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NONLINEAR INTERACTION OF WAVES IN BOUNDARY-LAYER FLOWS

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Abstract

First-order nonlinear interactions of Tollmien-Schlichting waves of different frequencies and initial amplitudes in boundary-layer flows are analyzed by using the method of multiple scales. For the case of two waves, a strong nonlinear interaction exists if one of the frequencies ω_2 is twice the other frequency ω_1. Numerical results for flow past a flat plate show that this interaction mechanism is strongly destabilizing even in regions where either the fundamental or its harmonic is damped in the absence of the interaction. For the case of three waves, a strong nonlinear interaction exists when $\omega_3 = \omega_2 - \omega_1$. This combination resonance causes the amplitude of the wave with the difference frequency ω_3 to multiply many times in magnitude in a short distance even if it is damped in the absence of the interaction. The initial amplitudes play a dominant role in determining the changes in the amplitudes of the waves in both of these mechanisms.

I. INTRODUCTION

One of the major roads from laminar to turbulent flow involves the initial linear amplification of disturbances, which might be present in the flow. However, as these disturbances grow to appreciable amplitudes, nonlinear effects set in. The nonlinear mechanisms that are activated depend on the spectrum of the disturbances. In this paper, we investigate two of these mechanisms.

In his experiments on the transition from laminar to turbulent flow in a separated shear layer, Sato[1] observed the appearance of the subharmonic of order one-half in addition to the higher harmonics of the fundamental wave. Wille[2] observed the development of subharmonic waves while investigating the stability of both circular and plane jets. Kachanov, et al[3] observed that, in addition to the higher harmonics of a fundamental wave, which was introduced in the flow by a vibrating ribbon, a subharmonic wave with one-half the frequency of the fundamental wave appeared downstream. Michalke[4] postulated that the subharmonic appears when two vortices rotate around each other in a fusion mating dance. Kelly[5] showed that the appearance of the subharmonic in a shear layer is due to a secondary linear instability associated with a time-dependent flow that consists of the superposition of the basic flow and a finite-amplitude fundamental wave. Nayfeh and Bozatli[6] investigated the appearance of the subharmonic in boundary layers by analyzing the instability associated with a time-dependent flow that consists of the superposition of the basic flow and a Tollmien-Schlichting wave. The results show that the amplitude of the fundamental wave must exceed a critical value to trigger this parametric instability. This value is proportional to a detuning parameter that is the real part of $k - 2K$, where k and K are the wavenumbers of the fundamental and its subharmonic, respectively. For the Blasius flow, the critical amplitude is approximately 29% of the mean flow. For other flows where the detuning parameter is small, such as free-shear layer flows, the critical amplitude can be small, thus the parametric instability might play a greater role. Since the analysis of Kelly[5] and Nayfeh and Bozatli[6] are linear, they do not account for the effect of the subharmonic wave on the fundamental wave. This effect may be small initially, but as the subharmonic grows appreciably, its effect on the fundamental cannot be neglected. One of the purposes of the present paper is to determine the nonlinear interaction of a Tollmien-Schlichting wave with its subharmonic.

Sato[7], Miksad[8], and Kachanov, et al[9] observed that the nonlinear development of the waves in the transition region depends on the initial and external disturbances. Sato[7] conducted an experiment on the stability of symmetric laminar wakes by exciting two unstable modes with the frequencies f_1 and f_2. He observed the generation of waves having the frequencies $f_2 \pm f_1$. Miksad[8] excited two unstable modes of a laminar asymmetric free-shear layer. He also observed nonlinear triggered instabilities of the difference mode $f_2 - f_1$, subharmonics, and higher harmonics of the fundamental waves. Kachanov, et al[9] introduced two Tollmien-Schlichting waves in the boundary layer on a flat plate by using two vibrating ribbons. They observed the appearance and growth of a Tollmien-Schlichting wave having the difference frequency $f_2 - f_1$. Norman[10] also observed the amplification of the difference harmonic of two introduced disturbance waves in his experimental study of secondary flows around and downstream of protuberances in laminar boundary layers. The second purpose of the present paper is to determine the nonlinear interaction of three Tollmien-Schlichting waves (combination resonance) in boundary layers and show that the difference frequency can be very unstable when generated by the nonlinearity, even though it is stable when introduced by itself in the boundary layer.

The problem is formulated in Sec. II. The analysis for the combination-resonance case is contained in Sec. III, while the results for the second-harmonic case are stated in Sec. IV. The numerical procedure is discussed in Sec. V, while the numerical results are presented in Sec. VI.

II. PROBLEM FORMULATION

We consider nonlinear interactions of wave packets in a two-dimensional steady incompressible boundary-layer. The equations describing the motion of the fluid are

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0, \tag{1}$$

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \tilde{u}}{\partial y} = -\frac{\partial \tilde{p}}{\partial x} + \frac{1}{R}\nabla^2 \tilde{u}, \tag{2}$$

$$\frac{\partial \tilde{v}}{\partial t} + \tilde{u}\frac{\partial \tilde{v}}{\partial x} + \tilde{v}\frac{\partial \tilde{v}}{\partial y} = -\frac{\partial \tilde{p}}{\partial y} + \frac{1}{R}\nabla^2 \tilde{v}, \tag{3}$$

$$\tilde{u} = \tilde{v} = 0 \quad \text{at} \quad y = 0, \tag{4}$$

$$\tilde{u} \to 1 \quad \text{as} \quad y \to \infty, \tag{5}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Here, x and y are made dimensionless by using a reference length δ_r, the time is made dimensionless by using δ_r/U_∞, and the velocities are made dimensionless by using the freestream velocity U_∞. The Reynolds number $R = U_\infty \delta_r/\nu$ with ν being the fluid kinematic viscosity.

The analysis is restricted to basic flows that are slightly nonparallel (i.e., vary slowly in the streamwise direction) and to disturbances that are small but finite. The slow variation is expressed by using the slow scale $x_1 = \epsilon_1 x$, where ϵ_1 is a small dimensionless quantity that characterizes the nonparallelism of the flow and can be related to R by $\epsilon_1 = R^{-1}$. The smallness of the amplitude of the disturbance is expressed by introducing the small dimensionless parameter ϵ. For a general solution, we assume that $\epsilon = O(\epsilon_1)$ so that the resulting expansion accounts simultaneously for the effects of nonparallelism and nonlinearity. When $\varepsilon \ll \varepsilon_1$, the nonlinear effects are negligible and the solution reduces to those obtained in Refs. 11 and 12. When $\varepsilon \gg \varepsilon_1$, the nonparallel effects are negligible and the solution reduces to equations with constant coefficients.

We assume that each flow quantity is the sum of a mean-flow quantity and an unsteady disturbance quantity, which is assumed to be much smaller than the mean-flow quantity. We can then express the velocity components and the pressure as

$$\tilde{u}(x,y,t) = U_0(x_1,y) + \varepsilon u(x,y,t), \tag{6}$$

$$\tilde{v}(x,y,t) = \varepsilon_1 V_0(x_1,y) + \varepsilon v(x,y,t), \tag{7}$$

$$\tilde{p}(x,y,t) = P\ (x_1) + \varepsilon p(x,y,t), \tag{8}$$

where U_0, V_0, and P_0 are the nonparallel basic-flow quantities. Substituting Eqs. (6) - (8) into Eqs. (1) - (5) and subtracting the basic-flow quantities, we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{9}$$

$$\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + v \frac{\partial U_0}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{R} \nabla^2 u = - \varepsilon_1 u \frac{\partial U_0}{\partial x_1} - \varepsilon_1 V_0 \frac{\partial u}{\partial y}$$

$$- \varepsilon u \frac{\partial u}{\partial x} - \varepsilon v \frac{\partial u}{\partial y} \tag{10}$$

$$\frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \frac{1}{R} \nabla^2 v = - \varepsilon_1^2 u \frac{\partial V_0}{\partial x_1} - \varepsilon_1 V \frac{\partial v}{\partial y} - \varepsilon_1 v \frac{\partial V_0}{\partial y}$$

$$- \varepsilon u \frac{\partial v}{\partial x} - \varepsilon v \frac{\partial v}{\partial y}, \tag{11}$$

$$u = v = 0 \quad \text{at} \quad y = 0, \tag{12}$$

$$u, v \quad \text{as} \quad y \to \infty. \tag{13}$$

Without loss of generality, we let $\varepsilon = \varepsilon_1$. To determine the wave-packet solutions of Eqs. (9) - (13), we use the method of multiple scales[13] and seek an expansion in the form

$$u = u_0(x_0,x_1,y,T_0,T_1) + \varepsilon u_1(x_0,x_1,y,T_0,T_1) + \ldots, \tag{14}$$

$$v = v_0(x_0,x_1,y,T_0,T_1) + \varepsilon v_1(x_0,x_1,y,T_0,T_1) + \ldots, \tag{15}$$

$$p = p_0(x_0,x_1,y,T_0,T_1) + \varepsilon p_1(x_0,x_1,y,T_0,T_1) + \ldots, \tag{16}$$

where $x_0 = x$, $T_0 = t$, and $T_1 = \varepsilon t$. Substituting Eqs. (14) - (16) into Eqs. (9) - (13) and equating coefficients of like powers of ε, we obtain

<u>Order ε^0</u>

$$\frac{\partial u_0}{\partial x_0} + \frac{\partial v_0}{\partial y} = 0, \tag{17}$$

$$\mathcal{L}_1(u_0,v_0,p_0) \equiv \frac{\partial u_0}{\partial T_0} + U_0 \frac{\partial u_0}{\partial x_0} + v_0 \frac{\partial U_0}{\partial y} + \frac{\partial p_0}{\partial x_0} - \frac{1}{R} \nabla_0^2 u_0 = 0, \tag{18}$$

$$\mathcal{L}_2(u_0,v_0,p_0) \equiv \frac{\partial v_0}{\partial T_0} + U_0 \frac{\partial v_0}{\partial x_0} + \frac{\partial p_0}{\partial y} - \frac{1}{R} \nabla_0^2 v_0 = 0, \tag{19}$$

$$u_0 = v_0 = 0 \quad \text{at} \quad y = 0 \tag{20}$$

$$u_0,\ v_0 \to 0 \quad \text{as} \quad y \to \infty \tag{21}$$

<u>Order ε</u>

$$\frac{\partial u_1}{\partial x_0} + \frac{\partial v_1}{\partial y} = - \frac{\partial u_0}{\partial x_1}, \tag{22}$$

$$\mathcal{L}_1(u_1,v_1,p_1) = - \frac{\partial u_0}{\partial T_1} - U_0 \frac{\partial u_0}{\partial x_1} - \frac{\partial p_0}{\partial x_1} + \frac{2}{R} \frac{\partial^2 u_0}{\partial x_0 \partial x_1} - u_0 \frac{\partial U_0}{\partial x_1}$$
$$- v_0 \frac{\partial u_0}{\partial y} - u_0 \frac{\partial u_0}{\partial x_0} - v_0 \frac{\partial u_0}{\partial y}, \tag{23}$$

$$\mathcal{L}_2(u_1,v_1,p_1) = - \frac{\partial v_0}{\partial T_1} - U_0 \frac{\partial v_0}{\partial x_1} + \frac{2}{R} \frac{\partial^2 v_0}{\partial x_0 \partial x_1} - V_0 \frac{\partial v_0}{\partial x_0} - v_0 \frac{\partial V_0}{\partial y}$$
$$- u_0 \frac{\partial v_0}{\partial x_0} - v_0 \frac{\partial v_0}{\partial y} \tag{24}$$

$$u_1 = v_1 = 0 \quad \text{at} \quad y = 0, \tag{25}$$

$$u_1, v_1 \to 0 \quad \text{as} \quad y \to \infty, \tag{26}$$

where

$$\nabla_0^2 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_2} .$$

In what follows, we describe the details of the analysis for the combination-resonance case and only state the results for the second-haromnic resonance case.

III. COMBINATION RESONANCES

A. First-Order Problem

For the case of combination resonances, we consider three wave-packets centered at the frequencies ω_1, ω_2, and ω_3, Then we examine the resonances that might exist among them. Thus, the solution of Eqs. (17) - (21) is expressed as a linear combination of three Tollmien-Schlichting waves; that is,.

$$u_0 = A_1(x_1,T_1)\zeta_{11}(y;x_1)\exp(i\theta_1) + A_2(x_1,T_1) \times$$

$$\zeta_{12}(y;x_1)\exp(i\theta_2) + A_3(x_1,T_1)\zeta_{13}(y;x_1)\exp(i\theta_3) + c.c., \quad (27)$$

$$v_0 = A_1(x_1,T_1)\zeta_{21}(y;x_1)\exp(i\theta_1) + A_2(x_1,T_1) \times$$

$$\zeta_{22}(y;x_1)\exp(i\theta_2) + A_3(x_1,T_1)\zeta_{23}(y;x_1)\exp(i\theta_3) + c.c., \quad (28)$$

$$p_0 = A_1(x_1,T_1)\zeta_{31}(y;x_1)\exp(i\theta_1) + A_2(x_1,T_1) \times$$

$$\zeta_{32}(y;x_1)\exp(i\theta_2) + A_3(x_1,T_1)\zeta_{33}(y;x_1)\exp(i\theta_3) + c.c., \quad (29)$$

where

$$\frac{\partial\theta_n}{\partial x_0} = k_n(x_1), \quad \frac{\partial\theta_n}{\partial T_0} = -\omega_n \qquad (n = 1, 2, 3) \quad (30)$$

with the ω_n being real constants. The quasi-parallel Orr-Sommerfeld problems for these waves are

$$M_1(\zeta_{1n}, \zeta_{2n}; k_n) \equiv D\zeta_{2n} + ik_1\zeta_{1n} = 0, \quad (31)$$

$$M_2(\zeta_{1n}, \zeta_{2n}, \zeta_{3n}; k_n, \omega_n) \equiv i(U_0k_n - \omega_n)\zeta_{1n} + \zeta_{2n}DU_0$$

$$+ ik_1\zeta_{3n} - \frac{1}{R}(D^2 - k_n^2)\zeta_{1n} = 0, \quad (32)$$

$$M_3(\zeta_{1n}, \zeta_{2n}, \zeta_{3n}; k_n, \omega_n) \equiv i(U_0 k_n - \omega_n)\zeta_{2n} + D\zeta_{3n}$$

$$- \frac{1}{R}(D^2 - k_n^2)\zeta_{2n} = 0, \qquad (33)$$

$$\zeta_{1n} = \zeta_{2n} = 0 \qquad \text{at} \qquad y = 0, \qquad (34)$$

$$\zeta_{1n}, \; \zeta_{2n}, \to 0 \qquad \text{as} \qquad y \to \infty, \qquad (35)$$

where $D = \partial/\partial y$.

B. Second-Order Problem

Substituting Eqs. (27) - (29) into Eqs. (22) - (26), we find that the inhomogeneous parts in Eqs. (22) - (26) contain terms proportional to

$$\exp(i\theta_1), \exp(i\theta_2), \exp(i\theta_3),$$

$$\exp[i(\theta_2 - \overline{\theta}_1)], \exp[i(\theta_1 + \theta_3)], \exp[i(\theta_2 - \overline{\theta}_3)]$$

where the overbar indicates the complex conjugate. The terms that are proportional to these exponential expressions will create secular terms in the particular solutions for u_1, v_1, and p_1 if $k_3 \simeq k_2 - k_1$ and $\omega_3 \simeq \omega_2 - \omega_1$; that is, when a combination resonance exists among the waves. To express quantitively the nearness of the above resonances, we introduce the two detuning parameters σ_1 and σ_2 defined by

$$\omega_3 - \omega_2 + \omega_1 = \varepsilon\sigma_1, \qquad (36)$$

$$\text{Real}(k_3 - k_2 + k_1) = \varepsilon\sigma_2, \qquad (37)$$

where $\sigma_n = O(1)$. Using Eqs. (36) and (37), we write

$$\theta_1 + \theta_3 = (k_1 + k_3)dx_0 - (\omega_1 + \omega_3)T_0 = \theta_2 + \phi \quad i \quad (k_{1i} + k_{3i} - k_{2i})dx_0, \tag{38}$$

$$\theta_2 - \theta_3 = \theta_1 - \phi + i \quad (k_{2i} + k_{3i} - k_{1i})dx_0, \tag{39}$$

$$\theta_2 - \overline{\theta}_1 = \theta_3 - \phi + i \quad (k_{1i} + k_{2i} - k_{3i})dx_0, \tag{40}$$

where k_{ni} stands for the imaginary part of k_n and

$$\phi = \int \sigma_2 dx_1 - \sigma_1 T_1 \tag{41}$$

To determine the A_n, we seek a particular solution for the second-order problem in the form

$$u_1 = \psi_{11}(y;x_1)\exp(i\theta_1) + \psi_{12}(y;x_1)\exp(i\theta_2) + \psi_{13}(y;x_1) \times \exp(i\theta_3) + \text{c.c.}, \tag{42}$$

$$v_1 = \psi_{21}(y;x_1)\exp(i\theta_1) + \psi_{22}(y;x_1)\exp(i\theta_2) + \psi_{23}(y;x_1) \times \exp(i\theta_3) + \text{c.c.}, \tag{43}$$

$$p_1 = \psi_{31}(y;x_1)\exp(i\theta_1) + \psi_{32}(y;x_1)\exp(i\theta_2) + \psi_{33}(y;x_1) \times \exp(i\theta_3) + \text{c.c.} \tag{44}$$

Substituting Eqs. (27) - (30) and (38) - (44) into Eqs. (22) - (26) and equating the coefficents of $\exp(i\theta_1)$, $\exp(i\theta_2)$, and $\exp(i\theta_3)$ on both sides, we obtain the following equations:

$$M_1(\psi_{1j}, \psi_{2j}; k_j) = d_{1j}, \tag{45}$$

$$M_2(\psi_{1j}, \psi_{2j}, \psi_{3j}; k_j, \omega_j) = d_{2j}, \tag{46}$$

$$M_3(\psi_{1j}, \psi_{2j}, \psi_{3j}; k_j, \omega_j) = d_{3j}, \tag{47}$$

$$\psi_{1j} = \psi_{2j} = 0 \qquad \text{at} \qquad y = 0, \tag{48}$$

$$\psi_{1j}, \psi_{2j} \quad \to \quad 0 \qquad \text{as} \qquad y \to \infty, \tag{49}$$

for $j = 1$, 2, and 3, where the d_{ij} are given in Appendix A.

C. Adjoint Problem

Since the homogeneous parts of Eqs. (45) - (49) are the same as Eqs. (31) - (35) and since the latter have a nontrivial solution, the inhomogeneous equations (45) - (49) have a solution if, and only if, the inhomogeneous parts are orthogonal to every solution of the adjoint homogeneous problem; that is,

$$\int_0^\infty (d_{1j}\zeta^*_{1j} + d_{2j}\zeta^*_{2j} + d_{3j}\zeta^*_{3j})dy = 0 \quad \text{for} \quad j = 1, 2, \text{ and } 3, \tag{50}$$

where the ζ^*'s are the solutions of

$$M^*_1(\zeta^*_{2j}, \zeta^*_{3j}; k_j) \equiv ik_j\zeta^*_{2j} - D\zeta^*_{3j} = 0, \tag{51}$$

$$M^*_2(\zeta^*_{1j}, \zeta^*_{2j}, \zeta^*_{3j}; k_j, \omega_j) \equiv i(U_0k_j - \omega_j)\zeta^*_{3j} + \zeta^*_{2j}DU_0$$
$$- D\zeta^*_{1j} - \frac{1}{R}(D^2 - k_j^2)\zeta^*_{3j} = 0, \tag{52}$$

$$M^*_3(\zeta^*_{1j}, \zeta^*_{2j}, \zeta^*_{3j}; k_j, \omega_j) \equiv i(U_0k_j - \omega_j)\zeta^*_{2j} + ik_j\zeta^*_{1j}$$
$$- \frac{1}{R}(D^2 - k_j^2)\zeta^*_{2j} = 0, \tag{53}$$

$$\zeta^*_{2j} = \zeta^*_{3j} = 0 \qquad \text{at} \quad y = 0, \tag{54}$$

$$\zeta^*_{1j}, \zeta^*_{2j} \quad \to 0 \qquad \text{as} \quad y \to \infty . \tag{55}$$

Substituting for the d_{ij} from Appendix A into Eq. (50) and defining

$$a_j = A_j \exp\left[\int_{\tilde{x}}^{x} k_{ji} dx\right], \tag{56}$$

we obtain the following differential equations for the evolution of a_1, a_2, and a_3:

$$\frac{1}{\omega_1'} \frac{\partial a_1}{\partial t} + \frac{\partial a_1}{\partial x} = (\varepsilon_1 \frac{h_{11}}{f_1} - k_{1i})a_1 + \varepsilon \frac{h_{123}}{f_1} a_2 \bar{a}_3 \exp(-i\phi), \tag{57}$$

$$\frac{1}{\omega_2'} \frac{\partial a_2}{\partial t} + \frac{\partial a_2}{\partial x} = (\varepsilon_1 \frac{h_{22}}{f_2} - k_{2i})a_2 + \varepsilon \frac{h_{213}}{f_2} a_1 a_3 \exp(i\phi), \tag{58}$$

$$\frac{1}{\omega_3'} \frac{\partial a_3}{\partial t} + \frac{\partial a_3}{\partial x} = (\varepsilon_1 \frac{h_{33}}{f_3} - k_{3i})a_3 + \varepsilon \frac{h_{312}}{f_3} a_2 \bar{a}_1 \exp(-i\phi), \tag{59}$$

where $\omega_n' = d\omega_n/dk_n$ is the group velocity and ϕ is defined in Eq. (41). For spatial modulation only, $\sigma_1 = 0$ and $\partial a_n/\partial t = 0$; all the calculations presented in this paper are for this case. We note that Eqs. (57) - (59) account for the combined effects of the nonparallelism (i.e., growth of the boundary layer) and the nonlinear interaction. If $\varepsilon \ll \varepsilon_1$, the nonlinear interactions can be neglected and the spatial variations in Eqs. (57) - (59) reduce to the nonparallel solutions of Refs. 11 and 12. When $\varepsilon_1 \ll \varepsilon$, the effects of the nonparallelism are negligible; that is, one can set $\varepsilon_1 = 0$ and all the coefficients in Eqs. (57) - (59) can be treated as constants.

IV. HARMONIC RESONANCE

The interaction between a fundamental Tollmien-Schlichting wave and its second harmonic is analyzed using a procedure similar to that outlined in the previous section. In this case, instead of Eqs. (57) - (59) we obtain

$$\frac{1}{\omega_1'}\frac{\partial a_1}{\partial t} + \frac{\partial a_1}{\partial x} = (\varepsilon_1\frac{h_{11}}{f_1} - k_{1i})a_1 + \varepsilon\frac{h_{12}}{f_1}a_2\bar{a}_1\exp(-i\phi), \tag{60}$$

$$\frac{1}{\omega_2'}\frac{\partial a_2}{\partial t} + \frac{\partial a_2}{\partial x} = (\varepsilon_1\frac{h_{22}}{f_2} - k_{2i})a_2 + \varepsilon\frac{h_{21}}{f_2}a_1^2\exp(i\phi), \tag{61}$$

where ϕ is defined in Eq. (41) and

$$\varepsilon\sigma_2 = \mathrm{Real}(k_2 - 2k_1), \quad \varepsilon\sigma_1 = \omega_2 - 2\omega_1, \tag{62}$$

and f_1, f_2, h_{11}, h_{22}, h_{12}, and h_{21} are given in Appendices B and C.

For spatial modulation only, $\sigma_1 = 0$ and $\partial a_n/\partial t = 0$. All the calculations presented in this paper are for this case.

V. COMPUTATION PROCEDURE

A. Solutions of First- and Second-Order Problems

The same procedure is followed in solving the first- and second-order problems for both harmonic and combination resonances. Therefore, only the computation methodology for the solution of the first-order problem for the first mode is outlined here.

Equations (31) - (33) are expressed as a system of first-order differential equation in the form

$$\frac{dz}{dx} = Gz, \tag{63}$$

where z is a 4 x 1 matrix with the elements

$$z_1 = \zeta_{11}(y;x_1),\ z_2 = D\zeta_{11}(y;x_1),\ z_3 = \zeta_{21}(y;x_1),\ z_4 = \zeta_{31}(y;x_1), \tag{64}$$

and G is a 4 x 4 matrix; its elements are given in Appendix D.

We start the integration of Eqs. (63) at $y = y_e$, where y_e is larger than the boundary-layer thickness. Hence, $U_0 = 1$, $DU_0 = 0$, and $D^2U_0 = 0$ at y_e. Then the matrix G has constant coefficients at $y = y_e$ and Eqs. (63) have solutions of the form

$$z_i = \sum_{j=1}^{4} c_{ij}\exp(\lambda_j y) \quad \text{for} \quad i = 1, 2, 3, \text{ and } 4, \tag{65}$$

where the c_{ij} are constants, the λ's are the solutions of

$$|G - \lambda I| = 0, \tag{66}$$

and I is the identity matrix. Equation (66) has the roots

$$\lambda_{1,2} = \pm k_1, \quad \lambda_{3,4} = \pm [k_1^2 + i(k_1 - \omega_1)R]^{1/2}, \tag{67}$$

Two of these roots have positive real parts that make the solution grow exponentially as $y \to \infty$; hence, they must be discarded to satisfy conditions (35). This leaves two linearly independent solutions that decay exponentially with y.

The eigenvalues are not known a priori and must be determined along with the eigenfunctions. For given values of ω_1 and R, we guess a value for k_1 and integrate Eqs. (63) from y_e to $y = 0$. If the quessed value of k_1 does not staisfy the boundary conditions at $y = 0$, k_1 is incremented by using a Newton-Raphson scheme and the procedure is repeated until the boundary conditions are satisfied to within a specified accuracy. The integration is done by using a computer code developed by Scott and Watts[14]. This technique orthonormalizes the solution of the set of equations whenever a loss of independence is detected.

B. Solution of Adjoint Problem

The solution procedure is exactly the same as that for the first-order problem. The coefficients of the z matrix are

$$z_1 = \overset{*}{\zeta}_{21}(y;x_1),\ z_2 = D\overset{*}{\zeta}_{21}(y;x_1),\ z_3 = \overset{*}{\zeta}_{31}(y;x_1),$$

$$z_4 = \overset{*}{\zeta}_{11}(y;x_1) \tag{68}$$

and the adjoint problem has the same eigenvalues as the first-order problem.

C. Solvability Conditions

The calculations are repeated at different streamwise locations to evaluate f_j, h_{jj}, k_j, and the other interaction integrals for a given frequency along the x-axis. A fourth-order fixed step-size Runge-Kutta integration scheme is used to solve either Ecs. (57) - (59) for combination resonances or Eqs. (60) and (61) for harmonic resonances to find the amplitudes of the waves for different initial amplitudes of the respective modes.

VI. Results and Discussion

The analysis presented in this paper is applicable to both two-and three-wave interactions. First, we present and discuss numerical results for the case of two-wave interactions. Then, we present and discuss numerical results for the interaction of three waves whose frequencies are such that $F_3 = F_2 - F_1$.

A. Two-Wave Interactions

The numerical results presented in Ref. 6 show that the amplitude of a wave $a_2^* = \varepsilon a_2$ must exceed a critical value before it can generate and amplify its subharmonic. For the Blasius flow, the critical value is approximately 29% of the mean flow. This is for the case when the subharmonic wave has an infinitesimal amplitude. When the amplitude $a_1^* = \varepsilon a_1$ of the subharmonic wave is not infinitesimal, its influence on a_2^* should be taken into account. The equations governing this influence are Eqs. (60) and (61) whose general solution is not available yet. The previous results of the parametric instability model[6] show that a_1^* oscillates about its non-interaction value until a_2^* reaches the critical value. Figures 1 and 2, obtained by numerically solving Eqs. (60) and (61), agree with this conclusion. Initially, a_2^* increases while a_1^* oscillates around its non-interaction value.

At $R \simeq 580$, Fig. 1 shows that a_1^* starts to deviate sharply from its non-interaction value, while it follows from Fig. 2 that $\ell n a_2^* \simeq -1.25$ or $a_2^* = 0.286$ at this location. Hence, when a_2^* is less than this critical value, a_1^* can be approximated by its non-interaction value; that is

$$\overset{*}{a}_1 = \overset{*}{a}_{10}\exp(-k_{1i}x + i\tau) \tag{69}$$

where $\overset{*}{a}_{10}$ and τ are the initial amplitude and phase of the subharmonic wave, respectively. If we substitute Eq. (69) into Eq. (61) and neglect the nonparallel effects, we obtain

$$\frac{d\overset{*}{a}_2}{dx} + k_{2i}\overset{*}{a}_2 = \frac{h_{21}}{f_2}\overset{*}{a}{}_{10}^{2}\exp[-(2k_{1i} + i\varepsilon\sigma_2)x + 2i\tau]. \tag{70}$$

The solution of Eq. (70) that satisfies the initial condition $\overset{*}{a}_2 = \overset{*}{a}_2(0)$ at $x = 0$ can be written as

$$\overset{*}{a}_2 = [\overset{*}{a}_2(0) + \frac{h_{21}}{f_2(2k_{1i} + i\varepsilon\sigma_2)}\overset{*}{a}{}_{10}^{2}\exp(2i\tau)]\exp(-k_{2i}x)$$
$$- \frac{h_{21}}{f_2(2k_{1i} + i\varepsilon\sigma_2)}\overset{*}{a}{}_{10}^{2}\exp[-(2k_{1i} + i\varepsilon\sigma_2)x + 2i\tau]. \tag{71}$$

Equation (71) represents an approximation to $\overset{*}{a}_2$ as long as it is less than the critical value needed to trigger the parametric instability in the subharmonic wave.

Next, we consider the generation and amplification of a second-harmonic wave by a fundamental Tollmien-Schlichting wave. We consider the following three cases: (i) fundamental wave is stable while its second harmonic is unstable, (ii) fundamental wave is unstable while its second harmonic is stable, (iii) both fundamental and second-harmonic waves are unstable.

When the fundamental wave is initially stable while its second harmonic is unstable, $\overset{*}{a}_1$ decays until it reaches the unstable region and then it increases as shown in Fig. 1. For Reynolds Number less than 560, $\overset{*}{a}_1$ oscillates around its non-interaction value, implying a small initial influence of its second harmonic on it. Thus, $\overset{*}{a}_2$ can be approximated initially by Eq. (71). Figure 2 shows that the values obtained from Eq. (71) are in good agreement with those obtained by numerically integrating Eqs. (60) and (61) for $R \leq 560$. After a short initial distance, the second term on the right-hand side of Eq. (71) decays. Then, a_2^* can be approximated by

$$a_2^* = \left[a_2^*(0) + \frac{h_{21}}{f_2(2k_{1i} + i\varepsilon\sigma_2)} a_{10}^{*2} \exp(2i\tau)\right] \exp(-k_{2i}x), \tag{72}$$

as long as a_2^* is less than the critical value.

Hence, the effect of the fundamental wave on its second harmonic is to increase its initial amplitude. However, as a_2^* attains large values, it strongly influences a_1^* which in turn strongly influences a_2^*. The result is an accelerated instability.

For the case when the fundamental wave is initially unstable while its second harmonic is stable, we performed calculations for waves with the frequencies $F_1 = 46.5 \times 10^{-6}$ and $F_2 = 93 \times 10^{-6}$. The fundamental wave is in the unstable region at $R = 950$ where the calculations are started. Thus, its unstable downstream of $R = 950$. Figure 3, obtained by numerically integrating Eqs. (60) and (61), shows that a_1^* hardly deviates from its non-interaction value. On the other hand, a_2^* increases many orders of magnitude even for small initial amplitudes of the fundamental wave as shown in Fig. 4. In these calculations, the initial amplitude of the second-harmonic wave is taken to be 0.1% while the initial amplitudes of the fundamental wave are 0.1% and 0.5%. Since a_1^* hardly deviates from its non-interaction value, Eq. (71) is expected to be a good approximation to a_2^*. Figure 4 shows that the values obtained from Eq. (71) oscillate about those obtained by numerically integrating Eqs. (60) and (61). Since the initial values are very small, a_2^* does not reach the critical value to influence a_1^*. After a short initial distance, the first term on the right-hand side of Eq. (71) decays and

a_2^* can be approximated by

$$a_2^* = -\frac{h_{21}}{f_2(2k_{1i} + i\varepsilon\sigma_2)} a_{10}^{*2}\exp[-(2k_{1i} + i\varepsilon\sigma_2)x + 2i\tau], \tag{73}$$

as long as a_2^* is less than the critical value. Consequently, the effect of the interaction is to produce a second-harmonic wave that grows approximately at a rate that is twice that of the fundamental wave.

For the case when both waves are unstable, we performed numerical calculations for waves having the frequencies $F_1 = 52 \times 10^{-6}$ and $F_2 = 104 \times 10^{-6}$ starting near $R \simeq 600$. Figure 5 shows that initially a_1^* deviates slightly from its non-interaction values. Hence, Eq. (71) is expected to be initially a good approximation to a_2^*. Figure 6 shows that the numerical values obtained from Eq. (71) are in good agreement with those obtained by numerically integrating Eqs. (60) and (61) when a_2^* is less than 0.29. Thus, in this case, the effect of the interaction on the second-harmonic wave is to increase its initial amplitude and to produce a term that grows at a rate that is twice the growth rate of the fundamental wave. Due to the fact that both waves are initially unstable, the interaction is more effective in this case than in the preceding two cases.

B. Three-Wave Interactions

In their experimental studies, Kachanov, et al[9], Miksad[8], Norman[10], and Sato[7] introduced two separate waves of different frequencies into the flow that was being studied. They observed the growth of a wave whose frequency is equal to the difference frequency. Kachanov et al used the frequency pairs $F_1 = 88 \times 10^{-6}$ and $F_2 = 104 \times 10^{-6}$ and $F_1 = 88 \times 10^{-6}$ and $F_2 = 120 \times 10^{-6}$ to analyze the growth of the associated difference-harmonic waves at $F_3 = 16 \times 10^{-6}$ (i.e. $F_3 = F_2 - F_1$) and $F_3 = 32 \times 10^{-6}$ in a boundary-layer flow over a flat plate. Using the same frequency pairs, we determined the amplitudes of the fundamental waves (a_1^* and a_2^*) and the difference-harmonic wave a_3^* by numerically solving Eqs. (57) - (59). Figures 7 and 8 show the large increases in the amplitudes of the above difference-harmonic waves due to the nonlinear interaction of the waves with the frequencies F_1 and F_2. The amplitude of the difference-harmonic wave at $R \cong 715$ is increased 120-200 times its non-interaction value.

However, the amplitudes of the fundamental waves change very little from their non-interaction values as shown in Fig. 9 for $F_1 = 88 \times 10^{-6}$. Thus, a_1^* and a_2^* can be approximated by

$$a_1^* = a_{10}^* \exp(-k_{1i}x + i\tau_1), \quad a_2^* = a_{20}^* \exp(-k_{2i}x + i\tau_2), \tag{74}$$

where a_{10}^* and a_{20}^* are the initial amplitudes and τ_1 and τ_2 are the initial phases of the fundamental waves. Substituting Eq. (74) into Eq. (59) and neglecting the nonparallel effects, we obtain

$$\frac{da_3^*}{dx} + k_{3i}a_3^* = \frac{h_{312}}{f_3} a_{10}^* a_{20}^* \exp[-(k_{1i} + k_{2i} + i\varepsilon\sigma_2)x + i(\tau_1 + \tau_2)] \tag{75}$$

The solution of Eq. (75) that satisfies the initial condition $a_3^* = a_3^*(0)$ at $x = 0$ can be expressed as

$$a_3^* = [a_3^*(0) + \frac{h_{312}}{f_3(k_{1i} + k_{2i} + i\varepsilon\sigma_2)} a_{10}^* a_{20}^* \exp(i\tau_1 + i\tau_2)] \times \exp(-k_{3i}x) - \frac{h_{312}}{f_3(k_{1i} + k_{2i} + i\varepsilon\sigma_2)} a_{10}^* a_{20}^* \exp[-(k_{1i} + k_{2i} + i\varepsilon\sigma_2)x + i(\tau_1 + \tau_2)]. \tag{76}$$

Equation (76) represents an approximation to a_3^* as long as the amplitudes of the fundamental waves do not deviate from their non-interaction values.

Figures 7 and 8 show that Eq. (76) is initially in good agreement with those obtained by numerically integrating Eqs. (57) - (59).

According to Eq. (76), the difference-harmonic wave grows at a rate that is the sum of the growth rates of the fundamental waves. Since the fundamental waves are unstable at R = 430, where the calculations are started, the difference-harmonic wave amplifies considerably, inspite of the fact that it is stable in the absence of the interaction. These results are in qualitative agreement with the experimental observations of Refs. 7 - 10.

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APPENDIX A

$$d_{1_j} = \left(\frac{\partial A_j}{\partial x_1} \zeta_{1_j} + A_j \frac{\partial \zeta_{1_j}}{\partial x_1} \right), \tag{A1}$$

$$d_{2_j} = d_{2_{j0}} + d_{2_{j1}}, \tag{A2}$$

$$d_{2_{j0}} = -(U_0\zeta_{1_j} + \zeta_{3_j} - \frac{2i}{R} k_j\zeta_{1_j} - V_0 D\zeta_{1_j}) \frac{\partial A_j}{\partial x_1} + [- U_0 \frac{\partial \zeta_{1_j}}{\partial x_1}$$

$$- \frac{\partial \zeta_{3_j}}{\partial x_1} + \frac{2i}{R} (k_j \frac{\partial \zeta_{1_j}}{\partial x_1} + \frac{dk_j}{dx_1}\zeta_{1_j}) - \frac{\partial U_0}{\partial x_1} \zeta_{1_j}]A_j, \tag{A3}$$

$$d_{211} = [i(k_2 - \overline{k}_3)\zeta_{12}\overline{\zeta}_{13} + \zeta_{22}D\overline{\zeta}_{13} + D\zeta_{12}\zeta_{23}]A_2\overline{A}_3 \times$$

$$\exp[-i\phi - \int(k_{2_i} + k_{3_i} - k_{1_i})dx], \tag{A4}$$

$$d_{221} = [i(k_1 + k_3)\zeta_{11}\zeta_{13} + \zeta_{21}D\zeta_{13} + D\zeta_{11}\zeta_{23}]A_1A_3 \times$$

$$\exp[i\phi - \int(k_{1_i} + k_{3_i} - k_{2_i})dx], \tag{A5}$$

$$d_{231} = (-i\overline{k}_1\overline{\zeta}_{11}\zeta_{12} + ik_2\zeta_{12}\overline{\zeta}_{11} + \zeta_{22}D\overline{\zeta}_{11} + D\zeta_{12}\overline{\zeta}_{21})A_2\overline{A}_1 \times$$

$$\exp[-i\phi - \int(k_{1_i} + k_{2_i} - k_{3_i})dx], \tag{A6}$$

$$d_{3_j} = d_{3_{j0}} + d_{3_{j1}}, \tag{A7}$$

$$d_{3_{j0}} = - (U_0\zeta_{2_j} - \frac{2i}{R} k_j\zeta_{2_j}) \frac{\partial A_j}{\partial x_1} - [U_0 \frac{\partial \zeta_{2_j}}{\partial x_1} + V_0 D\overline{\zeta}_{2_J} + \zeta_{2_j}DV_0$$

$$- \frac{2i}{R} (\frac{dk_j}{dx_1} \zeta_{2_j} + k_j \frac{\partial \zeta_{2_j}}{\partial x_1})]A_j, \tag{A8}$$

$$d_{311} = (-i\overline{k}_3\zeta_{12}\overline{\zeta}_{23} + ik_2\zeta_{22}\overline{\zeta}_{13} + \zeta_{22}D\overline{\zeta}_{23} + D\zeta_{22}\overline{\zeta}_{23})A_2\overline{A}_3 \times$$

$$\exp[-i\phi - \int(k_{2_i} + k_{3_i} - k_{1_i})dx], \tag{A9}$$

$$d_{321} = (ik_3\zeta_{11}\zeta_{23} + ik_1\zeta_{21}\zeta_{13} + \zeta_{21}D\zeta_{23} + D\zeta_{21}\zeta_{23})A_1A_3 \times$$

$$\exp[i\phi - \int(k_{1_i} + k_{3_i} - k_{2_i})dx], \tag{A10}$$

$$d_{331} = (-i\bar{k}_1\bar{\zeta}_{21} + ik_2\zeta_{22}\bar{\zeta}_{11} + \zeta_{22}D\bar{\zeta}_{21} + D\zeta_{22}\bar{\zeta}_{21})A_2\bar{A}_1 \times$$

$$\exp[-i\phi - \int(k_{1_i} + k_{2_i} - k_{3_i})dx]. \qquad \text{(A11)}$$

APPENDIX B

$$f_j = \int_0^\infty \Big\{ - \zeta_{1j}\zeta^*_{1j} - [(U_0 - \frac{2ik_j}{R})\zeta_{1j} + \zeta_{3j}]\zeta^*_{2j} - (U_0 - \frac{2ik_j}{R}) \times$$

$$\zeta_{2j}\zeta^*_{3j}\Big\} dy \qquad (j = 1, 2, 3), \tag{B1}$$

$$h_{jj} = \int_0^\infty \Big\{ \frac{\partial \zeta_{1j}}{\partial x_1} \zeta^*_{1j} + [U_0 \frac{\partial \zeta_{1j}}{\partial x_1} + \frac{\partial \zeta_{3j}}{\partial x_1} + V_0 D\zeta_{1j} - \frac{2i}{R}(k_j \frac{\partial \zeta_{1j}}{\partial x_1}$$

$$+ \frac{dk_j}{dx_1}\zeta_{1j}) + \frac{\partial U_0}{\partial x_1}\zeta_{1j}]\zeta^*_{2j} + [(U_0 - \frac{2ik_j}{R})\frac{\partial \zeta^*_{2j}}{\partial x_1} + V_0 D\zeta_{2j}$$

$$+ \zeta_{2j} DV_0 - \frac{2i}{R}\frac{dk_j}{dx_1}\zeta_{2j}]\zeta^*_{3j}\Big\} dy \qquad (j = 1, 2, 3), \tag{B2}$$

$$h_{123} = \int_0^\infty \Big\{ [i(k_2 - \overline{k}_3)\zeta_{12}\overline{\zeta_{13}} + \zeta_{22}D\overline{\zeta}_{13} + D\zeta_{12}\overline{\zeta}_{23}]\zeta^*_{21}$$

$$+ (-i\overline{k}_3\zeta_{12}\overline{\zeta}_{23} + ik_2\zeta_{22}\overline{\zeta}_{13} + \zeta_{22}D\overline{\zeta}_{23} + D\zeta_{22}\overline{\zeta}_{23})\zeta^*_{31}\Big\} dy, \tag{B3}$$

$$h_{213} = \int_0^\infty \Big\{ [i(k_1 + k_3)\zeta_{11}\zeta_{13} + \zeta_{21}D\zeta_{13} + D\zeta_{11}\zeta_{23}]\zeta^*_{22}$$

$$+ (ik_3\zeta_{11}\zeta_{23} + ik_1\zeta_{21}\zeta_{13} + \zeta_{21}D\zeta_{23} + D\zeta_{21}\zeta_{23})\zeta^*_{32}\Big\} dy, \tag{B4}$$

$$h_{312} = \int_0^\infty \Big\{ [i(k_2 - \overline{k}_1)\zeta_{12}\overline{\zeta}_{11} + \zeta_{22}D\overline{\zeta}_{11} + D\zeta_{12}\overline{\zeta}_{21}]\zeta^*_{23}$$

$$+ (-i\overline{k}_1\overline{\zeta}_{21}\zeta_{12} + ik_2\zeta_{22}\overline{\zeta}_{11} + \zeta_{22}D\overline{\zeta}_{21} + D\zeta_{22}\overline{\zeta}_{21})\zeta^*_{33}\Big\} dy. \tag{B5}$$

APPENDIX C

$$h_{12} = \int_0^\infty \Big\{ [\zeta_{22} D\bar{\zeta}_{11} + \bar{\zeta}_{21} D\zeta_{12} + i(k_2 - \bar{k}_1)\zeta_{12}\bar{\zeta}_{11}]\zeta_{21}^* + [ik_2\zeta_{22}\bar{\zeta}_{11} - i\bar{k}_1\bar{\zeta}_{21}\zeta_{12} + \zeta_{22} D\bar{\zeta}_{21} + \bar{\zeta}_{21} D\zeta_{22}]\zeta_{31}^* \Big\}\, dy \qquad (C1)$$

$$h_{21} = \int_0^\infty \Big\{ [(ik_1\zeta_{11}^2 + \zeta_{21} D\zeta_{11})\zeta_{22}^* + (ik_1\zeta_{11}\zeta_{21} + \zeta_{21} D\zeta_{21})\zeta_{32}^*] \Big\}\, dy \qquad (C2)$$

APPENDIX D

$$g_{11} = 0\ ,\ g_{12} = 1,\quad g_{13} = 0,\quad g_{14} = 0 \tag{D1}$$

$$g_{21} = i(U_0 k - \omega)R + k^2,\quad g_{22} = 0,\quad g_{23} = R\,\frac{dU_0}{y},\ g_{24} = ikR \tag{D2}$$

$$g_{31} = -\ ik,\quad g_{32} = g_{33} = g_{34} = 0 \tag{D3}$$

$$g_{41} = 0,\quad g_{42} = -ik/R,\quad g_{43} = -[i(U_0 k - \omega) + k^2/R],\ g_{44} = 0 \tag{D4}$$

Q is a 2 x 4 matrix consisting of the last two rows of the matrix B^{-1}. The matrix B has the elements:

$$b_{11} = b_{12} = b_{13} = b_{14} = 1 \tag{D5}$$

$$b_{21} = -k,\quad b_{22} = \tilde{k},\quad b_{23} = k,\quad b_{24} = -\ \tilde{k} \tag{D6}$$

$$b_{31} = i,\quad b_{32} = ik/\tilde{k},\quad b_{33} = -i,\quad b_{34} = -ik/\tilde{k} \tag{D7}$$

$$b_{41} = (\omega/k - 1),\quad b_{42} = 0,\ b_{43} = (\omega/k - 1),\ b_{44} = 0 \tag{D8}$$

where

$$\tilde{k} = [k^2 + i(k - \omega)R]^{1/2} \tag{D9}$$

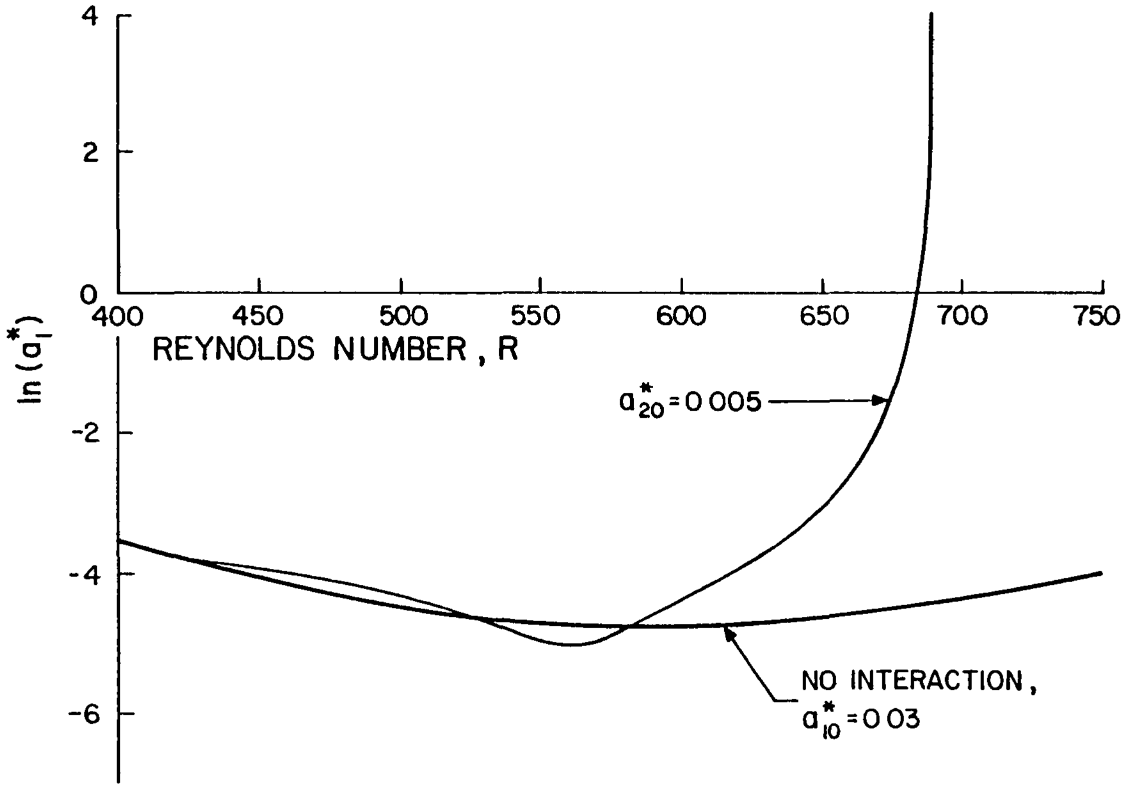

Figure 1. Amplitude of a wave at $F_1 = 52 \times 10^{-6}$ involved in a subharmonic resonance with a fundamental wave at $F_2 = 104 \times 10^{-6}$.

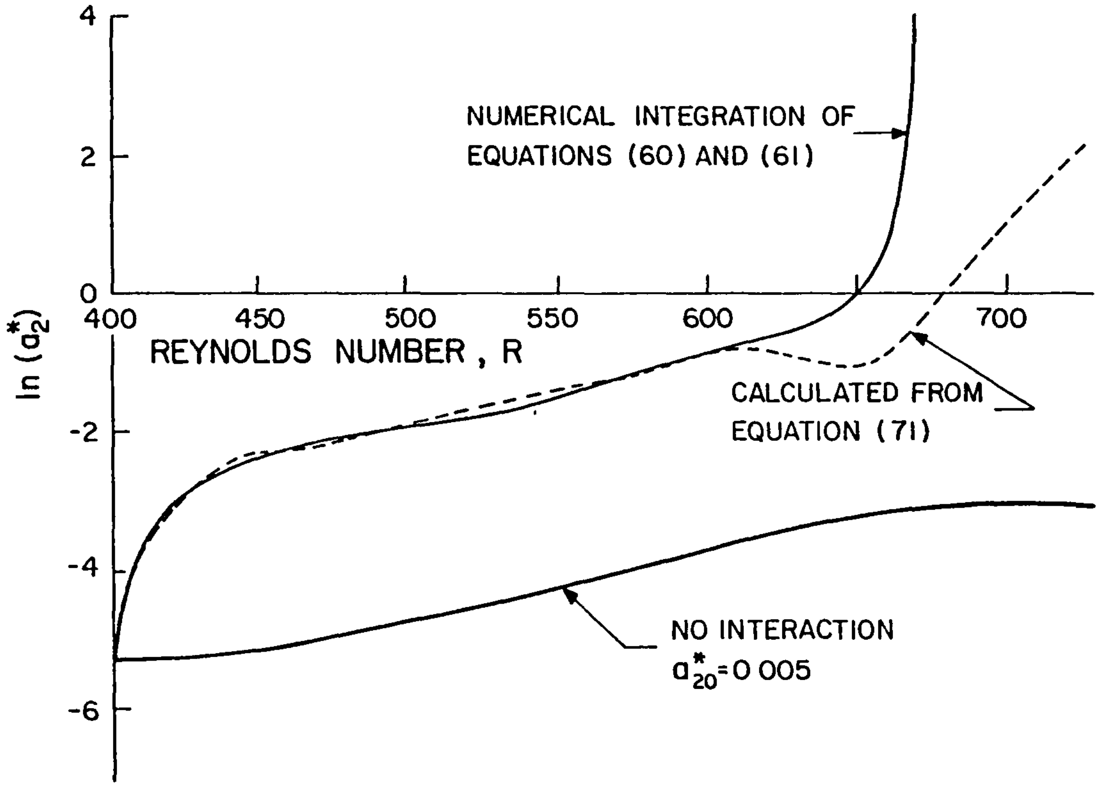

Figure 2. Amplitude of the fundamental wave at $F_2 = 104 \times 10^{-6}$ involved in resonance with its subharmonic at $F_1 = 52 \times 10^{-6}$.

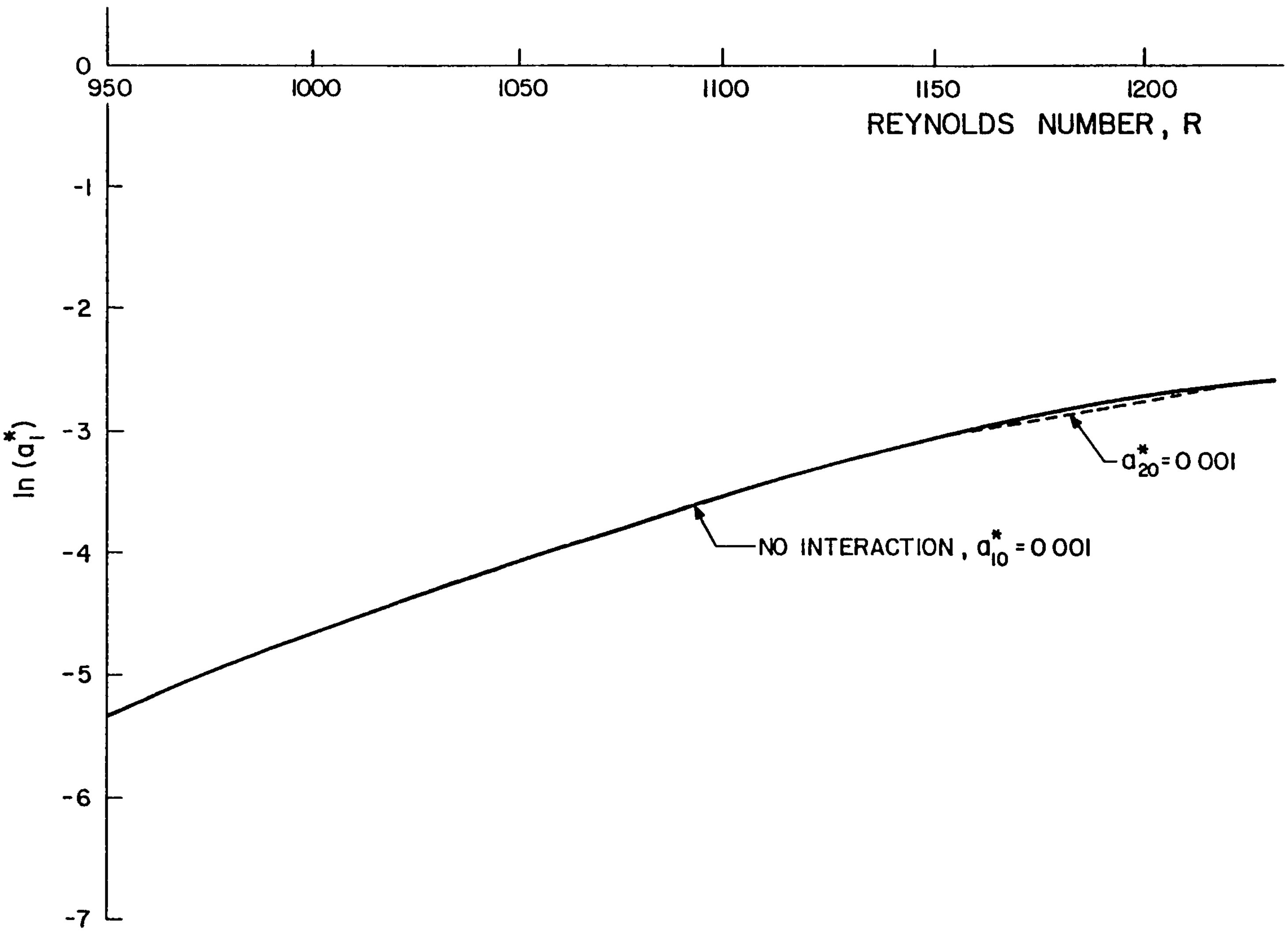

Figure 3. Amplitude of a fundamental wave at $F_1 = 46.5 \times 10^{-6}$ involved in resonance with its harmonic at $F_2 = 93 \times 10^{-6}$.

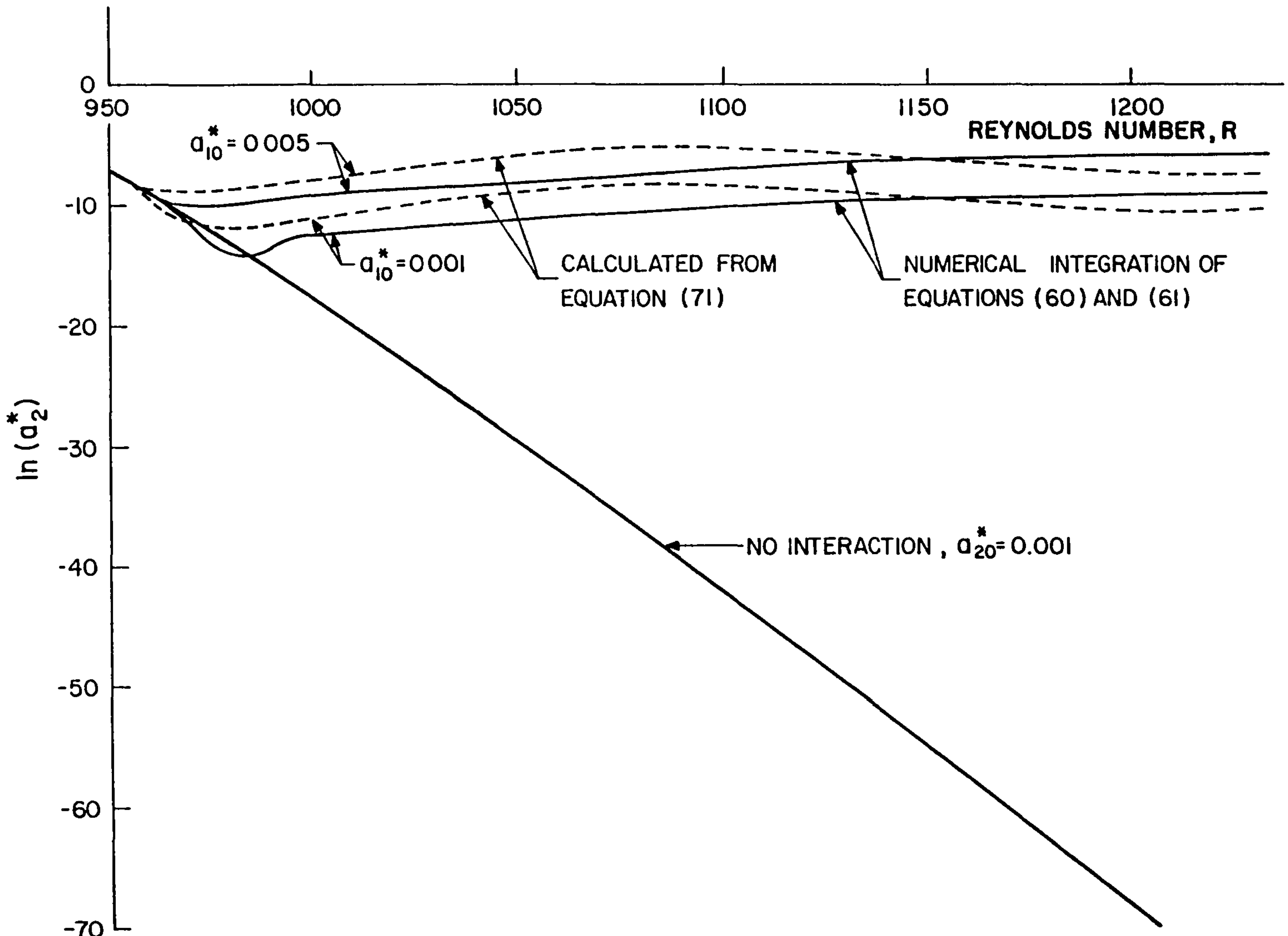

Figure 4. Amplitude of the second-harmonic wave at $F_2 = 93 \times 10^{-6}$ when involved in resonance with a wave at $F_1 = 46.5 \times 10^{-6}$.

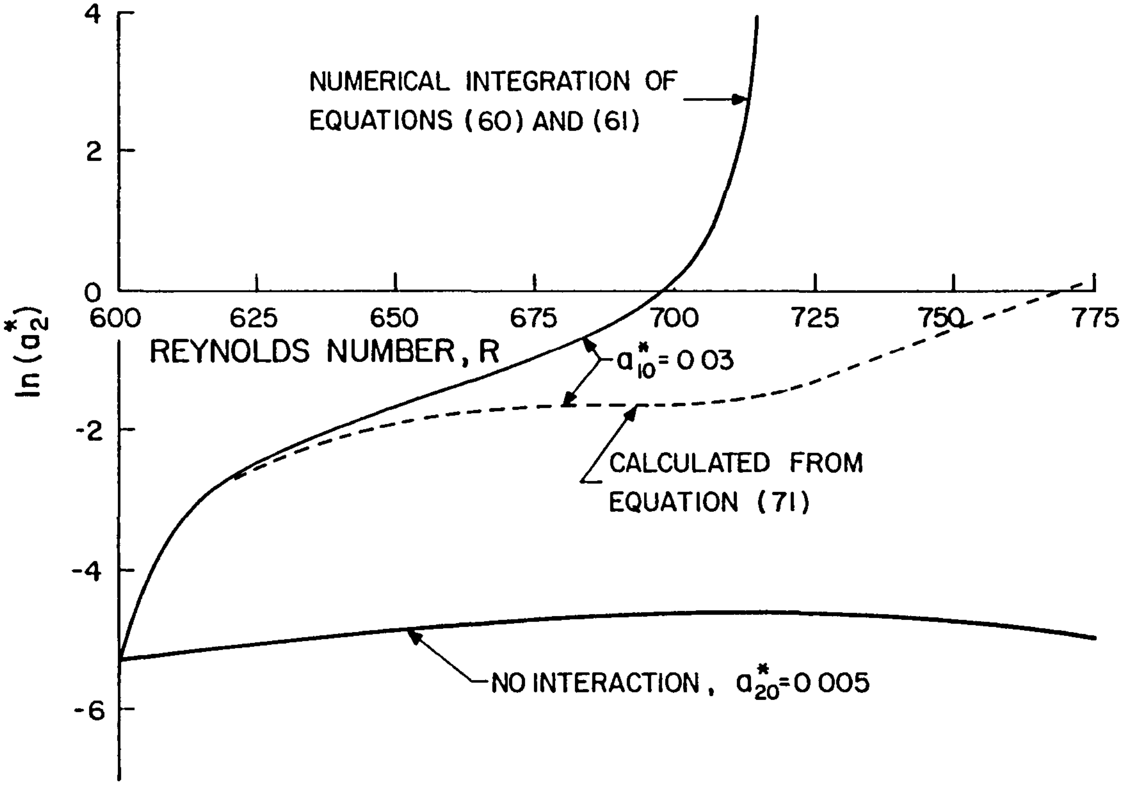

Figure 5. Amplitude of fundamental wave at $F_1 = 52 \times 10^{-6}$ involved in resonance with its harmonic at $F_2 = 104 \times 10^{-6}$.

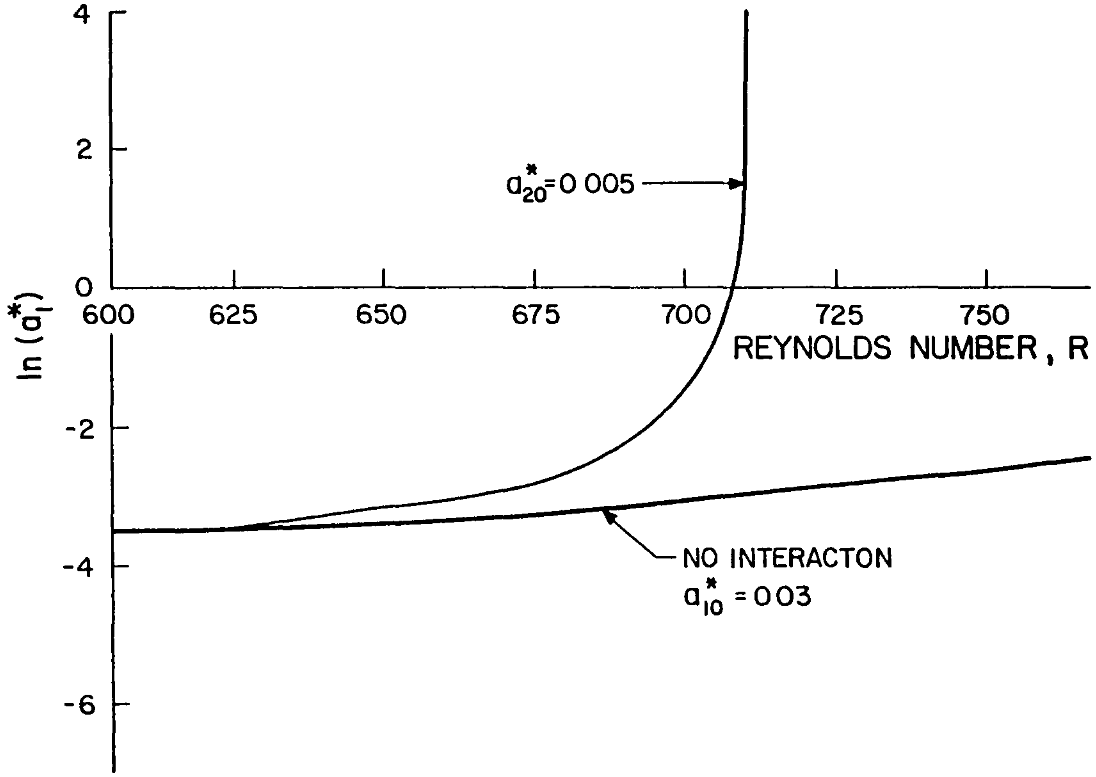

Figure 6. Amplitude of the second-harmonic wave at $F_2 = 104 \times 10^{-6}$ when involved in resonance with a wave at $F_1 = 52 \times 10^{-6}$.

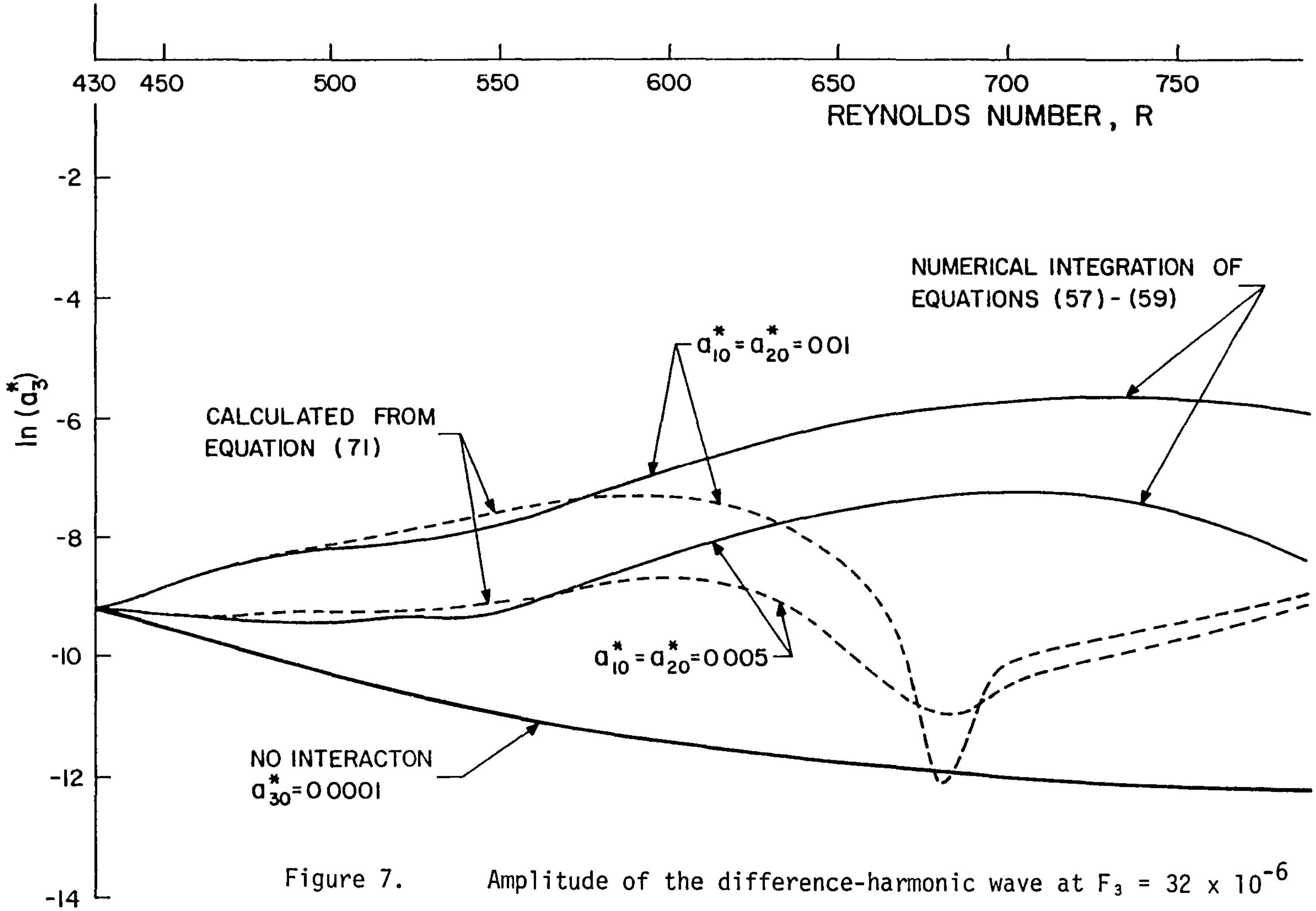

Figure 7. Amplitude of the difference-harmonic wave at $F_3 = 32 \times 10^{-6}$ generated from a combination resonance of $F_1 = 88 \times 10^{-6}$ and 120×10^{-6}.

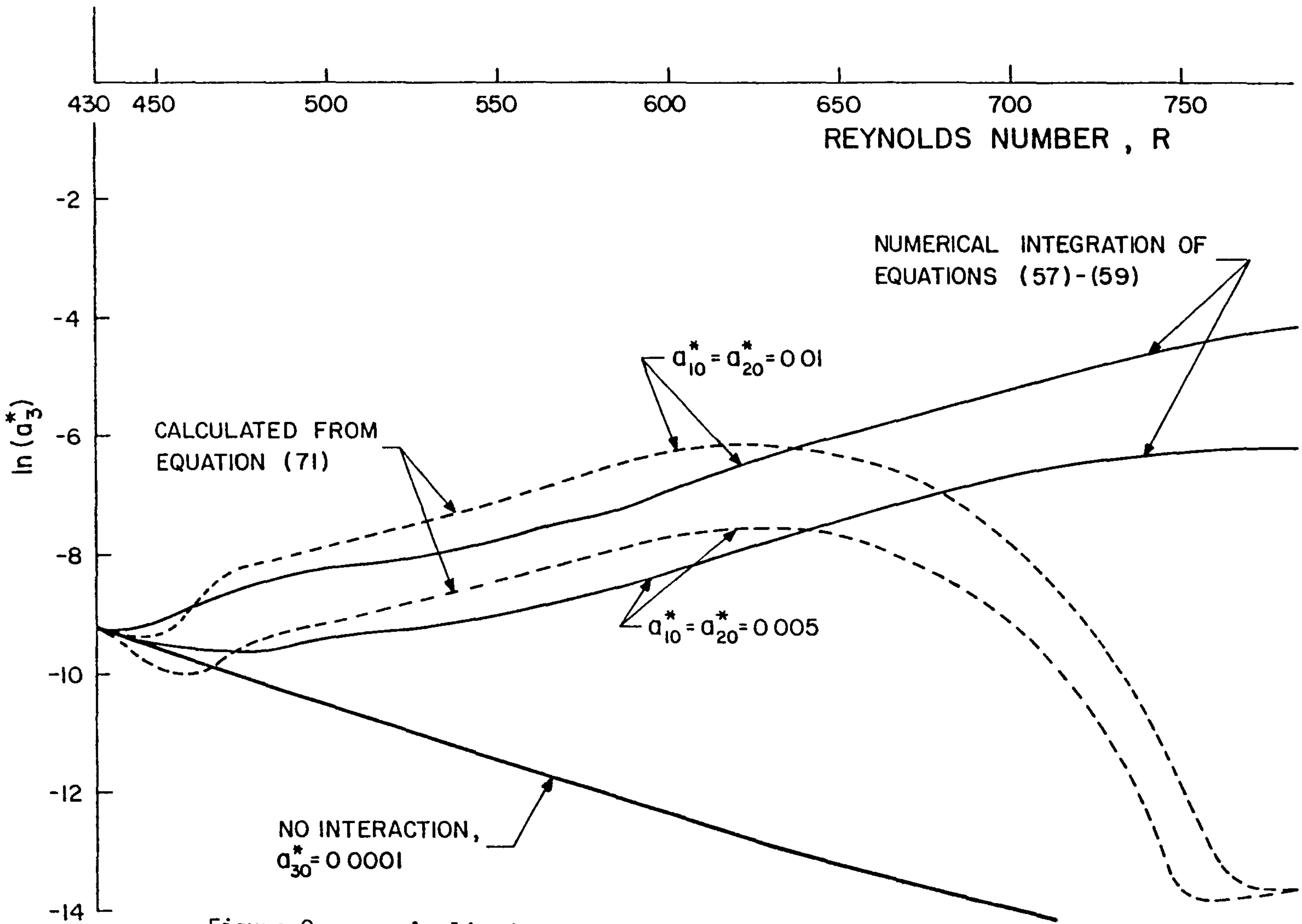

Figure 8. Amplitude of the difference-harmonic wave at $F_3 = 16 \times 10^{-6}$ generated from a combination resonance of $F_1 = 88 \times 10^{-6}$ and $F_2 = 104 \times 10^{-6}$.

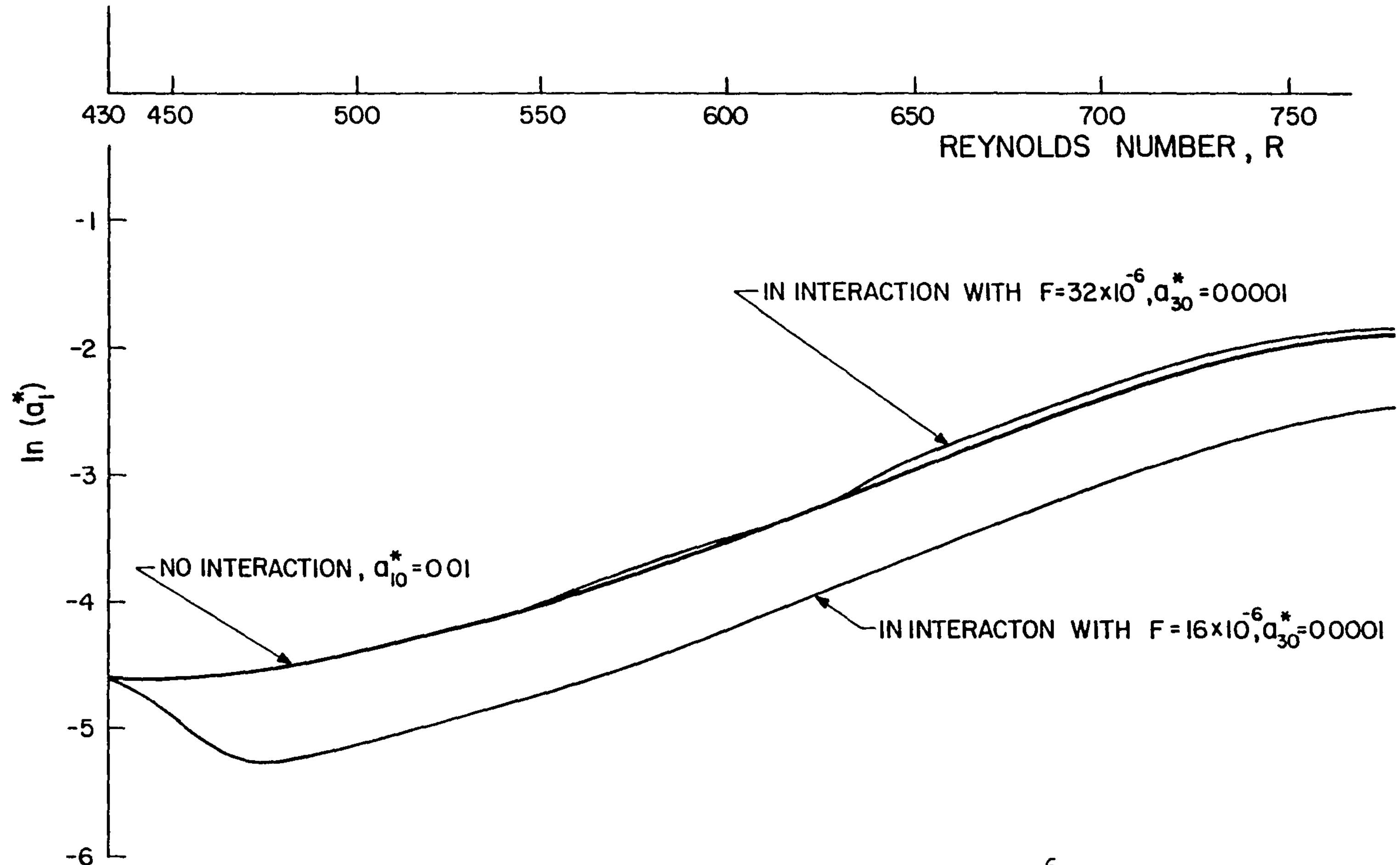

Figure 9. Amplitude of the fundamental wave at $F_1 = 88 \times 10^{-6}$ involved in a combination resonance with two waves at $F_2 = 104 \times 10^{-6}$ and $F_3 = 16 \times 10^{-6}$.

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