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FINAL REPORT

NASA GRANT: NSG-1414, Suppl. 1

THE DYNAMICS AND CONTROL OF LARGE FLEXIBLE SPACE STRUCTURES-II

PART A: SHAPE AND ORIENTATION CONTROL USING POINT ACTUATORS

HOWARD UNIVERSITY
SCHOOL OF ENGINEERING
DEPARTMENT OF MECHANICAL ENGINEERING
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THE DYNAMICS AND CONTROL OF LARGE FLEXIBLE SPACE STRUCTURES - II

PART A: SHAPE AND ORIENTATION CONTROL USING POINT ACTUATORS

by

Peter M. Bainum
Professor of Aerospace Engineering
Principal Investigator

and

A.S.S.R. Reddy
Graduate Research Assistant

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ABSTRACT

The equations of planar motion for a long, flexible free-free beam in orbit are developed and include the effects of gravity-gradient torques and control torques resulting from point actuators located along the beam. The actuators control both the orientation and the shape of the beam. Two classes of theorems are applied to the linearized form of these equations to establish necessary and sufficient conditions for controllability for preselected actuator configurations. It is seen that the number of actuators, if properly located, can be less than the number of modes in the model. After establishing the controllability of the system, the feedback gains are selected: (i) based on the decoupling of the original coordinates and to obtain proper damping and (ii) by applying the linear regulator problem to the individual modal coordinates separately. The linear control laws obtained using both techniques are then evaluated by numerical integration of the nonlinear system equations. Numerical examples are given considering pitch and various number of modes with different combination of actuator numbers and locations. The independent modal control concept used earlier with a discretized model of the thin beam in orbit is reviewed for the case where the number of actuators is less than the number of modes. It is seen that although the system is controllable it is not stable about the nominal (local vertical) orientation when the control is based on modal decoupling. An alternate control law not based on modal decoupling ensures stability of all the modes.

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List of Figures by Chapters

I. Introduction

The present grant represents a continuation of the effort attempted in the previous grant year (May 1977 - May 1978) and reported in Refs. 1 and 2.* In Ref. 1, a discretized planar model of a free-free beam in orbit was developed assuming the beam to be represented by a maximum of three point masses connected by idealized springs which accounted for the structural restoring effects. First order effects of gravity-gradient torques were included. It was assumed that two of the discrete masses were at the ends of the beam and that the third mass was at an interior position, later taken to be at the middle, when the beam was undeflected, and along the local vertical. Control was assumed to be realized by the action of one or two actuators located at the end masses, and implemented according to the concept of distributed modal control.[3] According to this concept, direct independent control in each mode considered is possible when the number of actuators is equal to the number of modes in the system model (neglecting the effects of higher modes not included in the model); when the number of actuators (P) is less than the number of modes (N) direct control of P modes may be implemented by proper selection of control law gains and the remaining N-P modes are effected according to the residual coupling in the control influence matrix according to the gains selected for the P actuators.

In Ref. 2, a mathematical model of a long, flexible free-free beam in orbit was obtained using the formulation developed by Santini[4] (in modified vector form) which develops the general equations of a flexible spacecraft in a gravitational field. The motion of a generic point in the body is described as a superposition of rigid body motion plus a combination of the flexible structural modes. The beam's center of mass was assumed to follow a circular orbit, the beam considered to be long and slender (shear deformation and rotational inertia effects neglected), and the axial deformation was assumed much smaller than the lateral deformation due to bending. In Ref. 2, the emphasis was placed on the analysis of the uncontrolled dynamics of this system where motion was restricted to occur only within the orbit plane; the equations of motion consisted of: a single equation describing the in plane (pitch) libration (rigid body rotational mode) and "n" generic modal equations expressed in terms of the vibrational modal amplitudes as the variables. For planar motion with only flexural vibrations, it was seen that the pitch motion was not influenced by the beam's elastic motion.

*For references cited in this report, please see reference list after each section.

For large values of the ratio of the structural modal frequency to the orbital angular rate, the elastic motion and pitch were decoupled; for small values of this ratio, the elastic motion was found to be governed by a Hill's three-term equation which could be approximated by a Mathieu equation, and the resulting stability considered by means of a Mathieu stability chart. Numerical simulations verified the possibility of vibrational instability for very long flexible beams in near-earth orbits.

In this report the control of an orbiting beam based on the continuum model of Ref. 2 with point actuators along the beam is examined. The format adapted in preparing this report is as follows: Two papers to be presented at the following conferences respectively, form the bases for Chapters II and III:

1. Second AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, June 21-23, 1979, Blacksburg, Va.

2. 1979 AAS/AIAA Astrodynamics Conference, Provincetown, Mass., June 25-27, 1979 (only the contributions by A.S.S.R. Reddy and P.M. Bainum are included here).

The first paper is concerned mainly with the modelling of point actuators, controllability conditions for a preselected set of actuators and a sample numerical case with one actuator and pitch plus two modes in the model. The second paper describes two control gain selection techniques using state variable feedback. The first technique uses decoupling of the original linearized equations of motion as a criteria to select gains and the second one applies the linear regulator problem to the system equations expressed in the independent modal coordinates.

In Chapter IV the independent modal control concept as applied to a discrete model of the orbiting beam developed earlier,[1] is reexamined according to controllability and stability considerations.

References are given separately for Chapters II, III, and IV. Symbols are defined in the text when and where they are used.

Chapter V describes the general conclusions together with recomendations for future work.

The introductions of Chapters II and III provide further details of the state of the art of beam modelling with relevant references.

I.1 References

1. Bainum, P.M. and Sellappan, R., "The Dynamics and Control of Large Flexible Space Structures," Final Report NASA Grant: NSG-1414, Part A: Discrete Model and Modal Control, Howard University, May 1978.

2. Bainum, P.M., Kumar, V.K., and James, P.K., "The Dynamics and Control of Large Flexible Space Structures," Final Report, NASA Grant: NSG-1414. Part B: Development of Continuum Model and Computer Simulation, Howard University, May 1978.

3. Advanced Tech. Lab. Program for Large Space Structures, Rockwell International SD-76-SA-0210, Nov. 1976, Appendix B, Modal Control of Flexible Spacecraft.

4. Santini, P. "Stability of Flexible Spacecrafts," Acta Astronautica, Vol. 3, pp. 685-713, 1976.

II. On the Controllability of Long Flexible Beam in Orbit.

Abstract

The equations of planar motion for a long, flexible free-free beam in orbit are developed and include the effects of gravity-gradient torques and control torques resulting from actuators assumed to be located at specific points along the beam. The control devices are used to control both the orientation as well as the shape of the beam. Application of two classes of theorems to the linearized form of these equations is used to establish necessary and sufficient conditions for controllability for different combinations of number/location of actuators with the number of modes contained in the mathematical model. It is seen that the number of actuators, if properly located, can be less than the number of modes in the system model. A numerical example illustrates the controlled response to an initial perturbation in both pitch angle as well as beam shape.

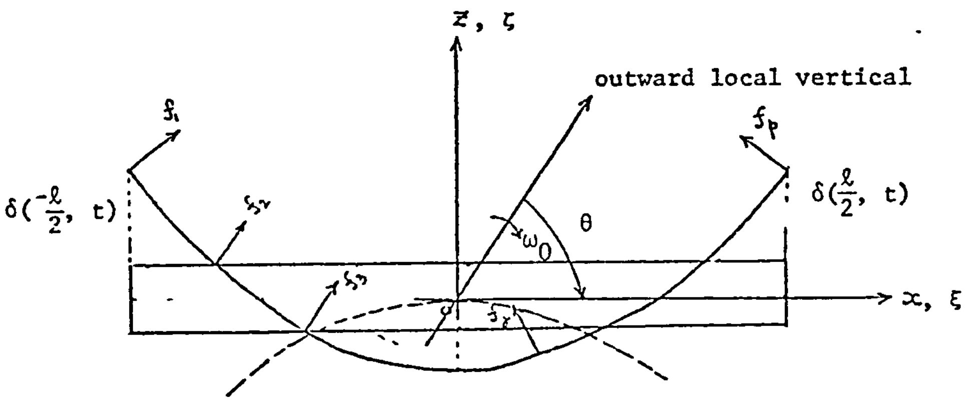
1. Introduction

Large, flexible space systems have been proposed for future use in communications, electronic orbital-based mail systems, and as possible collectors of solar energy for transmittal to power stations on the earth's surface.[1,2] Because of the inherent size and necessarily low weight to area ratio, the flexible parts of such systems become increasingly important and in some cases the entire system must be treated as being non-rigid. For meeting the requirements of these (and other) proposed missions, it will often be necessary to control both the geometrical shape as well as the orientation of the configuration.

Previously the formulation of the dynamics of a general flexible body in orbit was provided by Santini.[3] As a specific example, the equations of motion for an uncontrolled long, flexible uniform free-free beam in orbit were developed using a slightly modified version of the Santini formulation.[4] The motion of a generic point in the body was described as the superposition of rigid body motion plus a combination of the elastic modes. Further it was assumed that the system center of mass followed a circular orbit and that the pitch (rotation) and flexural deformations occurred within the orbital plane; also the elastic motion was assumed to be the result of only flexural vibrations.

The equations were linearized about a postion of zero structural deformation and alignment of the beam along either the local vertical or orbit tangent. It was seen that, in the absence of control, for small amplitude pitch, the pitch equation was uncoupled from the generic modal equations and that the generic modal equations were dynamically coupled with pitch only through a second order velocity term. Numerical simulations verified the possibility of vibrational instability for very long flexible beams in near-earth orbits.[4]

In the present paper, the uncontrolled system considered in Ref. 4 will be modified to include the effect of actuators located at specific point locations along the beam (Fig. 1). The modelling of actuator forces will be restricted to the case where the elastic displacements remain small as compared with typical beam dimensions of the order of hundreds of meters. For preselected sets of control devices (number and location) and the total number of modes in the system model, controllability conditions will be examined and some representative numerical results showing the controlled response of an initially perturbed system will be discussed.



$$\delta(x,t) = \sum_n A_n(t)\, \phi^n(x)$$

Figure 1: Beam Configuration with First Mode Deflection and p Actuators.

2. Mathematical Modelling

A. Equations of Motion for a Thin Beam in Orbit

The equations of motion for a thin homogeneous uniform beam whose center of mass is assumed to follow a circular orbit have been developed in Ref. 4. For the case where all rotations and transverse elastic displacements are assumed to occur within the plane of the orbit, and where the earth's gravitational field is considered to be spherically symmetric, these equations can be reduced to (Eqs. (24) - (25) of Ref. 4):

$$\frac{d^2\theta}{dt^2} + (3\omega_c^2 \sin 2\theta)/2 = T_p = N_p/J \tag{1}$$

$$\frac{d^2A_n}{dt^2} + \left[\omega_n^2 - \omega_c^2 (3\sin^2\theta - 1) - \left(\frac{d\theta}{dt} - \omega_c\right)^2\right] A_n = E_n/M_n \tag{2}$$

where

- $\theta(t)$ represents the pitch angle between the undeformed longitudinal axis and the local vertical
- $A_n(t)$ is the modal amplitude of the n^{th} generic mode
- ω_n is the n^{th} modal natural frequency
- ω_c is the orbital angular rate
- T_p is the external pitch acceleration, N_p/J
- E_n is the effect of external forces on the n^{th} generic mode
- M_n is the generalized mass of the beam in its n^{th} mode

It was further assumed that all elastic displacements are small as compared with the beam length. It can be concluded that there is no first order influence by the elastic motion on the rigid body pitch motion, but that the pitch motion affects the elastic motion due to higher order coupling. When the ratio of structural modal frequency to orbital rate is small and the pitch amplitude is small, it is shown[4] that the uncontrolled elastic motion can be approximated by a Mathieu equation, and with the aid of a Mathieu chart parametric instability regions can be readily identified.

For the development of the actuator modelling and subsequent consideration of controllability, Equations (1) and (2) will be linearized, and time and length will be nondimensionalized according to

$$\tau = \omega_c t \tag{3}$$

$$Z_n = A_n/\ell \tag{4}$$

where ℓ = length of the undeformed beam.

The resulting linearized system equations are:

$$\frac{d^2\theta}{d\tau^2} + 3\theta = T_p/\omega_c^2 \tag{5}$$

$$\frac{d^2Z_n}{d\tau^2} + \left(\frac{\omega_n}{\omega_c}\right)^2 Z_n = E_n/M_n\ell\omega_c^2 \; ; \; n = 1, 2, \ldots \tag{6}$$

By defining

$$\begin{array}{ll} \theta = X_1 & d\theta/d\tau = \dot{X}_1 = X_{n+1} \\ Z_1 = X_2 \quad \text{and} & dZ_1/d\tau = \dot{X}_2 = X_{n+2} \\ \vdots & \vdots \\ Z_{n-1} = X_n & dZ_{n-1}/d\tau = \dot{X}_n = X_{2n} \end{array} \tag{7}$$

eqs. (5) and (6) may be written in the standard form:

$$\dot{X} = \left[\begin{array}{c|c} 0 & I \\ \hline -A & 0 \end{array}\right] X + B_c U_c \tag{8}$$

where $X = [X_1, X_2, \ldots X_n, X_{n+1}, \ldots X_{2n}]^T$, state vector

0 = nxn null matrix

I = nxn identity matrix

$$A = \begin{bmatrix} 3 & & & \\ & (\omega_1/\omega_c)^2 & & 0 \\ & & \ddots & \\ 0 & & & (\omega_{n-1}/\omega_c)^2 \end{bmatrix}$$

$$B_c = \begin{bmatrix} - \\ I \end{bmatrix}$$

$U_c = [T_p/\omega_c^2, E_1/M_1 \ell\omega_c^2, \ldots E_{n-1}/M_{n-1}\ell\omega_c^2]^T$

and represents the control vector

B. Modelling of the Point Actuators along the Beam

It is assumed that p actuators be located along the beam at points $\xi_1, \xi_2, \ldots \xi_p$, where ξ lies along the beam's undeformed longitudinal axis and $\xi = 0$ corresponds to the mass center of the undeformed beam. The actual control forces associated with these actuators will be designated $f_1, f_2, \ldots f_j \ldots f_p$, respectively. For small elastic displacements the component of the control force, f_j, parallel to the ξ axis is very small and the component parallel to the ζ axis can be approximated by f_j. Thus the control torque due to the j th actuator may be expressed by

$$\overline{N}_{P_j} = \int \overline{r} \times \overline{f}_j dm \tag{9}$$

where

$$\overline{f}_j \simeq \overline{k}\, f_j\, \delta(\xi-\xi_j)$$

$$\overline{r} = \xi_j\, \overline{i} + \overline{q}_j\, (\xi_j)$$

$$\overline{q}_j = \sum_{n=1}^{\infty} \phi_z^n\, (\xi)\, A_n(t)\, \overline{k}$$

and ϕ_z^n is the z th component of the modal shape function corresponding to the n th mode

After integration there results

$$\overline{N}_{p_j} = -\overline{j}\, f_j \xi_j\, x(\text{const for a uniform beam}) \qquad (10)$$

For convenience the constant will subsequently be incorporated into f_j. It is then clear that for p actuators,

$$\overline{N}_p = \sum_j \overline{N}_{p_j} = -\overline{j}\, [f_1\xi_1 + f_2\xi_2 + \ldots + f_p\xi_p] \qquad (11)$$

and that this term divided by the pitch axis moment of inertia, J, provides the control acceleration for the pitch motion.

For the generic modal equations the control forces can be transformed into the corresponding modal forces by[3,4]

$$E_{n_j} = \int \overline{\phi}^{\,n} \cdot \overline{f}_j\, dm \qquad (12)$$

Under the assumptions previously stated,

$$E_{n_j} = f_j\, \phi_z^n\, (\xi_j)\, x(\text{const.}) \qquad (13)$$

As before, the constant will be incorporated into f_j so that the effect of all p thrusters on the n th generic mode can be expressed by

$$E_n = f_1\, \phi_z^n\, (\xi_1) + \ldots + f_p\, \phi_z^n\, (\zeta_p) \qquad (14)$$

The control vector, U_c, can now be related to the actuator forces, actuator locations, and modal shape functions by

$$U_c = \underbrace{\begin{bmatrix} -\xi_1/J\omega_c^2, & -\xi_2/J\omega_c^2, & \cdots & \cdots & , -\xi_p/J\omega_c^2 \\ \phi_z^1(\xi_1)/M_1\ell\omega_c^2, & \phi_z^1(\xi_2)/M_1\ell\omega_c^2, & \cdots & , & \phi_z^1(\xi_p)/M_1\ell\omega_c^2 \\ \vdots & & & & \vdots \\ \phi_z^{n-1}(\xi_1)/M_{n-1}\ell\omega_c^2, & & \cdots & , & \phi_z^{n-1}(\xi_p)/M_{n-1}\ell\omega_c^2 \end{bmatrix}}_{nxp} \underbrace{\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{bmatrix}}_{px1} \qquad (15)$$

If the nxp matrix appearing in Eq. (15) is denoted by B_{act}, then Eq. (8) may be expanded in the following form:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_{2n} \end{bmatrix} = \left[\begin{array}{c|c} \underset{nxn}{0} & \underset{nxn}{I} \\ \hline \begin{matrix} -3 & & \\ & -\left(\frac{\omega_1}{\omega_c}\right)^2 & \\ & & \ddots \\ & & & -\left(\frac{\omega_{n-1}}{\omega_c}\right)^2 \end{matrix} & \underset{nxn}{0} \end{array}\right] \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{2n} \end{bmatrix}$$

$$+ \left[\begin{array}{c} \underset{nxp}{0} \\ \hline \underset{nxp}{B_{act}} \end{array}\right] \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_p \end{bmatrix} \qquad (16)$$

The modal masses appearing in B_{act} can be evaluated for homogeneous free-free beams and shown to be independent of the mode number.[5,6] Specifically, $M_i = \rho\ell$ where ℓ is the undeformed length and ρ represents the mass density per unit length.

3. Controllability

A. Statement of the Controllability Theorems

Eq. (16) can be written as

$$\dot{X} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ B \end{bmatrix} f \qquad (17)$$

where

$X = [X_1, X_2, \ldots X_n, X_{n+1}, \ldots X_{2n}]^T$

$I =$ nxn identity matrix

$B = B_{act}$, nxp matrix

$f = [f_1, f_2 \ldots f_p]^T$

The system represented by Eq. (17) is controllable if the pair [A,B] is controllable[7] - i.e. it can be proven that the controllability matrix associated with the original state and control coefficient matrices

$$\begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} , \begin{bmatrix} 0 \\ B \end{bmatrix}$$

has rank 2n if and only if the controllability matrix associated with the pair of reduced state and control matrices,

$$[A , B]$$

has rank n.[7]

Furthermore, if the matrix A has eigenvalues of unit multiplicity (i.e. non-repeated eigenvalues), the system given by Eq. (17) is controllable if and only if each row of B has a non-zero entry.[7]

For the case where A has repeated eigenvalues (multiplicity greater than one), the reduced order state matrix, A, can be written in Jordan block matrix form. If the eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_m$ have multiplicity of $n_1, n_2, \ldots n_m$, respectively, with $\sum_{i=1}^{m} n_i = n$, then A can be transformed as

$$\begin{bmatrix} J_{n_1} & & & \\ & J_{n_2} & & \\ & & \ddots & \\ & & & J_{n_m} \end{bmatrix} \tag{18}$$

where

$$J_{n_j} = \underset{n_j \times n_j}{\begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}} \tag{19}$$

The system (17) can be divided into m subsystems. For the system (17) to be controllable, these m subsystems with their corresponding blocks in the B matrix must each be separately controllable.

B. Application of the Controllability Theorems

The theorems briefly outlined here will now be applied to several cases of interest for different combinations of numbers and locations of the actuators along the beam.

Case 1: One actuator at one end of the beam with pitch and two generic modes contained in the mathematical model.

The actuator is assumed to be located at the left end (Fig. 1) $\xi = -\ell/2$. The first and second modal shape functions for a free-free beam can be evaluated at that point to yield[8]

$$\phi_z^1(-\ell/2) = \phi_z^2(-\ell/2) = 2$$

and $M_1 = M_2$.[5]
The state equation for this system can be expanded in the form of Eq. (17) with the result that

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{bmatrix} \quad ; \quad B = \begin{bmatrix} a \\ b \\ b \end{bmatrix}$$

where $a = \ell/2J\omega_c^2$ $\qquad b = 2/M_1\ell\omega_c^2$

$c = (\omega_1/\omega_c)^2$ $\qquad d = (\omega_2/\omega_c)^2$

The controllability matrix based on the reduced system matrices, A and B, becomes

$$C = [B \mid AB \mid A^2B] = \begin{bmatrix} a & 3a & 9a \\ b & bc & bc^2 \\ b & bd & bd^2 \end{bmatrix}$$

For controllability the matrix, C, must have a rank of 3, or

$$\det C = -ab^2 (c-d)(c-3)(d-3) \neq 0.$$

The necessary and sufficient conditions for controllability become

$$\omega_1 \neq \omega_2 \quad \text{(trivial)} \; ; \; \omega_1 \neq \sqrt{3}\omega_c \; ; \; \omega_2 \neq \sqrt{3}\omega_c$$

The last two of these conditions will, in practice, place a lower bounds on the stiffness and/or an upper bounds on the length of such a long flexible structure in orbit.

Case 2: Three actuators two of which are assumed to be placed at the ends and one at the mid point of the beam, with the mathematical model containing only the first two generic modes.

For this case,

$$A = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \quad ; \quad B = \begin{bmatrix} a & a & -b \\ a & -a & 0 \end{bmatrix}$$

with

$$a = 2/M_1 \ell \omega_c^2 \quad b = \sqrt{2}/M_1 \ell \omega_c^2 \quad \text{with } M_1 = M_2$$

and c,d defined as in Case 1.

The controllability matrix may be calculated as

$$C = \left[\begin{array}{ccc|ccc} a & a & -b & ac & ac & -bc \\ a & -a & 0 & ad & -ad & 0 \end{array}\right]$$

It can be seen that since the B matrix itself has rank 2, then C will automatically have rank 2 and the system controllability is independent of the nature of the matrix A.

Case 3: Two actuators one each at the ends with pitch plus the first generic mode in the system model.

For this system the A and B matrices in Eq. (17) become

$$A = \begin{bmatrix} 3 & 0 \\ 0 & c \end{bmatrix} \qquad B = \begin{bmatrix} a & -a \\ b & b \end{bmatrix}$$

with

$$a = \ell/2J\omega_c^2 \qquad b = 2/M_1 \ell \omega_c^2 \qquad c = (\omega_1/\omega_c)^2$$

The resulting controllability matrix

$$C = \left[\begin{array}{cc|cc} a & -a & 3a & -3a \\ b & b & bc & bc \end{array}\right]$$

has rank 2 since the B matrix has rank 2. Thus, the system controllability is ensured.

Case 4: Three actuators two of which are assumed to be located at the ends and the remaining one at the mid point of the beam; the model contains pitch plus the first two generic modes.

For this case

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{bmatrix} \qquad B = \begin{bmatrix} a & -a & 0 \\ b & b & -\varepsilon \\ b & -b & 0 \end{bmatrix}$$

with c,d as defined in Case 1

and $a = \ell/2J\omega_c^2$; $b = 2/M_1\ell\omega_c^2$; $e = \sqrt{2}/M_1\ell\omega_c^2$
and $M_1 = M_2$.
The controllability matrix,

$$C = \left[\begin{array}{ccc|ccc|ccc} a & -a & 0 & 3a & -3a & 0 & 9a & -9a & 0 \\ b & b & -e & bc & bc & -ce & bc^2 & bc^2 & -c^2e \\ b & -b & 0 & bd & -bd & 0 & bd^2 & -bd^2 & 0 \end{array}\right]$$

must have rank 3 to ensure the system is controllable, which means that from the nine columns it must be shown that at least one 3x3 non-zero determinant exists.

If we arbitrarily select the first, third, and fourth columns, then

$$-eab\ (d-3) \neq 0$$

which is guaranteed if $\omega_2 \neq \sqrt{3}\omega_c$. Although this is a sufficient condition for controllability at this point we don't know whether it is also a necessary condition.

As an alternate, let us select the first, third, and sixth columns of C; then

$$ab^2\ (d-c)\ (c-3)\ (d-3) \neq 0$$

which will be ensured if the following sufficiency conditions are satisfied:

$$\omega_2 \neq \omega_1, \quad \omega_1 \neq \sqrt{3}\omega_c, \quad \text{and } \omega_2 \neq \sqrt{3}\omega_c$$

In order to establish necessary conditions, we will now assume that any two of the frequencies are the same and then apply the theorem for the case of repeated eigenvalues.

(a) First, if it is assumed that $c = d$ ($\omega_1 = \omega_2$), the corresponding subsystem matrices are

$$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \qquad B = \begin{bmatrix} b & b & -e \\ b & -b & 0 \end{bmatrix}$$

Since the B matrix has rank 2, the condition: $c \neq d$, is not necessary for controllability.

(b) If it is assumed $c = 3$, ($\omega_1 = \sqrt{3}\omega_c$) the corresponding subsystem matrices are

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} a & -a & 0 \\ b & b & -e \end{bmatrix}$$

Since B has rank 2, the system is controllable for this case.

(c) We now consider the case when $d = 3(\omega_2 = \sqrt{3}\omega_c)$ where the subsystem matrices are

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} a & -a & 0 \\ b & -b & 0 \end{bmatrix}$$

and

$$C = \left[\begin{array}{ccc|ccc} a & -a & 0 & 3a & -3a & 0 \\ b & -b & 0 & 3b & -3b & 0 \end{array}\right]$$

The controllability matrix has only one independent column and can not have rank 3 when $\omega_2 = \sqrt{3}\omega_c$.

(d) Finally we consider the case where $3 = c = d$ or $\omega_{Pitch} = \omega_1 = \omega_2$ (admittedly, of only academic interest)

Then

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} a & -a & 0 \\ b & b & -e \\ b & -b & 0 \end{bmatrix}$$

and

$$C = \left[\begin{array}{ccc|ccc|ccc} a & -a & 0 & 3a & -3a & 0 & 9a & -9a & 0 \\ b & b & -e & 3b & 3b & -3e & 9b & 9b & -9e \\ b & -b & 0 & 3b & -3b & 0 & 9b & -9b & 0 \end{array}\right]$$

It can be observed that the C matrix can not have rank 3.

In conclusion, we can say that only one of the three necessary conditions is also a sufficient condition for controllability, i.e.

$$\omega_2 \neq \sqrt{3}\omega_c$$

In actual practice the repeated frequencies associated with the modes included in the model will be known so that it is possible to verify in advance of the selection of the control law whether the particular choice of number and location of actuators will result in a controllable system.

4. Numerical Example

A numerical example of Case 1 is considered where it is assumed that the control force generated by the single actuator depends on only rate feed-back according to

$$f_1 = K_1\dot{x}_1 + K_2\dot{x}_2 + K_3\dot{x}_3$$

where the $\dot{x}_i$ are the pitch, and non-dimensionalized first, and second modal amplitude derivatives, respectively, with respect to the orbital time, τ.

It is assumed that the fundamental natural frequency of the free-free beam is 1/100 cps and that the c.m. moves in a 250 n. mile altitude circular orbit. For this case

$$(\omega_1/\omega_c)^2 = 3200 \quad \text{and} \quad (\omega_2/\omega_c)^2 \cong 28{,}800.$$

As an example, a 100m. long slender hollow tubular beam made of wrought aluminum (2014T6) and with an outside diameter of 10.79cm, and thickness of 1.06cm, would exhibit these frequencies

The completely nontrivial part of Eqs. (16) or (17) may be expanded to yield

$$\ddot{x}_1 + 3x_1 - 59.52K_1\dot{x}_1 - 59.52K_2\dot{x}_2 - 59.52K_3\dot{x}_3 = 0$$

$$\ddot{x}_2 + 3200x_2 - 20.0K_2\dot{x}_2 - 20.0K_1\dot{x}_1 - 20.0K_3\dot{x}_3 = 0$$

$$\ddot{x}_3 + 28{,}800x_3 - 20.0K_3\dot{x}_3 - 20.0K_2\dot{x}_1 - 20.0K_2\dot{x}_2 = 0$$

If we arbitrarily select $K_1 = -0.00577$, $K_2 = -0.05656$, and $K_3 = -0.01695$ (note these gains would correspond to much less than critical damping if the other coupling terms in rates did not appear) and assume that the initial conditions are $x_1(0) = x_2(0) = x_3(0) = 0.01$ and all initial $\dot{x}_i(0) = 0$, the controlled response is illustrated in Fig. 2. The relatively long response time with the relatively low level of peak thrust should be noted.

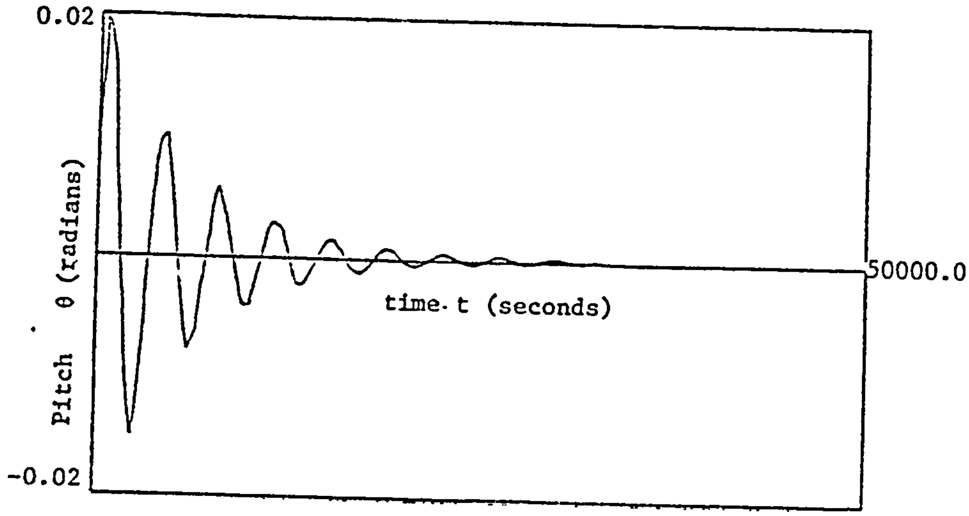


Fig. 2A

Figure 2: Case 1 - Controlled Response, Pitch + Two Modes with One Actuator at Left End.

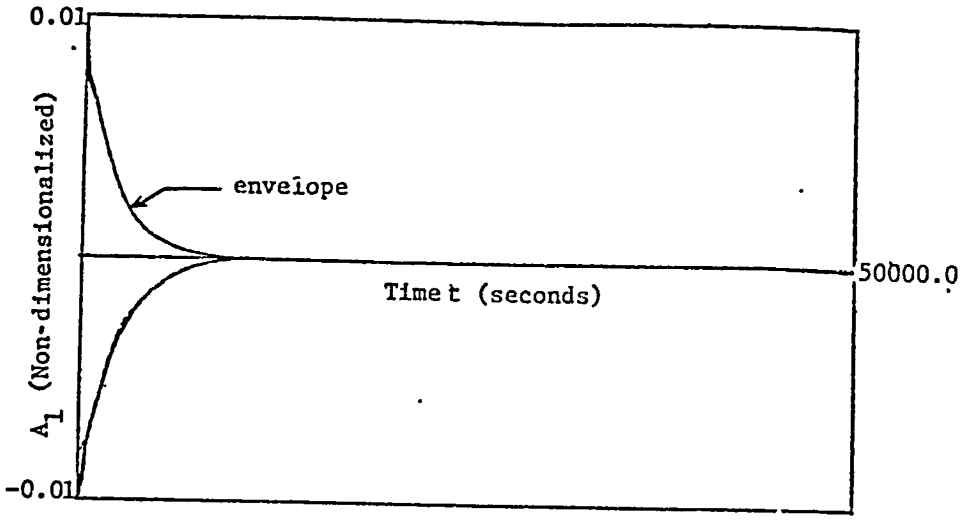


Fig. 2B

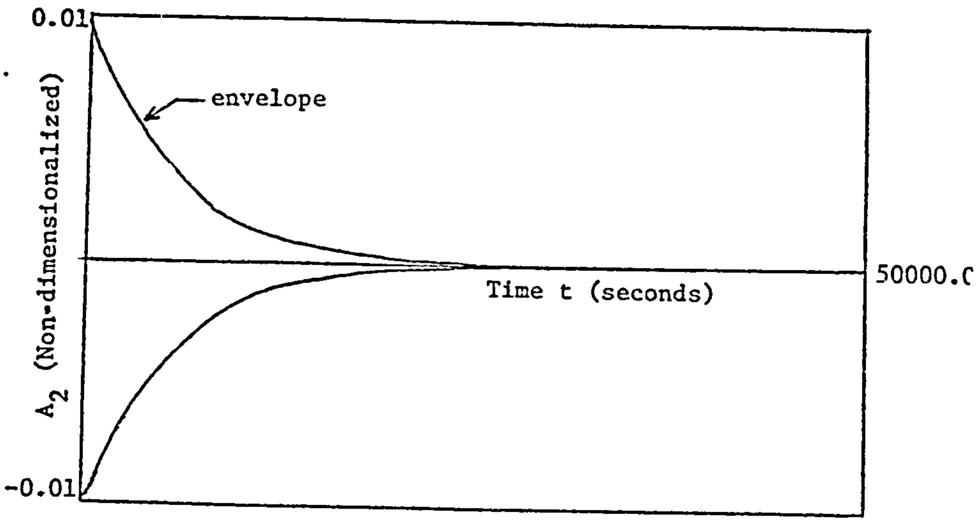


Fig. 2C

Figure 2: Case 1 - Controlled Response, Pitch + Two Modes with One Actuator at Left End.

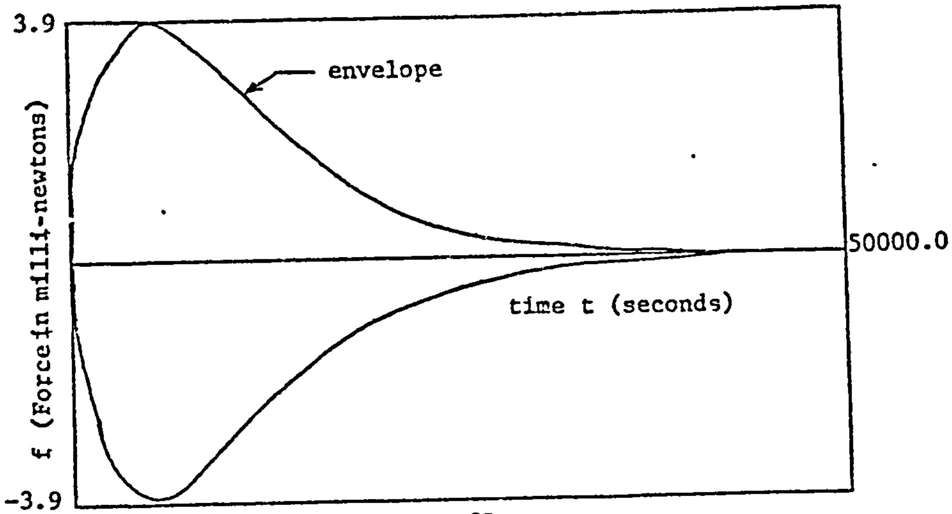


Fig. 2D

Figure 2: Case 1 - Controlled Response, Pitch + Two Modes with One Actuator at Left End.

This example is presented as a verification that the system in Case 1 with the number of actuators less than the number of modes is controllable. In a related paper methods of selecting control law gains based on decoupling considerations is discussed.[9] Control gains are selected based on the following two criteria: (i) decoupling of the linearized system equations with appropriate state variable feedback; and (ii) applying the linear regulator problem to the n modal coordinates separately and thus selecting the gains by solving groups of n two dimensional matrix Riccati equations.[9]

5. Concluding Remarks

In the present paper a model is developed for predicting the dynamics of a long, flexible free-free beam in orbit under the influence of control devices which are considered to act at specific points along the beam. Application of two classes of theorems establishes the necessary and sufficient conditions for controllability and clearly demonstrates that the number of actuators, if properly located, can be less than the number of modes in the system model.

The insight gained by this preliminary study will be useful in analyzing the dynamics and control of more complicated structures such as of a large flexible plate in orbit, which more adequately represents a large flexible orbiting platform. Another possible extension to the current work would be a study of the effect of using control devices which are distributed along the beam instead of being treated as point actuators.

Acknowlegment

This research was supported by the National Aeronautics and Space Administration Grant No. NSG-1414, Suppl. 1. Special appreciation to Mr. H.A. Hamer and Dr. L. Pinson of the Langley Research Center for their interest and helpful suggestions.

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III. Decoupling Control of a Long Flexible Beam in Orbit

Control of large flexible systems using state variable feedback is presented with a long flexible beam in orbit as an example. Once the controllability of the system is established, the feedback gains are selected: (i) based on the decoupling of the original coordinates and to obtain proper damping and (ii) by applying the linear regulator problem to the individual modal coordinates separately. The linear control laws obtained using both techniques are then evaluated by numerical integration of the non-linear system equations. The response of the state together with resulting beam deflection and actuator force (s) required are obtained as functions of time for different combinations of the number/location of actuators and the number of modes in the system model. Also included are results showing the effects (control spillover) on the uncontrolled modes when the number of controllers is less than the number of modes, and the effects of inaccurate knowlege of the control influence coefficients which lead to errors in the calculated feedback gains.

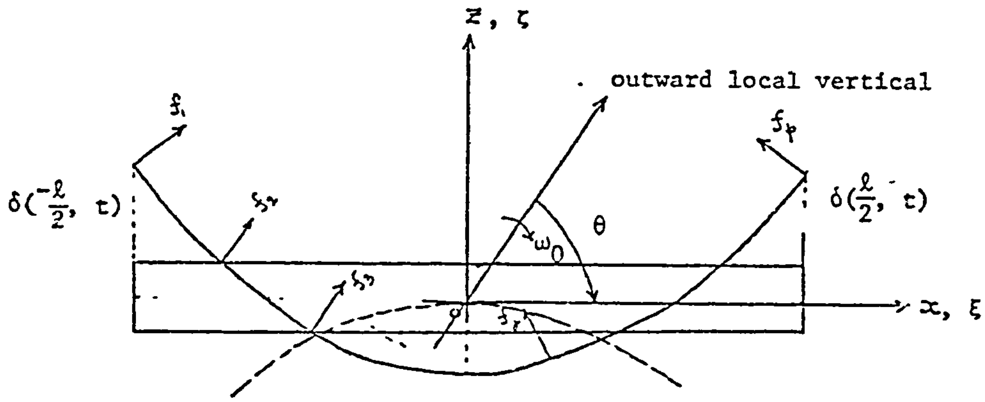
1. Introduction

Future proposed space missions would involve large, inherently flexible systems for use in communications, as collectors of solar energy, and in electronic, orbital-based mail systems.[1,2] For the first time the flexible parts, and in some cases the entire system, due to its size, must be modelled as being completely flexible. In order to satisfy the requirements of such missions, it will be necessary to control not only the orientation of the system but also the geometrical shape of the configuration.

As a specific example of the general formulation of the dynamics of an arbitrary flexible body in orbit developed by Santini[3], the uncontrolled motion of a long, flexible beam was investigated.[4] The motion of a generic point in the body was described as the superposition of rigid body motion plus a combination of the elastic modes.

Further it was assumed that the system center of mass followed a circular orbit and that the pitch (rotation) and flexural deformations occured within the orbital plane. For this planar motion, it was seen that the pitch motion was not influenced by the beam's elastic motion. The decoupling of pitch and the elastic modes was observed for large values of the ratio of the structural modal frequency to the orbital angular rate. When the values of this ratio are small the elastic motion is governed by a Hill's three-term equation which could be approximated by a Mathieu equation, and the resulting stability was considered by means of a Mathieu stability chart. Numerical simulations verified the possibility of vibrational instability for a very long uncontrolled flexible beam in near-earth orbits.[4]

The controllability of a long flexible beam with point actuators located along the beam is considered in Ref. 5 for the case of small amplitude flexural deformations (Fig. 1). Necessary and sufficient conditions for controllability with preselected locations of actuators are derived using theorems developed in Ref. 6. Once controllability is assured, values for the gains in the control laws are selected on an arbitrary basis, and only for one combination of actuator location and number of flexural modes.[5]



$$\delta(x,t) = \sum_n A_n(t)\, \phi^n(x)$$

Fig. 1. Beam Configuration with First Mode Deflection and p Actuators.

In the present paper selection of control gains for any large flexible system using the following two criteria is discussed: (i) decoupling of the linearized system equations with appropriate state variable feedback[7]; and (ii) applying the linear regulator problem to the modal coordinates (n) separately and, thus, selecting gains by solving groups of "n" two by two matrix Riccati equations.[8,9,10,11]

A long flexible beam in orbit is taken as an example with the model developed in Refs. 4 and 5. Gains are selected using the two techniques and numerical simulation of the non linear equations is employed to predict the responses for sample cases. The deflection of the controlled beam at various instants of time is also illustrated.

2. Decoupling by State Variable Feedback

After appropriate linearization the dynamic model for any flexible system can be represented by

$$A\ddot{X} + B\dot{X} + CX = DU \tag{1}$$

where
- A is an nxn non singular matrix
- B,C are nxn matrices
- D is an nxm matrix
- X is an nx1 state vector representing deflections in addition to the rigid body rotations.
- U is an mx1 control vector

Equation (1) can be written in more standard state space form by defining

$X = X_1$, $\dot{X} = X_2 = \dot{X}_1$ as

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A^{-1}C & -A^{-1}B \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ A^{-1}D \end{bmatrix} U \tag{2}$$

Equation (1) takes into account the modelling of any structural damping inherently present in the system. Controllability of the systems represented by Equation (2) can not be obtained using theorems developed in Ref. 6, unless B = 0. In cases where B ≠ 0, the controllability matrix of the pair

$$\begin{bmatrix} 0 & I \\ -A^{-1}C & -A^{-1}B \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ A^{-1}D \end{bmatrix} \quad \text{must have rank } 2n.$$

After choosing a state variable feedback control law of the form

$$U = K\dot{X} + LX \tag{3}$$

where

K = mxn rate feedback gain matrix
L = mxn position feedback gain matrix

Equation (1) can be rewritten as:

$$\ddot{X} + (A^{-1}B-A^{-1}DK)\dot{X} + (A^{-1}C-A^{-1}DL)X = 0 \tag{4}$$

For decoupling of the states $X = (x_1, x_2, \ldots, x_n)^T$ the matrices $(A^{-1}B-A^{-1}DK)$ and $(A^{-1}C-A^{-1}DL)$ must be diagonal.

i.e. $$A^{-1}B-A^{-1}DK = \zeta \tag{5}$$

$$A^{-1}C-A^{-1}DL = \omega \tag{6}$$

where

$$\zeta = \begin{bmatrix} \zeta_{11} & 0 & \cdots & 0 \\ 0 & \zeta_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \zeta_{nn} \end{bmatrix} \quad \text{and} \quad \omega = \begin{bmatrix} \omega_{11} & 0 & & 0 \\ 0 & \omega_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \omega_{nn} \end{bmatrix} \qquad (7)\ (8)$$

Redefining

$$A^{-1}B = E\ , \quad A^{-1}C = F$$
$$A^{-1}D = G \tag{9}$$

we have

$$E - GK = \zeta \tag{10}$$

$$F - GL = \omega \tag{11}$$

with

K_i = i^{th} column of the K matrix
L_i = i^{th} column of the L matrix

$E_i' = i^{th}$ column of $(E-\zeta)$ matrix

$F_i' = i^{th}$ column of $(F-\omega)$ matrix

Equations (10) and (11) can be written as 2n sets of algebraic equations of the form:

$$GK_i = E_i' \qquad i = 1,2, \ldots n \tag{12}$$
$$GL_i = F_i'$$

Consider one of the above sets of linear algebraic equations for the case where $i = 1$,

$$GK_1 = E_1' \tag{13}$$

There are n equations and m unknowns (the elements in the first column of the K matrix). The fundamental theorem for a set of n linear equations with m unknowns is now applied[12,13]:

For a unique solution:

Case 1: If $n > m$ (more equations and less unknowns) the rank of G and the augumented matrix $[G \mid E_1']$ must be n.

Case 2: If $n = m$ (number of equations = number of unknowns) the rank of G and the augumented matrix $[G \mid E_1']$ must be m (or n) - i.e. G must be non-singular.

Case 3: If $n < m$ (less equations, more unknowns) no unique solution exists.

For non-trivial solution:

Case 1: If $n > m$ (more equations and less unknowns), the rank of G and the augumented matrix $[G \mid E_1'] \leq m$.

Case 2: If $n = m$ (number of equations = number of unknowns), the rank of G and the augumented matrix $[G \mid E_1'] \leq m$. If the rank = m = n, a unique solution exists.

Case 3: If $n < m$ (less equations and more unknowns) the rank of G and the agumented matrix $[G \mid E_1'] \leq n$.

These conditions must be satisfied by the 2n subsystems given by (12) for decoupling to be implemented and, thus, dictate the choice of the actuators.

For cases where the number of actuators are more or less than the number of original coordinates in the system, the controllability conditions and the conditions to be satisfied for decoupling may pose numerical (computational) problems, especially when the order of the system is large.

When the number of actuators equals the number of original coordinates all controllability and decoupling conditions depend on the non singularity of matrix, G. This matrix is an nxn matrix and can be made non-singular by properly selecting the location of the actuators.

If there are 6 original coordinates and 3 actuators, then the controllability matrix has 36 columns out of which 12 columns must be independent to have rank 12. The maximum number of determinants to be evaluated are

$$^{36}C_{12} = \frac{36!}{12!24!} = 1.2516775\times10^{9} \text{ determinants.}$$

Assuming a 1 sec. computational time required for the evaluation of each twelth order determinant, the examination of all possible combinations would involve 347,688 hrs. of computer time.

Specific cases, where controllability of this system is examined when the number of actuators differs from the number of modes in the system model are presented in Ref. 5, but only for a low order system ($n_{max} = 3$).

3. Linear Regulator Problem

Using modal analysis[8,9,10,11] dynamical systems represented by (a):

$$M\ddot{q} + G\dot{q} + C\dot{q} + Kq = u(t) \tag{14}$$

where
- q = n dimensional vector describing angular and elastic displacments
- M,K = positive definite mass and stiffness matrice
- G = gyroscopic antisymmetric matrix
- C = pervasive damping, either positive definite or semidefinite matrix
- $u(t)$ = control vector

or by (b):

$$M\ddot{q} + G\dot{q} + Kq = u(t) \tag{15}$$

with $M = M^T$, $G = -G^T$ and $K = K^T$

can be transformed to

$$\text{(a)} \quad \ddot{X} + D\dot{X} + FX = Eu = u' \tag{16}$$

or

$$\text{(b)} \quad \ddot{X} + F\dot{X} = Eu = u' \tag{17}$$

with D and F representing the diagonal transformed matrices. The left hand sides of Eqs. (14) and (15) represent either damped or undamped harmonic oscillator These oscillators can be controlled optimally and independently with one control force for every independent modal coordinate.

The actual control, $u(t)$, in the original coordinates, q_i, can be calculated from the control, $u'(t)$, in the decoupled (modal) coordinates, χ_i, by

$$u = E^{-1}u' \tag{18}$$

The i^{th} component of the control vector, u', can be calculated as follows:

The i^{th} independent modal coordinate is governed by

$$\ddot{\chi}_i + D_i\dot{\chi}_i + F_i\chi_i = u'_i \tag{19}$$

where $D_i = i^{th}$ diagonal element of the D matrix

$F_i = i^{th}$ diagonal element of the F matrix

Equation (19) can be written in state space form by defining

$$\chi_i = \chi_{i_1},\ \dot{\chi}_i = \chi_{i_2} = \dot{\chi}_{i_1}$$

$$\begin{bmatrix} \dot{\chi}_{i_1} \\ \dot{\chi}_{i_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -F_i & -D_i \end{bmatrix} \begin{bmatrix} \chi_{i_1} \\ \chi_{i_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i' \tag{20}$$

similar to the standard form

$$\dot{\chi} = A\chi + Bu \tag{21}$$

A performance index for the i^{th} modal coordinate is defined as

$$J_i = \int_0^\infty (\chi_i^T Q_i \chi_i + (u_i')^2 R_i)dt \tag{22}$$

where

$$Q_i = \begin{bmatrix} Q_{i_1} & 0 \\ 0 & Q_{i_2} \end{bmatrix}$$ and R_i is a scalar.

The control vector is given by

$$u_i' = -R_i^{-1}B_i^T S_i \chi_i \quad \text{where} \quad \chi_i = [\chi_{i_1}\ \chi_{i_2}]^T \tag{23}$$

Where S_i is the symmetric matrix solution of the two dimensional Riccati equation:

$$-S_iA_i - A_i^TS_i + S_iB_iR^{-1}B_i^TS_i - Q = 0 \tag{24}$$

with

$$A_i = \begin{bmatrix} 0 & 1 \\ -F_i & -D_i \end{bmatrix} \qquad B_i^T = [0,\ 1]$$

$$R = R_i \text{ and } Q = Q_i$$

For the second order system represented by Eq. (24), the elements of S_i may be solved in closed form with the results

$$S_{i_{12}} = R_i \left[-F_i \pm \sqrt{F_i^2 + Q_{i_1}/R_i}\right] \tag{25}$$

$$S_{i_{22}} = R_j \left[-D_i \pm \sqrt{D_i^2 + \frac{1}{R_i}\left(Q_{i_2} + 2S_{i_{12}}\right)}\right] \tag{26}$$

$$S_{i_{11}} = F_i\, S_{i_{22}} + D_i S_{i_{12}} + \frac{1}{R_i} S_{i_{12}} S_{i_{22}} \tag{27}$$

where the signs of the radicals are selected such that S_i is a positive definite matrix.

Then

$$u_i' = -\left[S_{i_{12}},\ S_{i_{22}}\right] \begin{bmatrix} X_{i_1} \\ X_{i_2} \end{bmatrix} \tag{28}$$

4. Numerical Example

A long flexible free-free beam in orbit is considered to demonstrate the two gain selection techniques described earlier. The model (Fig. 1) including point actuators is taken from Refs. 4 and 5 and is based on the assumption that all rotations and deformations occur only within the orbital plane. The equations of motion are given by[4,5]

$$\frac{d^2\theta}{dt^2} + 3\theta = \frac{T_P}{J\omega_c^2} \tag{29}$$

$$\frac{d^2 z_n}{dt^2} + \left(\frac{\omega_n}{\omega_c}\right)^2 z_n = \frac{E_n}{M_n \ell \omega_c^2} \tag{30}$$

$$n = 1, 2, \ldots$$

where

θ = pitch angle relative to local vertical
$\tau = \omega_c t$, normalized time
$z_n = A_n/\ell$, non-dimensional modal amplitudes
ℓ = length of the beam
ω_c = orbital angular rate

After defining

$$\theta = \chi_1 \qquad d\theta/dt = \dot{\chi}_1 = \chi_{n+1}$$

$$z_1 = \chi_2 \qquad dz_1/dt = \dot{\chi}_2 = \chi_{n+2}$$

$$z_{n-1} = \chi_n \qquad dz_{n-1}/dt = \dot{\chi}_n = \chi_{2n}$$

Equations (29) and (30) can be written as

$$\dot{X} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} X + B_c u_c \tag{31}$$

where

$X = [\chi_1, \chi_2, \ldots, \chi_n, \chi_{n+1}, \ldots, \chi_{2n}]^T$ state vector

0 = nxn null matrix

I = nxn identity matrix

$$A = \begin{bmatrix} -3.0 & & 0 \\ & -\left(\frac{\omega_1}{\omega_c}\right)^2 & \\ & & \ddots \\ 0 & & -\left(\frac{\omega_{n-1}}{\omega_c}\right)^2 \end{bmatrix}$$

$$B_c = \left[\frac{0}{I}\right]$$

$$u_c = \left[\frac{T_p}{\omega_c^2},\ \frac{E_1}{M_1 \ell \omega_c^2},\ \ldots,\ \frac{E_{n-1}}{M_{n-1}\ell\omega_c^2}\right]$$

with (Ref 5) p actuators located at $(\xi_1, \xi_2, \ldots, \xi_p)$

$$T_p \doteq -\frac{1}{J}\left[f_1\xi_1 + \ldots + f_p\xi_p\right] \tag{32}$$

$$E_n = f_1\phi_z^n(\xi_1) + f_2\phi_z^n(\xi_2) + \ldots + f_p\phi_z^n(\xi_p) \tag{33}$$

Then

$$u_c = \begin{bmatrix} -\frac{\xi_1}{J\omega_c^2} & -\frac{\xi_2}{J\omega_c^2} & \cdots & -\frac{\xi_p}{J\omega^2} \\ \frac{\phi_z^1(\xi_1)}{M_1\ell\omega_c^2} & \frac{\phi_z^1(\xi_2)}{M_1\ell\omega_c^2} & \cdots & \frac{\phi_z^1(\xi_p)}{M_1\ell\omega_c^2} \\ \vdots & \vdots & & \vdots \\ \frac{\phi_z^{n-1}}{M_{n-1}\ell\omega_c^2} & \frac{\phi_z^{n-1}(\xi_2)}{M_{n-1}\ell\omega_c^2} & \cdots & \frac{\phi_z^{n-1}(\xi_p)}{M_{n-1}\ell\omega_c^2} \end{bmatrix} \tag{34}$$

Case 1: pitch + 2 modes are considered in the model with actuators located at $(-\ell/2, \ell/2, \ell/4)$

$$A = \begin{bmatrix} -3.0 & & \\ & -3200.0 & \\ & & -28800.0 \end{bmatrix} \qquad B_c u_c = \begin{bmatrix} 59.52 & -59.52 & 29.76 \\ 20.0 & 20.0 & -2.0 \\ 20.0 & -20.0 & 9.3 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

The gains are selected for decoupling and critical damping in the decoupled modes (only rate feedback is considered here since the uncontrolled system is already decoupled). The required control forces are given by:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} -0.011 & -2.835 & -5.2757 \\ 0.017 & -2.835 & 3.5171 \\ 0.0602 & 0.0 & -17.5855 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

The time response of pitch, nondimensionalized modal amplitudes, and forces required are plotted in the following figures, (Fig. 2). As the mode number (and frequency) increase the decay time decreases. The pitch takes a relatively long time to decay since its natural frequency is very low and only rate feedback is considered here. The maximum amplitude of the forces are of the order of newtons for this application.

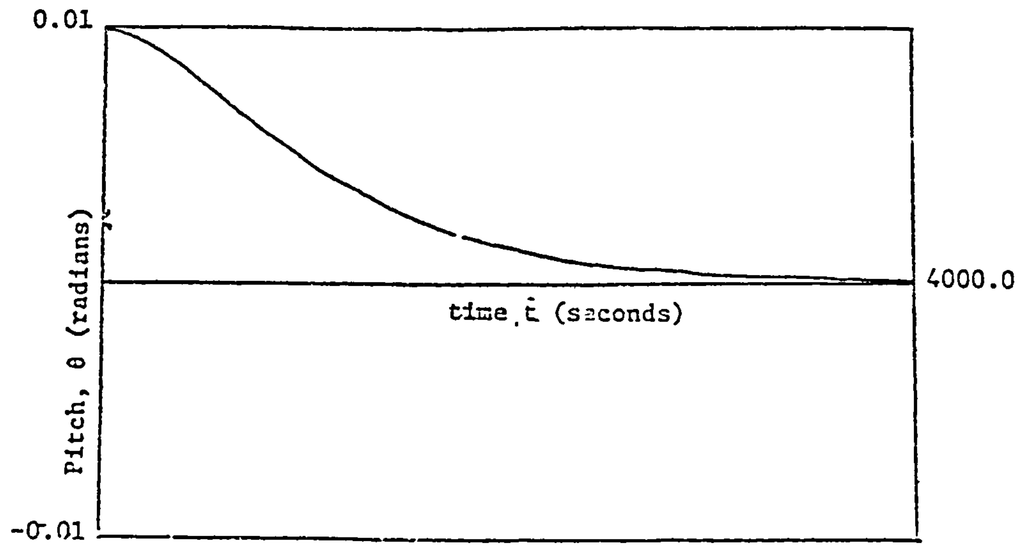


Fig. 2A

Fig. 2. Decoupled Controlled Response - Pitch + Two Modes with Three Actuators $(-\ell/2, \ell/2, \ell/4)$.

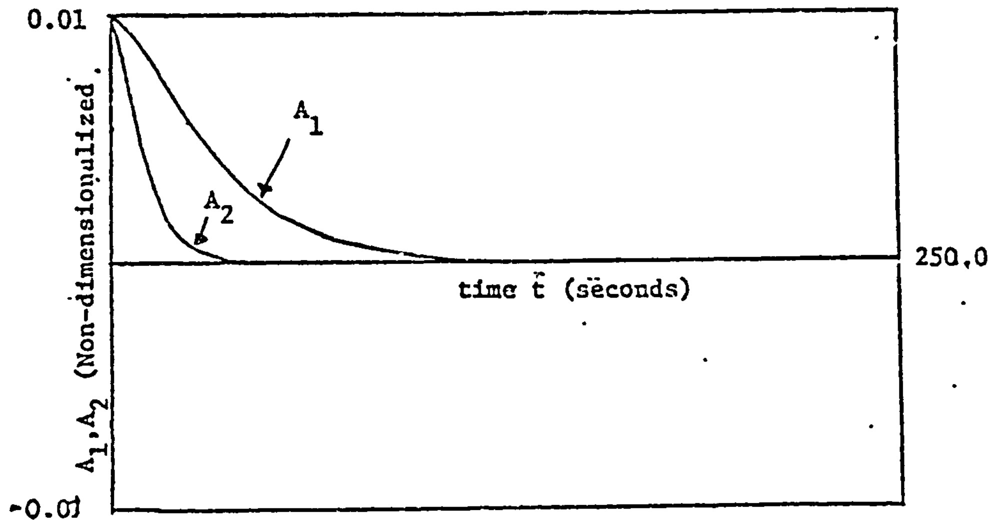


Fig. 2B

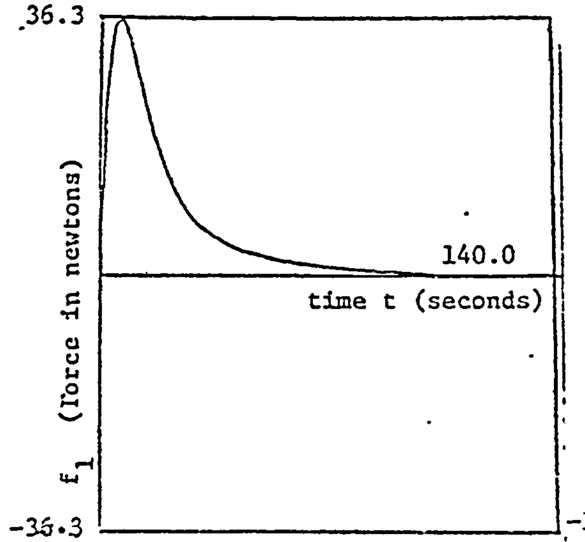


Fig. 2C

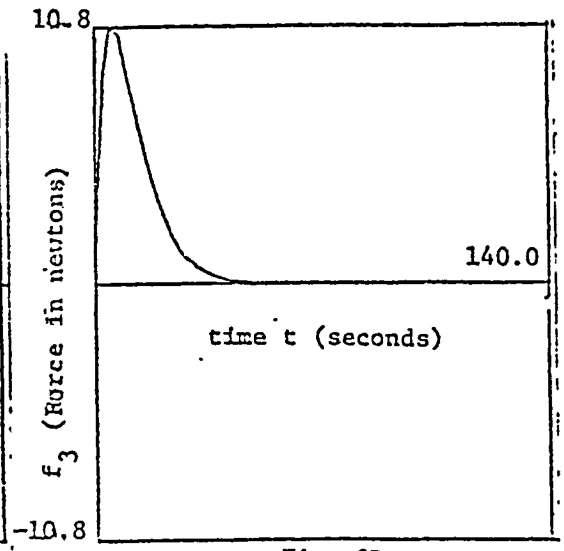


Fig. 2D

Fig. 2. Decoupled Controlled Response - Pitch + Two Modes with Three Actuators ($-\ell/2$, $\ell/2$, $\ell/4$).

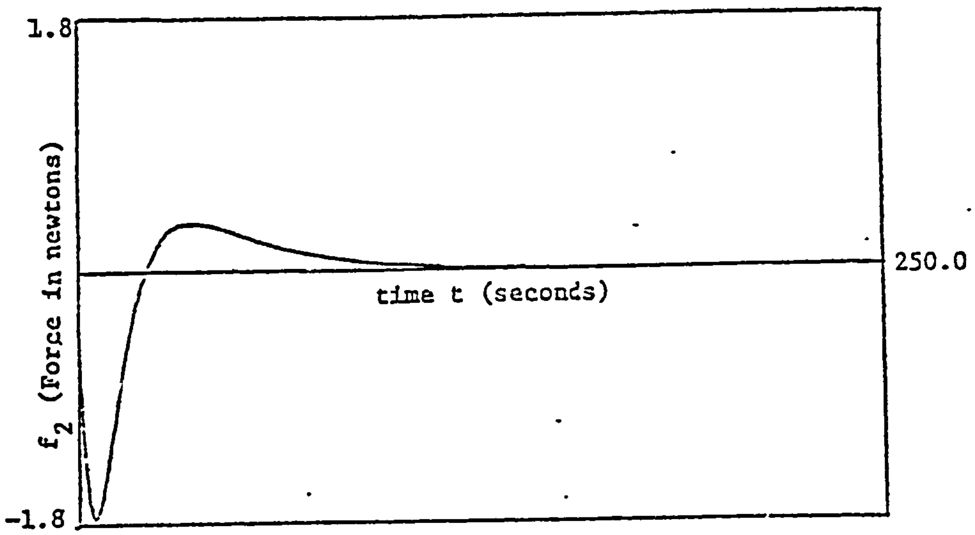


Fig. 2E

Fig. 2. Decoupled Controlled Response - Pitch + Two Modes with Three Actuators ($-\ell/2$, $\ell/2$, $\ell/4$).

Case 2: pitch + 4 modes considered with actuators located at ($-\ell/2, -\ell/4$, 0, $\ell/4$, $\ell/2$)

$$A = \begin{bmatrix} -3.0 & 0 & 0 & 0 & 0 \\ 0 & -3200.0 & 0 & 0 & 0 \\ 0 & 0 & -28800.0 & 0 & 0 \\ 0 & 0 & 0 & -93079.50 & 0 \\ 0 & 0 & 0 & 0 & -255331.40 \end{bmatrix}$$

and

$$B_c u_c = \begin{bmatrix} 59.52 & 29.76 & 0.0 & -29.76 & -59.52 \\ 20.0 & -1.98 & -14.0 & -1.99 & 20.0 \\ 20.0 & -11.698 & 0.0 & 11.685 & -20.0 \\ 20.0 & -12.437 & 14.18 & -12.37 & 20.0 \\ 20.0 & -5.1266 & 0.0 & 5.01 & -20.0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

For case 2, the gains are selected using both decoupling and an application of the linear regulator problem to the independent modal coordinates.

Using decoupling (pitch and the first four modes are critically damped), the control forces are given by

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} -1.199 & -1.423 & -789.03 & -7.578 & 3365.27 \\ -3.329 & 0.0 & -2183.39 & 0.0 & 9392.17 \\ -2.443 & 4.015 & -1620.70 & -21.649 & 6946.89 \\ -3.278 & 0.0 & -2200.36 & 0.0 & 9397.81 \\ -1.669 & -1.423 & -780.55 & -7.578 & 3362.45 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix}$$

The response to an assumed perturbation of 0.01 in all the position coordinates, including pitch, is shown in Fig. 3. The maximum amplitudes of forces are of the order of hundreds of newtons. Fig. 3, also illustrates how the initially deformed beam is straightened out under the influence of the controllers. It is seen that after 36 secs the beam is essentially straight, but continues to exhibit a pitch displacement until about 4000 secs.

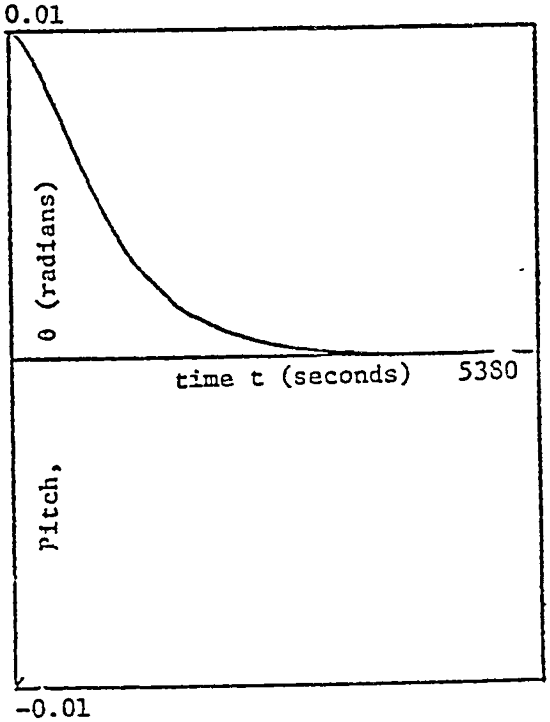


Fig. 3A

Fig. 3. Decoupled Controlled Response - Pitch + Four Modes with Five Actuators $(-\ell/2, -\ell/4, 0, \ell/4, \ell/2)$.

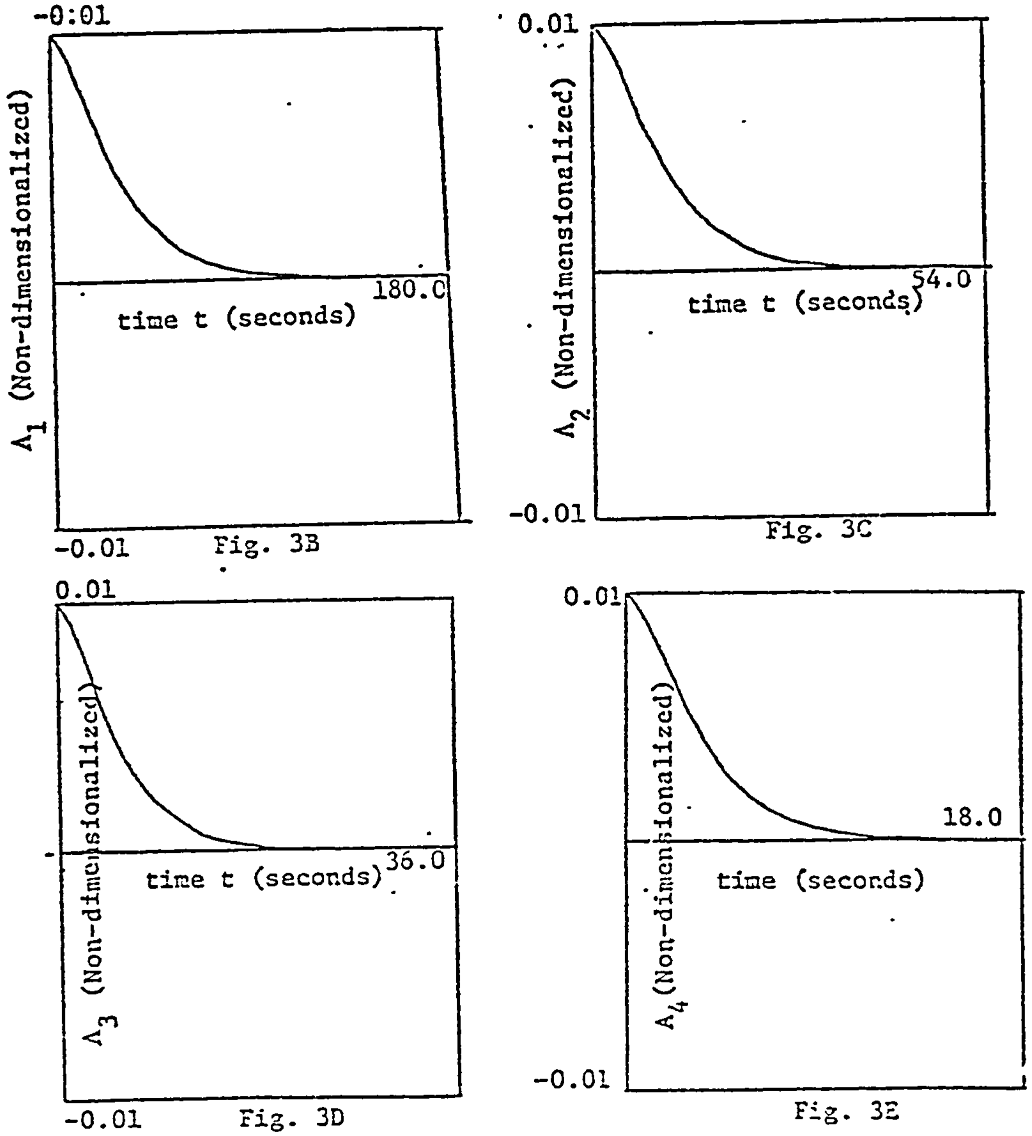


Fig. 3. Decoupled Controlled Response - Pitch + Four Modes with Five Actuators ($-\ell/2$, $-\ell/4$, 0, $\ell/4$, $\ell/2$)

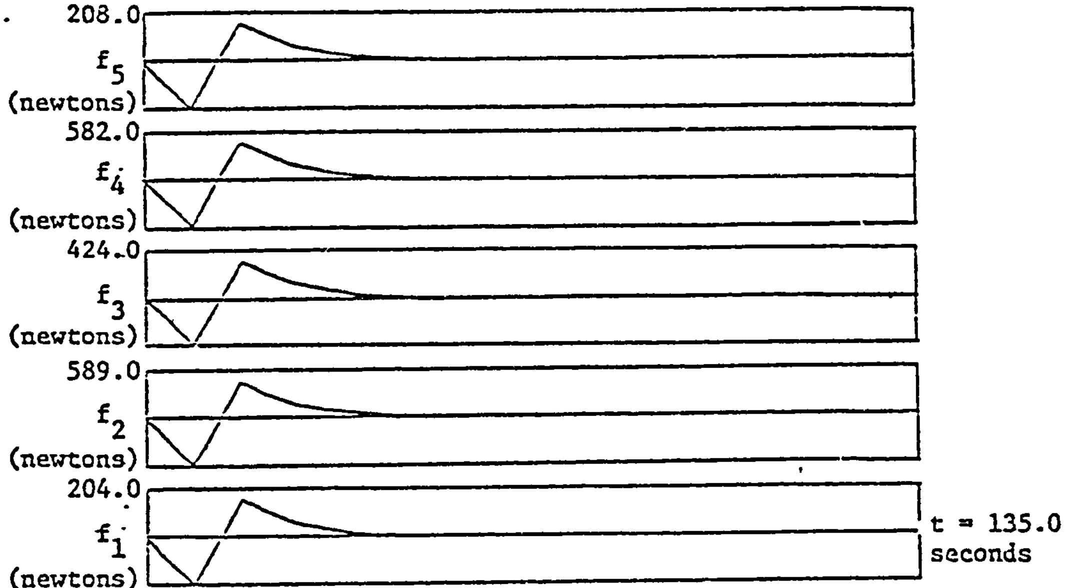


Fig. 3F. Time History of Required Actuator Forces.

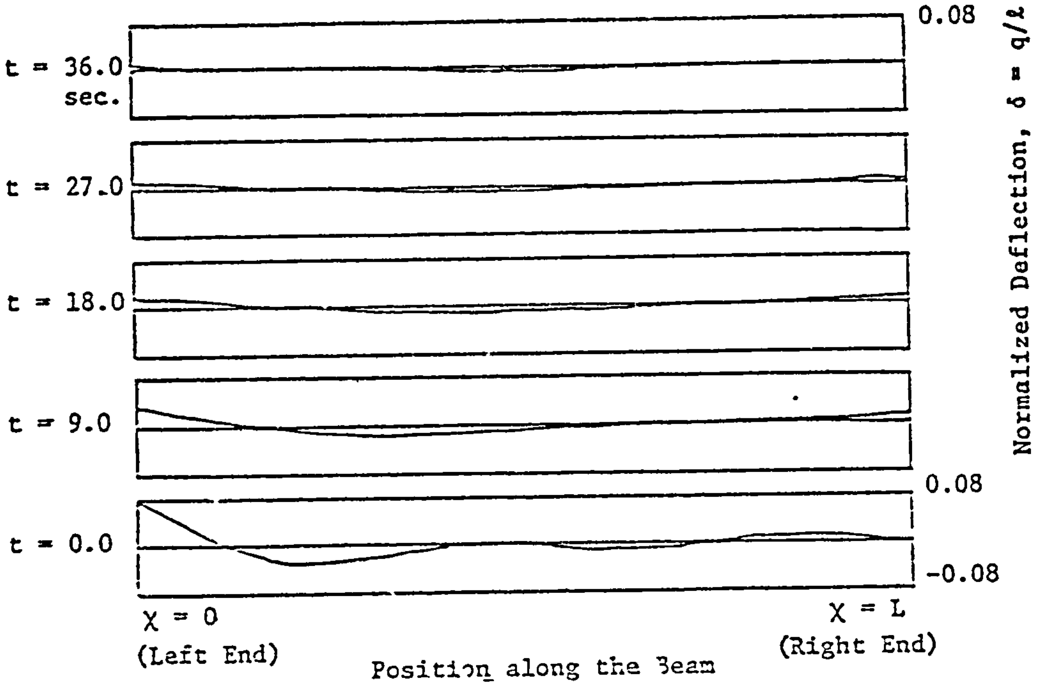


Fig. 3G. Beam Deflection with Time.

Fig. 3. Decoupled Controlled Response - Pitch + Four Modes with Five Actuators ($-\ell/2$, $-\ell/4$, 0, $\ell/4$, $\ell/2$)

The linear regulator problem is applied to the system in the decoupled (modal) coordinates, where the non-dimensionalized rates are penalized by a factor of frequency squared as compared with the non-dimensionalized position coordinates. After solution of the five two dimensional matrix Riccati equations, the actual control forces are given in terms of the following gain matrix.

-0.056	1.258×10^{-6}	0.0	0.0	0.0	-0.6312	-0.712	-334.52	-4.348	1582.6
-0.1559	0.0	0.0	0.0	0.0	-1.7527	0.0	-109.17	0.0	4696.0
-0.1144	3.548×10^{-6}	0.0	0.0	0.0	-1.2859	2.007	-810.35	-6.022	3473.4
-0.1535	0.0	0.0	0.0	0.0	-1.7255	0.0	-1100.18	0.0	4698.9
-0.0545	0.0	0.0	0.0	0.0	-0.6142	-0.712	390.27	-4.348	1631.2

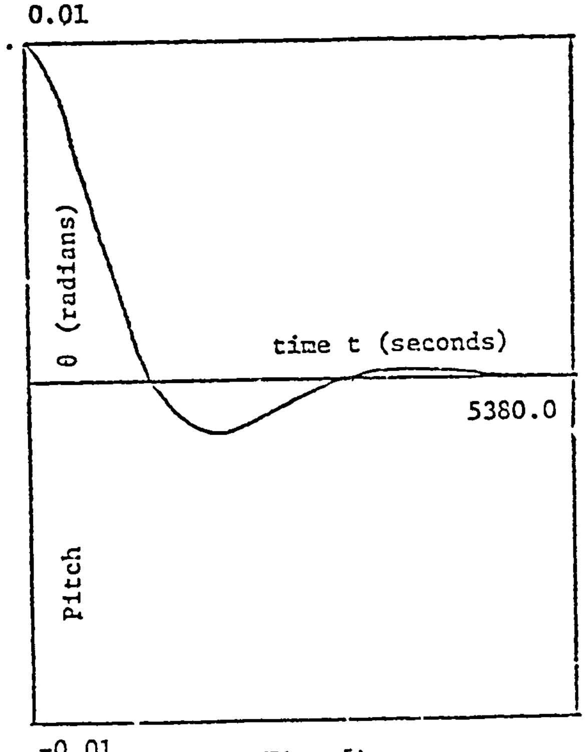


Fig. 4A

Fig. 4. Linear Regulator Application Controlled Response - Pitch + Four Modes with Five Actuators ($-\ell/2$, $-\ell/4$, 0, $\ell/4$, $\ell/2$).

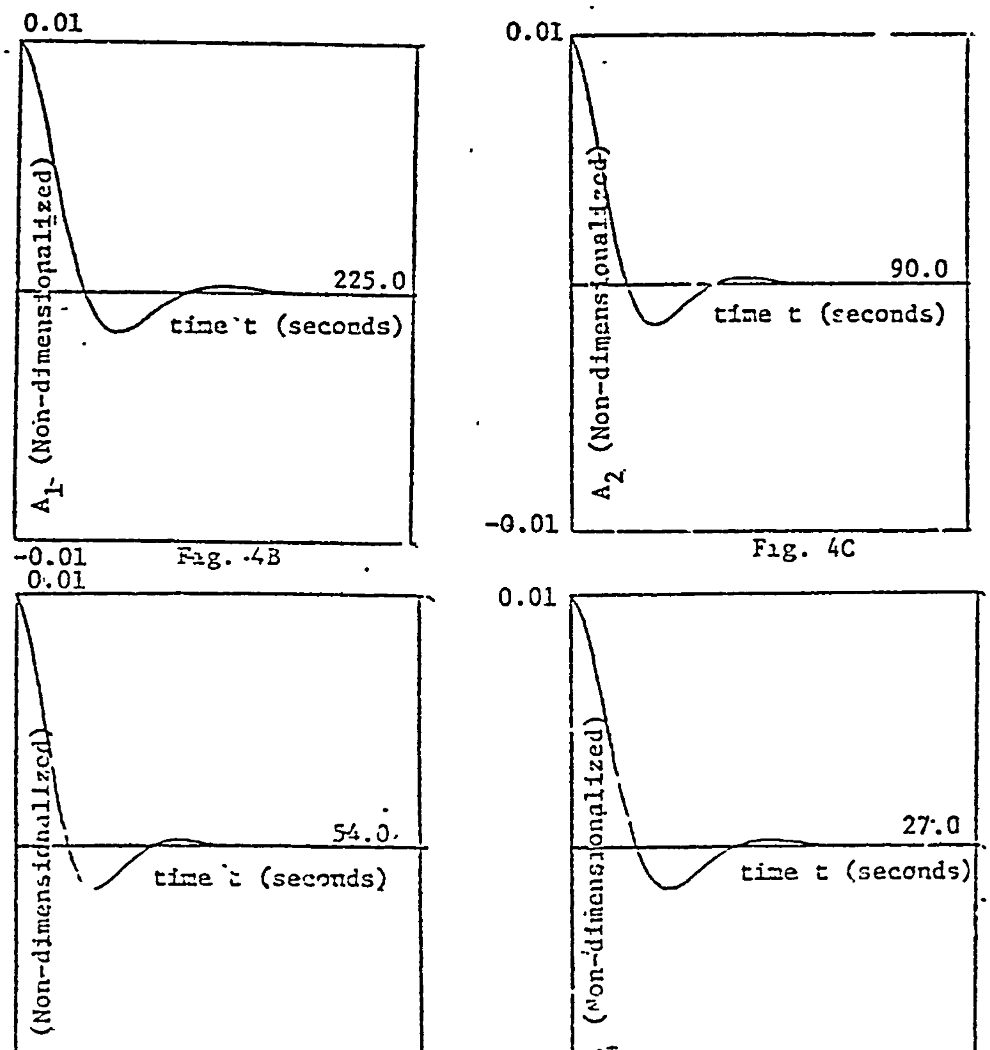


Fig. 4. Linear Regulator Application Controlled Response-Pitch + Four Modes with Five Actuators ($-\ell/2$, $-\ell/4$, 0, $\ell/4$, $\ell/2$)

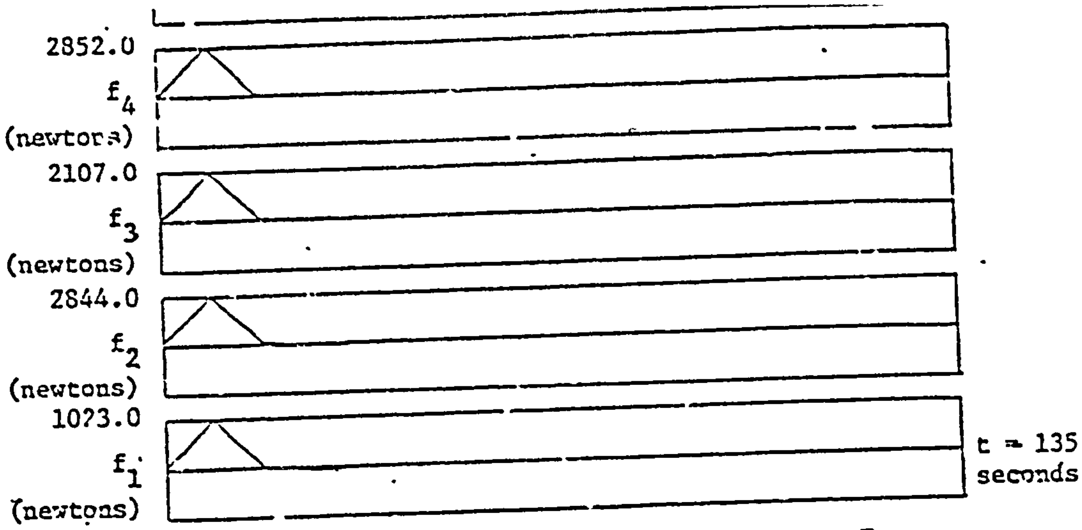


Fig. 4F. Time History of Required Actuator Forces

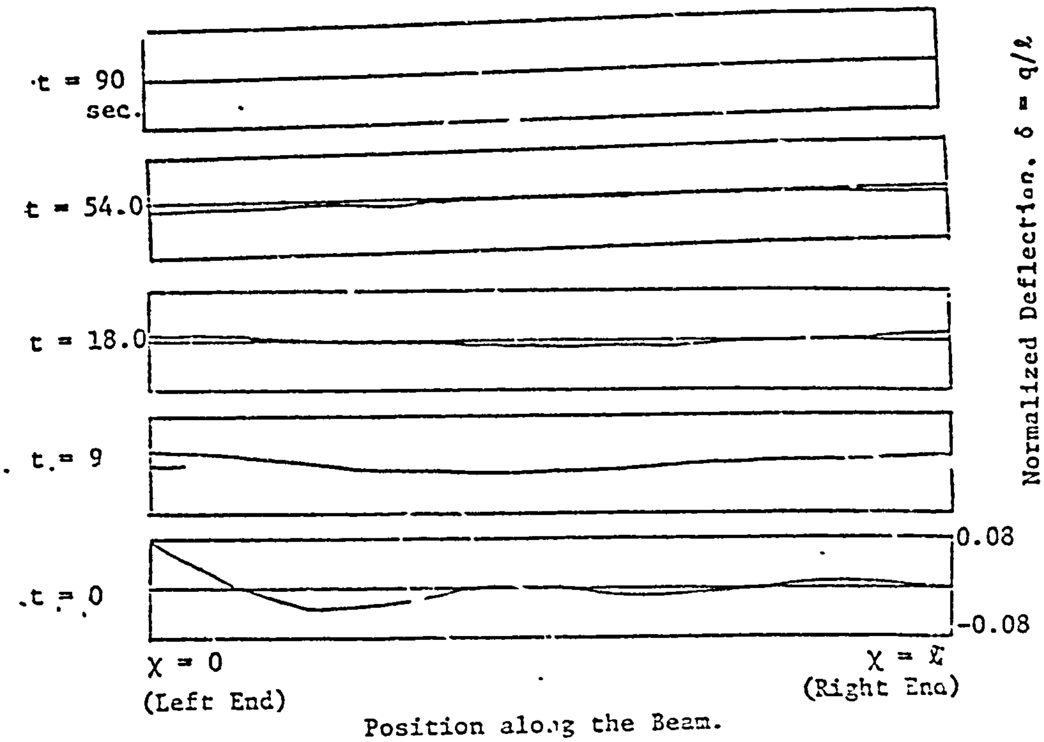


Fig. 4G. Beam Deflection with Time.

Fig. 4. Linear Regulator Application Controlled Response - Pitch + Four Modes with Five Actuators ($-\ell/2, -\ell/4, 0, \ell/4, \ell/2$)

The response to an assumed initial perturbation of 0.01 in all the position coordinates is shown in Fig. 4. As the rates are penalized heavily when compared to the positions, the controlled system frequencies are not changed appreciably and the damping obtained in the individual modes is less than critical. The very small numbers and zeros (which are in reality $< 10^{-7}$ in the second to fifth columns of the position feedback portion of the gain matrix) are due to unit weighting of the positions in the Q_1 matrix. It can be shown that the forces required have a maximum amplitude of the order of thousands of newtons. When compared with Fig. 3, the maximum amplitude of the forces required here are approximately two orders of magnitude larger. This can be explained by the fact that the model used here includes the third and fourth higher frequency modes, and it has been assumed that all four modes and pitch were initially excited equally.

5. Conclusion

A technique for selecting control system gains based on the decoupling of the original linear system equations of motion is presented. This avoids use of modal analysis and does not require system matrices to be symmetric or skew symmetric. When the number of actuators is equal to the number of modes, a unique solution for the control gains depends on the non-singularity of a matrix based on (modal shape functions evaluated at) actuator locations. When the number of actuators is less than the number of modes and the order of the system is high, implementation of decoupling control may be limited by the computational capacity.

The linear regulator problem can be applied to the decoupled modal coordinates only when the number of actuators is equal to the number of modes. Otherwise instead of solving n second order matrix Riccati equations, a 2nx2n matrix Riccati equation has to be solved.

6. References - Chapter III

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IV. Comparison with Results Based on Independent Modal Control

In Refs. 1 and 2 the control of the planar motion of a long flexible beam in orbit was studied based on the concept of distributed modal control. The control forces generated based on this concept provide a means of controlling each system mode independently of all other modes as long as the number of modes in the system mathematical model is the same as the number of actuators. For the case where the number of modes (N) is greater than the number of actuators (P) independent control of P modes is possible, and the response of the remaining (N-P) modes depends on the residual coupling due to the P actuators.

The mathematical model used in Refs. 1 and 2 is based on a three-mass discretization of the free-free beam, with two of the masses assumed to be at the ends of the beam and the third mass at an interior point which later was selected at the center of the undeformed beam. The beam was represented by two hinged cantilever-type members consisting of the end mass connected by (assumed) massless springs which were responsible for the structural restoring forces (Fig. 1). One of the results from Refs. 1 and 2 indicates that the beam, represented by two degrees of freedom, and containing a single actuator at one end, when given an initial perturbation, will not return to the equilibrium position when the control is based on the concept of independent modal control. In the present study (Chapter II), it is clearly shown that a beam with a single actuator at one end and with pitch and two generic modes in the model can be controlled and will return to a desired undeformed alignment with the local vertical.

In an effort to resolve this apparent ambiguity, we will return to the previously developed discretized model and examine both the controllability and stability of the system when $P = 1 < N = 2$.

The linearized equations of motion are [Eq. (3.18) of Ref. 1 or Eq. (38) of Ref. 2]:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} F_{v_1} \\ F_{v_2} \end{bmatrix} \qquad (1)$$

where

$$a = M^* (1 + \overline{m}_0); \quad b = M^*; \quad c = 3\omega_0^2 M^* (2 + \overline{m}_0) + k$$

and

$M^* = m^2/\overline{M}$

$\overline{m}_0 = m_0/m$

m = mass of each end mass

m_0 = mass of interior mass

k = elastic restoring constant ($=3EI/\ell^3$ for assumed cantilever members)

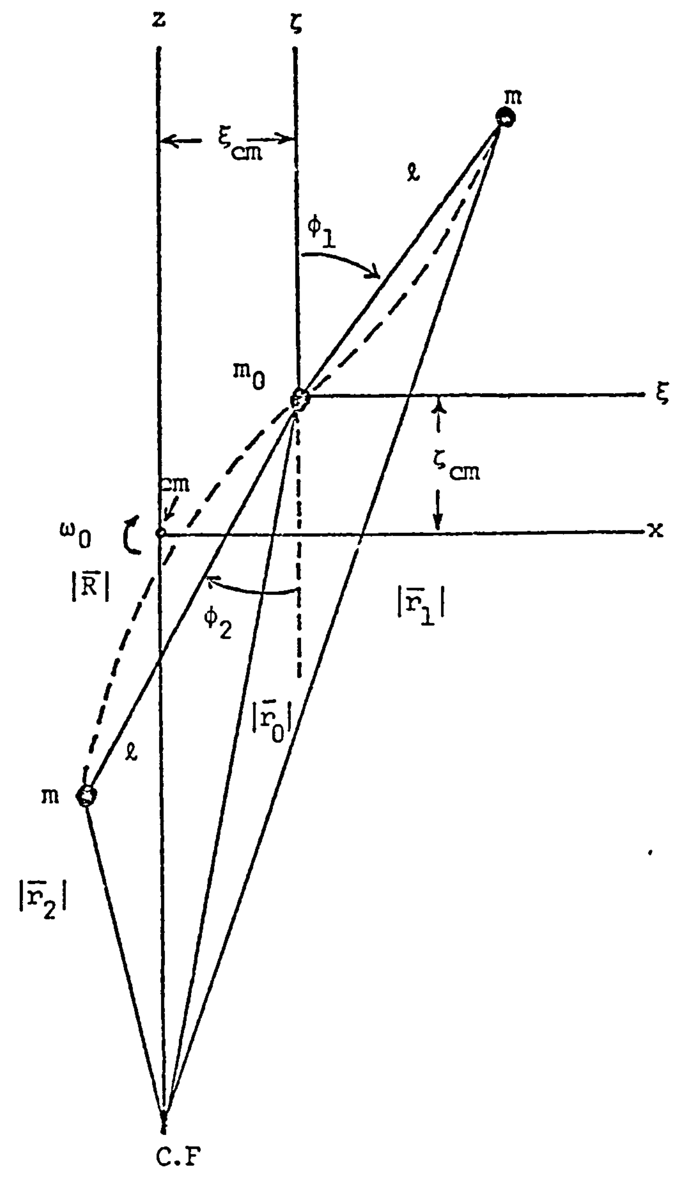

Fig. 1. Three-Mass System Configuration

ℓ = length of each member (one-half the undeformed beam length)

ω_0 = orbital angular velocity

$v_{1,2}$ = linear deflection of each end mass

$F_{v_{1,2}}$ = control forces due to actuators

Eq. (2) can be rewritten as

$$\begin{bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{bmatrix} = -\frac{1}{a^2-b^2}\begin{bmatrix} ac & -bc \\ -bc & ac \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \frac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}\begin{bmatrix} F_{v_1} \\ F_{v_2} \end{bmatrix} \qquad (2)$$

If

$$v_1 = X_1 \qquad \dot{v}_1 = X_3 = \dot{X}_1$$
$$v_2 = X_2 \qquad \dot{v}_2 = X_4 = \dot{X}_2$$

then Eq. (2) can be written in standard state space form as:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-ac}{a^2-b^2} & \frac{bc}{a^2-b^2} & 0 & 0 \\ \frac{bc}{a^2-b^2} & \frac{-ac}{a^2-b^2} & 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \frac{1}{a^2-b^2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ a & -b \\ -b & a \end{bmatrix}\begin{bmatrix} F_{v_1} \\ F_{v_2} \end{bmatrix} \qquad (3)$$

Eq. (3) is now written in the form:

$$\dot{X} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ B \end{bmatrix} f \qquad (4)$$

so that according to the controllability theorem, the system represented by Eq. (4) is controllable if and only if the controllability matrix, C, associated with the pair of reduced state and control matrices, [A,B], is controllable. In this case

$$C = \frac{1}{a^2-b^2}\left[\begin{array}{cc|cc} a & -b & \frac{-c(a^2+b^2)}{a^2-b^2} & \frac{2abc}{a^2+b^2} \\ -b & a & \frac{2bc}{a^2-b^2} & \frac{-c(a^2+b^2)}{a^2-b^2} \end{array}\right] \qquad (5)$$

Since $\det \begin{bmatrix} a & -b \\ -b & a \end{bmatrix} \neq 0$, in general,
C has rank 2 and the system (2) is controllable.

If only one actuator is assumed to be present (i.e. $F_{v_2} = 0$), then Eq. (3) can be written

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-ac}{a^2-b^2} & \frac{bc}{a^2-b^2} & 0 & 0 \\ \frac{bc}{a^2-b^2} & \frac{-ac}{a^2-b^2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \frac{1}{a^2-b^2} \begin{bmatrix} 0 \\ 0 \\ a \\ -b \end{bmatrix} F_{v_1} \tag{6}$$

The reduced order controllability matrix is:

$$C = \left[\begin{array}{c|c} a/(a^2-b^2) & -c(a^2+b^2)/(a^2-b^2)^2 \\ -b/(a^2-b^2) & 2abc/(a^2-b^2)^2 \end{array} \right]$$

and its determinant,

$\det C = bc/(a^2-b^2)^2 \neq 0$, since in general, $a \neq b$.

Thus, the system is controllable with a single actuator present.

The stability of system (6) will now be examined using the particular control law used in Refs. 1 and 2. The linear equations of motion (1) or (2) can be transformed into the modal coordinates, q_1 and q_2, and for the case where only one actuator is present have the form:

$$\ddot{q}_1 + \lambda_1 q_1 = u_1 \tag{7}$$

$$\ddot{q}_2 + \lambda_2 q_2 = [(a+b)/(a-b)]u_1 = g\, u_1 \tag{8}$$

where

$\lambda_1 = c/(a+b) \qquad \lambda_2 = c/(a-b)$

and $v_1 = q_1 + q_2 \qquad v_2 = q_1 - q_2$

Following Refs. 1 and 2, in accordance with the concept of independent modal control, the control in the modal coordinates was selected as

$$u_1 = -f_1 q_1 - f_2 \dot{q}_1 \tag{9}$$

Then Eqs. (7) and (8) can be written as:

$$\ddot{q}_1 + f_2\dot{q}_1 + (f_1+\lambda_1)\, q_1 = 0 \tag{10}$$

$$g(f_2\dot{q}_1+f_1q_1) + \ddot{q}_2 + \lambda_2 q_2 = 0 \tag{11}$$

The characteristic equation for the system described by Eqs. (10) and (11) can be developed with the result:

$$(s^2+\lambda_2)\ (s^2+f_2s+f_1\lambda_1) = 0 \tag{12}$$

An undamped mode remains at frequency $\sqrt{\lambda_2}$ and is not affected by the feedback gains, f_1 and f_2. After the control removes the initial perturbation in q_1, in general, the system will continue to oscillate at the second (uncontrolled) modal frequency. The system is unstable about $q_1 = q_2 = 0$ in the (strong) sense of Routh-Hurwitz where the control law for the single actuator has the form of Eq. (9). An example is illustrated by Fig. 9 of Ref. 2 for this case where $f_1 = f_2 = 1.0$, and demonstrates the basic phenomenon of control spillover.

Instead of selecting the control law based on the independent al control concept, suppose that a coupled rate feedback control law is employed having the form:

$$u_1 = -K_1\dot{q}_1 - K_2\dot{q}_2 \tag{13}$$

Eqs. (7) and (8) can then be expressed as:

$$\ddot{q}_1 + K_1\dot{q}_1 + \lambda_1 q_1 + K_2\dot{q}_2 = 0 \tag{14}$$

$$g\, K_1\dot{q}_1 + \ddot{q}_2 + gK_2\dot{q}_2 + \lambda_2 q_2 = 0 \tag{15}$$

with the associated characteristic equation

$$s^4 + (K_1+gK_2)s^3 + (\lambda_1+\lambda_2)s^2 + (\lambda_2K_1+\lambda_1gK_2)s + \lambda_1\lambda_2 = 0 \tag{16}$$

If the rate feedback gains, K_1 and K_2, are selected to be positive, the system will be stable about $q_1 = q_2 = 0$ according to the Routh - Hurwitz criteria, noting that $g = (a+b)/(a-b) > 0$.

Numerical Example

Following Refs. 1 and 2, the total mass is selected to be 1000 kg, equally divided between the two end masses and central mass, m_0. Thus,

$$\overline{M} = 1000 \text{ kg}$$

$$m = \overline{M}/3 = m_0$$

$$\overline{m}_0 = m_0/m = 1$$

$$M^* = m^2/\overline{M} = 111.11 \text{ kg}$$

$$a = M^*(1+\overline{m}_0) = 222.22 \text{ kg}$$

$$b = M^* = 111.11 \text{ kg}$$

$k = 3EI/\ell^3 = 0.18497$ N/m for a cylindrical wrought aluminum tubular beam, 100m. long ($L = 2\ell$)

$$\omega_0 = 1.115 \times 10^{-3} \text{ rad/sec}$$

$$c = 0.1862522$$

$$\lambda_1 = 5.5876\text{x}10^{-4}$$

$$\lambda_2 = 1.67627\text{x}10^{-3}$$

$$g = 3$$

If the feedback rate gains are selected as

$$K_1 = 0.4728$$

$$K_2 = 0.4000$$

(such as to produce less than critical damping in each of the two normal modes) then the roots of the characteristic equation (16) as solved by a computerized polynomial root - finding routine are:

$$-6.4019\text{x}10^{-4}$$

$$-8.58139\text{x}10^{-4} \pm j\, 2.952816\text{x}10^{-2}$$

$$-1.670404$$

verifying the stabili.y of the system.

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V. General Conclusions and Recommendations

A model is developed for predicting the dynamics of a long, flexible free-free beam in orbit under the influence of control devices which are considered to act at specific points along the beam. Two classes of theorems are applied to the system model to establish necessary and sufficient conditions for controllability depending on whether the system possesses non-repeated or repeated eigenvalues. It is observed that with a proper selection of the location and number of actuators along the beam, a lesser number of actuators than the number of modes in the model can control and stabilize the system.

After establishing the controllability of the system, control gains are selected using the following two criteria: (i) decoupling of the linearized system equations with appropriate state variable feedback; and (ii) applying the linear regulator problem to the modal coordinates, and thus, selecting gains by solving groups of "n" two by two matrix Riccati equations.

The decoupling technique avoids modal analysis and is computationally simple when the number of actuators is equal to the number of modes. However gain selection is possible even when the number of actuators is different from the number of modes. The linear regulator application described in this report depends on an a priori modal analysis and the number of actuators must be equal to the number of modes. When the number of actuators is not equal to the number of modes the general linear regulator problem can still be applied and a 2nx2n matrix Riccati equation has to be solved for a system containing n modes.

The independent modal control concept used earlier for a long flexible beam modelled by three discrete masses is reviewed for stability when the number of actuators is less than the number of modes. For this case, it is seen that even though the system is controllable, it is not stable about the zero state vector (gives rise to a simple example of control spillover). It is observed that a proper control law not based on modal decoupling ensures stability of all the modes.

In the present study control and observation spillover are not directly considered and all states are assumed to be available (noise free). Selection of modes for the mathematical model is done on an arbitrary basis. Only point actuators are modelled.

As an extension of this study, control gain selection using pole allocation can be investigated. Model reduction using energy or shape of the structure as a criteria may be studied. Distributed actuators can also be modelled and their effectiveness can be compared with that of the point actuators. Control and observation spillover can be taken into account in designing state estimators and reduced order controller designs.

Evaluation of Modal Mass (M_n):

The shape function $[\phi_r(\chi)]$ of a free-free beam satisfy

$$\phi_r^{IV} = \lambda_r^4 \phi_r \tag{A-1}$$

(where $\phi_r^{IV} = \frac{d^4\phi_r}{d\chi^4}$ and a similar notation denotes other ordered derivatives)

with boundary conditions

$$\phi_r''(0) = \phi_r'''(0) \tag{A-2}$$

$$\phi_r''(\ell) = \phi_r'''(\ell)$$

From consideration of Eqs. (A-1) and (A-2), the shape function is given by

$$\phi_r(\chi) = \cosh \lambda_r\chi + \cos \lambda_r\chi - \sigma_r (\sinh \lambda_r\chi + \sin \lambda r\chi) \tag{A-3}$$

where λ_r is given by the solution of the transcendental equation

$$\cos \lambda_r\ell \cosh \lambda_r\ell - 1 = 0 \tag{A-4}$$

and

$$\sigma_r = \frac{\cosh \lambda_r\ell - \cos \lambda_r\ell}{\sinh \lambda_r\ell - \sin \lambda_r\ell} \tag{A-5}$$

We have for two different shape functions ϕ_r, ϕ_s corresponding to λ_r and λ_s

$$\phi_r^{IV} = \lambda_r^4 \phi_r \tag{A-6}$$

$$\phi_s^{IV} = \lambda_s^4 \phi_s \tag{A-7}$$

Eqs. (A-6) and (A-7) can be combined as

$$\phi_r\phi_s (\lambda_r^4 - \lambda_s^4) = \phi_s \phi_r^{IV} - \phi_r \phi_s^{IV} \tag{A-8}$$

Integrating (A-8) by parts,

$$\int_0^\ell \phi_r \phi_s \, d\chi = \frac{1}{(\lambda_r^4-\lambda_s^4)} \int_0^\ell [\phi_s \phi_r^{IV} - \phi_r \phi_s^{IV}] d\chi$$

$$= \frac{1}{(\lambda_r^4-\lambda_s^4)} [\phi_s \phi_r''' - \phi_s' \phi_r''' - \phi_r \phi_s'''$$

$$+ \phi_r' \phi_s''']_0^\ell = 0 \text{ for } r \neq s \qquad \text{(A-9)}$$

When $r = s = n$ the above integral is defined as modal mass (M_n) per unit density per length:

$$M_n/\rho = \int_0^\ell \phi_n^2 \, d\chi \qquad \text{(A-10)}$$

$$= \frac{1}{\lambda_n^4} \int_0^\ell \phi_n \phi_n^{IV} \, d\chi$$

$$= \frac{1}{\lambda_n^4} [\phi_n \phi_n'''\Big|_0^\ell - \int_0^\ell \phi_n' \phi_n''' \, d\chi]$$

$$= -\frac{1}{\lambda_n^4} \int_0^\ell \phi_n' \phi_n''' \, d\chi \qquad \text{(A-11)}$$

$$= -\frac{1}{\lambda_n^4} [\phi_n' \phi_n''\Big|_0^\ell - \int_0^\ell \phi_n'' \phi_n'' \, d\chi]$$

$$= \frac{1}{\lambda_n^4} \int_0^\ell \phi_n''^2 \, d\chi \qquad \text{(A-12)}$$

So

$$4M_n = \int_0^\ell [\phi_n^2 + \frac{1}{\lambda_n^4} (\phi_n''^2 - 2\phi_n' \phi_n''')] \, d\chi \qquad \text{(A-13)}$$

After substitution for ϕ_n and its derivatives into Eq. (A-13).

$$M_n = \rho \int_0^\ell d\chi = \rho\ell.$$

References:

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2. Private discussion with Mark J. Balas.

Appendix B

The program described in this appendix solves the equations of the form

$$\dot{X} = AX + BU \qquad \text{(B-1)}$$

where

X is an n dimensional state vector
A is an nxn matrix
B is an nxm matrix
U is a mx1 control vector.

U is obtained using state variable feedback

i.e. $U = GX$ (B-2)

using (B-2), (B-1) can be written as

$$\dot{X} = (A + BG)\, X$$

One can either give (A + BG) as a single matrix to the program or A, B, G as separate matrices. The solution is obtained using the state transition matrix technique. The plotting is incorporated in the progra using the separate computer algorithm REDOK-PLOT (see program listing which follows).

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*                              HOWARD UNIVERSITY -- SCHOOL OF ENGINEERING --
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6/6/79   15:22:17
!JOB [READ IN AT 15:20:53]   REDOK
!FORT/A/B/E/P/S FORT.LS/L
!LISTING

C....+<-------------------- FORTRAN  STATEMENT  -------------------->

      DIMENSION B(10,10),G(10,10),F(10,10)
      COMMON/REDDY1/P(10,10,10),NMC(10),ALPHA(10),BETA(10)
      DIMENSION A(10,10),EIGR(10),EIGI(10),C(11),AINV(10,10),NAME(5)
      DIMENSION AK(10,10),U(10),DUMMY(10,105),Z(105),FORCE(5,105)
      DIMENSION X(10),XI(10)
      CALL INOUT(5,8)
113   WRITE(8,1)
1     FORMAT(5X,'A MATRIX IN THE EQUN. X=AX+BU')
C     N=DIMENSION OF  'A' MATRIX ,NP=COLUMNS OF 'B' MATRIX
C     IF  NPCP=1 EQUATION IS  'X=AX' AND NPCP=0 IF X=(A+BG)X'
C     CIF NPCP=1, B AND G MATRICES NEED NOT BE GIVEN
      READ(5,2) N,NP,NPCP
2     FORMAT(3I2)
      WRITE(8,3) N,NP
3     FORMAT(2X,' DIMENSION OF A=',I2,5X,'COLUMNS OF B=' ,I2)
C     READING A MATRIX ROWWISE
      DO 4 I=1,N
      READ(5,5) (A(I,J),J=1,N)
5     FORMAT(8F10.0)
4     WRITE(8,6) (A(I,J),J=1,N)
6     FORMAT(2X,10(F10.4,2X))
C     READING B MATRIX ROWWISE
      IF(NPCP.GT.0) GOTO 1234
      DO 7 I=1,N
      READ(5,5) (B(I,J),J=1,NP)
7     WRITE(8,6) (B(I,J),J=1,NP)
C     READING G MATIX OF U=GX ,ROWWISE
      DO 8 I=1,NP
      READ(5,5) (G(I,J),J=1,N)
8     WRITE(8,6) (G(I,J),J=1,N)
      DO 9 I=1,N
      DO 9 J=1,N
      SUM=0.0
      DO 100 M=1,NP
100   SUM=SUM+B(I,M)*G(M,J)
9     F(I,J)=SUM
      DO 11 I=1,N
11    WRITE(8,6) (F(I,J),J=1,N)
      DO 12 I=1,N
      DO 12 J=1,N
12    A(I,J)=F(I,J)+A(I,J)
      WRITE(8,138)
13,   FORMAT(5X,'MATRIX A=A+BC WHERE U=CX ')
```

```
      DO 120 I=1,N
 20   WRITE(8,6) (A(I,J),J=1,N)
      BASIC MATRIX PROGRAMME
 234  CONTINUE
      NAME OF PROGRAMME IN 'A' FORMAT
      READ(5,2001) (NAME(I),I=1,5)
'001  FORMAT(5A4)
:     KEEP A BLANK CARD TO GET ALL OPTIONS OF THE PROGRAMME
:     IDET=0  PRINTS DETERMINANT VALUE
:     INV=0 PRINTS INVERSE OF A MATRIX
:     NRM=0 PRINTS RESOLVENT MATRIX
:     ICP=0  PRINTS CHARCTERISTIC POLYNOMIAL
:     IEIG=0  PRINTS EIGEN VALUES
:     ISTM=0  PRINTS STATE TRANSITION MATRIX
:     IF ABOVE PARAMETERD ARE NOT ZERO THEN CORRESPONDING VALUES ARE NOT
      READ(5,201 3) IDET,INV,NRM,ICP,IEIG,ISTM
      WRITE(8,2008)
'008  FORMAT(1H1,5X,' BASIC MATRIX PROGRAM')
      WRITE(8,2009) (NAME(I),I=1,5)
 2009 FORMAT(6X,' PROBLEM INDENTIFICATION:',5X,5A4)
      WRITE(8,2012)
'012  FORMAT(1H0,45(1H*))
      IF(IDET.NE.0) GOTO 14
      D=DET(A,N)
      WRITE(8,2010)
'010  FORMAT(1H0,5X,' DETERNIMANT OF THE  MATRIX')
      WRITE(8,200 3) D
'00 3 FORMAT(1P6E20.7)
 4    IF(IN V.NE.0) GOTO 15
      WRITE(8,2011)
'011  FORMAT(1H0,5X,' THE INVERSE OF MATRIX ')
      CALL SIMEQ(A,C,N,AINV,C,IERR)
      IF(IERR.EQ.0) GOTO 15
      DO  208  I=1,N
:08   WRITE(8,2003) (AINV(I,J),J=1,N)
.5    CALL CHREQ(A,N,C,NRM)
      CALL PROOT(N,C,EIGR,EIGI,+1)
      IF(ICP.NE.0) GOTO  308
      WRITE(8,2012)
      WRITE(8,2005)
:005  FORMAT(1H0,5X,' THE CHARCTERSTIC POLYNOMIAL-IN ASCENDING POWERR
     1 OF S')
      NN=N+1
      WRITE(8,200 3) (C(I),I=1,NN)
308   IF(IEIG.NE.0  GOTO  35
      WRITE(8,2012)
      WRITE(8,2006)
'006  FORMAT(1H0,5X,' EIGEN VALUES OF A MATRIX ')
:007  FORMAT(9X,'REAL PART',8X,'IMAGINARY PART')
'013  FORMAT(6I1)
      WRITE(8,2007)
      DO 16 I=1,N
 6    WRITE(8,200 3) EIGR(I),EIGI(I)
 35   IF(ISTM.NE.0) GOTO 25
      CALL STAST(N,A,EIGR,EIGI,ISTM)
```

```
C....+<--------------------  FORTRAN  STATEMENT  -------------------->
25     CONTINUE
C      N=DIMENSION OF A MATRIX
C      M=COLUMNS OF B MATRIX
C      NC=DIMENSION OF FEEDBACK STATES
C      NWRITE=0 IF PRINTING IS NEEDED
C      NPLOT=0 IF PLOTTING IS NEEDED
C      NSELCT=0 ALWAYS
C      T=INITIAL TIME
C      TMAX=FINAL TIME
C      XMIN=MINIMUM OF SUM OF STATES
       READ(5,1875) N,M,NC,NWRITE,NPLOT,NSELCT
1875   FORMAT(2I2,I4,3I1)
       READ(5,20)(XI(I),I=1,N)
       READ(5,20) T,TMAX,H,XMIN
       IF(NSELCT.GT.0) GOTO 1724
       READ(5,20) (ALPHA(I),BETA(I),I=1,N)
1724   DO 1268 I=1,M
1268   READ(5,20) (AK(I,J),J=1,N)
       WRITE(8,40) (XI(I),I=1,N)
       WRITE(8,40) T,TMAX,H,XMIN
       WRITE(8,40) (ALPHA(I),BETA(I),I=1,N)
       WRITE(8,1270)
       DO 1269 J=1,N
1269   WRITE(8,40) (AK(I,J),I=1,M)
1270   FORMAT(2X,'K MATRIX WRITTEN COLUMN WISE')
       IF(NSELCT.GT.0) GOTO 1725
       READ(5,80) (NMC(I),I=1,N)
       DO 30 I=1,N
       DO 30 J=1,N
30     READ(5,20) (P(I,J,K),K=1,N)
40     FORMAT(2X,1P6E20.7)
80     FORMAT(80I1)
20     FORMAT(8F10.0)
1725   DO 1726  NMK=1,N
       DO 1726  NML=1,N
1726   WRITE(8,40) (P(NMK,NML,NMM),NMM=1,N)
       NMN=1
50     DO 10 J=1,N
       X(J)=0.0
       DO 101 I=1,N
       ABP=ALPHA(I)*T
       IF(ABP.GT.100.0.OR.ABP.LT.-100.0) GOTO 101
       DO 102  K=1,N
       IF(NMC(I).EQ.0) GOTO 13
       X(J)=X(J)+P(I,J,K)*EXP(ALPHA(I)*T)*COS(BETA(I)*T)*XI(K)
       GOTO 102
13     X(J)=X(J)+P(I,J,K)*EXP(ALPHA(I)*T)*SIN(BETA(I)*T)*XI(K)
102    CONTINUE
101    CONTINUE
10     CONTINUE
       DO 1271 I=1,M
       U(I)=0.0
       DO 1271 J=1,N
1271   U(I)=X(J)*AK(I,J)+U(I)
       DO 1371 K=1,M
```

```
C....+<--------------------  FORTRAN  STATEMENT  -------------------->
1371  FORCE(K,NMN)=U(K)
      IF(NWRITE.GT.0)  GOTO 1400
      WRITE(8,103) T,(X(J),J=1,N),(U(I),I=1,M)
103   FORMAT(2X,1P8E16.5)
1400  CONTINUE
      DO 1272 I=1,N
1272  DUMMY(I,NMN)=X(I)
      SUM=0.0
      DO 70 MM=1,N
70    SUM=SUM+X(MM)**2
      T=T+H
      NMN=NMN+1
      IF(SUM.LT.XMIN.OR.T.GT.TMAX) GOTO 60
      GOTO 50
60    CONTINUE
      CALL FOPEN(1,"DP0:REDOK")
      WRITE BINARY(1)N,NC,M
      WRITE BINARY(1) ((DUMMY(I,J),J=1,NC),I=1,N)
      WRITE BINARY(1) ((FORCE(I,J),J=1,NC),I=1,M)
      CALL FCLOSE(1)
      GOTO 11 3
1420  STOP
      END
```

```
C....+<-------------------  FORTRAN  STATEMENT  ------------------->
      SUBROUTINE CHREQ(A,N,C,NRM)
C     THIS SUBROUTINE FINDS THE COEFFICIENTS OF THE CHARACTERISTIC POLY
C     NOMIAL USING THE LEVERRIER ALGORITHM
      COMMON ZED(10,10,10)
      DIMENSION A(10,10),C(11),ATEMP(10,10),PROD(10,10)
1000  FORMAT(1H0,5X,'THE MATRIX COEFFICENTS OF THE NUMERATOR OF THE RESO
     1LVENT MATRIX ')
1001  FORMAT(1H0,5X,'THE MATRIX COEFFICENTS OF S',I1/)
1002  FORMAT(1P6E20.7)
1003  FORMAT(1H0,45(1H*))
C     REPLACING THR DATA CARD DATA ATEMP/100*0.0/
      DO 1315 I=1,10
      DO 1315 J=1,10
1315  ATEMP(I,J)=0.0
      CALL CHM    ,N,C)
      DO 65 I=1
65    ATEMP(I,    0
70    DO 80 I=1,N
      DO 80 J=1,N
80    ZED(N,I,J)=ATEMP(I,J)
      IF(NRM.NE.0) GOTO 71
      WRITE(8,100 3)
      WRITE(8,1000)
      M=N-1
      WRITE(8,1001) M
      DO  35 I=1,N
 35   WRITE(8,1002) (ATEMP(II,,J)Y9J=1,N)
SYNTAX IN ERROR, PUNCTUATION MISSING, OR IDENTIFIER OF WRONG VARIETY------
71    DO 40 I=1,N
      DO 40 J=1,N
40    ATEMP(I,J)=A(I,J)
      DO 10 I=1,N
      NNN=N-I
      IF(I.EQ.1) GOTO 55
      IF(NRM.NE.0) GOTO 60
      WRITE(8,1001) NNN
      DO 45 J=1,N
45    WRITE(8,1002) (ATEMP(J,K),K=1,N)
60    NP=NNN+1
      DO 90  II=1,N
      DO 90 J=1,N
90    ZED(NP,II,J)=ATEMP(II,J)
      DO 15 J=1,N
      DO 15 K=1,N
      PROD(J,K)=0.0
      DO  15 L=1,N
15    PROD(J,K)=PROD(J,K)+(A(J,L)*ATEMP(L,K))
      DO 1 3 J=1,N
      DO 1 3 K=1,N
13    ATEMP(J,K)=PROD(J,K)
55    DO 10 J=1,N
10    ATEMP(J,J)=ATEMP(J,J)+C(N-I+1)
      RETURN
      END
```

```
....+<--------------------  FORTRAN  STATEMENT  -------------------->
      SUBROUTINE CHMRQA(A,N,C)
      DIMENSION J(11),C(11),B(10,10),A(10,10),D( 300)
      NN=N+1
      DO 20 I=1,NN
'0    C(I)=0.0
      C(NN)=1.0
      DO 14 M=1,N
      K=0
      L=1
      J(1)=1
      GOTO 2
      J(L)=J(L)+1
'     IF(L-M)  3,5,50
3     MM=M-1
      DO 4 I=L,MM
      II=I+1
!     J(II)=J(I)+1
;     DO 10 I=1,M
      DO 10 KK=1,M
      NR=J(I)
      NC=J(KK)
.0    B(I,KK)=A(NR,NC)
      K=K+1
      D(K)=DET(B,M)
      DO 6 I=1,M
      L=M-I+1
      IF(J(L)-(N-M+L)) 1,6,50
>     CONTIN UE
      M1=N-M+1
      DO 14 I=1,K
   14 C(M1)=C(M1)+D(I)*(-1.0)**M
      RETURN
   50 WRITE(8,2000)
?000  FORMAT(1H0,5X,' ERROR IN CHREQA ')
      RETURN
      END
```

```
...+<--------------------- FORTRAN  STATEMENT  -------------------->

      FUNCTION DET(A,KC)
      THIS FUNCTION SUBPROGRAM FINDS THE DETERMINANT OF A MATRIX
      USING DIAGONALISATION PROCEDURE
      DIMENSION A(10,10),B(10,10)
      IREV=0
      DO 1 I=1,KC
      DO 1 J=1,KC
      B(I,J)=A(I,J)
      DO 20 I=1,KC
      K=I
      IF(B(K,I)) 10,11,10
      K=K+1
      IF(K-KC) 9,9,51
      IF(I-K) 12,14,51
      DO 1 3 M=1,KC
      TEMP=B(I,M)
      B(I,M)=B(K,M)
      B(K,M)=TEMP
      IREV=IREV+1
      II=I+1
      IF(II.GT.KC) GOTO 20
      DO 17 M=II,KC
      IF(B(M,I)) 19,17,19
      TEMP=B(M,I)/B(I,I)
      DO 16 N=I,KC
      B(M,N)=B(M,N)-B(I,N)*TEMP
      CONTINUE
      CONTINUE
      DET=1.
      DO 2 I=1,KC
      DET=DET*B(I,I)
      DET=(-1)**IREV*DET
      RETURN
1     DET=0.0
      RETURN
      END
```

```
C....+<--------------------  FORTRAN  STATEMENT  -------------------->

      SUBROUTINE PROOT(N,A,U,V,IR)
C     THIS SUBROUTINE USES A MODIFIED BARSTOW METHOD TO FIND THE ROOTS
C     OF A POLYNOMIAL.
      DIMENSION A(20),U(20),V(20),H(21),B(21),C(21)
      IREV=IR
      NC=N+1
      DO 1 I=1,NC
1     H(I)=A(I)
      P=0.0
      Q=0.
      R=0.
3     IF(H(1)) 4,2,4
2     NC=NC-1
      V(NC)=0.
      U(NC)=0.
      DO 1002 I=1,NC
1002  H(I)=H(I+1)
      GOTO 3
4     IF(NC-1) 5,100,5
5     IF(NC-2) 7,6,7
6     R=-H(1)/H(2)
      GOTO 50
7     IF(NC-3) 9,8,9
8     P=H(2)/H(3)
      Q=H(1)/H(3)
      GOTO 70
9     IF(ABS(H(NC-1)/H(NC))-ABS(H(2)/H(1))) 10,19,19
10    IREV=-IREV
      M=NC/2
      DO 11 I=1,M
      NL=NC-I+1
      F=H(NL)
      H(NL)=H(I)
11    H(I)=F
      IF(Q) 13,12,13
12    P=0.
      GOTO 15
13    P=P/Q
      Q=1./Q
15    IF(R) 16,19,16
16    R=1./R
19    E=5.E-10
      B(NC)=H(NC)
      C(NC)=H(NC)
      B(NC+1)=0.
      C(NC+1)=0.
      NP=NC-1
20    DO 49 J=1,1000
      DO 21 I1=1,NP
      I=NC-I1
      B(I)=H(I)+R*B(I+1)
21    C(I)=B(I)+R*C(I+1)
      IF(ABS(B(1)/H(1))-E) 50,50,24
24    IF(C(2)) 23,22,23
```

```
22    R=R+1
      GOTO 30
23    R=R-B(1)/C(2)
30    DO 37 I1=1,NP
      I=NC-I1
      B(I)=H(I)-P*B(I+1)-Q*B(I+2)
37    C(I)=B(I)-P*C(I+1)-Q*C(I+2)
      IF(H(2)) 32,31,32
31    IF(ABS(B(2)/H(1))-E) 33,33,34
32    IF(ABS(B(2)/H(2))-E) 33,33,34
33    IF(ABS(B(1)/H(1))-E) 70,70,34
34    CBAR=C(2)-B(2)
      D=C(3)**2-CBAR*C(4)
      IF(D) 36,35,36
35    P=P-2
      Q=Q*(Q+1.)
      GOTO 49
36    P=P+(B(2)*C(3)-B(1)*C(4))/D
      Q=Q+(-B(2)*CBAR+B(1)*C(3))/D
49    CONTINUE
      E=E*10.
      GOTO 20
50    NC=NC-1
      V(NC)=0.
      IF(IREV) 51,52,52
51    U(NC)=1./R
      GOTO 53
52    U(NC)=R
53    DO 54 I=1,NC
54    H(I)=B(I+1)
      GOTO 4
70    NC=NC-2
      IF(IREV) 71,72,72
71    QP=1./Q
      PP=P/(Q*2.0)
      GOTO 73
72    QP=Q
      PP=P/2.
73    F=(PP)**2-QP
      IF(F) 74,75,75
74    U(NC+1)=-PP
      U(NC)=-PP
      V(NC+1)=SQRT(-F)
      V(NC)=-V(NC+1)
      GOTO 76
75    IF(PP) 81,80,81
80    U(NC+1)=-SQRT( F)
      GOTO 82
81    U(NC+1)=-(PP/ABS(PP))*(ABS(PP)+SQRT(F))
82    CONTINUE
      V(NC+1)=0.
      U(NC)=QP/U(NC+1)
      V(NC)=0.
76    DO 77 I=1,NC
77    H(I)=B(I+2)
```

```
:....+<-------------------  FORTRAN  STATEMENT  ------------------->
      GOTO 4
00    RETURN
      END
```

```
...+<--------------------  FORTRAN  STATEMENT  -------------------->
      SUBROUTINE STMST(N,A,EIGR,EIGT,IKNOW)
      THIS SUBROUTINE DETERMINES THE STATE TRANSITON MATRIX USING
      SYLVESTER'S EXPANSION THEOREM
      COMMON CHI(10,10,10)
      COMMON/REDDY1/P(10,10,10),NMC(10),ALPHA(10),BETA(10)
      DIMENSION A(10,10),EIGR(10),EIGI(10),SPS(10,10)
      COMPLEX CA(10,10),CA1(10,10),CA2(10,10),TCA(10,10),DENOM(10),CEIG(
     110)
      IMN=1
)00   FORMAT(1H0,5X,'THE ELEMENTS OF THE STATE TRASITION MATRIX')
)01   FORMAT(1H0,5X,'THE MATRIX COEFFICENTS OF  EXP(',1PE13.6,'T*COS(',
     11PE13.6,')T')
)02   FORMAT(1P6E20.7)
)03   FORMAT(1H0-5X,'THE MATRIX COEFFICIENT OFEXP(',1PE13.6,')T*SIN(',1P
     1E13.6,')T')
)04   FORMAT(1H0,5X,'THE MATRIX COEFFICIENT OF EXP(',1PE13.6,')T')
)05   FORMAT(1H0,45(1H*))
      IF(IKNOW.NE.0) GOTO 800
      WRITE(8,1005)
)0    DO 10 K=1,N
      CEIG(K)=CMPLX(EIGR(K),EIGI(K))
      DO 10 L=1,N
)     CA(K,L)=CMPLX(A(K,L),0.0)
      I=1
      IF(IKNOW.NE.0) GOTO 700
      WRITE(8,1000)
00    DO 15 K=1,N
5     DENOM(K)=CEIG(I)-CEIG(K)
      DO 500 J=1,N
      IF(J-I) 100,500,200
00    IF(J-1) 110,110,150
00    IF(I-1) 300,300,400
00    IF(J-I-1) 110,110,150
00    IF(J-I-1) 110,150,150
10    DO 5 K=1,N
      DO 5 L=1,N
      CA1(K,L)=CA(K,L)
      DO 20 K=1,N
      CA1(K,K)=CA(K,K)-CEIG(J)
      DO 20 L=1,N
0     CA1(K,L)=CA1(K,L)/DENOM(J)
      GOTO 500
50    DO 40 K=1,N
      DO 40 L=1,N
0     CA2(K,L)=CA(K,L)
      DO 25 K=1,N
      CA2(K,K)=CA(K,K)-CEIG(J)
      DO 25 L=1,N
5     CA2(K,L)=CA2(K,L)/DENOM(J)
      DO 30 K=1,N
      DO 30 L=1,N
      TCA(K,L)=(0.0,0.0)
      DO 30 M=1,N
0     TCA(K,L)=TCA(K,L)+CA1(K,M)*CA2(M,L)
```

```
C....+<--------------------  FORTRAN  STATEMENT  -------------------->
      DO 35 K=1,N
      DO 35 L=1,N
35    CA1(K,L)=TCA(K,L)
300   CONTINUE
      IF(AIMAG(CEIG(I))) 45,50,45
45    IM=I
      I=I+1
      ALPHA(IMN)=EIGR(I)
      BETA(IMN)=EIGI(I)
      NMC(IMN)=1
      IF(IKNOW.NE.0) GOTO 801
      WRITE(8,1001) EIGR(I),EIGI(I)
801   DO 65 K=1,N
      DO 65 L=1.N
65    SPS(K,L)=REAL(CA1(K,L))*2.0
      DO 66 K=1,N
      DO 66 L=1,N
      CHI(IM,K,L)=SPS(K,L)
66    CONTINUE
      DO 1100  J=1,N
      DO 1100 K=1,N
1100  P(IMN,J,K)=SPS(J,K)
      IMN=IMN+1
      IF(IKNOW.NE.0) GOTO 802
      DO 80 K=1,N
80    WRITE(8,1002) (SPS(K,L),L=1,N)
      ALPHA(IMN)=EIGR(I)
      BETA(IMN)=EIGI(I)
      NMC(IMN)=0
      WRITE(8,1003) EIGR(I),EIGI(I)
802   DO 55 K=1,N
      DO 55 L=1,N
55    SPS(K,L)=AIMAG(CA1(K,L))*2.0
      DO 56 K=1,N
      DO 56 L=1,N
      CHI(I,K,L)=SPS(K,L)
56    CONTINUE
      DO 1110 J=1,N
      DO 1110 K=1,N
1110  P(IMN,J,K)=SPS(J,K)
      IMN=IMN+1
      IF(IKNOW.NE.0) GOTO 600
      DO 85 K=1,N
85    WRITE(8,1002) (SPS(K,L),L=1,N)
      GOTO 600
50    CONTINUE
      ALPHA(IMN)=EIGR(I)
      BETA(IMN)=0.0
      NMC(IMN)=1
      IF(IKNOW.NE.0) GOTO 804
      WRITE(8,1004) EIGR(I)
804   DO 60 K=1,N
      DO 60 L=1,N
60    SPS(K,L)=REAL(CA1(K,L))
      DO 61 K=1,N
```

```
C....+<--------------------  FORTRAN  STATEMENT  --------------------->
      DO 61 L=1,N
      CHI(I,K,L)=SPS(K,L)
61    CONTINUE
      DO 1120  J=1,N
      DO 1120  K=1,N
1120  P(IMN,J,K)=SPS(J,K)
      IMN=IMN+1
      IF(IKNOW.NE.0) GOTO 600
      DO 75 K=1,N
75    WRITE(8,1002) (SPS(K,L),L=1,N)
600   IF(I.GE.N) RETURN
      I=I+1
      GOTO 700
      END
```

```
C....+<--------------------  FORTRAN  STATEMENT  -------------------->
      SUBROUTINE SIMEQ(A,XDOT,KC,AINV,X,IERR)
C     THIS SUBROUTINE FINDS THE INVERSE OF THE MATRIX A USING
C     DIAGONALIZATION PROCEDURES
      DIMENSION A(10,10),B(10,10),XDOT(11),X(11),AINV(10,10)
      N=1
      IERR=1
      DO 1 I=1,KC
      DO 1 J=1,KC
      AINV(I,J)=0.
1     B(I,J)=A(I,J)
      DO 2 I=1,KC
      AINV(I,I)=1.
2     X(I)=XDOT(I)
      DO 3 I=1,KC
      COMP=0.
      K=I
6     IF(ABS(B(K,I))-ABS(COMP)) 5,5,4
4     COMP=B(K,I)
      N=K
5     K=K+1
      IF(K-KC) 6,6,7
7     IF(B(N,I)) 8,51,8
8     IF(N-I) 51,12,9
9     DO 10 M=1,KC
      TEMP=B(I,M)
      B(I,M)=B(N,M)
      B(N,M)=TEMP
      TEMP=AINV(I,M)
      AINV(I,M)=AINV(N,M)
10    AINV(N,M)=TEMP
      TEMP=X(I)
      X(I)=X(N)
      X(N)=TEMP
12    X(I)=X(I)/B(I,I)
      TEMP=B(I,I)
      DO 13 M=1,KC
      AINV(I,M)=AINV(I,M)/TEMP
13    B(I,M)=B(I,M)/TEMP
      DO 16 J=1,KC
      IF(J-I) 14,16,14
14    IF(B(J,I)) 15,16,15
15    X(J)=X(J)-B(J,I)*X(I)
      TEMP=B(J,I)
      DO 17 N=1,KC
      AINV(J,N)=AINV(J,N)-TEMP*AINV(I,N)
17    B(J,N)=B(J,N)-TEMP*B(I,N)
16    CONTINUE
3     CONTINUE
      RETURN
51    WRITE(8,52)
52    FORMAT(6X,'THE MATRIX IS SINGULAR ')
      IERR=0
      RETURN
      END
```

```
*
*****************************************************************************

2/28/79   12:12:22
!JOB [READ IN AT 12:9:33]   REDOK-PLOT PROGRAM
!FORT/A/B/E/P/S FORT.LS/L
!LISTING

C....+<--------------------  FORTRAN  STATEMENT  -------------------->

C       IN THE DATA CARD /READ STATEMENT MA DECIDES BEGINING OF X AND XDOT
C       CURVES ,N=0 SKIPS ALL X-CURVES MA DECIDES BEGINING OF FORCE CURVES
C       SKIPS ALL FORCE CURVES. IF N.NE.0  AND M.NE./  DECIDES HOW MANY FO
C       DEFLECTION CURVES THEY NEED
        DIMENSION DUMMY(10,105),FORCE(5,105),Z(105)
        CALL INOUT(2,8)
        CALL FOPEN(1,"DP0:REDOK")
        READ BINARY(1) NX,NC,MX
        READ BINARY(1) ((DUMMY(I,J),J=1,NC),I=1,NX)
        READ BINARY(1) ((FORCE(I,J),J=1,NC),I=1,MX)
        CALL FCLOSE(1)
        READ(2,2675) MA,N,MA,M
 2675   FORMAT(4I2)
        IF(N.EQ.0) GOTO 2676
        DO 1274 I=1,N
        DO 1275 J=1,NC
 1275   Z(J)=DUMMY(I,J)
        ZMAX=Z(1)
        ZMIN=Z(1)
        DO  1276  IJ=1,NC
        IF(ZMAX.LT.Z(IJ+1))  GOTO 1277
        GOTO 1288
 1277   ZMAX=Z(IJ+1)
 1288   IF(ZMIN.LT.Z(IJ+1)) GOTO 1276
        ZMIN=Z(IJ+1)
 1276   CONTINUE
        IF(ABS(ZMAX).GT.ABS(ZMIN))  GOTO 6378
        ZMAX=ABS(ZMIN)
        GOTO 6379
 6378   ZMIN=-ABS(ZMAX)
 6379   CONTINUE
        WRITE(8,1501) ZMAX,ZMIN
        CALL PGRID(0,2)
        CALL PLOX(ZMIN,Z,ZMAX,NC)
        CALL PLOGO(0.0,6.0)
 1274   CONTINUE
 2676   CONTINUE
        IF(M.EQ.0) GOTO 1410
        DO 1376 I=1,M
        DO 1377 J=1,NC
 1377   Z(J)=FORCE(I,J)
        ZMAX=Z(1)
        ZMIN=Z(1)
```

```
.....+<---------------------   FORTRAN  STATEMENT  -------------------
      DO 1378 IJK=1,NC
      IF(ZMAX.LT.Z(IJK+1)) GOTO 1379
      GOTO 1380
 1379 ZMAX=Z(IJK+1)
 1380 IF(ZMIN.LT.Z(IJK+1)) GOTO 1378
         ZMIN=Z(IJK+1)
 1378 CONTINUE
      IF(ABS(ZMAX).GT.ABS(ZMIN))  GOTO 6380
      ZMAX=ABS(ZMIN)
      GOTO 6381
 6380 ZMIN=-ABS(ZMAX)
 6381 CONTINUE
      WRITE(8,1501) ZMAX,ZMIN
 1501 FORMAT(2X,'**ZMAX=',E20.7,  3X,'**ZMIN=',E20.7)
      CALL PGRID(0,2)
      CALL PLOX(ZMIN,Z,ZMAX,NC)
      CALL PLOGO(0.0,6.0)
 1376 CONTINUE
 1410 CONTINUE
      CALL EXIT
      END
```

Appendix C

The program described in this appendix solves the non-linear equations of motion of the beam in orbit incorporating the control laws obtained using the linearized model.

It can plot deflection of the beam at various instants of time with control and the time history of actuator forces required.

Data cards to be given are explained in the program by means of comment cards. A listing of this program follows.

```
 ******************************************************************
                  HOWARD UNIVERSITY -- SCHOOL OF ENGINEERING --
*******************************************************************

6/79   15:33:8
OB [READ IN AT 15:32:16]        NON-LINEAR
ORT/A/B/E/P/S FORT.LS/L
ISTING

...+<-------------------- FORTRAN  STATEMENT  -------------------->

      EXTERNAL FCT,OUTP
      COMMON/REDD/AOMEGA(10),DAMP(10),WC,NMODES,LNEAR,NACT,NSELCT,ISTATE
      COMMON/RED/Z(10,20)
       COMMON/REDDO/FREQ(10),Q(20),NPONTS,AL(20),ISELCT,NPLOT,NWRITE
      COMMON/RAJ/QMAX1,QMIN1
      COMMON/BEAM/PITCH(100),NPK,QSMALL
      COMMON/REJA/AK(5,10),F(5,100),FF(100)
      COMMON/KIM/IMULT
      DIMENSION Y(20),DERY(20),AUX(8,20),A(4),B(4),C(4),PRMT(5)
      DIMENSION SIZE(10)
      CALL INOUT(5,8)
41    CONTINUE
      CALL PLOGO(0.0,6.0)
      QMAX1=0.0
      NPK=0
      QMIN1=0.0
      WC=ORBITAL FRQUENCY,TOL=TOLRERANCE FOR RUNGE-KUTTA SUBROUTINE,
      QSMALL=SMALLEST VALUE OF DEFLECTION ALONG THE BEAM TO STOP PLOTTIN
      SIZE(I)-MAXIMUM VALUES OF STATES FOR RUNGE-KUTTA ROUTINE
      AOMEGA(I) __ FRQUENCY VALUES
      PRMT(1)=INITIAL TIME, PRMT(2)=FINAL TIME,PRMT(3)=INCREMENT
      Z(M,N)=FEEDBACK GAIN MATRIX
      FREQ(I)__FREQUENCY VALUES
      AL(I)___POSITIONS ALONG THE BEAM AT WHICH DEFLECTION IS CALUCULATE
      READ(5,9000) WC,TOL,QSMALL

      NSELCT=0 IF DEFLECTION PLOTS ARE NEEDED AT ONE PLACE OTHERWISE ONE
      NMODES=NUMBER OF MODES CONSIDERED INCLUDING PITCH
      NPONTS=NUMBER OF POINTS ALONG THE BEAM
      ISELCT=TIME INTERVAL SELCTION OF PLOTING
      NPLOT=1 IF PLOTTING IS NEEDED OTHERWISE ZERO
      NWRITE=1 IF WRITING IS NEEDED OTHERWISE ZERO
      NLNEAR=1 IF EQUATIONS ARE NON LINEAR OTHERWISE ZERO
      QSMALL=QUANTITY DEFINING THE SMALLEST MAXIMUM DEFLECTION ONE WANTS
      TO PLOT (DEFINING PRACTICAL ZERO)
      NACT=NUMBER OF ACTUATORS
      NACT=0 THEN DATA CARDS FOR AK(I,J) NEED NOT BE SUPPLIED
      CONTROL FORCES CAN NOT BE PLOTTED IF NACT=0
      ISTATE=NO. OF. STATES CONSIDERED FOR FEEDBACK

      READ(5,9014) NMODES,NPONTS,ISELCT,NPLOT,NWRITE,NLNEAR,NACT,NSELCT
     1,ISTATE,IMULT
      IKK=2*NMODES
```

```
C....+<--------------------   FORTRAN  STATEMENT   -------------------->
9000   FORMAT(3F10.0)
9003   FORMAT(2X,I2,2(E13.6,2X)/)
9004   FORMAT(2X,6(E13.6,2X)/)
9005   FORMAT(50X,'SIZE VALUES')
9006   FORMAT(50X,'FRQUENCIES')
9007   FORMAT(20X,'PARAMETERS:INITIAL TIME,FINAL TIME,INTERVAL')
9008   FORMAT(50X,'INITIAL VALUES')
9011   FORMAT(20X,'Z-BOTTOM PART OF A-MATRIX')
9123   FORMAT(50X,'NUMBER OF BISECTIONS',I2)
9001   FORMAT(8F10.0)
       READ(5,9001) (SIZE(I),I=1,IKK)
       READ(5,9001) (AOMEGA(I),I=1,NMODES)
       READ(5,9001) (PRMT(I),I=1,3)
       READ(5,9001) (Y(I),I=1,IKK)
       DO 108 M=1,NMODES
108    READ(5,9001) (Z(M,N),N=1,IKK)
9014   FORMAT(2I2,I4,6I2,I4)
       READ(5,9001) (FREQ(I),I=1,NMODES)
       READ(5,9001) (AL(I),I=1,NPONTS)
       WRITE(8,9031)
9031   FORMAT(10X,'VALUES OF K MATRIX')
       IF(NACT.EQ.0) GOTO 9037
       DO 9030  I=1,NACT
       READ(5,9001)  (AK(I,J),J=1,ISTATE)
9030   WRITE(8,9004) (AK(I,J),J=1,ISTATE)
9037   CONTINUE
       WRITE(8,9003) NMODES,NC,TOL
       WRITE(8,9005)
       WRITE(8,9004) (SIZE(I),I=1,IKK)
       WRITE(8,9006)
       WRITE(8,9004) (AOMEGA(I),I=1,NMODES)
       WRITE(8,9007)
       WRITE(8,9004) (PRMT(I),I=1,3)
       WRITE(8,9008)
       WRITE(8,9004) (Y(I),I=1,IKK)
       WRITE(8,9011)
       DO 9013 M=1,NMODES
       WRITE(8,9004) (Z(M,N),N=1,IKK)
9013   CONTINUE
       IF(NSELCT.EQ.1) GOTO 9042
       CALL PGRID(0,2)
9042   CONTINUE
       CALL RKSCL(IKK,SIZE,DERY,TOL,PRMT)
       CALL RKGS(PRMT,Y,DERY,IKK,IHLF,FCT,OUTP,AUX)
       PMAX=PITCH(1)
       PMIN=PITCH(1)
       DO 9015 IL=2,NPK
       IF(PMAX.LT.PITCH(IL)) GOTO 9016
       GOTO 9017
 9016  PMAX=PITCH(IL)
9017   IF(PMIN.LT.PITCH(IL)) GOTO 9015
       PMIN=PITCH(IL)
9015   CONTINUE
       IF(ABS(PMAX).GT.ABS(PMIN)) GOTO 9018
       PMAX=ABS(PMIN)
```

```
C....+<--------------------  FORTRAN  STATEMENT  -------------------->
      GOTO 9019
9018  PMIN=-PMAX
9019  CONTINUE
      WRITE(8,9020) PMAX,PMIN
9020  FORMAT(10X,'**PMAX=',E13.6,5X,'**PMIN=',E13.6)
      IF(NSELCT.EQ.1) GOTO 9043
      CALL PLOGO(0.0,4.75)
9043  CALL PLOGO(0.0,1.25)
      CALL PGRID(0,2)
      CALL PLOX(PMIN,PITCH,PMAX,NPK)
      WRITE(8,9123) IHLF
      FFMAX1=0.0
      FFMIN1=0.0
      IF(NACT.EQ.0) GOTO 9036
      DO 9021 I=1,NACT
      DO 9022 J=1,NPK
9022  FF(J)=F(I,J)
      FFMAX=FF(1)
      FFMIN=FF(1)
      DO 9023 II=1,NPK
      IF(FFMAX.LT.FF(II+1)) GOTO 9024
      GOTO 9025
9024  FFMAX=FF(II+1)
9025  IF(FFMIN.LT.FF(II+1)) GOTO 9023
      FFMIN=FF(II+1)
9023  CONTINUE
      IF(ABS(FFMAX).GT.ABS(FFMIN)) GOTO 9026
      FFMAX=ABS(FFMIN)
      GOTO 9027
9026  FFMIN=-ABS(FFMAX)
9027  CONTINUE
      WRITE(8,9035) FFMAX,FFMIN
9035  FORMAT(2X,'**FFMAX=',E13.6,5X,'**FFMIN=',E13.6)
      IF(NSELCT.EQ.1) GOTO 9044
      CALL PLOGO(0.0,4.75)
9044  CALL PLOGO(0.0,1.25)
      CALL PGRID(0,2)
      CALL PLOX(FFMIN,F",FFMAX,NPK)
9021  CONTINUE
      WRITE(8,9033)
9033  FORMAT(2X,'VALUES OF FORCES STARTING AT TMIE=0.0')
      DO 9032 J=1,NPK
9032  WRITE(8,9004) ( F(I,J),I=1,NACT)
9036  CONTINUE
      GOTO 9041
9040  CONTINUE
      STOP
      END
      ISTATE=NO. OF. STATES CONSIDERED FOR FEEDBACK

      READ(5,9014) NMODES,NPONTS,ISELCT,NPLOT,NWRITE,NLNEAR,NACT,NSELCT
     1,ISTATE,IMULT
      IKK=2*NMODES
```

```
:....+<--------------------  FORTRAN  STATEMENT  -------------------->
      SUBROUTINE FCT(X,Y,DERY)
      COMMON/REDD/AOMEGA(10),DAMP(10),WC,NMODES,LNEAR,NACT,NSELCT
      COMMON/RED/Z(10,20)
      COMMON/REJA/AK(5,10),F(5,100),FF(100)
      COMMON/KIM/IMULT
      DIMENSION Y(20),DERY(20),AUX(8,20),A(4),B(4),C(4),PRMT(5)
      IF(NLNEAR.EQ.0) GOTO 14
      Z(1,1)=-1.5*SIN(2.*Y(1))
      DO 10 I=2,NMODES
      Z(I,I)=-((AOMEGA(I))**2/(WC**2)-(3.*((SIN(Y(1)))**2)-1.)-((Y(NMODE
     1S+1)/WC-1)**2))
10    CONTINUE
      GOTO 15
4     Z(1,1)=-3.0*Y(1)
      DO 16 I=2,NMODES
6     Z(I,I)=-(AOMEGA(I))**2/(WC**2)
5     CONTINUE
      DO 11 I=1,NMODES
1     DERY(I)=Y(I+NMODES)
      PITCH EQUATION

      NKC=NMODES+1
      DERY(NKC)=Z(1,1)+Z(1,NKC)*Y(NKC)
      DO 12 I=2,NMODES
      IJ=I+NMODES
      IK=2*NMODES
      DERY(IJ)=0.0
      DO 13 J=1,IK
      DERY(IJ)=DERY(IJ)+Z(I,J)*Y(J)
3     CONTINUE
2     CONTINUE
      RETURN
      END
```

```
....+<--------------------  FORTRAN  STATEMENT  ------------------->
      SUBROUTINE OUTP(X,Y,DERY,IHLF,NDIM,PRMT)
      LOGICAL RKNXT
      COMMON/REDD/AOMEGA(10),DAMP(10),WC,NMODES,LNEAR,NACT,NSELCT,ISTATE
       COMMON/REDDO/FREQ(10),Q(20),NPONTS,AL(20),ISELCT,NPLOT,NWRITE
      COMMON/RAJ/QMAX1,QMIN1
      COMMON/BEAM/PITCH(100),NPK,QSMALL
      COMMON/REJA/AK(5,10),F(5,100),FF(100)
      COMMON/KIM/IMULT
      DIMENSION Y(20),DERY(20),AUX(8,20),A(4),B(4),C(4),PRMT(5)
      IF(.NOT.RKNXT(IHLF)) GOTO 2
      IF(NWRITE.EQ.0) GOTO 13
      WRITE(8,901) X,(Y(I),I=1,NDIM)
01    FORMAT(2X,F10.3,6(E13.6,2X))
3     CONTINUE
      IJK=2*NMODES
      ISS=ISTATE/2
      IF(NPLOT.EQ.0) GOTO 12
      IMULT=IMULT+1
      IF(IMULT.EQ.1) GOTO 11
      IX=IMULT/ISELCT
      IM=IX*ISELCT
      IF(IM.EQ.IMULT) GOTO 11
      GOTO 12
   11 CONTINUE
      NPK=NPK+1
      PITCH(NPK)=Y(1)
      IF(NACT.EQ.0) GOTO 16
      DO 14 IAC=1,NACT
      F(IAC,NPK)=0.0
      DO 15 JK=1,ISS
      F(IAC,NPK)=F(IAC,NPK)+AK(IAC,JK)*Y(JK)
5     CONTINUE
      DO 19 JKN=1,ISS
      F(IAC,NPK)=F(IAC,NPK)+AK(IAC,JKN+ISS)*Y(JKN+NMODES)
9     CONTINUE
4     CONTINUE
5     CONTINUE
      DO 4 J=1,NPONTS
      Q(J)=0.0
      DO 3 I=2,NMODES
      COSHW=(EXP(FREQ(I))+EXP(-FREQ(I)))/2.
      SINHW=(EXP(FREQ(I))-EXP(-FREQ(I)))/2.
      COSHWL=(EXP(FREQ(I)*AL(J))+EXP(-FREQ(I)*AL(J)))/2.
      SINHWL=(EXP(FREQ(I)*AL(J))-EXP(-FREQ(I)*AL(J)))/2.
      DUP1=(COS(FREQ(I))-COSHW)/(SINHW-SIN(FREQ(I)))
      DUP2=SIN(FREQ(I)*AL(J))+SINHWL
      DUP3=COS(FREQ(I)*AL(J))+COSHWL
      DUP=DUP1*DUP2+DUP3
      Q(J)=Q(J)+Y(I)*DUP
      CONTINUE
      CONTINUE
      QMAX=Q(1)
      QMIN=Q(1)
      DO 5 M=1,NPONTS
```

```
:....+<--------------------  FORTRAN  STATEMENT  -------------------->
      IF(QMAX.LT.Q(M+1)) GOTO 6
      GOTO 7
)     QMAX=Q(M+1)
'     IF(QMIN.LT.Q(M+1)) GOTO 5
      QMIN=Q(M+1)
;     CONTINUE
      WRITE(8,18) QMAX,QMIN
 8    FORMAT(20X,'QMAX=',E13.6,'QMIN=',E13.6,'**IN THIS CASE')
      IF(ABS(QMAX).GT.ABS(QMIN)) GOTO 8
      QMAX=ABS(QMIN)
      GOTO 9
;     QMIN=-ABS(QMAX)
'     CONTINUE
      IF(ABS(QMAX).LT.QSMALL) GOTO 12
      IF(QMAX.GT.QMAX1) GOTO 1910
      QMAX=QMAX1
      QMIN=-QMAX1
      GOTO 1911
910   QMAX1=QMAX
911   CONTINUE
      WRITE(8,10) QMAX,QMIN
0     FORMAT(10X,'**QMAX=',E13.6,5X,'**QMIN=',E13.6)
      IF(NSELCT.EQ.0) GOTO 17
      CALL PLOGO(0.0,1.25)
      CALL PSIZE(8.0,1.0)
      CALL PGRID(0,2)
7     CONTINUE
      CALL PLOX(QMIN,Q,QMAX,NPONTS)
2     CONTINUE
      CONTINUE
      RETURN
      END
```

END

DATE

FILMED

AUG 15 1979

End of Document