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# LONG PERIOD PERTURBATIONS OF EARTH SATELLITE ORBITS



**A**NALYTICAL AND  
**C**OMPUTATIONAL  
**M**ATHEMATICS, INC.

LONG PERIOD PERTURBATIONS  
OF  
EARTH SATELLITE ORBITS

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1.0 INTRODUCTION

In reference 1, Scheifele and Graf introduced a complete first order solution for the orbital motion of a satellite perturbed by earth oblateness. This solution was expressed in the DS $\phi$  elements. In reference 2, Bond and Scheifele expressed the first order short period and secular  $J_2$  solution in the non-singular PS $\phi$  elements. This theory was implemented in an operational computer program named ASOP described in reference 3. In references 4 and 5, the PS $\phi$  analytical theory was updated to include the drag effects. In reference 6, the theory was developed to account for the time dependent gravitational harmonics. The drag and time dependent geopotential terms have also been included in ASOP .

Bond also extended the PS $\phi$  theory to include the first order long period terms and second order secular perturbations due to  $J_2$  ,  $J_3$  ,  $J_4$  and  $J_5$  . However, no documentation of the equations was ever published. In reference 7, Mueller developed a recursive theory to include the first order long period terms and second order secular perturbations due to zonal harmonics of any order. Mueller's theory plus the second order  $J_2$  theory developed by Bond have now been implemented in ASOP .

The purpose of this report is to document all the equations involved in extending the  $PS\phi$  solution to include the long periodic and second order secular effects of the zonal harmonics.

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## 2.0 METHOD OF SOLUTION

### 2.1 Notation

The DS $\phi$  elements are a set of eight variables which have the following description:

Angle Elements:

$$\begin{aligned}\alpha_1 &= \phi && \text{true anomaly} \\ \alpha_2 &= g && \text{argument of pericenter} \\ \alpha_3 &= h && \text{longitude of ascending node} \\ \alpha_4 &= \ell && \text{time element}\end{aligned}$$

Action Elements:

$$\begin{aligned}\beta_1 &= \phi && \text{related to two body energy} \\ \beta_2 &= G && \text{total angular momentum} \\ \beta_3 &= H && \text{z-component of the angular momentum} \\ \beta_4 &= L && \text{total energy}\end{aligned}$$

These may be canonically transformed to the PS $\phi$  elements by the following relations:

$$\begin{aligned}\sigma_1 &= \phi + g + h \\ \sigma_2 &= -\sqrt{2(\phi - G)} \sin(g + h) \\ \sigma_3 &= -\sqrt{2(G - H)} \sin(h) \\ \sigma_4 &= \ell \\ \rho_1 &= \phi \\ \rho_2 &= \sqrt{2(\phi - G)} \cos(g + h) \\ \rho_3 &= \sqrt{2(G - H)} \cos(h) \\ \rho_4 &= L\end{aligned}\tag{1}$$

The DS $\phi$  Hamiltonian for the zonal oblateness problem is given by:

$$F = F_0 + \epsilon F_1 + \epsilon^2 F_2 \quad (2)$$

where

$$F_0 = \phi - \frac{\mu}{\sqrt{2L}} \quad (\text{two body contributions})$$

$$F_1 = \frac{1}{qr} \left[ \left( \frac{x_3}{r} \right)^2 - \frac{1}{3} \right] \quad (J_2 \text{ contribution})$$

$$F_2 = \frac{1}{q} \sum_{n=3}^N \hat{J}_n \frac{1}{r^{n-1}} P_n \left( \frac{x_3}{r} \right) \quad (\text{higher zonal harmonics})$$

$$\epsilon = \frac{3}{2} J_2 \frac{R_e^2}{a^3}$$

$P_n$  are the Legendre polynomials,  $R_e$  is the mean equatorial radius of earth, and  $J_2$  and  $\hat{J}_n$  are oblateness coefficients.

## 2.2 Solution Algorithm

Von Zeipel's method of elimination of the short and long periodic terms is used. The solution first requires the transformation to eliminate the short periodic terms due to  $J_2$ . The generating function is assumed to be of the form

$$S = S_0 + \epsilon S_1$$

$S_0$  give the identity transformation,  $S_1$  is so chosen that the new Hamiltonian is no longer a function of short period variable  $\phi$ . The Hamiltonian has the form

$$F'(\beta', g') = F'_0 + \epsilon F'_1 + \epsilon^2 F'_2$$

A more thorough discussion of the elimination of short periodic terms can be found in reference 2.

An additional transformation must be made to eliminate the long periodic terms from  $F'$ . This transformation is defined by the generating function

$$S^* = S_0^* + \epsilon S_1^*$$



Again,  $S_0^*$  gives the identify transformation,  $S_1^*$  is chosen such that the long period variable  $g'$  is eliminated from the Hamiltonian. The new Hamiltonian has the form

$$F''(\beta'') = F_0'' + \epsilon F_1'' + \epsilon^2 F_2''$$

A more thorough discussion of the elimination of the  $J_2$  and higher order zonal perturbation long periodic terms can be found in references 1 and 7.

The solution algorithm can be divided into three steps:

(1) Initialize the primed variables

$$\begin{aligned} \sigma'_{k,0} &= \sigma_{k,0} + \epsilon \left( \frac{\partial S_1}{\partial \rho_{k,0}} + \frac{\partial S_1^*}{\partial \rho_{k,0}} \right) \\ \rho'_{k,0} &= \rho_{k,0} - \epsilon \left( \frac{\partial S_1}{\partial \sigma_{k,0}} + \frac{\partial S_1^*}{\partial \sigma_{k,0}} \right) \quad k = 1, 2, 3, 4 \end{aligned} \quad (3)$$

(2) Analytical integration of primed variables

$$\begin{aligned} \sigma'_1 &= \sigma'_{1,0} + A_1 \tau \\ \sigma'_2 &= \sigma'_{2,0} \cos(A_2 \tau) - \rho'_{2,0} \sin(A_2 \tau) \\ \sigma'_3 &= \sigma'_{3,0} \cos(A_3 \tau) - \rho'_{3,0} \sin(A_3 \tau) \\ \sigma'_4 &= \sigma'_{4,0} + A_4 \tau \\ \rho'_1 &= \rho'_{1,0} \\ \rho'_2 &= \rho'_{2,0} \cos(A_2 \tau) + \sigma'_{2,0} \sin(A_2 \tau) \\ \rho'_3 &= \rho'_{3,0} \cos(A_3 \tau) + \sigma'_{3,0} \sin(A_3 \tau) \\ \rho'_4 &= \rho'_{4,0} \end{aligned} \quad (4)$$

The definitions of  $A_1, A_2, A_3, A_4$  are given in section 3.0 of this report, The relation between time  $t$  and the new independent variable  $\tau$  is given by  $\frac{dt}{d\tau} = r^2/q$ , the definition of  $q$  is also given in section 3.0

(3) Back transformation

$$\sigma_k = \sigma'_k - \epsilon \left( \frac{\partial S_1}{\partial \rho_k} + \frac{\partial S_1^*}{\partial \rho_k} \right)$$

$$\rho_k = \rho'_k + \epsilon \left( \frac{\partial S_1}{\partial \sigma_k} + \frac{\partial S_1^*}{\partial \sigma_k} \right)$$

$$k = 1, 2, 3, 4$$

(5)

### 3.0 EQUATIONS FOR ELIMINATION OF LONG PERIODIC TERMS AND ANALYTICAL INTEGRATION OF PRIMED VARIABLES

A detailed description of generating function  $S_1$  and derivatives of  $S_1$  with respect to the PS $\phi$  elements can be found in Appendix F of Reference 3. In this section a detailed description of generating function  $S_1^*$  and derivatives of  $S_1^*$  with respect to the PS $\phi$  elements will be given. The derivatives of  $F_2''$  with respect to the DS $\phi$  elements will also be given.

#### 3.1 Generating Function $S_1^*$

From Reference 1 we have:

$$S_1^* = \frac{1}{\left(\frac{\partial F_1}{\partial G}\right)_q} \left[ \hat{S} - \frac{1}{2} \frac{f^2}{48} (2 - 3b + 6qB)e^2 b \sin(2g) \right] \quad (6)$$

$\hat{S}$  are terms related to higher order zonal perturbations. A detailed description of  $\hat{S}$  can be found in reference 7.

Now we introduce  $\sin(g)$  and  $\cos(g)$

$$\begin{aligned} \sin(g) &= \frac{1}{CD} (\sigma_6 \sigma_3 - \sigma_7 \sigma_2) \\ \cos(g) &= \frac{1}{CD} (\sigma_6 \sigma_7 + \sigma_2 \sigma_3) \end{aligned} \quad (7)$$

where

$$C = \sqrt{2(\phi - G)}$$

$$D = \sqrt{2(G - H)}$$

To write  $S_1^*$  in terms of PS $\phi$  elements we introduce the following abbreviations

$$Q = \left\{ \frac{\sigma_8}{\mu^2} \left[ \frac{2\mu}{\sqrt{2\sigma_8}} - \frac{1}{2} (\sigma_2^2 + \sigma_6^2) \right] \right\}^{\frac{1}{2}} \quad (8)$$

$$p = \frac{1}{\mu} \left[ -\frac{1}{2} (\sigma_2^2 + \sigma_6^2) + \frac{\mu}{\sqrt{2\sigma_8}} \right]^2 \quad (9)$$

$$e = \left( 1 - \frac{2\sigma_8}{\mu} p \right) = QD \quad (10)$$

$$b = 1 - \frac{G^2}{H^2} \quad (11)$$

$$\chi = eb^{\frac{1}{2}} \sin(g) = \frac{1}{2} \frac{Q}{G} \sqrt{2(G+H)} (\sigma_6 \sigma_3 - \sigma_7 \sigma_2) \quad (12)$$

$$\psi = eb^{\frac{1}{2}} \cos(g) = \frac{1}{2} \frac{Q}{G} \sqrt{2(G+H)} (\sigma_6 \sigma_7 + \sigma_2 \sigma_3) \quad (13)$$

$$\theta = e^2 b \sin(2g) = 2\chi\psi \quad (14)$$

Now we have

$$S_1^* = \frac{1}{\left(\frac{\partial F_1}{\partial G}\right)_q} \left[ \hat{S} - \frac{1}{2} \frac{f^2}{48} (2 - 3b + 6qB)\theta \right] \quad (15)$$

where

$$q = -\frac{1}{2} (\sigma_6^2 + \sigma_2^2 - \sigma_5) + \frac{h}{\sqrt{2\sigma_8}} \quad (16)$$

$$f = \frac{1}{pq} \quad (17)$$

$$B = \frac{2H^2}{G^3} \quad (18)$$

$$\frac{\partial F_1}{\partial G} = \frac{1}{2} f \left[ f \left( \frac{2}{\mu} qd + p \right) \left( \frac{2}{3} - b \right) + B \right] \quad (19)$$

and

$$d = (p\mu)^{\frac{1}{2}} \quad (20)$$

Let

$$T_a = -\frac{1}{2} \frac{f^2}{48}$$

$$T_b = (2 - 3b + 6qB)T_a \quad (21)$$

$$T_c = \theta T_b$$

$$T = \hat{S} + T_c$$

then

$$S_1^* = \frac{T}{\left(\frac{\partial F_1}{\partial G}\right)_q} \quad (22)$$

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### 3.2 Derivatives of $S_1^*$

Let

$$S_{1k}^* = \frac{\partial S_1^*}{\partial \sigma_k}, \quad k = 1, 2, 3, \dots, 8$$

From now on the subscript  $k$  represents partial derivatives with respect to the 8  $PS\phi$  elements, unless otherwise specified.

$$\bullet \quad S_{1k}^* = - \frac{1}{\left(\frac{\partial F_1}{\partial G}\right)_q} \left\{ T \left[ \frac{\left(\frac{\partial F_1}{\partial G}\right)_k}{\left(\frac{\partial F_1}{\partial G}\right)} + \frac{q_k}{q} \right] - T_k \right\} \quad (23)$$

$$\bullet \quad \left(\frac{\partial F_1}{\partial G}\right)_k = \frac{r_k}{2} \left[ r \left( \frac{2qd}{\mu} + p \right) \left( \frac{2}{3} - b \right) + B \right] + \frac{r}{2} \left[ B_k - r \left( \frac{2qd}{\mu} + p \right) b_k \right. \\ \left. + r \left( \frac{2}{3} - b \right) p_k + r \left( \frac{2}{3} - b \right) \frac{2}{\mu} (dq_k + qd_f) \right] \quad (24)$$

$$\bullet \quad T_k = \left( \frac{\partial \hat{S}}{\partial p} - \frac{2}{p} T_c \right) p_k + \left( \frac{\partial \hat{S}}{\partial b} - 3T_a \right) b_k + \frac{\partial \hat{S}}{\partial e^2} e_k^2 + \left( \frac{\partial \hat{S}}{\partial \psi} + 2T_b \chi \right) \psi_k \\ + \left( \frac{\partial \hat{S}}{\partial \chi} + 2T_b \psi \right) \chi_k - \left( \frac{2}{q} T_c - 6BT_a \right) q_k + 6T_a q \theta B_k \quad (25)$$

$$\bullet \quad p_k = 0 \quad \text{for } k = 1, 3, 4, 5, 7$$

$$p_2 = -2 \frac{\sqrt{\mu p}}{\mu} \sigma_2 \quad (26)$$

$$p_6 = -2 \frac{\sqrt{\mu p}}{\mu} \sigma_6$$

$$p_8 = -2 \frac{\sqrt{\mu p}}{(2\sigma_8)^{3/2}}$$

$$\bullet \quad q_k = 0 \quad \text{for } k = 1, 3, 4, 6, 7$$

$$q_2 = -\frac{\sigma}{2} \quad (27)$$

$$q_5 = \frac{1}{2}$$

$$q_8 = -\frac{\mu}{2} \frac{1}{(2\sigma_8)^{3/2}}$$

- $G_k = 0$  for  $k = 1, 3, 4, 7, 8$  (28)

$$G_2 = -\sigma_2$$

$$G_5 = 1$$

$$G_6 = -\sigma_6$$

- $H_k = G_k$  for  $k = 1, 2, 4, 5, 6, 8$  (29)

$$H_3 = -\sigma_3$$

$$H_7 = -\sigma_7$$

- $f_k = -f \left( \frac{p_k}{p} + \frac{q_k}{q} \right)$  (30)

- $b_k = -\frac{2H}{G^2} \left( H_k - \frac{H}{G} G_k \right)$  (31)

- $d_k = \frac{1}{2} \left( \frac{\mu}{\nu} \right)^{\frac{1}{2}} p_k$  (32)

- $B_k = \frac{H}{G^3} \left( 4H_k - \frac{H}{G} G_k \right)$  (33)

- $e_k^2 = -2\sigma_8 \mu p_k$  for  $k = 1, 2, 3, 4, 5, 6, 7$  (34)

$$e_8^2 = -2\sigma_8 \mu p - 2\mu p$$

- $$X_k = (\sigma_6 \sigma_3 - \sigma_7 \sigma_2) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_k - \frac{Q}{G} G_k + \frac{Q}{2(G+H)} (G_k + H_k) \right]$$
  
 $k = 1, 4, 5, 8$  (35)

$$X_2 = (\sigma_6 \sigma_3 - \sigma_7 \sigma_2) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_2 - \frac{Q}{G} G_2 + \frac{Q}{2(G+H)} (G_2 + H_2) \right]$$

$$- \sigma_7 \frac{\sqrt{2(G+H)}}{2G}$$

$$X_3 = (\sigma_6 \sigma_3 - \sigma_7 \sigma_2) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_3 - \frac{Q}{G} G_3 + \frac{Q}{2(G+H)} (G_3 + H_3) \right]$$

$$+ \sigma_6 \frac{Q\sqrt{2(G+H)}}{2G}$$

$$\chi_6 = (\sigma_6\sigma_3 - \sigma_7\sigma_2) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_6 - \frac{Q}{G} G_6 + \frac{Q}{2(G+H)}(G_6 + H_6) \right] + \frac{\sigma_3 Q \sqrt{2(G+H)}}{2G}$$

$$\chi_7 = (\sigma_6\sigma_3 - \sigma_7\sigma_2) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_7 - \frac{Q}{G} G_7 + \frac{Q}{2(G+H)}(G_7 + H_7) \right] - \frac{\sigma_2 Q \sqrt{2(G+H)}}{2G}$$

$$\psi_k = (\sigma_6\sigma_7 + \sigma_2\sigma_3) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_k - \frac{Q}{G} G_k + \frac{Q}{2(G+H)}(G_k + H_k) \right]$$

$$k = 1, 4, 5, 8$$

(36)

$$\psi_2 = (\sigma_6\sigma_7 + \sigma_2\sigma_3) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_2 - \frac{Q}{G} G_2 + \frac{Q}{2(G+H)}(G_2 + H_2) \right] + \frac{\sigma_3 Q \sqrt{2(G+H)}}{2G}$$

$$\psi_3 = (\sigma_6\sigma_7 + \sigma_2\sigma_3) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_3 - \frac{Q}{G} G_3 + \frac{Q}{2(G+H)}(G_3 + H_3) \right] + \frac{\sigma_2 Q \sqrt{2(G+H)}}{2G}$$

$$\psi_6 = (\sigma_6\sigma_7 + \sigma_2\sigma_3) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_6 - \frac{Q}{G} G_6 + \frac{Q}{2(G+H)}(G_6 + H_6) \right] + \frac{\sigma_7 Q \sqrt{2(G+H)}}{2G}$$

$$\psi_7 = (\sigma_6\sigma_7 + \sigma_2\sigma_3) \frac{\sqrt{2(G+H)}}{2G} \left[ Q_7 - \frac{Q}{G} G_7 + \frac{Q}{2(G+H)}(G_7 + H_7) \right] + \frac{\sigma_6 Q \sqrt{2(G+H)}}{2G}$$

The partial derivatives of  $\hat{S}$  with respect to  $p$ ,  $b$ ,  $e^2$ ,  $\psi$  and  $\chi$  can be found in Reference 3.

### 3.3 Derivative of $F_2''$ with Respect to DS $\phi$ Elements

From Reference 1, one can find that

$$F_2'' = \frac{f^2}{288} \delta + \hat{H}/4 \quad (37)$$

$$\delta = \frac{e^2}{q} (-3b^2 + 24b - 8) + 18 \frac{b^2}{q} - \frac{1}{\mu} d \left( \frac{e^2}{p} + \frac{L}{\mu} \right) \quad (38)$$

$$(60b^2 - 96b + 32) - Bb(24e^2 + 36)$$

$\hat{H}$  is the Hamiltonian of higher harmonics, see Reference 7 for detailed description. Because the new Hamiltonian is a function of only action DS $\phi$  elements, from now on the subscript  $k$  represents partial derivative with respect to those DS $\phi$  action elements.

$$\begin{aligned} \delta_k = & \frac{1}{q^2} [(e_k^2 q - e^2 q_k)(-3b^2 + 24b - 8)] + \frac{e^2}{q} [(-6b + 24)b_k] \\ & + \frac{36b}{q^2} (b_k q - \frac{bq_k}{2}) - \frac{1}{\mu} \left\{ \left[ d_k \left( \frac{e^2}{p} + \frac{L}{\mu} \right) + d \left( \frac{e_k^2 p - e^2 p_k}{p^2} \right) \right. \right. \\ & \left. \left. + \frac{L_k}{\mu} \right] (60b - 96b + 32) + d \left( \frac{e^2}{p} + \frac{L}{\mu} \right) (120bb_k - 96b_k) \right\} \\ & - (24e^2 + 36) (B_k b + Bb_k) - 24e_k^2 Bb \end{aligned} \quad (39)$$

where

$$B_1 = 0 \quad (40)$$

$$B_2 = - \frac{6H^2}{G^4}$$

$$B_3 = \frac{4H}{G^3}$$

$$B_4 = 0$$



$$\begin{aligned}d_1 &= -1. \\d_2 &= 1. \\d_3 &= 0. \\d_4 &= -\mu(2L)^{-3/2}\end{aligned}\tag{41}$$

$$\begin{aligned}p_1 &= -2\left(\frac{p}{\mu}\right)^{1/2} \\p_2 &= -p_1 \\p_3 &= 0. \\p_4 &= -2(p\mu)^{1/2}(2L)^{-3/2}\end{aligned}\tag{42}$$

$$\begin{aligned}e_1^2 &= -\frac{2L}{\mu} p_1 \\e_2^2 &= -e_1^2 \\e_3^2 &= 0. \\e_4^2 &= -\frac{2}{\mu}(p + Lp_4)\end{aligned}\tag{43}$$

$$\begin{aligned}q_1 &= -0.5 \\q_2 &= 1.0 \\q_3 &= 0 \\q_4 &= -0.5\mu(2L)^{-3/2}\end{aligned}\tag{44}$$

$$\begin{aligned}b_1 &= 0 \\b_2 &= \frac{2}{G}\left(\frac{H}{G}\right)^2 \\b_3 &= -\frac{2}{G}\left(\frac{H}{G}\right) \\b_4 &= 0.\end{aligned}\tag{45}$$

$$\begin{aligned}L_1 &= 0 \\L_2 &= 0 \\L_3 &= 0 \\L_4 &= 1\end{aligned}\tag{46}$$

$$F''_{2k} = \frac{f}{288} (2f_k \delta + f \delta_k) + \frac{1}{q} (\hat{H}_k - \frac{q_k}{q} \hat{H}) \quad (47)$$

$$\hat{H}_k = \frac{\partial \hat{H}}{\partial p} p_k + \frac{\partial \hat{H}}{\partial e^2} e_k^2 + \frac{\partial \hat{H}}{\partial b} b_k \quad (48)$$

where

$$\begin{aligned} f_1 &= \frac{f^2}{\mu} \left( \frac{1}{2} \mu p + 2q \sqrt{\mu p} \right) \\ f_2 &= - \frac{f^2}{\mu} \left( \mu p + 2q \sqrt{\mu p} \right) \\ f_3 &= 0 \\ f_4 &= - \frac{f^2}{(2\sigma_8)^{3/2}} \left( \frac{1}{2} \mu p + 2q \sqrt{\mu p} \right) \end{aligned} \quad (49)$$

Now the abbreviations  $A_1, A_2, A_3, A_4$  in the expressions of analytical integration will be given:

$$\begin{aligned} A_4 &= \frac{\epsilon}{2} f_4 (b - 2/3) + \mu(2L)^{-3/2} + \frac{\epsilon^2}{288} f (2f_4 \delta + f \delta_4) \\ &\quad + \frac{\epsilon^2}{q} \left( \hat{H}_4 - \frac{q_4}{q} \hat{H} \right) \end{aligned} \quad (50)$$

$$A_3 = \frac{\epsilon}{2} f b_3 + \frac{\epsilon^2}{288} f^2 \delta_3 + \frac{\epsilon^2}{q} \left( \hat{H}_3 - \frac{q_3}{q} \hat{H} \right) \quad (51)$$

$$\begin{aligned} A_2 &= \frac{\epsilon}{2} \left[ f_2 (b - 2/3) + f b_2 \right] + \frac{\epsilon^2}{288} f (2f_2 \delta + f \delta_2) \\ &\quad + \frac{\epsilon^2}{q} \left( \hat{H}_2 - \frac{q_2}{q} \hat{H} \right) + A_3 \end{aligned} \quad (52)$$

$$\begin{aligned} A_1 &= 1 + \frac{\epsilon}{2} f_1 (b - 2/3) + \frac{\epsilon^2}{288} f (2f_1 \delta + f \delta_1) \\ &\quad + \frac{\epsilon^2}{q} \left( \hat{H}_1 - \frac{q_1}{q} \hat{H} \right) + A_2 \end{aligned} \quad (53)$$

#### 4.0 CONCLUSIONS

The equations described in this report have been implemented into the ASOP program. The program has been checked out and verified with results documented in reference 8. Comparisons with numerical integrations show the long period theory to be accurate to within several meters after 800 revolutions. The extension of ASOP to include the long period terms, allows the solution to maintain a high degree of accuracy even for extremely long prediction intervals.

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**APPENDIX**

**COMPUTATIONAL PROCEDURE**

The computational procedure for elimination of long periodic terms and analytical integration of primed variables are described below. First subroutine LONGPP(NN) (long period perturbations) is called with parameter 0 , it will return initialized primed variable. During the procedure subroutine DETERM is called to compute terms related to the higher order harmonics. Then subroutine will be called again with parameter 1 , this time it will return the partial derivatives of primed Hamiltonian with respect to the  $DS\phi$  elements. During the procedure subroutine FPRIME is called to compute derivatives of higher order harmonics. The sequence of computation will be given below. The left column gives the quantity to be computed, and the right column references the equation number in the text.

LONGPP(0)

Computating Sequence	From Equation
d	(20)
B	(18)
e	(10)
$\frac{\partial F_1}{\partial G}$	(19)
X	(12)
$\psi$	(13)
$\theta$	(14)
$\hat{S}_k$	subroutine DETERM
$f_k$	(30)
$b_k$	(31)
$d_k$	(32)

Computating Sequence (continued)

From Equation  
(continued)

$B_k$	(33)
$e_k^2$	(34)
$\left(\frac{\partial F_1}{\partial G}\right)_k$	(24)
$\chi_k$	(35)
$\psi_k$	(36)
$T_k$	(25)
$S_{1k}^*$	(23)
$\sigma'(0), \rho'(0)$	(3)

LONGPP(1)

$d$	(20)
$B$	(18)
$e$	(10)
$B_k$	(40)
$d_k$	(41)
$p_k$	(42)
$e_k^2$	(43)
$q_k$	(44)
$\hat{H}_k$	subroutine FPRIME
$\delta_k$	(39)
$F_{2k}''$	(47)
$A_1, A_2, A_3, A_4$	(50) - (53)