014524-8-T
SEL 135

Phased Models for Evaluating the Performability of Computing Systems

L. T. WU
J. F. MEYER

July 1979

Prepared for

National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23365

G. E. Migneault, Technical Officer

NASA Grant NSG 1306

DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING
SYSTEMS ENGINEERING LABORATORY
THE UNIVERSITY OF MICHIGAN, ANN ARBOR

014525-8-T
SEL 135

PHASED MODELS FOR EVALUATING THE PERFORMABILITY OF COMPUTING SYSTEMS

L. T. Wu
J. F. Meyer

Prepared For
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23365
G. E. Migneault, NASA Technical Officer
NASA Grant NSG 1306

John F. Meyer, Principal Investigator
Department of Electrical and Computer Engineering
Systems Engineering Laboratory
The University of Michigan
Ann Arbor, Michigan 48109

July 1979

014525-8-T
SEL 135

PHASED MODELS FOR EVALUATING
THE PERFORMABILITY OF COMPUTING SYSTEMS

L. T. Wu
J. F. Meyer

Prepared For
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23365
G. E. Migneault, NASA Technical Officer
NASA Grant NSG 1306

John F. Meyer, Principal Investigator
Department of Electrical and Computer Engineering
Systems Engineering Laboratory
The University of Michigan
Ann Arbor, Michigan 48109

July 1979

TABLE OF CONTENTS

PHASED MODELS FOR EVALUATING THE

PERFORMABILITY OF COMPUTING SYSTEMS

by

L. T. Wu and J. F. Meyer

The University of Michigan
Ann Arbor, MI 48109

Abstract - On-line control applications of fault-tolerant computers often require the computers to execute different sets of computational tasks during different phases of a control process. To evaluate the system's "ability to perform", a phase-by-phase modeling technique is introduced. Intraphase processes are allowed to differ from phase to phase, where the probabilities of interphase state transitions (which occur at the time of a phase change) are specified by interphase transition matrices. Based on constraints imposed on the intraphase and interphase transition probabilities, various iterative solution methods are developed for calculating system performability.

I. INTRODUCTION

During recent years, the use of probabilistic models as a basis for evaluating the performance and reliability of computing systems has become increasingly widespread. Typically, the models employed are Markov processes (e.g., [1]) or queueing models (e.g., [2]) which can often be analyzed in terms of imbedded Markov processes. However, it is usually assumed that the underlying process is "time-homogeneous" in the sense that state transition probabilities are invariant with time. (By "state" here we mean the state of the "total system", i.e., the state of the computing system and its environment.) Although this assumption of time-homogeneity is appropriate for certain applications, there are many situations where the user's demands on the computing system can change appreciably during different phases of its utilization. This is particularly true for real-time control applications in which the computing system is required to execute different sets of computational tasks during different phases of a control process.

One approach to dealing with a time-varying environment is to decompose the system's utilization period into consecutive time periods (usually referred to as a decomposition of the system's "mission" into "phases"; see [3]-[5], for example). Demands on the system are then allowed to vary from phase to phase; within a given phase, however, they are assumed to be time-invariant. This permits intraphase behaviors to be evaluated in terms of conventional time-homogeneous models, but raises the interesting question of how the intraphase results are combined. This is the essential question addressed in investigations of "phased mission" reliability evaluation methods (e.g., [3]-[5]) where the problem has been constrained as follows. It is assumed, first, that a "success criterion" (formulated, say, by a "structure function"; see [5] for example) can be established for each phase, where the cirterion is independent of what occurs during other phases. It is required further that successful performance of the system be identified with success during all phases, that is, the system performs successfully if and only if, for each phase, the corresponding success criterion is satisfied throughout that phase.

Although the above constraints are reasonable for certain types of systems, they exclude systems where successful performance involves nontrivial interaction among the phases of the mission. In more exact terms, it has been shown (see [6], Theorem 6) that such "structure-based" formulations of success are possible if and only if the phases are "functionally independent" in a precisely defined manner. What we wish to do, therefore, is to examine the utility of "phased models" in a less restricted context.

In addition to removing the above constraints, we extend the domain of application to include evaluations of computing system "performability" [7], [8]. (Although performability concepts will be introduced as needed in the presentation that follows, some prior familiarity with this background may improve the reader's perspective of what is being accomplished.) Finally, unlike the models used in phased mission reliability evaluation, we permit the state sets of the intraphase models to differ from phase to phase. Thus, the modeling of a particular phase can be tailored not only to the computational demands of that phase but also to the relevant properties of the total system which influence performance during that phase.

II. PHASED MODELS

Generally, in modeling the performability of a computing system C in some specified computational environment E (see [7], [8]), the most detailed view of the total system $S = (C,E)$ is represented by a stochastic process X_S referred to as the base model of S. X_S is defined over a time interval T called the utilization period and each random variable X_t ($t \varepsilon T$) takes values in a state space Q, i.e., with respect to a common "description space" Ω, $X_t : \Omega \to Q$. In general, a state $q \varepsilon Q$ represents a particular status of both the computer C and its environment E. Moreover, the computer coordinate of q may include both the structural state of C and the internal state of the structure.

An instance of the base model's behavior corresponding to an outcome ω is a <u>state trajectory</u> (or "sample path") u_ω where $u_\omega: T \to Q$ with $u_\omega(t) = X_t(\omega)$, for all $t \in T$. The collection

$$U = \{u_\omega | \omega \in \Omega\}$$

is referred to as the <u>trajectory space of S</u>. At a higher, less detailed level of description, the user's view of total system behavior is modeled by a random variable Y_S called the <u>performance of S</u>. Y_S takes values in a set A of <u>accomplishment levels</u> where it is assumed that X_S is refined enough to support Y_S, i.e., there exists a function $\gamma_S: U \to A$, called the <u>capability function of S</u>, such that, for all $\omega \in \Omega$,

$$\gamma_S(u_\omega) = Y_S(\omega).$$

Finally, the <u>performability of S</u> is taken to be the probability distribution function of the performance variable Y_S or, in case A is discrete, the probability mass function p_S where, for all $a \in A$

$$p_S(a) = \text{the probability that S performs at level a.}$$

To generalize the notion of a "phased mission" in the context of performability modeling, let us suppose that the utilization period T is the continuous interval $T = [0,h]$. Suppose further that T is divided into a finite number of consecutive <u>phases</u> (time intervals) $T_1 = [t_0, t_1]$, $T_2 = [t_1, t_2], \ldots, T_m = [t_{m-1}, t_m]$ where $0 = t_0 < t_1 < \ldots < t_m = h$. During phase T_k, we assume that the system can be modeled in the manner described earlier for the entire period T, i.e., by a (continuous time) stochastic process

$$X^k = \{X_t^k | t \in T_k\}$$

where each random variable X_t^k takes values in the <u>phase k state space</u> Q_k ($X_t^k: \Omega \to Q_k$). X^k is referred to as the <u>intraphase process</u> (of phase k) and, combining these processes, we obtain the process

$$X_S = \bigcup_{k=1}^{m} X^k = \bigcup_{k=1}^{m} \{X_t^k | t \varepsilon T_k\}.$$

On examining X_S we see that it is similar to a base model except that, for each time instant t_k, $1 \leq k \leq m-1$, the state of the system is represented by two random variables $X_{t_k}^k$ and $X_{t_k}^{k+1}$ whose values, respectively, are the final state of the k^{th} phase and the initial state of the $k+1^{th}$ phase (see Figure 1). However, if we consider an augmented utilization period

$$\hat{T} = T \cup \{t_k' | k=1,2,\ldots,m-1\}$$

(where t_k' can be interpreted as the initial time of pahse k+1), then X_S can be expressed as

$$X_S = \{X_t | t \varepsilon \hat{T}\}$$

where

$$X_t = \begin{cases} X_0^1 & \text{if } t = 0 \\ X_t^k & \text{if } t \varepsilon (t_{k-1}, t_k] \\ X_{t_k}^{k+1} & \text{if } t = t_k'. \end{cases} \tag{1}$$

If, further, we regard the state space of X_S as the union

$$Q = \bigcup_{k=1}^{m} Q_k$$

then X_S is a base model in the sense defined above. When X_S is so constructed from intraphase processes, we will refer to it as a _phased_ base model.

Let us suppose now that the base model X_S of a performability model is phased and that X_S "supports" the capability function γ_S in the sense that the end-of-pahse "samples" of a state trajectory u uniquely determine the accomplishment level $\gamma_S(u)$. More precisely, a phased model X_S <u>supports</u> γ_S if

$$u(t_k) = u'(t_k), \text{ for all } k, \text{ implies } \gamma_S(u) = \gamma_S(u'). \qquad (2)$$

Given that X_S supports γ_S, the performability model can then be simplified as follows. The simplified base model is taken to be the imbedded discrete-time process

$$\overline{X}_S = \{Z_k | k=0,1,\ldots,m\}$$

where $Z_0 = X_0$ and, for each $k \geq 1$,

$$Z_k = X^k_{t_k} \quad . \qquad (3)$$

Since Z_0 is required only for the initial state distribution, the trajectory space $\overline{U}$ of $\overline{X}_S$ can be effectively regarded as the product space

$$\overline{U} = Q_1 \times Q_2 \times \ldots \times Q_m$$

(where Q_k is the state space of phase k). The corresponding simplification of γ_S is the capability function

$$\overline{\gamma}_S : \overline{U} \rightarrow A$$

where if $u(t_k) = q_k$, for $k=1,2,\ldots,m$, then

$$\overline{\gamma}_S(q_1,q_2,\ldots,q_m) = \gamma_S(u) \quad .$$

Then, by (2), it follows that, for all $a \in A$,

$$p_S(a) = \Pr(\gamma_S^{-1}(a)) = \Pr(\overline{\gamma}_S^{-1}(a)) \qquad (4)$$

and hence the performability model $(\overline{X}_S, \overline{\gamma}_S)$ can be used to evaluate the performability of S. We will thus refer to $(\overline{X}_S, \overline{\gamma}_S)$ as being <u>equivalent</u> to the model (X_S, γ_S).

Although the concept of "support" (2) might appear to be somewhat restrictive, this is not the case when we look at what is typically done in reliability modeling. Given a traditional single-phase reliability model, the system reliability can often be determined by sampling the state of the system at the end of its utilization period. Such single phase equivalents (or multiple phase equivalents in the case of phased models) exist whenever traditional reliability modeling assumptions are made with regard to the intraphase processes.

To illustrate this point, consider a continuous time Markov model of a TMR (Triple-Modular-Redundancy) system with a perfect voter where the simplex system has failure rate λ, i.e., the base model $X_S = \{X_t | t \varepsilon T\}$ is represented by the graph

$$\begin{tikzcd} 1 \arrow[d, "3\lambda"] \\ 2 \arrow[d, "2\lambda"] \\ 3 \end{tikzcd} .$$

If the utilization period is $T = [t_0, t_1]$ and the accomplishment set is $A = \{a_0, a_1\}$ (where a_0 = success and a_1 = failure) then a_0 is accomplished if and only if the system is in state 1 or 2 throughout T. Thus the capability function is

$$\gamma_S(u) = \begin{cases} 1 \text{ if } u(t) \varepsilon \{1,2\}, & \text{for all } t \varepsilon T \\ 0 \text{ otherwise} & \end{cases}$$

and accordingly the performability at a_0 (i.e., the reliability) is

$$p_S(a_0) = Pr(\gamma_S^{-1}(a_0)) = Pr(\{u | \gamma_S(u) = a_0\}).$$

Since state 3 is absorbing, it follows that $Pr(\{u | \gamma_S(u) \neq a_0$ and $u(t_1) \varepsilon \{1,2\}\}) = 0$, and hence

$$\begin{aligned} p_S(a_0) &= Pr(\{u | \gamma_S(u) = a_0\}) + Pr(\{u | \gamma_S(u) \neq a_0 \text{ and } u(t_1) \varepsilon \{1,2\}\}) \\ &= Pr(\{u | u(t_1) \varepsilon \{1,2\}\}) \\ &= Pr(X_{t_1} \varepsilon \{1,2\}) . \end{aligned}$$

Accordingly, the equivalent base model is a pair of random variables

$$\bar{X}_S = \{Z_0, Z_1\}$$

describing the state of the original model at the beginning and the end of the utilization period, i.e., $Z_0 = X_{t_0}$ and $Z_1 = X_{t_1}$. The corresponding equivalent capability function is the structure function $\bar{\gamma}_S : Q \to A$ where

$$\bar{\gamma}_S(q) = \begin{cases} a_0 & \text{if } q \varepsilon \{1,2\}, \\ a_1 & \text{otherwise.} \end{cases}$$

III. PROBABILITY COMPUTATION OF CARTESIAN TRAJECTORY SETS

If $(\bar{X}_S, \bar{\gamma}_S)$ is a performability model of S, then the performability of S is determined by the probabilities of the trajectory sets $\bar{\gamma}_S^{-1}(a) \subseteq U$ (see (4)) where, for each $a \varepsilon A$, $\bar{\gamma}_S^{-1}(a)$ is the set of all state trajectories of $\bar{X}_S$ corresponding to accomplishment level a. Generally, the evaluation of $Pr(\bar{\gamma}_S^{-1}(a))$ requires a detailed knowledge of how intra-phase processes cooperate to accomplish level a, i.e., a thorough understanding of their "functional dependencies" (see [6]). The difficulties are further aggravated by statistical dependencies between phases. However, we have found that when a trajectory set $V \subseteq \bar{U}$ is <u>Cartesian</u> in the sense that, for every phase k, there exists $R_k \subseteq Q_k$ such that $V = R_1 \times R_2 \times \ldots \times R_m$, then Pr(V) can be determined iteratively using matrix multiplications. Moreover, given this ability to compute the probabilities of Cartesian sets, the probabilities of more general sets can be determined by decomposing them into disjoint unions of Cartesian components. (The latter is taken care of automatically by algorithms which determine $\bar{\gamma}_S^{-1}(a)$; see [8]). Hence, the problem reduces to that of computing the probabilities of Cartesian trajectory sets.

If X_S is the phased model from which $\overline{X}_S$ is derived, for each phase k, let $Y_k = X_{t'_{k-1}}$ be the initial state of the k^{th} intraphase process and let n_k be the number of states in Q_k. Then, for a Cartesian trajectory set $V = R_1 \times R_2 \times \ldots \times R_m$, the <u>conditional intraphase transition matrix</u> of the k^{th} phase is the $n_k \times n_k$ matrix $P_{V,k}$ where, for all $i,j \in Q_k$,

$$P_{V,k}(i,j) = Pr(Z_k=j \mid Y_k=i, Z_{k-1}\varepsilon R_{k-1}, \ldots, Z_1 \varepsilon R_1).$$

In other words, $P_{V,k}(i,j)$ is the initial-to-final state transition probability of the k^{th} intraphase process, conditioned by the first k-1 components of V. Similarly, for all but the first phase, the <u>conditional interphase transition matrix</u> is the $n_{k-1} \times n_k$ matrix $H_{V,k}$ where, for all $i\varepsilon Q_{k-1}$ and $j\varepsilon Q_k$,

$$H_{V,k}(i,j) = Pr(Y_k=j \mid Z_{k-1}=i, Z_{k-2}\varepsilon R_{k-2}, \ldots, Z_1 \varepsilon R_1).$$

In other words, $H_{V,k}(i,j)$ is the probability that the k^{th} phase initiates in state j given that the final state of the $k-1^{th}$ phase is i, conditioned by the first k-2 components of V. For consistency, we let $H_{V,1}$ be the $n_1 \times n_1$ identity matrix. Finally for each phase, the <u>characteristic matrix</u> of the k^{th} phase is the $n_k \times n_k$ matrix $G_{V,k}$ where

$$G_{V,k}(i,j) = \begin{cases} 1 \text{ if } i=j \text{ and } i\varepsilon R_k \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

In terms of the above matrices, we are able to establish the following matrix formula for computing the probability of a Cartesian trajectory set V. Given $\overline{X}_S$, let I(0) denote its initial state distribution, i.e., $I(0) = [p_1, p_2, \ldots, p_n]$ where $p_i = Pr(Z_0=i) = Pr(X_0^1=i)$; and let F_k denote the $n_k \times 1$ column matrix with "1" in each entry. Then by induction on k, it can be established that

Theorem 1:

If $V = R_1 \times \ldots \times R_k \times Q_{k+1} \times \ldots \times Q_m$ then

$$Pr(V) = I(0) \cdot [\prod_{\ell=1}^{k} H_{V,\ell} \cdot P_{V,\ell} \cdot G_{V,\ell}] \cdot F_k .$$

Proof:

For k=1,

$$I(0) \cdot H_{V,1} \cdot P_{V,1} = I(0) \cdot P_{V,1} = [a_1, \ldots, a_j, \ldots, a_{n_1}]$$

where

$$a_j = \sum_{i=1}^{n_1} Pr(Z_0=i) \cdot Pr(Z_1=j \mid Z_0=i)$$

$$= \sum_{i=1}^{n_1} Pr(Z_0=i, Z_1=j) = Pr(Z_1=j).$$

Multiplied by $G_{V,1}$ and F_1,

$$I(0) \cdot H_{V,1} \cdot P_{V,1} \cdot G_{V,1} \cdot F_1$$

$$= \sum_{j \varepsilon R_1} Pr(Z_1=j) = Pr(Z_1 \varepsilon R_1)$$

$$= Pr(Z_1 \varepsilon R_1, Z_1 \varepsilon Q_2, \ldots, Z_m \varepsilon Q_m)$$

$$= Pr(V).$$

Suppose that the formula holds for $k \leq m$, then

$$I(0) \cdot [\prod_{\ell=1}^{k+1} H_{V,\ell} \cdot P_{V,\ell} \cdot G_{V,\ell}] \cdot F_{k+1}$$

$$= I(0) \cdot [\prod_{\ell=1}^{k} H_{V,\ell} \cdot P_{V,\ell} \cdot G_{V,\ell}] \cdot H_{V,k+1} \cdot P_{V,k+1} \cdot G_{V,k+1} \cdot F_{k+1}$$

$$= A_1 \cdot H_{V,k+1} \cdot P_{V,k+1} \cdot G_{V,k+1} \cdot F_{k+1}$$

where

$$A_1 = [b_1,\ldots,b_j,\ldots;b_{n_1}]$$

and

$$b_j = \begin{cases} Pr(Z_k=j, Z_{k-1}\varepsilon R_{k-1},\ldots,Z_1\varepsilon R_1) \text{ if } j\varepsilon R_k \\ \\ 0 \text{ otherwise,} \end{cases}$$

by applying the equation for k.

When we iteratively compute the matrix product, beginning from the left, the first two terms become

$$A_2 = A_1 \cdot H_{V,k+1} = [c_1,\ldots,c_j,\ldots,c_{n_{k+1}}]$$

where

$$c_j = \sum_{i=1}^{n_k} b_i \cdot H_{V,k+1}(i,j)$$

$$= \sum_{i\varepsilon R_k} Pr(Z_k=i, Z_{k-1}\varepsilon R_{k-1},\ldots,Z_1\varepsilon R_1)\cdot$$

$$Pr(Y_{k+1}=j \mid Z_k=i, Z_{k-1}\varepsilon R_{k-1},\ldots,Z_1\varepsilon R_1)$$

$$= \sum_{i\varepsilon R_k} Pr(Y_{k+1}=j, Z_k=i, Z_{k-1}\varepsilon R_{k-1},\ldots,Z_1\varepsilon R_1)$$

$$= Pr(Y_{k+1}=j, Z_k\varepsilon R_k,\ldots,Z_1\varepsilon R_1).$$

The next partial product is the result of multiplying A_2 by the transition matrix $P_{V,k+1}$ which yields:

$$A_3 = A_2 \cdot P_{V,k+1} = [d_1,\ldots,d_j,\ldots,d_{n_{k+1}}]$$

where

$$d_j = \sum_{i=1}^{n_{k+1}} c_i \cdot P_{V,k+1}(i,j)$$

$$= \sum_{i=1}^{n_{k+1}} Pr(Y_{k+1}=i, Z_k \varepsilon R_k, \ldots, Z_1 \varepsilon R_1)$$

$$\cdot Pr(Z_{k+1}=j \,|\, Y_{k+1}=i, Z_k \varepsilon R_k, \ldots, Z_1 \varepsilon R_1)$$

$$= \sum_{i=1}^{n_{k+1}} Pr(Z_{k+1}=j, Y_{k+1}=i, Z_k \varepsilon R_k, \ldots, Z_1 \varepsilon R_1)$$

$$= Pr(Z_{k+1}=j, Z_k \varepsilon R_k, \ldots, Z_1 \varepsilon R_1) \;.$$

The product is completed by multiplying A_3 by the characteristic matrix $G_{V,k+1}$ of the $k+1^{th}$ phase and the summing vector F_{k+1}, that is,

$$I(0) \cdot [\prod_{\ell=1}^{k+1} H_{V,\ell} \cdot P_{V,\ell} \cdot G_{V,\ell}] \cdot F_{k+1}$$

$$= A_3 \cdot G_{V,k+1} \cdot F_{k+1}$$

$$= \sum_{i \varepsilon R_{k+1}} Pr(Z_{k+1}=i, Z_k \varepsilon R_k, \ldots, Z_1 \varepsilon R_1)$$

$$= Pr(Z_{k+1} \varepsilon R_{k+1}, Z_k \varepsilon R_k, \ldots, Z_1 \varepsilon R_1)$$

$$= Pr(Z_1 \varepsilon R_1, \ldots, Z_{k+1} \varepsilon R_{k+1}, Z_{k+2} \varepsilon Q_{k+2}, \ldots, Z_m \varepsilon Q_m)$$

$$= Pr(R_1 \times \ldots \times R_{k+1} \times Q_{k+2} \times \ldots \times Q_m) \,.$$

Accordingly, the equation holds for all $k \leq m$, which completes the proof of Theorem 1.

Corollary:

For any Cartesian set $V = R_1 \times R_2 \times \ldots \times R_m$,

$$Pr(V) = I(0) \cdot [\prod_{\ell=1}^{m} H_{V,\ell} \cdot P_{V,\ell} \cdot G_{V,\ell}] \cdot F_m . \tag{5}$$

To illustrate this method, consider a system with three identical subsystems M_1, M_2 and M_3. During the first phase $T_1 = [t_0, t_1]$, each of the subsystems is dedicated to different computational tasks.

However, during the second phast $T_2 = [t_1, t_2]$, the system is reconfigured into a TMR configuration. The system is capable of degraded performance which occurs when (i) at least one subsystem has failed during phase 1 and at least two subsystems are functional through phase 2, or (ii) no failures occur during phase 1 and the system functions as a simplex system at the end of phase 2. Suppose that each of M_1, M_2 and M_3 fail permanently with a constant failure rate λ and failure characteristics of the subsystems are statistically independent. Then the probabilistic nature of phase 1 and phase 2 can be represented, respectively, by finite-state time-homogeneous Markov processes with transition graphs as illustrated in Figure 2.

Based on the above description of the system, three accomplishment levels can be established, i.e., $A = \{a_0, a_1, a_2\}$ where a_0 = no failure, a_1 = degraded performance and a_2 = failure. When expressed in terms of the state trajectories

$$\overline{\gamma_S}^{-1}(a_0) = \{(1,1),(1,2)\} = \{1\}\times\{1,2\}$$

$$\overline{\gamma_S}^{-1}(a_1) = \{2,3,4\}\times\{1,2\}\cup\{1\}\times\{3\}$$

$$\overline{\gamma_S}^{-1}(a_2) = \{2,3,4,5\}\times\{3,4\}\cup\{1\}\times\{4\}\cup\{5\}\times\{1,2\} \quad .$$

Then, solving the intraphase probabilities which, in this case, are the same for all Cartesian sets V,

$$P_{V,1} = \begin{bmatrix} r^3 & r^2s & r^2s & r^2s & 3rs^2+s^3 \\ 0 & r^2 & 0 & 0 & 2rs+s^2 \\ 0 & 0 & r^2 & 0 & 2rs+s^2 \\ 0 & 0 & 0 & r^2 & 2rs+s^2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} r = e^{-\lambda(t_1-t_0)} \\ s = 1 - r \end{array}$$

and

$$P_{V,2} = \begin{bmatrix} p^3 & 3p^2q & 3pq^2 & q^3 \\ 0 & p^2 & 2pq & q^2 \\ 0 & 0 & p & q \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{l} p = e^{-\lambda(t_2-t_1)} \\ q = 1 - p \end{array} .$$

Suppose that the initial state distribution of $\overline{X}_S$ is $I(0) = [1,0,0,0,0]$. If at time t_1, the system is in state 5 with respect to the phase 1 model (i.e., at least two subsystems have failed) then, depending on the exact number of subsystems failed, the state of the system with respect to phase 2 model is either 3 or 4.

By applying the definition of interphase transition probability,

$$H_{V,2}(5,3) = \frac{\Pr(\text{two failures before } t_1)}{\Pr(\text{two or three failures before } t_1)}$$

$$= \frac{3e^{-\lambda(t_1-t_0)}}{1 + 2e^{-\lambda(t_1-t_0)}} = c_1$$

and

$$H_{V,2}(5,4) = \frac{1 - e^{-\lambda(t-t_0)}}{1 + 2e^{-\lambda(t_1-t_0)}} = c_2 .$$

Transitions from states other than 5 happen to be deterministic, and thus we obtain the following interphase transition matrix

$$H_{V,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_1 & c_2 \end{bmatrix} .$$

$H_{V,1}$, by definition, is the 5×5 identity matrix.

Using these intraphase and interphase matrices, the probability of $V = \overline{\gamma_S}^{1}(a_0)$ can be computed using equation 5, i.e., since

$$G_{V,1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } G_{V,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we have

$$\begin{aligned} Pr(V) &= I(0)\cdot H_{V,1}\cdot P_{V,1}\cdot G_{V,1}\cdot H_{V,2}\cdot P_{V,2}\cdot G_{V,2}\cdot F_2 \\ &= r^3p^3 + 3r^3\ p^2q\ . \end{aligned}$$

Similarly,

$$Pr(\overline{\gamma_S}^{1}(a_1)) = 3r^2sp^2 + 3r^3pq^2$$

and

$$Pr(\overline{\gamma_S}^{1}(a_2)) = s^3+3rs^2+3pqr^2s+3qr^2s+r^3q^3.$$

Although equation 5 provides us with a general formula for computing the probability of a Cartesian set, its disadvantages derive from the fact that $H_{V,k}$ and $P_{V,k}$ matrices may be difficult to obtain in practical applications. In particular, these matrices will generally depend on V as well as X_S and, moreover, will generally depend on the history of X_S prior to phase k. However, the latter objections disappear when the transition probabilities are "memoryless". More precisely, let the (unconditional) <u>intraphase transition matrix</u> of the k^{th} phase to be the $n_k \times n_k$ matrix P_k where, for all $i,j \varepsilon Q_k$,

$$P_k(i,j) = Pr(Z_k=j\,|\,Y_k=i),$$

i.e., the probability that the k^{th} intraphase process ends up in state j given that it initiates in state i. Similarly, let the (unconditional) <u>interphase transition matrix</u> be the $n_{k-1} \times n_k$ matrix H_k where, for all

$i\varepsilon Q_{k-1}$ and $j\varepsilon Q_k$,

$$H_k(i,j) = Pr(Y_k=j|Z_{k-1}=i),$$

i.e., the probability that the k^{th} intraphase process initiates in state j given that the $k\text{-}1^{th}$ intraphase process ends up in state i. Then the intraphase transitions of X_S are <u>memoryless for V at phase k</u> if

$$P_{V,k} = P_k.$$

Similarly, the interphase transitions of X_S are <u>memoryless for V at phase k</u> if

$$H_{V,k} = H_k.$$

Accordingly, when transitions are memoryless through phase k, by the definitions and Theorem 1 we obtain

<u>Theorem 2</u>:

If $V = R_1 \times R_2 \times \ldots \times R_k \times Q_{k+1} \times \ldots \times Q_m$ and the intraphase and interphase transitions of X_S are memoryless for V through phase k, then

$$Pr(V) = I(0)\cdot[\prod_{\ell=1}^{k} H_\ell \cdot P_\ell \cdot G_{V,\ell}]\cdot F_k \ .$$

<u>Corollary</u>:

For any Cartesian set V, if the intraphase and interphase transitions of X_S are memoryless for V for all phases, then

$$Pr(V) = I(0)\cdot[\prod_{\ell=1}^{m} H_\ell \cdot P_\ell \cdot G_{V,\ell}]\cdot F_m \ .$$

When V is a Cartesian set and $R_\ell = Q_\ell$, for $\ell = 1,2,\ldots,k-1$, then the intraphase and interphase transitions of X_S are memoryless for V through phase k. Moreover, $G_{V,\ell}$ is an identity matrix for $\ell = 1,2,\ldots,$ k-1. Accordingly, applying Theorem 2, we obtain the following formula for the probability of the trajectory set $V = Q_1 \times \ldots \times Q_{k-1} \times R_k \times Q_{k+1} \times \ldots \times Q_m$

which, alternatively, is the probability of the event "$Z_k \varepsilon R_k$".

Theorem 3:

If $V = Q_1 \times \ldots \times Q_{k-1} \times R_k \times Q_{k+1} \times \ldots \times Q_m$, then

$$Pr(V) = I(0) \cdot [\prod_{\ell=1}^{k} H_\ell \cdot P_\ell] \cdot G_{V,k} \cdot F_k \ .$$

When Theorem 3 is specialized to singleton sets $R_k = \{i\}$, where $i \varepsilon Q_k$, it permits us to compute the probability of the event "$Z_k = i$". More generally, if we denote the probability distribution of the random variable Z_k by the n_k-dimensional vector $I(k) = [p_1, p_2, \ldots, p_{n_k}]$ where $p_i = Pr(Z_k = i)$, then

Theorem 4:

$$I(k) = I(0) \cdot [\prod_{\ell=1}^{k} H_\ell \cdot P_\ell].$$

Proof:

By Theorem 3,

$$Pr(Z_k = i) = Pr(Q_1 \times \ldots \times Q_{k-1} \times \{i\} \times Q_{k+1} \times \ldots \times Q_m)$$

$$= I(0) \cdot [\prod_{\ell=1}^{k} H_\ell \cdot P_\ell] \cdot G_{V,k} \cdot F_k$$

where, by the nature of V, $G_{V,k} \cdot F_k = E_i$, i.e., the $n_k \times 1$ column matrix with "1" on the i^{th} entry and "0" elsewhere. Thus, $Pr(Z_k = i)$ is equal to the i^{th} entry of the n_k-dimensional row vector

$$I(0) \cdot [\prod_{\ell=1}^{k} H_\ell \cdot P_\ell].$$

By Theorems 2-4, when certain intraphase and interphase transitions are memoryless for V, the probability of a Cartesian set V is relatively easily obtained. However, such results may still be difficult to use due to the fact that, even though the transitions are memoryless for V, they may not be memoryless for other Cartesian sets. Accordingly, we have sought to identify stronger conditions under which the formulas will hold for all Cartesian trajectory sets. First, by extending previous definitions, the intraphase (interphase) transitions of X_S are <u>memoryless at phase k</u> if they are memoryless for all Cartesian sets V at phase k; the intraphase (interphase) transitions of X_S are <u>memoryless</u> if they are memoryless at all phases. The advantage of memoryless transitions are obvious, for by their definition and the corollary to Theorem 2, we have

<u>Theorem 5</u>:

If X_S is a phased model and the intraphase and interphase transitions of X_S are memoryless then, for all Cartesian sets V,

$$Pr(V) = I(0)\cdot[\prod_{\ell=1}^{m} H_\ell\cdot P_\ell\cdot G_{V,\ell}]\cdot F_m. \tag{6}$$

Moreover, we find that the memoryless property is relatively easy to characterize, that is, we are able to show the following characteristic conditions for the memoryless property. It is important to note that the conditions do not involve any specific Cartesian sets.

<u>Theorem 6</u>:

(1) The intraphase transitions of X_S are memoryless at phase k if and only if, for all $i_\ell \varepsilon Q_\ell$ $(\ell=1,2,\ldots,k-1)$,

$$Pr(Z_k=j|Y_k=i, Z_{k-1}=i_{k-1},\ldots,Z_1=i_1) = P_k(i,j). \tag{7}$$

(2) The interphase transitions of X_S are memoryless at phase k if and only if, for all $i_\ell \varepsilon Q_\ell$ $(\ell=1,2,\ldots,k-2)$,

$$Pr(Y_k=j|Z_{k-1}=i, Z_{k-2}=i_{k-2},\ldots,Z_1=i_1) = H_k(i,j). \tag{8}$$

<u>Proof</u>:

Suppose $P_{V,k}$ is memoryless for all Cartesian sets $V = R_1\times R_2\times\ldots\times R_m$. By taking R_ℓ to be the singleton set $\{i_\ell\}$, $\ell = 1,2,\ldots,k-1$,

$$\begin{aligned} P_{V,k}(i,j) &= Pr(Z_k=j|Y_k=i, Z_{k-1}=i_{k-1},\ldots,Z_1=i_1) \\ &= P_k(i,j). \end{aligned}$$

Now, suppose that, for all $i_\ell \varepsilon Q_\ell$ $(\ell = 1,2,\ldots,k-1)$,

$$Pr(Z_k=j|Y_k=i,Z_{k-1}=i_{k-1},\ldots,Z_1=i_1)$$

$$= Pr(Z_k=j|Y_k=i).$$

Then, for any Cartesian set $V = R_1 \times R_2 \times \ldots \times R_m$,

$$P_{V,k}(i,j) = Pr(Z_k=j|Y_k=i,Z_{k-1}\varepsilon R_{k-1},\ldots,Z_1\varepsilon R_1)$$

$$= \frac{\sum_{i_1\varepsilon R_1,\ldots,i_{k-1}\varepsilon R_{k-1}} [Pr(Z_k=j|Y_k=i,Z_{k-1}=i_{k-1},\ldots,Z_1=i_1)\cdot Pr(Y_k=i,Z_{k-1}=i_{k-1},\ldots,Z_1=i_1)]}{Pr(Y_k=i,Z_{k-1}\varepsilon R_{k-1},\ldots,Z_1\varepsilon R_1)}$$

Thus, by the assumption, $P_{V,k}(i,j)$ is equal to

$$\frac{\sum_{i_1\varepsilon R_1,\ldots,i_{k-1}\varepsilon R_{k-1}} Pr(Z_k=j|Y_k=i)\cdot Pr(Y_k=i,Z_{k-1}=i_{k-1},\ldots,Z_1=i_1)}{Pr(Y_k=i,Z_{k-1}\ R_{k-1},\ldots,Z_1\ R_1)}$$

Factoring out the term $Pr(Z_k=j|Y_k=i)$, we have

$$P_{V,k}(i,j) = Pr(Z_k=j|Y_k=i)\cdot 1 = P_k(i,j)$$

which completes the proof for part (1) of the theorem. Part (2) is proven in a like manner.

Finally, when the transitions of X_S are memoryless, the equivalent model $\overline{X}_S$ is a time-varying Markov chain. This can be demonstrated as follows.

By Theorem 6 and the definition of memoryless transition, we have

$$Pr(Z_k=i_k|Y_k=j,Z_{k-1}=i_{k-1},\ldots,Z_1=i_1)$$

$$= Pr(Z_k=i_k|Y_k=j)$$

$$= Pr(Z_k=i_k|Y_k=j,Z_{k-1}=i_{k-1}).$$

Accordingly, for each $k \leq m$ and $i_\ell \in Q_\ell$ $(\ell = 1,2,\ldots,k)$,

$$\Pr(Z_k=i_k|Z_{k-1}=i_{k-1},\ldots,Z_1=i_1)$$

$$= \sum_{j\in Q_k} \Pr(Z_k=i_k|Y_k=j,Z_{k-1}=i_{k-1},\ldots,Z_1=i_1)\cdot$$

$$\Pr(Y_k=j|Z_{k-1}=i_{k-1},\ldots,Z_1=i_1)$$

$$= \sum_{j\in Q_k} \Pr(Z_k=i_k|Y_k=j,Z_{k-1}=i_{k-1})\cdot\Pr(Y_k=j|Z_{k-1}=i_{k-1})$$

$$= \sum_{j\in Q_k} \Pr(Z_k=i_k,Y_k=j|Z_{k-1}=i_{k-1})$$

$$= \Pr(Z_k=i_k|Z_{k-1}=i_{k-1}).$$

Hence, $\overline{X}_S = \{Z_k|k=0,1,\ldots,m\}$ satisfies the Markov properties.

Moreover, the transition probabilities of $\overline{X}_S$ associated with phase k can be expressed as a matrix.

$$\overline{P}_k = [\overline{P}_k(i,j)]$$

where

$$\overline{P}_k(i,j) = \Pr(Z_k=j|Z_{k-1}=i)$$

$$= \sum_{h\in Q_k} \Pr(Z_k=j|Y_k=h,Z_{k-1}=i)\cdot\Pr(Y_k=h|Z_{k-1}=i)$$

$$= \sum_{h\in Q_k} \Pr(Z_k=j|Y_k=h)\cdot\Pr(Y_k=h|Z_{k-1}=i)$$

$$= \sum_{h\in Q_k} P_k(h,i)\cdot H_k(i,h).$$

Accordingly, in terms of matrix multiplication,

$$\overline{P}_k = H_k\cdot P_k$$

and equation 6 can be represented in a more convenient form:

$$\Pr(V) = I(0).[\prod_{\ell=1}^{m} \overline{P}_\ell\cdot G_{V,\ell}]\cdot F_m.$$

IV. SUMMARY

It has been shown that the concept of a "phased mission" can be extended to performability models via the notion of a "phased" base model X_S. Under reasonable conditions, X_S yields an equivalent performability model $(\overline{X}_S, \overline{\gamma}_S)$ and, as demonstrated by the results of the paper, the intraphase and interphase probabilities of X_S suffice to determine the probabilistic nature of $\overline{X}_S$. In particular, it has been shown that, for any trajectory set V of $\overline{X}_S$ with a "Cartesian" structure, the probability of V can be computed as a product of matrices (Theorem 1). In general, each matrix depends on X_S and V but, as established in subsequent results (Theorems 2-5), the formulations may be simplified when certain phases of X_S are "memoryless for V." Finally, it has been demonstrated (Theorem 6) that transitions which are memoryless for all Cartesian sets V are characterized by a "Markovian property" relative to preceding end-of-phase observations of the phased base model X_S.

REFERENCES

[1] A. Costes, C. Landrault and J.-C. Laprie, "Reliability and availability models for maintained systems featuring hardware failures and design faults," IEEE Trans. Comput., Vol. C-27, pp. 548-560, June 1978.

[2] H. Kobayashi, Modeling and Analysis: An Introduction to System Performance Evaluation Methodology. Reading, MA: Addison-Wesley, 1978.

[3] H. S. Winokur, Jr. and L. J. Goldstein, "Analysis of mission-oriented systems," IEEE Trans. Reliability, Vol. R-18, No. 4, pp. 144-148, Nov. 1969.

[4] J. L. Bricker, "A unified method for analyzing mission reliability for fault tolerant computer systems," IEEE Trans. Reliability, Vol. R-22, No. 2, pp. 72-77, June 1973.

[5] J. D. Esary and H. Ziehms, "Reliability analysis of phased missions," Reliability and Fault Tree Analysis, Philadelphia, PA: SIAM, pp. 213-236, 1975.

[6] R. A. Ballance and J. F. Meyer, "Functional dependence and its application to system eveluation," Proc. of the 1978 Conf. on Information Sciences and Systems, Johns Hopkins University, Baltimore, MD, pp. 280-284, March 1978.

[7] J. F. Meyer, "On evaluating the performability of degradable computing systems," Proc. 1978 Int'l Symp. on Fault-Tolerant Computing, Toulouse, France, pp. 44-49, June 1978.

[8] J. F. Meyer, D. G. Furchtgott and L. T. Wu, "Performability evaluation of the SIFT computer," Proc. 1979 Int'l Symp. on Fault-Tolerant Computing, Madison, WI, June 1979.

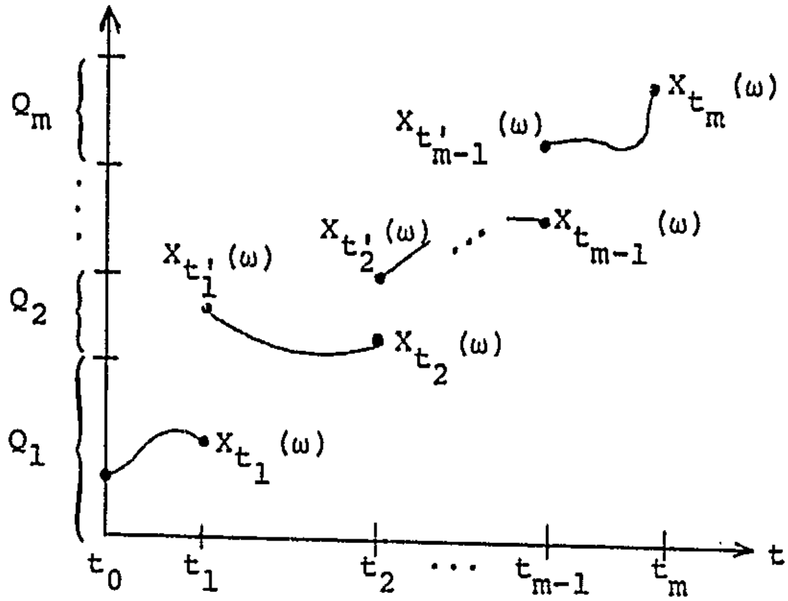

Figure 1
A state trajectory of X_S

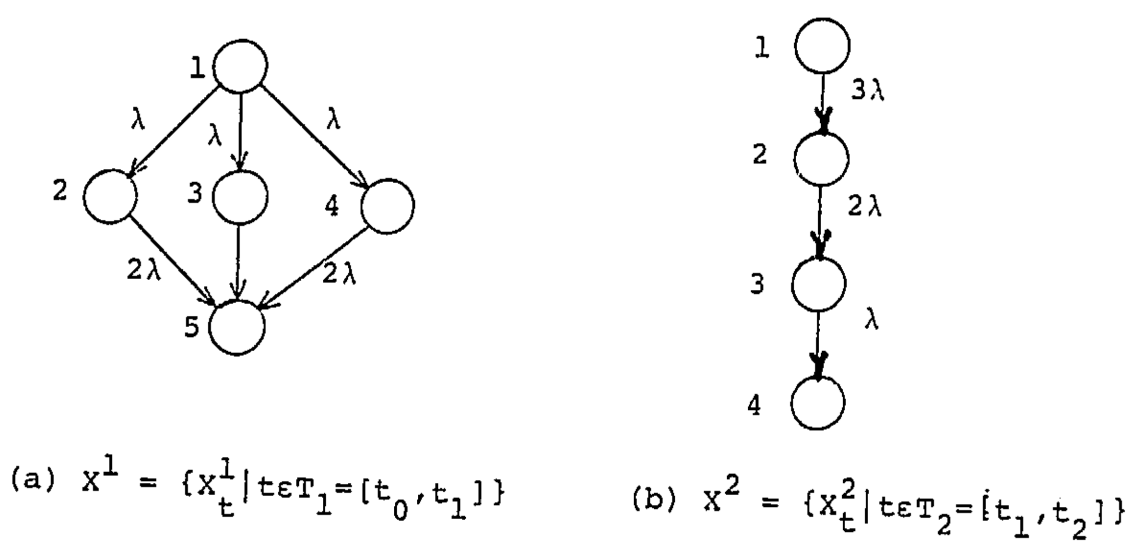

Figure 2
Markov model transition graphs for X_S