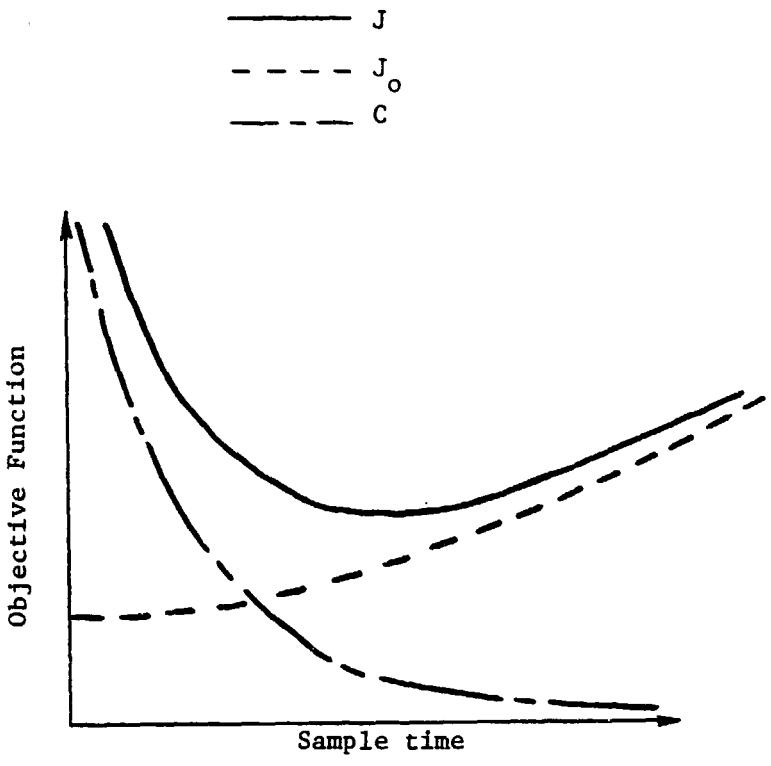


Figure 5.5.1. Qualitative Effect of Information Exchange Rate (Sample Rate) on the Stability and Control Objective Function J_o, the Information Exchange Penalty Function C, and the Total Design Objective Function J.

The first term forces the algorithm to obtain a desired system performance and the second term penalizes high information exchange rates.

For the purpose of optimizing the total objective function, J, the sampling rate τ is assumed to be in the range of usable values. The range of usable τ-values is limited from below by computer I/O capabilities and is limited from above by stability requirements. For the case of a linear plant of the form

$$\dot{x} = Ax(t) + Bu(t) \tag{5.5.2}$$

$$u(t) = u(k) \tag{5.5.3}$$

for $t_k \leq t < t_{k+1}$, k = 1, 2, . . . with the total objective function

$$J = \frac{1}{2} \int_o^\infty (x^T(t)\ Q_c x(t) + u^T(t)\ R_c u(t)\ dt + f(\tau) \tag{5.5.4}$$

When the plant (5.5.2) is subject to the constraint (5.5.3), the plant dynamics have the following equivalent discrete-time representation

$$x(k+1) = \Phi(k+1,\ k)\ x(k) + \Gamma(k+1,\ k)\ u(k) \tag{5.5.4}$$

with the cost function

$$J = \sum_{k=1}^{\infty} [x(k)\ Q(k)\ x(k) + 2x(k)\ M(k)\ x(k) + u(k)\ R(k)\ u(k)] + f(\tau) \tag{5.5.5}$$

where $\Phi(k+1,\ k)$, $\Gamma(k+1,\ k)$, Q(k), M(k) and R(k) are functions

of τ. The minimization of J should be taken over the set of sequences $U = \{u(k);\ k = 1, \ldots, \infty\}$. This, however, leads to a complex analytical problem. The analytical complexity can be reduced by considering a restricted class of information patterns P_i where the data bus is assumed to be sampled at uniform intervals. If the data bus is, in fact, assumed to be sampled at uniform intervals, τ, the solution to the optimization problem over the sequence u is the feedback control law:

$$u^*(k) = F(\tau)\ x(k) \tag{5.5.6}$$

where

$$F(\tau) = -\ (R + \Gamma^T K \Gamma)^{-1}(\Gamma^T K \Phi + M^T) \tag{5.5.7}$$

and K is the solution to the matrix Riccati equation

$$K = \Phi^T K \Phi + Q - (\Gamma^T K \Phi + M^T)^T (R + \Gamma^T K \Gamma)^{-1}\ (\Gamma^T K \Phi + M^T) \tag{5.5.8}$$

Note that F and K, as well as Q, R and M are functions of τ. When the sequence $u^*(k)$ is applied, the cost function J becomes

$$J = x^T(o)\ K(\tau)\ x(o) + f(\tau) \tag{5.5.9}$$

which is a function of the initial state x(o) and sample time τ. If the number of usable sampling intervals is small and the initial state is known, the minimal J can be determined by evaluating Equation (5.5.9) for all possible values of τ. This is not

feasible for an on-line application if the number of usable sampling intervals is large.

A different approach involves computing τ by optimizing J over a sample interval $(t_k, t_k + \tau)$. The cost accrued over that interval is

$$\Delta J(\tau) = x^T(k)\, H(\tau)\, x(k) + f(\tau) \tag{5.5.10}$$

where

$$H(\tau) = Q(\tau) + 2M(\tau)\, F(\tau) + F^T(\tau)\, R(\tau)\, F(\tau) \tag{5.5.11}$$

The functions $Q(\tau)$, $M(\tau)$ and $R(\tau)$ are analytic in τ and, under the assumption that $f(\tau)$ is also analytic, $\Delta J(\tau)$ can be expanded in a Taylor series about the sample time which was last used, viz., τ_i. This results in

$$\Delta J(\tau) = f + x^T(k)[H + H'(\tau-\tau_i) + \tfrac{1}{2} H''(\tau-\tau_i)^2]\, x(k)$$
$$+ f'(\tau-\tau_i) + \tfrac{1}{2} f''(\tau-\tau_i)^2 \tag{5.5.12}$$

where e.g., $H' \triangleq \frac{\partial H}{\partial \tau}$ and all functions are evaluated at the last sample time, τ_i. The minimization can be simplified by neglecting higher-order terms (higher than $(\tau-\tau_i)^2$). This results in

$$\tau - \tau_i = - \frac{f'(\tau_i) + x^T(k)\, H'(\tau_i)\, x(k)}{f''(\tau_i) + x^T(k)\, H''(\tau_i)\, x(k)} \tag{5.5.12}$$

For on-line implementation, it is only necessary to store f', f", H' and H".

Using the above adaptive-sampling algorithm, the control system will have random sampling intervals. Unless the sampling process can be modeled as a semi-Markov process, there is currently no control law which can account for the random sampling intervals. In order to use Equations (5.2.3-5.2.6) to calculate the control law, we must know the transition probabilities. On the other hand, in order to determine the transition probabilities in simulation, we must first have the control law. To avoid this dilemma, a heuristic method is introduced in the flow chart shown in Figure (5.5.2). In this approach the steady-state gain is calculated for each sampling interval assuming that the gain will be used in a uniformly-sampled system. These gains are then applied to the simulation of the adaptively-sampled system. These gains are, of course, not optimal in the adaptively-sampled system. From the simulation the Markov transition probabilities between adjacent sampling intervals are determined. Using these probabilities, the optimal quasi-steady-state gains can be determined from Equations (5.2.3-5.2.6). We will then iterate between generating the new transition probabilities and calculating the quasi-steady gains until the algorithm converges. Since this method is heuristic in nature, the convergence of the algorithm is not guaranteed. However, if the algorithm does converge, the quasi-steady-state gains should be very closed to the optimal solution.

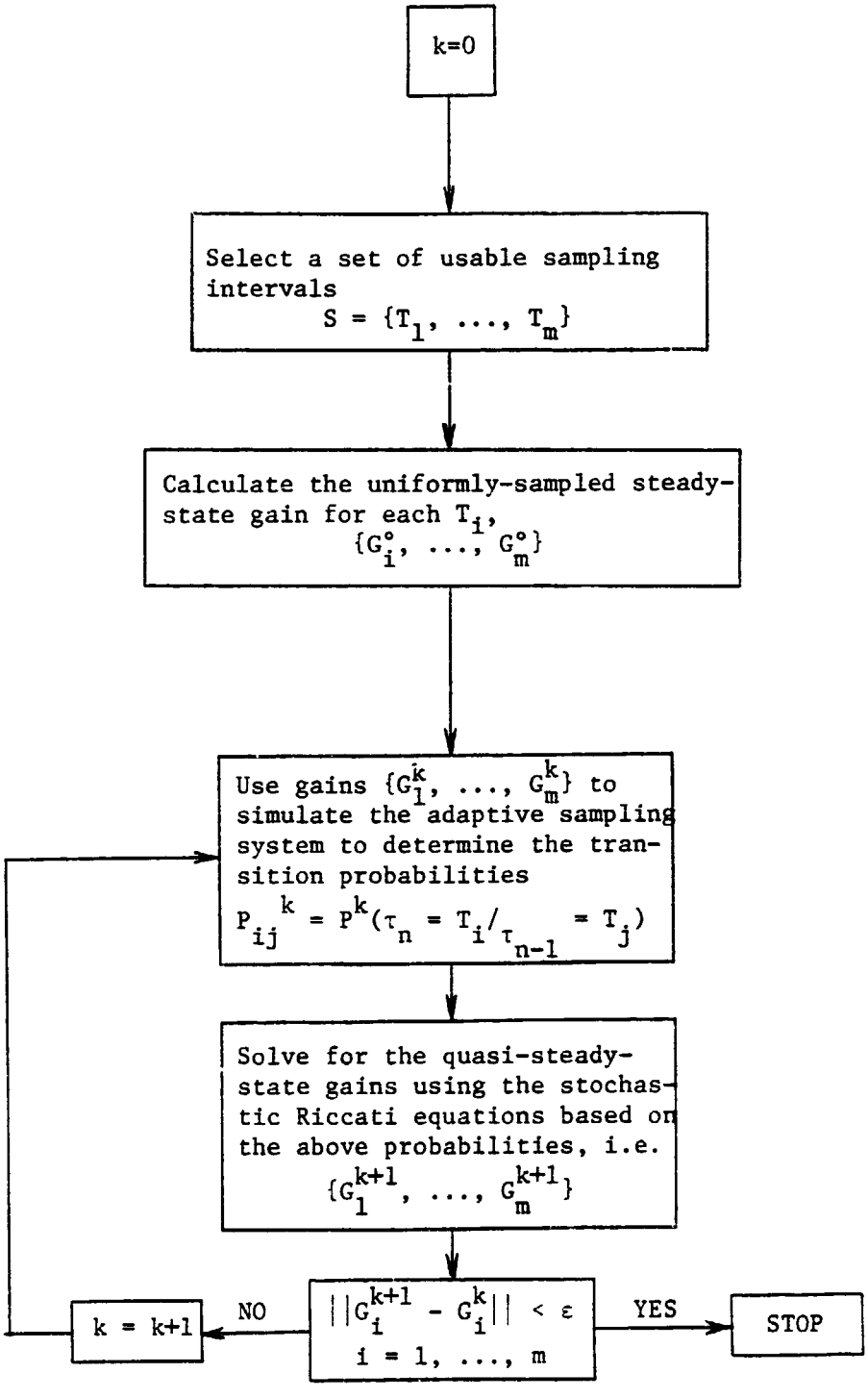


Figure 5.5.2. A Heuristic Approach to Determine the Control Law for an Adaptively-Sampled System.

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