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SEMIDIRECT COMPUTATION OF THREE-DIMENSIONAL VISCOUS FLOWS OVER SUCTION HOLES IN LAMINAR FLOW CONTROL SURFACES

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FINAL REPORT

SEMIDIRECT COMPUTATION OF THREE-DIMENSIONAL VISCOUS FLOWS OVER SUCTION HOLES IN LAMINAR FLOW CONTROL SURFACES

by

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Abstract

This report summarizes the attempts made to apply semidirect methods to the calculation of three-dimensional viscous flows over suction holes in laminar flow control surfaces. The attempts were all unsuccessful, due to either (1) lack of resolution capability, (2) lack of computer efficiency, or (3) instability.

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INTRODUCTION

Laminar flow control technology has the potential for significantly increasing the range and reducing the fuel consumption of commercial aircraft. With laminar flow control, the viscous drag of large surfaces such as wings is reduced by stabilizing the boundary layer, preventing or greatly delaying the transition from laminar to turbulent flow.

On the conventional laminar flow airfoil, boundary layer stability is enhanced by overall design of the surface so as to avoid pressure peaks and their inevitable concomitant, adverse pressure gradients. In the unconventional LFC technology, further stabilization is achieved by using a suction surface on the wing or fuselage surface to withdraw part of the boundary layer. The suction may be accomplished through long spanwise slots, a porous surface, or discrete cylindrical holes.

The first-order prediction of the stabilization resulting from the surface suction is well understood by the classical boundary-layer stability theory based on the parallel-flow Orr-Sommerfeld equation, associated with C. C. Lin, Tollmien, Schlichting, and others (e.g., Refs. 1, 2). Second-order effects, which could be significant for slots, may require the inclusion of x-dependency, as shown by Saric and Nayfeh (Ref. 3). The discrete hole surface can be manufactured by electron-beam technology and has structural advantages over slots and economic advantages over porous surfaces manufactured by sintering techniques.

However, the complicated three-dimensional flow over the cylindrical suction hole could possibly develop a secondary instability which would not be revealed by the boundary layer stability analysis of the unperturbed boundary layer flow.

In order to study this secondary instability theoretically, an accurate representation of the three-dimensional flow in the neighborhood of the suction hole is required. Particularly needed are accurate values of first and second partial derivatives in the y-direction normal to the surface and in the spanwise z-direction. Parametric studies require variation of the inflow boundary-layer thickness and profile, pressure gradients, hole diameter and spanwise spacing, skin thickness, and suction rates, as well as purely numerical parameters such as mesh-size.

The objective of the present work was to develop a computer code to rapidly and accurately calculate the three-dimensional viscous flow over the suction holes, using the complete compressible Navier-Stokes equations. The intended technique was to be an application of the semidirect methods previously developed by the Principal Investigator.

PREVIOUSLY USED METHODS

The classical method of analysis for viscous flows involves the calculation of an inviscid flow followed by a boundary layer calculation. This technique has been developed to a high level of sophistication in two dimensions and in three dimensions (e.g., Refs. 5, 6). With the inclusion of viscous iteration, it can even be used to calculate mildly separating flows in two dimensions (Ref. 7). The boundary-layer method appears to be the method of choice for the suction slot problem. However, the discrete suction hole introduces added ellipticity in the spanwise direction, and the boundary layer equations become less dependable and more difficult to calculate.

The next alternative is to use the parabolic marching equations. These equations are intermediate between the classical boundary layer equations and the full Navier-Stokes equations. This approach has been successfully applied to simple flows in constant-area ducts (Refs. 8, 9) and is quite economical compared to time-dependent Navier-Stokes solutions. However, there remains some questions as to the validity of the procedure in regard to the treatment of longitudinal pressure gradient terms. The practical accuracy is suspect in the present case wherein those terms may vary significantly in the immediate neighborhood of the hole, which region is the most likely candidate for secondary instability (Ref. 4).

A third alternative is to use the full Navier-Stokes equations with no approximations, solved with a conventional time-dependent or time-like iteration scheme (e.g., Ref. 10), obtaining the desired steady-state solution as the asymptotic limit of the time-dependent problem. However, three-dimensional time-dependent calculations of viscous flows are notoriously expensive, especially for the high resolution in y required for the present problem.

SEMIDIRECT METHODS

The present work involved "semidirect methods" for solving the full Navier-Stokes equations for the steady-state solution, using iterative procedures which are not time-dependent or even time-like. The principal investigator had previously developed several such methods with considerable success. (See Refs. 11-15. See also Ref. 16, and similar methods for the transonic inviscid equations in Refs. 17 and 18.)

The basic concept in semidirect methods is to use the recently developed fast (direct, or non-iterative) linear equation solvers to solve linearized steady-state equations, which are then iterated to remove the non-linearities. This iteration is not time-like, as shown in Refs. 11-13. Convergence can be obtained in as few as 6 to 8 iterations for some problems, and more typically in 10 to 15 iterations for problems with flow-through computational boundaries like the ones of interest here. Each of these iterations requires computer time comparable to that for a single time-step in an efficient time-dependent method, i.e. one which uses a direct elliptic solver for the Poisson equation. If the more common iterative methods (ADI, SOR, etc.) are used for the Poisson equation in a time-dependent method, then the semidirect methods can sometimes attain a complete steady-state solution with computer time comparable to that of a single time step.

The basic concept of the semidirect iteration scheme can be described by reference to a two-dimensional laminar flow problem using the vorticity transport equation,

$$\zeta_t = -\mathrm{Re}\ \nabla\cdot\vec{V}\zeta + \nabla^2\zeta$$

where ζ is vorticity, Re is the Reynolds number, and $\vec{V}$ is the vector velocity. A steady state is assumed, so the time derivative ζ_t is set = 0. In the Split NOS method proposed here, the velocity $\vec{V}^k$ at the k-th iteration is split into an initial guess $\vec{V}^o$ and a perturbation $\vec{V}'$, not necessarily small. That is,

$$\vec{V}^k = \vec{V}^o + \vec{V}'$$

Then the steady-state form of the vorticity transport equation is solved for the k-th iteration using $O(\Delta x^2)$ centered differences, as

$$L(\zeta^k) = \mathrm{Re}\ \nabla\cdot(V'\zeta)^{k-1}$$

where the linear operator L is defined by

$$L(\zeta) = \nabla^2\zeta - \mathrm{Re}\ \nabla\cdot(V^o\zeta)$$

For the present 3D problem, the treatment of the spanwise convection terms prevented a successful calculation, as will be described later.

PARTICULARS OF THE LFC PROBLEM FORMULATION

For the presently considered LFC problems, the free-stream pressure is prescribed, so that the primitive (velocity-pressure) system of variables is recommended. The principal investigator had also applied the Split NOS method to primitive variables in incompressible flow with success in 2D. The additional equation for energy required for the compressible flow solution does not present any special difficulties, since the boundary conditions (on heat flux or temperature) are well specified. (Free-convection problems have recently been calculated by the P.I. using semidirect methods.) The compressibility correction terms were to be lagged in the iterations, as are the velocity perturbation terms above. The work of Ref. 17 on inviscid flow indicates that compressibility effects in subsonic and slightly supersonic flows are compatible with a semidirect iteration scheme.

Because the subject problem involved flow-through computational boundaries, rapid convergence requires (Refs. 11-15) that the linear solver used can accomodate first-derivative (convection) terms at least in the primary flow direction. Also, accurate resolution of derivatives in y requires a fine mesh and possibly coordinate stretching in y. These requirements dictated that the commonly available fast Poisson solvers are not applicable to the proposed solution method.

The EVP method previously developed by the Principal Investigator (Refs. 19, 20) and recently extended to three dimensions

(Ref. 20-23) is applicable to this problem. Three different 3D linear solvers were utilized in the present investigation; simple 3D marching, 3D marching with a banding approximation, and the 3D EVP/FFT method. It was intended that the solution to the title problem would be obtained with the full Navier-Stokes equations, rather than with some parabolized approximation; this will likely be significant near the suction hole. The constant-property approximation was used. This assumption is not necessary for the semidirect method, but the simplification helped the code debugging process.

The equations used were the three momenta equations and the total energy equation, all in conservation form, and a modified Poisson equation for pressure. It is believed that the use of the pressure equation, rather than a steady-state continuity equation for density, would improve the accuracy for the slightly compressible flows of interest, would be compatible with the specification of the pressure in the external inviscid flow, and would circumvent the problem associated with the semidirect iteration of the continuity equation as described in Ref. 12, page 194. The surface boundary conditions on temperature could be either specified temperature or heat transfer boundary conditions, including adiabatic walls. These and other boundary conditions are the same as would be used in a conventional time-dependent approach. In this regard, the conventional extrapolation conditions at outflow boundaries (e.g., Ref. 10) are the simplest to program and to adapt to a vectorized computer.

The most recent work on semidirect methods for viscous flows (Ref. 15) indicated no numerical stability limitation on the 2D iteration scheme up to Re = 10^7. The steady-state solutions obtained are the same as those obtained by the usual time-dependent methods using the same spatial differences, same variables, and same mesh.

Iterative convergence rates would be improved (even in 2D) if all the terms possible were retained in the linear operators of the Split NOS method. However, it was decided to lag in the iterations the terms in the momentum equations which are the departure from incompressibility and constant viscosity, and those in the energy (temperature, not total energy) equation which arise from dissipation effects. This would result in some decreased iterative performance at high M, but would save considerably in storage. If all the possible terms were included in the linear operator, 5 large matrices at each of the z-wave numbers would have to be stored and solved by LU decomposition, one for each elliptic equation for u, v, w, T and P. With the scheme described above, the same matrix would serve for u, v, w and T, with a separate matrix for P. Thus, only 2 matrices would serve for each of the five variables at each wave number. Also, the 3D EVP/FFT method itself has a savings of almost ½ because of the symmetry in the z-wave number (Ref. 23). Thus, the storage penalty for this method would only be about 20% of the basic storage required for u, v, w, T and P. This could be further reduced, by using Hockney's method for the pressure equation, to about 10%. For large problems to be run on the STAR 100, the savings in storage would translate into savings in time spent for page transfers, and would likely decrease actual computation time more than enough to compensate for the increased number of iterations.

The choice for the mesh for the pressure equation was made. After considerable investigation, it was decided that a staggered grid for the pressure is very desirable, even though a non-staggered grid (in which all variables are defined at the same locations) is typically used in compressible flow solutions. These conventional solutions invariably utilize an explicit or an inherent artificial damping for the purpose of shock-capturing and stabilization. This artificial damping also depresses the parasitic mode that arises from the pressure solution. In the present case, the solutions would not contain artificial damping. However, the MAC-type grid was also rejected because of its complexity in 3D, which is especially difficult for the interpolative coupling between the flow above the plate and the flow in the suction hole. Also, the MAC grid would require different operators for u, v, w and T, with the resulting storage and time penalties mentioned above. Therefore, it was decided to use a grid in which only pressure is staggered, while u, v, w and T are all defined on the same grid. There is some penalty in operation count, but the interpolative coupling problem is tractable and the parasitic solution mode does not exist.

It was intended to perform the calculations in several grids, which would be patched together either iteratively or directly.

A cylindrical coordinate grid was to be used inside the suction hole. Above the wing surface, a fine-grid and a coarse-grid solution would be obtained in rectangular coordinates.

The use of the rectangular coordinate system above the wing surface would allow the streamlines to roughly follow coordinates, resulting in better accuracy and aiding (it was thought) convergence of the LAD-like iteration in z, as previously described.

The several methods of solution attempted in this work will be described in detail in the following sections. The motivations and decisions about these methods are somewhat esoteric because of their intimate connection with the direct methods used for the linearized equations. The characteristics of these linear solvers are described in detail in Refs. 21-23. It should be realized that those methods are not easy to program in 3D, and that most of the P.I.'s time was spent in coding and debugging.

3D MARCHING: FULL MATRIX AND BANDED APPROXIMATION

The "simple" 3D marching method (Refs. 20, 23) was coded and tested. Previously, the 3D EVP method had been worked out theoretically but had not been validated in an actual computation.

The operation count for the 3D EVP method in its simplest form is very bad with initialization for an MxMxM problem requiring $O(M^6)$ operations. It would be necessary to use a banding approximation to the full influence coefficient matrix in 3D to make the method economical. This banding approximation had been proven in 2D and worked out theoretically in 3D; it was now proven in 3D. The 2D experience carried over with no surprises to 3D, with only the usual programming difficulties. The result is the only 3D elliptic solver with optimal operation count, $0(M^3)$, for repeat solutions. Initialization requires $0(M^4)$ operations.

The validating experiments with various bandwidths were first performed by simply zeroing the terms in the full influence coefficient matrix which were outside the band; i.e., the calculations were performed with a full matrix solution routine with null calculations. Later the code was actually converted to a banded matrix solver to realize the advantages in operation count and storage. The relative advantages of using a simple banded matrix solver vs. a block-banded solver were studied. The former appeared to be preferable. The available banded matrix solvers were studied and the IMSL programs were rejected in favor of the LINPACK programs written by Prof. Cleve Moler of the University of New Mexico Math Department. His programs give an estimate of the condition number of the matrix problem, and are well-designed to avoid unnecessary accumulation of round-off error, which is critical in the large 3D problems.

After considerable difficulty with operating systems, the LINPACK system was set up on both the NOS system and the batch CDC 6600 system. The latter proved to be necessary because of the stringent storage limitation on the NOS system.

A large parametric study was then performed of the accuracy of the solutions. Both the condition number of the matrices and the maximum error in the field were investigated as functions of cell aspect ratio, independent mesh dimentions I, J and K, band width, number of corrective iterations, and dimensionality. Also studied were the solution time for initialization, solution time for repeats, and storage.

The adequacy of the banding approximation is much more sensitive to J, the problem size in the marching direction, then had been previously realized. Also, with KBAND = 1 (a minimal approximation which includes only the nearest neighbors in the banding approximation) a 31x31x31 problem does not fit into core in a CDC 6600, since the storage penalty is about $3 \cdot KBAND \cdot M^3$. A 26x26x26 problem fits, but the condition of the matrix is so poor, even at cell aspect ratios = 10, that accurate solutions are not possible. The best problem that fits is 21x21x21 with KBAND = 3, which gives excellent accuracy of the finite difference solution. However, it does not seem that this resolution would be adequate for the LFC problem.

A scheme was then analyzed for effectively doubling the mesh size by solving on a submesh and using iterative corrections. The method appears to be useable, but doubles the mesh resolution only in one direction. It would also be very difficult to code. Further, a discussion on 2/16/78 with D. Bushnell indicated that an $O(\Delta^4)$ solution even in a 31x31x41 mesh, while adequate for initial studies, would not be adequate for final calculations. The methods considered are mesh-size limited, i.e., the submesh scheme cannot be applied recursively in 3D. It thus became evident that high mesh resolution with the semidirect methods in 3D was going to be very difficult. High resolution is not a difficulty in 2D, and the codes in 2D can be arranged to systematically refine the mesh, but the techniques are not simply extendable to 3D. Thus, the 3D solutions require different nonlinear iteration methods dictated by the linear solvers.

3D EVP/FFT LINEAR SOLVER

The remaining nonlinear iterative methods which were attempted used the 3D EVP/FFT linear solver. The peculiar properties of this solver limit the success of the nonlinear iterations, so it is necessary to describe its characteristics here.

The 3D EVP/FFT method is a combination of the original two-dimensional EVP method (Ref. 19) and Hockney's method (Ref. 24). The dependent variable is first Fourier transformed in the z direction using the Fast Fourier Transform or FFT. Then the EVP method is used to solve the two-dimensional problem in x and y for each Fourier component, followed by a reverse transformation.

The operation count for this method for an IxJxK problem is $0(I \cdot J \cdot K \cdot \ln K)$ for repeats. Although the Fourier-transformed variable is complex, the required influence coefficient matrix CI

(Ref. 23) is real. Further, it is symmetric in the wave number k_z, thus reducing the required storage and operation count by almost a factor of ½. The three-dimensional equations require storage and initialization of only two CI's, one for the three velocity components and one for the pressure. The error propogation characteristics of this three dimensional problem were worked out in detail (Ref. 23) and found to be applicable to the coordinate and mesh systems of the present problem. This EVP/FFT direct linear solver retains the ability of the EVP method to handle variable-coefficient first- and cross-derivative terms in the x-y plane, although it is restricted to constant-coefficient second-derivatives in z, as are the more common methods.

TIME-LIKE AND LOCALLY-ONE-DIMENSIONAL METHODS

In the interest of solving the LFC problem as efficiently as possible, several schemes were considered which fell somewhere between the semidirect methods and the more conventional time-like methods.

An obvious approach is to use 2D methods in x-y, iteratively coupled to the terms in z, by way of a tridiagonal or hopscotch solution in z. However, there is no reason to expect such a planar ADI or planar hopscotch method to converge any faster than a 2D ADI solution, so this approach was discarded, even though the ease of coding was attractive. Another method which would be easy to code is a BIR (block-implicit relaxation) method extended to 3D. Here, the requirement for high resolution would again slow convergence, so this approach was discarded. Several variants of a locally-one-dimensional correction scheme were devised and tested. The first scheme devised was an iteration scheme which can be derived as a LOD method with an infinite time step. The first ½-step uses the 3D EVP/FFT linear solution. Then, instead of merely lagging all the cross flow advection terms, these are used in the next ½-step in a 1D correction which includes advection. Part of the z-diffusion term is needed in the second ½-step in order for that term to be elliptic. The method was programmed and tested with a general weighting factor W in the usual manner, with $0 \leq W \leq 1$, to split the z-diffusion terms between the first and second half-steps. The code also allowed for finite Δt, for different Δt's in the first and second ½-steps, and for different numbers of cycles in the first and second ½-steps. For example, the code accepts a single cycle at Δt_1 for the first ½-step, and n cycles at $\Delta t_2 = \Delta t_1/2$ for the second ½-step. The experimentation was performed on a linear constant coefficient 3D advection-diffusion equation, with input parameters of directional Reynolds numbers, Rx, Ry, and Rz.

The results were uniformly negative. Extensive parametric tests were run without discovering a stable combination, even for all Reynolds numbers = 0. This was surprising, and there is always the possibility of coding errors, but the store of debugging techniques and much of the computing budget was exhausted, and the work proceeded on the premise that these LOD methods are in fact unstable.

SEMIDIRECT ITERATION WITH LAGGING SPANWISE - ADVECTION TERMS

The final method which was coded and tested was a combination of the Split NOS and LAD methods. The linear solver used was the 3D EVP/FFT method, which allows only $\partial^2/\partial z$ terms in z, and limits the advection coefficients in the linear operator to u(x,y) and v(x,y). In earlier published results on the LAD method in 2D (which contains no advection terms in the linear operator) convergence was limited to low Re. However, this method was analyzed (under separate funding) and modified to achieve convergence at high Re, although at a loss in convergence speed.

The 3D EVP/FFT routine was coded into a CALLable subroutine, and used as the basis of a fully 3D fluid dynamics solution procedure using the velocity and pressure variables. Extensive experimentation was carried out with this code and with a related 3D code, developed under separate contract, using the vorticity-velocity variables. With both codes, tests were conducted on the simple problem of perturbed 3D Couette flow.

It became evident that the 3D EVP/FFT method would not be useable for 3D fluid dynamics solutions using semidirect methods.

Instability resulted at all non-zero Reynolds numbers in both codes, using a variety of boundary conditions. When the pressure solution was removed from the nonlinear iteration procedure, stability was slightly improved, but still not useable. With only the u-component of velocity active, stability was achieved only through Re = 0(10). For Re = 15, convergence was obtained but very slowly and erratically. There always remains the possibility that the failure was due to coding errors, but the test for only the u-component is fairly straightforward; i.e., there were no changes in boundary conditions (except when periodic conditions were used in the third direction) and no interactions with other velocity components or pressure. When the equation was linearized, correct solutions resulted. Also, in 2D, the method is convergent even for high Re, for the nonlinear equations. It therefore appears that the instability is inherent in the 3D solution using the 3D EVP/FFT direct solver. It is conjectured at this time that the instability is related to the sensitivity of the FFT to small disturbances.

OTHER LFC APPLICATIONS

The semidirect solution methods still have these possible applications to the LFC suction boundary-layer control problem. (1) 2D solutions can be obtained very rapidly, and with high resolution. (2) The linear 3D EVP/FFT method can be used for the direct solution of the pressure field for use with a time-dependent or time-like solution procedure in 3D.

PAPERS AND PRESENTATIONS PARTIALLY FUNDED UNDER THIS CONTRACT

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