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16. Abstract

In treating the recorded results from detectors at different locations in space, the analysis of the time delays of signals is crucial to locating the sources of detected radiation. Because the correlation method requires the manipulation of awkward matrices to evaluate its accuracy, the present work outlines a solution based on minimizing the sum of the squares of signal deviations, and presents the algorithms for evaluating the resulting error.

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# ALGORITHMS FOR THE DETERMINATION OF THE TIME DELAYS OF THE SIGNAL WHEN USING UNEQUAL DETECTORS

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In determining the time delays in the reception of signals from detectors located in space, it is often advisable to take account of the circumstance that the detectors differ both in sensitivity and in time resolution. /2*

The present work contains a method for determining delays which was developed for just such a case. The method is based on minimizing the sum of the squares of the discordances of the signals recorded by different detectors. The results of an investigation of this method are presented by the method of maximizing the correlation of the signals. The computed algorithms for the evaluation of the accuracy of the solution are also presented, once obtained by the means shown in the work.

1. There is a whole series of well-known problems whose solutions require the determination of the time delays of signals detected by several receivers. A related problem is the localization in the celestial sphere of the sources of gamma-ray bursts [1]. /3

In reference [2] the method for determining delays and the procedure of evaluating accuracy were developed for the case of equivalent detectors. As in [2], we will deal here with detectors having discrete signals (counters), but, in

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* Numbers in the margin indicate pagination in the foreign text.

contrast to [2], we shall take into account that the detectors may vary in sensitivity (that is, in the probability of recording) and in time resolution.

Let $\{n_j^{(i)}\}$ $i=1,2$; $j=1,2,\ldots M_i$ represent the sequence $M_i$ of readings of the $i^{th}$ detector with a time resolution $\Delta t^{(i)}$. We will interpret this sequence as generated by a random Poisson process with several variable intensities $\lambda^{(i)}(\cdot)$ which are unknown beforehand. It is clear that the results produced by the functions $\lambda^{(i)}(\cdot)$ assumed above will differ for different i in displacement of the argument and in factors; that is, there also exist a certain range of $\tau$ and $\beta$ such that all values fulfill the equality

$$\lambda(u+\tau)=\beta\lambda^{(1)}(u) \quad . \tag{1}$$

Frequently the range of $\tau$ can be limited to a certain region

$$|\tau| \leqslant \tau_0 , \tag{2}$$

where $\tau_0$ is a known value. We shall assume the existence of just such a situation. /4

The problem of finding the time delays therefore reduces to determining the values of $\tau$ satisfying equation (1) by using the measurements $\{n_j^{(1)}\}$ for an unknown $\beta$ and fulfilling equation (2).

Let us examine the possible approaches to the solution to the stated problem. We notice that any method must be based on a comparison of the measurements $\{n_j^{(1)}\}$ and $\{n_j^{(2)}\}$ for different values of the indices, and this reduces to the range of those values $\ell^*$ of the index $\ell$ at which the sequences $\{n_j^{(1)}\}$ and $\{n_{j+\ell}^{(2)}\}$ correlate, in some sense, best with each other.

Here the value of the delay which is sought is obviously determined exactly at the value $\Delta t^{(2)}$ by the expression $\tau^* = \ell^* \Delta t^{(2)}$.

We will not yet take into account the possibility of different time resolutions in the detectors, but assume here that $\Delta t^{(2)} = \Delta t^{(1)} = \Delta t$, $M_1 = M_2 = M$, $k = \left[\frac{\tau_0}{\Delta t}\right]$ (where $[\cdot]$ designates the whole integer part of any number), and $m = M - 2k$.

Since equation (1) contains the unknown factor $\beta$, it is natural to use the method of maximum correlation over the range of $\tau$. Let the index $\ell$ vary from 1 to $2k + 1$. Designate the coefficient of correlation corresponding to a given $\ell$ by $r_\ell$. Then

$$r_\ell = \frac{\sum_{j=1}^{m} n_{j+k}^{(1)} n_{j+k}^{(2)}}{\sqrt{\sum_{j=1}^{m} n_{j+k}^{(1)2}} \sqrt{\sum_{j=1}^{m} n_{j+\ell}^{(2)2}}} \qquad (3)$$

If $\max r_\ell = r_{\hat{\ell}}$, then the solution to the problem using /5
the method of maximum correlation is $\hat{\tau} = \hat{\ell}\Delta t$.

The requirement of evaluating the accuracy of the solution arrived at, however, complicates the use of this method. In fact, since $r_\ell$ is the realization of a random value, in order to evaluate the accuracy of the solution one must know its probability value. Analytically obtaining such characteristics is a very complicated problem, however. A study of the distribution of $r_\ell$ by means of statistical modeling is also obviously unrealistic because of the large (2M) number of parameters to be defined. These circumstances force us to turn to other methods of locating $\tau$.

If we try to find the solution to the problem by the method of least squares, this reduces to the minimization in the given case of the expression

$$z_\ell(b) = \sum_{j=1}^{m}\left(n_{k+j}^{(1)} - b n_{j+\ell}\right)^2 \qquad (4)$$

with respect to $\ell$ and $b$. Let us first prove that the solution to the problem (1), (2) by the method of least squares is equivalent to that of the method of maximum correlation. That is, the same value of the index $\ell$ represents the maximum and minimum of the respective expressions (3) and (4).

At the maximum of the range of $z_\ell(b)$ the condition is

$$\sum_{j=1}^{m} n_{j+k}^{(1)} n_{j+\ell}^{(2)} - b\sum_{j=1}^{m} n_{j+\ell}^{(2)\,2} = 0 .$$

Whence

$$b = \frac{\sum_{j=1}^{m} n_{j+k}^{(1)} n_{j+\ell}^{(2)}}{\sum_{j=1}^{m} n_{j+\ell}^{(2)\,2}}$$

Substituting this expression into (4), we arrive at the problem /6

$$\sum_{j=1}^{m}\left(n_{j+k}^{(1)} - \frac{\sum_{s=1}^{m} n_{s+k}^{(1)} n_{s+\ell}^{(2)}}{\sum_{s=1}^{m} n_{s+\ell}^{(2)\,2}}\, n_{j+\ell}^{(2)}\right)^2 \xrightarrow[\ell]{} \min$$

or

$$\sum_{j=1}^{m} n_{j+k}^{(1)2} - \frac{2\left(\sum_{j=1}^{m} n_{j+k}^{(1)} n_{j+\ell}^{(2)}\right)^2}{\sum_{j=1}^{m} n_{j+\ell}^{(2)2}} + \frac{\left(\sum_{j=1}^{m} n_{j+k}^{(1)} n_{j+\ell}^{(2)}\right)^2}{\sum_{j=1}^{m} n_{j+\ell}^{(2)2}} \longrightarrow \min_{\ell} .$$

The latter is equivalent to the problem

$$\frac{\sum_{j=1}^{m} n_{j+k}^{(1)} n_{j+\ell}^{(2)}}{\sum_{j=1}^{m} n_{j+k}^{(1)} \sum_{j=1}^{m} n_{j+\ell}^{(2)}} \longrightarrow \max_{\ell} .$$

Comparing this expression with (3), we may conclude that the value of the index $\ell$ which represents a minimum in (4) is equal to $\hat{\ell}$.

The use of the method of least squares makes it possible to evaluate the accuracy of the solution in the following manner: replace [illegible] in (4) with the expression

$$\frac{\sum_{j=1}^{m} n_{j+k}^{(1)} n_{j+\hat{\ell}}^{(2)}}{\sum_{j=1}^{m} n_{j+\hat{\ell}}^{(2)2}}$$

and base the subsequent procedure of evaluating accuracy on the assumption that

$$\beta = \frac{\sum_{j=1}^{m} n_{j+k}^{(1)} n_{j+\hat{\ell}}^{(2)}}{\sum_{j=1}^{m} n_{j+\hat{\ell}}^{(2)2}} .$$ /7

We note that it is also reasonable to represent $\beta$ with the expression $\frac{\sum_{j=1}^{M} n_j^{(1)}}{\sum_{j=1}^{M} n_j^{(2)}}$. Hereafter we will assume that one of these substitutions has been performed.

Let us now consider the effect of differences in the time resolutions of the detectors on the procedure of determining the delays and the evaluation of the accuracy. We assume that $\Delta t^{(1)} = r\Delta t^{(2)}$ (5), where r is an integer. We note that the more general case, where r is not an integer, can be reduced to the procedure indicated above. Henceforth $M_1$ shall be designated M and $M_1-2k$ as m. We designate $\sum_{i=(j-1)r+\ell}^{jr+\ell-1} n_i^{(2)}$ as $n_{\ell_j}$ for j = 1,2,..., m.

It is clear that, in order to satisfy relationship (5) and to use the method of least squares, the procedures indicated above must be based on the minimization of, rather than the expression (4), of the expression

$$\sum_{j=1}^{m} \left(n_{j+k}^{(1)} - \beta n_{\ell_j}\right)^2.$$

2. We now pass to the problem of evaluating the accuracy of the solution. For subsequent analysis it is desirable to have the normalized random values which, as in reference [2], we obtain by normalizing the sequences $\{n_j^{(1)}\}$ and $\{n_{\ell_j}\}$ by multiplying by $1/\sqrt{\hat{n}_j^{(1)}}$ and $1/\sqrt{\hat{n}_{\ell_j}}$, respectively. Here $\hat{n}_j^{(1)}$ is the averaged value of $n_j^{(1)}$, and $\hat{n}_{\ell_j}$ is the averaged value of $n_{\ell_j}$.

Using these normalized quantities, we then write

$$\chi(\ell) = \frac{1}{1+\beta} \sum_{j=1}^{m} \left( \frac{n_{j+k}^{(1)}}{\sqrt{\hat{n}_{j+k}^{(1)}}} - \sqrt{\beta}\, \frac{n_{\ell_j}}{\sqrt{\hat{n}_{\ell_j}}} \right) \qquad (6)$$ /8

and adopt its use in studying probability characteristics.

We shall designate random values with capital latin letters. Let the parameter of the Poisson value $N_j^{(2)}$ be equal to $\mu_j^{(2)}$, the parameter of $N_j^{(1)}$ be equal to $\beta\mu_j^{(1)}$, and the parameter of $N_{\ell j}$ be equal to $\mu_{\ell j}$. Consequently

$$\mathcal{E}N_j^{(2)} = \mu_j^{(2)} \; ; \qquad \mathcal{D}N_j^{(2)} = \mu_j^{(2)} \; ;$$
$$\mathcal{E}N_j^{(1)} = \beta\,\mu_j^{(1)} \; ; \qquad \mathcal{D}N_j^{(1)} = \beta\,\mu_j^{(1)} \; ;$$
$$\mathcal{E}N_{\ell_j} = \mu_{\ell_j} \; ; \qquad \mathcal{D}N_{\ell_j} = \mu_{\ell_j}$$

and

$$\mu_{\ell_j} = \sum_{i=(j-1)r+\ell-1}^{jr+\ell-1} \mu_i^{(2)} .$$

Whence

$$\mathcal{E}\frac{N_j^{(1)}}{\sqrt{\hat{n}_j^{(1)}}} = \sqrt{\beta}\sqrt{\mu_j^{(1)}} \; ; \qquad \mathcal{D}\frac{N_j^{(1)}}{\sqrt{\hat{n}_j^{(1)}}} = 1 \; ;$$
$$\mathcal{E}\frac{\sqrt{\beta}\,N_{\ell_j}}{\sqrt{\hat{n}_{\ell_j}}} = \sqrt{\beta}\sqrt{\mu_{\ell_j}} \; ; \qquad \mathcal{D}\frac{\sqrt{\beta}\,N_{\ell_j}}{\sqrt{\hat{n}_{\ell_j}}} = \sqrt{\beta}$$

and the value $\chi(\ell)$, defined by equation (6), is the realization of the random values with a non-central chi-square distribution with m degrees of freedom and the parameter of non-centrality:

$$\nu_\ell = \frac{\beta}{1+\beta}\sum_{j=1}^{m}\left(\sqrt{\mu_j^{(1)}} - \sqrt{\mu_{\ell_j}}\right)^2 .$$

The value of the parameter $\nu_\ell$ can serve as a measure of the accuracy of the determination of $\tau$. Indeed, we will assume the relationship

$$(\ell^* - 1)\Delta t^{(2)} \leqslant \tau \leqslant (\ell^* + 1)\Delta t^{(2)}$$

is fulfilled for $\ell^*$ satisfying a $\min\limits_{\ell} \nu_\ell$ .

Let us now examine the consequences implied by the use of counting devices. For this determination we assume that /9 such a device is equipped with a second detector and that the coefficient of counting is k.

We will assume [3] that

$$\mathcal{E}N_j^{(2)} = \frac{\mu_j}{k}, \quad \mathcal{D}N_j^{(2)} = \frac{\mu_j}{k^2}$$

Then

$$\mathcal{E}N_{\ell_j} = \frac{\mu_{\ell_j}}{k}, \quad \mathcal{D}N_{\ell_j} = \frac{\mu_{\ell_j}}{k}$$

$$\mathcal{E}\frac{\sqrt{\beta k}\, N_{\ell_j}}{\sqrt{\hat{n}_{\ell_j}}} = \sqrt{\beta}\, \mu_{\ell_j}, \quad \mathcal{D}\frac{\sqrt{\beta k}\, N_{\ell_j}}{\sqrt{\hat{n}_{\ell_j}}} = \beta .$$

Clearly, with a counting device the expressions

$$\frac{\sum_{j=1}^{m} n_{j+k}^{(1)} n_{\ell_j}}{\sum_{j=1}^{m} n_{\ell_j}^2} \quad \text{and/or} \quad \frac{\sum_{j=1}^{M_1} n_j^{(1)}}{\sum_{j=1}^{M_1} n_{\ell_j}}$$

give an estimate for $\beta_k$, not for $\beta$. Therefore

$$\chi(\ell) = \frac{1}{1+\frac{\beta}{k}} \sum_{j=1}^{m} \left( \frac{n_j^{(1)}}{\sqrt{\hat{n}_j^{(1)}}} - \sqrt{\beta}\, \frac{n_{\ell_j}}{\sqrt{\hat{n}_{\ell_j}}} \right)^2 . \tag{7}$$

just as equation (6), is the realization of the random values with a non-central chi-square distribution with m degrees of freedom. Thus, the use of counting devices leads merely to the appearance of the factor 1/k in the second element of the denominator of the coefficient in front of the summation sign in the expression for the quantity $\chi(\ell)$. Henceforth, we will use expression (7) for $\chi(\ell)$, assuming k equivalent units in the absence of a counting device.

Let us return to the problem of evaluating the accuracy of the solution. As in reference [2], we will use two methods to evaluate the accuracy of the determination of $\tau$. The procedure of the first method requires an insignificant replacement in the method of the reference mentioned. The interval of confidence in $\nu_\ell$ is determined by the formula /10

$$\varkappa_\ell^{+,-} = \chi(\ell) - M + 2\xi_\alpha^2 \pm 2\xi_\alpha \sqrt{\xi_\alpha^2 + \chi(\ell) - \frac{M}{2}} \tag{8}$$

where $\varkappa_\ell^{+,-}$ are the upper and lower limits of confidence of the non-centrality parameter, $\nu_\ell$, $\zeta_\alpha$ is a quantum on the order of $(2-\alpha)/2$ of the normal distribution, and $1-\alpha$ is the coefficient of confidence.

We now form the system of intervals $\mathcal{T}_\ell = \{\tau\ (\ell-1)\Delta t^{(2)} \leq \tau \leq (\ell+1)\Delta t^{(2)}\}$ and examine $\mathcal{T}^{(1)} = \bigcup_s \mathcal{T}_s$, which is the union of all those $J_s$ for which the condition

$$\varkappa_s^- \leq \varkappa_\ell^+ \tag{9}$$

is fulfilled.

The error, $\Delta\tau$, when this method of evaluating accuracy is used, is determined by the condition of affiliation of $\tau$ to the expression $J^{(1)}$.

The calculation formulas of the second method for evaluating the accuracy of the determination of $\tau$ are substantially more complicated than those in reference [2]. We introduce the designations:

$$n^{(1)} = \mathrm{col}\left(n^{(1)}_{1+k}, n^{(1)}_{2+k}, \ldots, n^{(1)}_{m+k}\right); \quad n^{(2)} = \mathrm{col}\left(n^{(2)}_1, n^{(2)}_2, \ldots, n^{(2)}_M\right),$$

$$\underset{m\times m}{P^{(1)}} = \mathrm{diag}\left\{\frac{1}{\sqrt{\hat{n}^{(1)}_j}}\right\}; \quad \underset{m\times m}{P^{(\ell)}} = \mathrm{diag}\left\{\frac{1}{\sqrt{\hat{n}_{\ell_j}}}\right\}; \; j=1,2,\ldots,m.$$

$$\underset{m\times M}{I_\ell} = \begin{pmatrix} 00\ldots0\overbrace{II\ldots I}^{\ell}00\ldots0\ldots\ldots00\ldots000\ldots0 \\ 00\ldots000\ldots0II\ldots I\ldots\ldots00\ldots000\ldots0 \\ \text{- - - - - - - - - - - - - - - - - - - - - -} \\ 00\ldots000\ldots000\ldots0\ldots\ldots II\ldots I00\ldots0 \end{pmatrix}$$

Using these quantities, the quantity $\chi(\ell)$ can be rewritten in the form

$$\chi(\ell) = \frac{1}{1+\frac{\beta}{k}} \left[\begin{array}{c} P^{(1)} n^{(1)} \\ \hline -\sqrt{\beta}\, P^{(\ell)} I_\ell\, n^{(2)} \end{array}\right]^T \left[\begin{array}{c} P^{(1)} n^{(1)} \\ \hline -\sqrt{\beta}\, P^{(\ell)} I_\ell\, n^{(2)} \end{array}\right] \qquad /11$$

In addition, we introduce the designations:

$$\overset{o}{n}{}^{(1)}_j = \frac{n^{(1)}_j}{\sqrt{\hat{n}^{(1)}_j}}; \quad \overset{o}{n}{}^{(2)}_j = \frac{n^{(2)}_j}{\sqrt{\hat{n}^{(2)}_j}};$$

$$\overset{o}{n}{}^{(1)} = \mathrm{col}\left(\overset{o}{n}{}^{(1)}_{1+k}, \ldots, n^{(1)}_{m+k}\right); \quad \overset{o}{n}{}^{(2)} = \mathrm{col}\left(n^{(2)}_1, n^{(2)}_2, \ldots, n^{(2)}_M\right),$$

$$\underset{m\times m}{Q^{(1)}} = \mathrm{diag}\left\{\sqrt{\hat{n}_j}\right\} \; j=1,2,\ldots,m; \; \underset{M\times M}{Q^{(2)}} = \mathrm{diag}\left\{\sqrt{\hat{n}^{(2)}_j}\right\} \; j=1,2,\ldots,M.$$

Since $P^{(1)}Q^{(1)} = E$, where E is the unit matrix, in the new terminology

$$\chi(\ell) = \frac{1}{1+\frac{\beta}{k}} \left[\begin{array}{c} \overset{o}{n}{}^{(1)} \\ \hline -\sqrt{\beta}\, P^{(\ell)} I_\ell\, Q^{(2)} \overset{o}{n}{}^{(2)} \end{array}\right]^T \left[\begin{array}{c} \overset{o}{n}{}^{(1)} \\ \hline -\sqrt{\beta}\, P^{(\ell)} I_\ell\, Q^{(2)} \overset{o}{n}{}^{(2)} \end{array}\right],$$

that is,

$$\chi(\ell) = \frac{1}{1+\frac{\beta}{k}} \left[\begin{array}{c} \overset{o}{n}{}^{(1)} \\ \hline \overset{o}{n}{}^{(2)} \end{array}\right]^T \left[\begin{array}{c|c} E & -\sqrt{\beta}\, P^{(\ell)} I_\ell\, Q^{(2)} \\ \hline -\sqrt{\beta}\, Q^{(2)T} I_\ell^T P^{(\ell)T} & \beta Q^{(2)T} I_\ell^T P^{(\ell)T} P^{(\ell)} I_\ell Q^{(2)} \end{array}\right] \left[\begin{array}{c} \overset{o}{n}{}^{(1)} \\ \hline \overset{o}{n}{}^{(2)} \end{array}\right]$$

So we obtain for $\chi(\ell)$ the expression

$$\chi(\ell) = \frac{1}{1+\frac{\delta}{k}} \overset{o}{n}^{T} A_{\ell} \overset{o}{n}$$

where

$$A_{\ell} = \left[\begin{array}{c|c} E & -\sqrt{\delta}\, P^{(\ell)} I_{\ell}\, Q^{(2)} \\ \hline -\sqrt{\delta}\, Q^{(2)T} I_{\ell}^{T} P^{(\ell)T} & \delta Q^{(2)T} I_{\ell}^{T} P^{(\ell)T} P^{(\ell)} I_{\ell}\, Q^{(2)} \end{array}\right], \text{ a } \overset{o}{n} = \left[\begin{array}{c} \overset{o}{n}^{(1)} \\ \hline \overset{o}{n}^{(2)} \end{array}\right]$$

is the realization of the random normal vector with components /12
of unit dispersion. Henceforth, elements of the diagonal matrix $P^{(\ell)}$ will be designated as $p_i^{(\ell)} = 1,2,\ldots,m$, and the elements of the diagonal matrix $Q^{(2)}$ will be designated $q_i = 1,2,\ldots,M$.

The evaluation of the accuracy in determining $\tau$ is then performed using the statistic

$$t_{\ell}(n) = \frac{\overset{o}{n}^{T} A_{\ell} \overset{o}{n} - \overset{o}{n}^{T} A_{\hat{\ell}} \overset{o}{n}}{\left(1+\frac{\delta}{k}\right)\sqrt{\overset{o}{n}^{T}\left(A_{\ell}-A_{\hat{\ell}}\right)^{2}\overset{o}{n} - S_{p}\left(A_{\ell}-A_{\hat{\ell}}\right)^{2} + \frac{1}{2}\sum_{i=1}^{m+M}\left(a_{ss}^{(\ell)}-a_{ss}^{(\hat{\ell})}\right)^{2}}} \tag{10}$$

where $a_{ss}^{(\ell)}$ and $a_{ss}^{(\hat{\ell})}$ are diagonal elements of the matrices $A_{\ell}$ and $A_{\hat{\ell}}$, respectively. Let $\zeta_{\alpha}$ be a quantum on the order of $1-\alpha$ of the normal distribution, $\alpha$ is the error of the second order, and $J_s$, as in the first method of evaluating accuracy, is the interval

$$\left\{\tau \ (s-1)\Delta t^{(2)} \leq \tau \leq (s+1)\Delta t^{(2)}\right\}$$

If

$$t_{s}(n) \leq \zeta_{\alpha} \tag{11}$$

then the corresponding interval $J_s$ will be included in the union $\mathcal{J}^{(2)} = \bigcup_s \mathcal{J}_s$ , and the error $\Delta\tau$ is determined by the condition of affiliation of $\tau$ to the expression $J^{(2)}$.

3. Since the dimensions of the matrices $A_\ell$ and $A_{\hat{\ell}}$ in expression (10) - 2M - are very large numbers, operating with these matrices immediately is difficult. In this connection we can obtain simple expressions describing the dependence of the statistic $\tau_\ell$ on the sequential measurements $\{\eta^{(1)}\}$ and $\{\eta^{(2)}\}$.

We designate $\min\{\ell,\hat{\ell}\}$ as $\ell_1$ and $\max\{\ell,\hat{\ell}\}$ as $\ell_2$, so that $\ell_2 \geqq \ell_1$. Clearly,

$$A_\ell - A_{\hat{\ell}} = \left[\begin{array}{c|c} 0 & \sqrt{\beta}\left(P^{(\ell_1)} I_{\ell_1} - P^{(\ell_2)} I_{\ell_2}\right) Q^{(2)} \\ \hline \sqrt{\beta}\, Q^{(2)T}\left(I_{\ell_1}^T P^{(\ell_1)T} - I_{\ell_2}^T P^{(\ell_2)T}\right) & \beta Q^{(2)T}\left(I_{\ell_2}^T P^{(\ell_2)2} I_{\ell_2} - I_{\ell_1}^T P^{(\ell_1)2} I_{\ell_1}\right) Q^{(2)} \end{array}\right] \qquad /13$$

If the non-zero blocs of this matrix are designated $A^{(1)}$, $A^{(1)T}$ and $A^{(2)}$, the matrix $A_\ell - A_{\hat{\ell}}$ is rewritten in the form

$$A_\ell - A_{\hat{\ell}} = \left[\begin{array}{c|c} 0 & A^{(1)} \\ \hline A^{(1)T} & A^{(2)} \end{array}\right]$$

And if we introduce the designation V for $A^{(1)}\overset{\circ}{\eta}{}^{(2)}$ and W for $A^{(1)T}\overset{\circ}{\eta}{}^{(1)} + A^{(2)}\overset{\circ}{\eta}{}^{(2)}$ , then the quadratic form $\overset{\circ}{\eta}{}^T (A_\ell - A_{\hat{\ell}})^2 \overset{\circ}{\eta}$ from expression (10) is obviously written in the form $VV^T + WW^T$. The formulas for the calculation of this quadratic form depend on the relationships among $\ell_1$, $\ell_2$, M, m and r.

Let $\ell_0 = \left[\frac{\ell_2 - \ell_1}{r}\right] r$ where $[\cdot]$ is the integral part of the number, and $S = \ell_2 - \ell_1 - \ell_0$.

1) If the relationship $\ell_1 + rm - 1 \geqq \ell_2$ is fulfilled, then

$$W^T = 6\sum_{k=1}^{m}\Bigg(\Bigg(P_k^{(1)}\sum_{j=r(k-1)+1}^{rk}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1} - P_k^{(2)}\sum_{j=r(k-1)+1}^{rk}\overset{o(2)}{n}_{\ell_1+\ell_0+S+j}\times q_{\ell_1+\ell_0+S+j-1}\Bigg)^2\Bigg) \tag{12}$$

$$WW^T = 6\sum_{k=1}^{\ell_0/r}\Bigg(P_k^{(1)2}\sum_{i=(k-1)r+\ell_1}^{kr-1+\ell_1} q_i^2\Bigg(\overset{o(1)}{n} - \sqrt{6}\,P_k^{(1)}\Bigg(\sum_{j=(k-1)r+1}^{rk}\overset{o(2)}{n}_{\ell_1+j-1}\times q_{\ell_1+j-1}\Bigg)\Bigg)^2\Bigg) +$$

$$+ 6P^{(1)2}_{\ell_0/r+1}\sum_{i=\ell_1+\ell_0}^{\ell_1+\ell_0+S-1} q_i^2\Bigg(\overset{o(1)}{n}_{\ell_0/r+1} - \sqrt{6}\,P^{(1)}_{\ell_0/r+1}\sum_{j=\ell_0+1}^{\ell_0+r}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1}\Bigg)^2 +$$

$$+ 6\sum_{k=1}^{m-\ell_0/r}\ \sum_{i=\ell_1+\ell_0+S+r(k-1)}^{\ell_1+\ell_0+rk-1} q_i^2\Bigg(P^{(1)}_{k+\ell_0/r}\,\overset{o(1)}{n}_{k+\ell_0/r} - P_k^{(2)}\,\overset{o(1)}{n}_k + \sqrt{6}\Bigg(-P^{(1)2}_{\ell_0/r+k}\times$$

$$\times\sum_{j=\ell_0+1+r(k-1)}^{\ell_0+S+r(k-1)}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1} + \Big(P_k^{(2)2} - P^{(1)2}_{\ell_0/r+k}\Big)\sum_{j=\ell_0+S+1+r(k-1)}^{\ell_0+rk}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1} +$$

$$+ P_k^{(2)2}\sum_{j=\ell_0+rk+1}^{\ell_0+rk+S}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1}\Bigg)\Bigg)^2 + 6\sum_{k=1}^{m-\ell_0/r-1}\Bigg(\sum_{i=\ell_0+\ell_1+rk}^{\ell_1+\ell_0+rk+S-1} q_i^2\Bigg(P^{(1)}_{k+\ell_0/r+1}\,\overset{o(1)}{n}_{k+\ell_0/r+1} -$$

$$- P_k^{(2)}\,\overset{o(1)}{n}_k + \sqrt{6}\Bigg(P_k^{(2)}\sum_{j=\ell_0+r(k-1)+S+1}^{\ell_0+rk}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1} + \Big(P_k^{(2)2} - P^{(1)2}_{\ell_0/r+k+1}\Big)\times$$

$$\times\sum_{j=\ell_0+rk+1}^{\ell_0+rk+S}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1} - 6P^{(2)2}_{m-\ell_0/r}\sum_{i=\ell_1+rm}^{\ell_1+rm+S-1} q_i^2\Bigg(-\overset{o(1)}{n}_{m-\ell_0/r} + \sqrt{6}\,P^{(2)}_{m-\ell_0/r}\times$$

$$\times\sum_{j=(m-1)+S+1}^{rm+S}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1}\Bigg)\Bigg)^2\Bigg) + 6\sum_{k=1}^{\ell_0/r}\Bigg(P^{(2)2}_{m-\ell_0/r+k}\sum_{i=(k-1)r+rm+S+\ell_1}^{kr-1+rm+S+\ell_1} q_i^2\times$$

$$\times\Bigg(-\overset{o(1)}{n}_{m-\ell_0/r+k} + \sqrt{6}\,P^{(2)}_{m-\ell_0/r+k}\Bigg(\sum_{j=(k-1)r+rm+S+1}^{kr+rm+S}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1}\Bigg)\Bigg)^2\Bigg)\ . \tag{13}$$

2) If the relationship $\ell_1 + rm - 1 < \ell_2$ is fulfilled, then

$$
VV^{T}=b\sum_{i=1}^{m}\left(\left(P_i^{(1)}\sum_{j=1}^{r}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1}-P_i^{(2)}\sum_{j=1}^{r}\overset{o(2)}{n}_{\ell_1+\ell_0+s+j-1}\times\right.\right.
$$

$$
\left.\left.\times\, q_{\ell_1+\ell_0+s+j-1}\right)^{2}\right) \tag{14}
$$

$$
WW^{T}=b\sum_{k=1}^{m}\left(P_k^{(1)^2}\sum_{i=\ell_1+(k-1)r}^{\ell_1+kr-1}q_i^2\left(\overset{o(1)}{n}_k-\sqrt{b}\,P_k^{(1)}\sum_{j=(k-1)r}^{kr}\overset{o(2)}{n}_{\ell_1+j-1}\times\right.\right.
$$

$$
\left.\left.\times\, q_{\ell_1+j-1}\right)^{2}\right)+b\sum_{k=1}^{m}\left(P_k^{(2)^2}\sum_{i=\ell_1+\ell_0+s+(k-1)r}^{\ell_1+\ell_0+s+kr-1}q_i^2\times\left(\overset{o(1)}{n}_k-\sqrt{b}\times\right.\right.
$$

$$
\left.\left.\times\, P_k^{(2)}\sum_{j=\ell_0+s+1+(k-1)r}^{\ell_0+s+kr}\overset{o(2)}{n}_{\ell_1+j-1}\,q_{\ell_1+j-1}\right)\right) \tag{15}
$$

Let us pass to the calculation of $S_p(A_\ell-A_{\hat{\ell}})^2$ from expression (10). As is well known, any quadratic matric B of dimension n with elements $b_{ij}$ fulfills the condition

$$
S_p B^2=\sum_{i,j=1}^{n}|b_{ij}|^2
$$

It follows from this equation that

$$
S_p\left(A_\ell-A_{\hat{\ell}}\right)^2=\sum_{i,j=1}^{m+M}\delta a_{ij}^2=2\sum_{j=1}^{M}\sum_{i=1}^{m}a_{ij}^{(1)^2}+\sum_{i,j=1}^{M}a_{ij}^{(2)^2}
$$

Here $\delta a_{ij}$ are the elements of the matrix $A_\ell - A_{\hat{\ell}}$, $a_{ij}^{(1)}$ are elements of the matrix $A^{(1)}$, and $a_{ij}^{(2)}$ are elements of the matrix $A^{(2)}$.

We obtain from the last equation the following: /16

$$S_n\left(A_l-A_\ell\right)^2=b^2\sum_{k=1}^{l_0/r}\left(P_k^{(1)^4}\sum_{i,j=l_1+(k-1)r}^{l_1+rk-1} q_i^2 q_j^2+b^2\sum_{k=1}^{m-l_0/r}\times\right.$$

$$\times\sum_{i=l_1+l_0+s+r(k-1)}^{l_1+l_0+rk-1}\left(P_{l_0/r+k}^{(1)^4}\sum_{j=l_1+l_0+r(k-1)}^{l_1+l_0+r(k-1)+s-1} q_i^2 q_j^2+\left(P_k^{(2)^2}-P_{l_0/r+k}^{(1)^2}\right)^2\times\right.$$

$$\left.\times\sum_{j=l_1+l_0+r(k-1)+s}^{l_1+l_0+rk-1} q_i^2 q_j^2+P_k^{(2)^2}\sum_{j=l_1+l_0}^{l_1+l_0+rk+s-1} q_i^2 q_j^2\right)+\sum_{k=1}^{m-l_0/r-1}\times$$

$$\times\sum_{i=l_1+l_0+r(k-1)+s}^{l_1+l_0+rk-1}\left(P_{l_0/r+k+1}^{(1)^4}\sum_{j=l_1+l_0+s+r(k-1)}^{l_1+l_0+rk-1} q_i^2 q_j^2+\left(P_k^{(2)^2}-P_{l_0/r+k+1}^{(1)^2}\right)\times\right.$$

$$\times\sum_{j=l_1+l_0+rk}^{l_1+l_0+rk+s-1} q_i^2 q_j^2+P_k^{(2)^4}\sum_{j=l_1+l_0+rk+s}^{l_1+l_0+r(k+1)-1} q_i^2 q_j^2+2b\sum_{i=1}^{m} P_i^{(1)^2}\times$$

$$\left.\times\sum_{j=l_1+r(i-1)}^{l_1+ri-1} q_j^2+\sum_{i=1}^{m} P_i^{(2)^2}\sum_{j=l_1+l_0+r(i-1)+s}^{l_1+l_0+ri-1+s} q_j^2\right) \tag{16}$$

for $\ell_1 + rm - 1 \geq \ell_2$, and

$$S_n\left(A_l-A_\ell\right)^2=2b\left(\sum_{k=1}^{m} P_k^{(1)^2}\sum_{j=l_1+(k-1)r}^{l_1+rk-1} q_j^2+\sum_{k=1}^{m} P_k^{(2)}\sum_{j=l_1+l_0+s+(k-1)r}^{l_1+l_0+s+kr-1} q_j^2\right)+$$

$$+b^2\left(\sum_{k=1}^{m} P_k^{(1)^4}\sum_{i,j=l_1+(k-1)r}^{l_1+rk-1} q_i^2 q_j^2+\sum_{k=1}^{m} P_k^{(2)^4}\sum_{i,j=l_1+l_0+s+(k-1)r}^{l_1+l_0+s+rk-1} q_i^2\times\right.$$

$$\left.\times\, q_j^2\right) \tag{17}$$

for $\ell_1 + rm - 1 \leq \ell_2$.

Finally, for the final factor in brackets in expression (10) the correct formulas are /17

$$\frac{1}{2}\sum_{i=1}^{m+M}\left(a_{ss}^{(\ell)}-a_{ss}^{(\hat{\ell})}\right)^2=\frac{1}{2}\left(\sum_{j=1}^{\ell_0/r}P_j^{(1)^4}\sum_{i=\ell_1+(j-1)r}^{\ell_1+jr-1} q_i^4 + P_{\ell_0/r+1}^{(1)^4}\times\right.$$
$$\times\sum_{i=\ell_1+\ell_0}^{\ell_1+\ell_0+s-1} q_i^4+\sum_{k=1}^{m-\ell_0/r}\left(P_k^{(2)^2}-P_{\ell_0/r+k}^{(1)^2}\right)\sum_{i=\ell_1+\ell_0+s}^{\ell_1+\ell_0+r-1} q_i^4+\sum_{k=1}^{m-\ell_0/r-1}\left(P_k^{(2)^2}-\right.$$
$$\left.-P^{(1)^2}_{\ell_0/r+k+1}\right)\sum_{i=\ell_1+\ell_0+s+r(k-1)}^{\ell_1+\ell_0+rk-1} q_i^4 + P^{(2)^4}_{m-\ell_0/r}\sum_{i=\ell_1+rm}^{\ell_1+rm+s-1} q_i^4 +$$
$$\left.+\sum_{k=m-\ell_0/r+1}^{m} P_k^{(2)^4}\sum_{i=\ell_0+\ell_1+r(k-1)+s}^{\ell_0+\ell_1+rk-1+s} q_i^4\right) \tag{18}$$

for $\ell_1 + rm \geqq \ell_2$ and

$$\frac{1}{2}\sum_{i=1}^{m+M}\left(a_{ss}^{(\ell)}-a_{ss}^{(\hat{\ell})}\right)^2=$$
$$=\frac{1}{2}\left(\sum_{k=1}^{m} P_k^{(1)^4}\sum_{j=\ell_1+r(k-1)}^{\ell_1+rk-1} q_i^4+\sum_{k=1}^{m} P_k^{(2)^4}\sum_{j=\ell_1+\ell_0+s+r(k-1)}^{\ell_1+\ell_0+s+rk-1} q_i^4\right) \tag{19}$$

for $\ell_1 + rm < \ell_2$.

Thus, the algorithm of the accuracy of the second methods consists of calculating the statistic (10) for all values of $\ell$, using formulas (12), (13), (16) and (18) for $\ell_1 + rm \geqq \ell_2$ and (14), (15), (17) and (19) for $\ell_1 + rm < \ell_2$, while checking that condition (11) is fulfilled.

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