General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.

- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

NASA

Technical Memorandum 79679

Multivariate Approximation Methods and Applications to Geophysics and Geodesy

August 1979

National Aeronautics and Space Administration

Goddard Space Flight Center
Greenbelt, Maryland 20771

TM 79679

MULTIVARIATE APPROXIMATION METHODS AND APPLICATIONS IN GEOPHYSICS AND GEODESY

Marie-Jeanne Munteanu

August 1979

GODDARD SPACE FLIGHT CENTER

Greenbelt, Maryland

MULTIVARIATE APPROXIMATION METHODS AND APPLICATIONS IN GEOPHYSICS AND GEODESY

Marie-Jeanne Munteanu

PREFACE

This is the first report in a series which is intended to be written by the author with the purpose of treating a class of approximation methods of functions in one and several variables and ways of applying them to Geophysics and Geodesy.

The first report is divided in three parts and is devoted to the presentation of the mathematical theory and formulas. Some of my unpublished results are being summarized, together with my recent communications and well known results of other authors in the field.

We try to give a unified feeling about the topics presented not always insisting about certain details which can be found easily in the other author's papers.

We discuss various optimal ways of representing functions in one and several variables and the associated error when we have information about the function such as satellite data of different kinds.

The framework chosen is Hilbert spaces, which are very important for their simplicity in obtaining optimal solutions, for their links with the statistical interpretations and inverse theory.

Experiments have been performed on satellite altimeter data and on satellite to satellite tracking data. In both cases the results are very satisfactory and they are going to be presented in the next report.

CONTENTS

MULTIVARIATE APPROXIMATION METHODS AND APPLICATIONS IN GEOPHYSICS AND GEODESY

Marie-Jeanne Munteanu

INTRODUCTION

The purpose of the present survey is to make the reader aware of the existence of the various representations of a function of several variables and a common source of their derivation.

I will discuss the numerical experience obtained with them in the satellite data analysis in a second report.

The main theme of the class of methods is the representation of a function in a Hilbert space as a solution of a boundary value problem for a differential equation using Green's function. One part of the solution can be viewed as an approximation of the function and the second part as the corresponding remainder.

The choice of the differential equation is very important and should be related to the specific application.

In the next report we will discuss such a choice in connection with the statistical interpretation and with the links between different methods such as collocation, kernel functions, spline functions and finite elements.

There is a wealth of connections between different approaches which should give a perspective on the choice of the appropriate method for the specific problem and also to lead us to the desired interpretations.

The importance of the representations derived in this report is that:

1. They furnish an approximation of a given function and the corresponding remainder.

2. An inner product for the construction of a Hilbert space with reproducing kernel is provided.

3. Hilbert spaces with reproducing kernels generated in this manner are very interesting in optimization theory, namely numerically efficient optimal solutions are obtained via projection on different subspaces.

4. Computationally very efficient optimal numerical formulas for interpolation, filtering, differentiation, integration etc. together with their error representations are obtained very elegantly.

5. Error estimates for the optimal formulas in Hilbert spaces are given by the hypercycle inequality.

6. The reproducing kernel of a Hilbert space provides us with a covariance function and therefore leads us to statistical interpretation.

7. Hilbert space treatment seems to be an alternative to the statistical approach.

8. Generalized inverse problems which are so vitally important in geophysics have many times as a framework the Hilbert spaces approaches. (See Bachus and Gilbert papers.) All these ideas will be discussed in detail in future reports.

Concerning the numerical experience with these methods, together with Carl Siefring I have experimented with the tensor product representation and more specifically with the tensor product of univariate B-spline functions.

In the application of the tensor products one needs a first stage, namely generating the values of the unknown function on the rectangular grid with, for example, least squares and orthogonal polynomials.

We are applying this scheme to sea surface topography.

On the test data the tensor product of splines works outstandingly well, the error associated with the method is extremely small. What is extremely interesting about the tensor product of B-splines is that the univariate B-splines are functions with support of small size and outstanding numerical properties so that the above two dimensional scheme can be viewed as a very efficient finite element formula. This finite element scheme can be used not only in the best approximation context such as optimal filtering or interpolation but also in optimally solving partial differential equations as well as optimization problems of all kinds. In geophysical problems such as for example modelling the SAN ANDREAS fault or in the variational approach of whole-earth dynamics, etc. we can use these functions as trial functions. Our numerical experimentation with satellite data such as altimeter data is going to be included in the next report and hopefully be presented at the AGU meeting this spring. Of course the nature of for example altimeter data also calls for the application of other schemes such as blending spline functions. This is the next step in sequence. The reason is that tensor product spline functions interpolate data on a rectangular grid, while blending spline functions interpolate data along curves in space which in our case are the satellite tracks.

In addition if we use B-spline and least squares in the blending schemes one can compress data significantly. We have experimented very succesfully with satellite to satellite tracking data. compressing around 300 data to 32 coefficients.

The FORTRAN subroutines used and the graphs obtained are available and will be included in a note together with all the other experiments.

PART I

The first part is concerned with the presentation of a general method in order to obtain representations of functions in several variables. This uses, essentially, the form of the solution of a boundary value problem for an ordinary differential equation and the isomorphism between the lattice of a collection of linear projection operators and the lattice of corresponding image spaces. New representations, well known formulas such as the Taylorian representation of D.D. Stancu [29] and Sard [26] as well as the very interesting and elegant results concerning multivariate functions obtained by Gordon [19], Gordon and Birkhoff [7] are obtained using intuitive diagrams. The reproducing kernel for a Hilbert space of bivariate functions which have Taylor representations can then be constructed (see Nielson [24], Mansfield [21], and Ritter [25]).

One can generalize their method for more general cases than bivariate spaces of functions possesing Taylor expansions and their variations. Once the reproducing kernels are at hand the question of optimal approximation of linear functionals in Hilbert spaces is reduced to requiring that the approximation be exact for the representers of approximating functionals.

PART II

In the second part approximation formulas and the corresponding remainders are characterized as solutions of certain partial differential problems. These formulas furnish representations of functions belonging to a certain space of bivariate functions and provide in some cases an inner product. The space can be completed in the sense of this inner product, reproducing kernels kernels can be constructed and consequently optimal approximations of linear functionals and other very useful quantities in numerical analysis can be derived. One can derive an entire variety of differential problems giving intermediate approximations and the corresponding remainders, maximal and minimal approximations being the two extreme cases.

The isomorphism between the lattice of a class of differential problems and the corresponding lattice of linear projection operators which are solutions of the differential problems is used in the main. If, in addition, the isomorphism to the corresponding lattice generated by the projectors is used the theory of approximation of multivariate functions can be unified in a beautiful manner.

At the end of the second part of the report the tensor product splines of Ahlberg, Nilson and Walsh, Gordon's blending splines interpolating a network of curves and generally splines in several variables interpolating hypersurfaces and corresponding remainders are introduced. It is well known that blending splines in two variables have been used with great success in problems of computer-aided design, airplane fuselages, automobile exteriors, etc. Generally speaking these multivariate splines can be used in the problem of modeling smooth surfaces.

The major role in the construction of these multivariate spline functions is played by univariate spline functions. Using Gordon's language the univariate spline functions generate a distributive lattice of projectors which has a "maximal" projector and a "minimal" projector. For the bivariate case the "minimal" projector is the tensor product spline and the "maximal" projector is the blending spline mentioned above.

The lattice of corresponding remainders gives the errors associated with the maximal, minimal and generally all intermediate approximations.

The above mentioned isomorphism between lattices suggests the Euler-Lagrange differential equations satisfied by these splines. Consequently, the minimization properties of these functions can be derived.

In a later report several approaches to defining multivariate splines will be presented. Of course, one important approach will be the variational one, starting with the minimization property of spline functions, deducing the Euler-Lagrange differential equations and finally obtaining the analytical form of certain types of multivariate splines.

1. REPRESENTATIONS OF FUNCTIONS IN SEVERAL VARIABLES

1.1 The Bivariate Case

1.1.1 W. J. Gordon's Maximal and Minimal Approximations

Consider the space $C^p[I']$, the set of real-valued functions, defined and continuous on the interval $I' = [0,1]_x$ such that all derivatives of order less than or equal to p are continuous on I'. Similarly we define the space $C^q[I'']$ where $I'' = [0,1]_y$. Consider also the space $C^{p,q}[R_u]$ of bivariate functions, defined on $R_u = I' \times I''$, real-valued, and having derivatives

$D^{(i,j)}F(x,y) \in C^0(R_0)$ $0 \le i \le p$, $0 \le j \le q$, where

$$D^{(i,j)} = \frac{\partial^{i+j}}{\partial x^i \partial y^j}$$

Let D_x and D_y be the linear differential operators

$$D_x = a_p(x)\frac{d^p}{dx^p} + \ldots + a_1(x)\frac{d}{dx} + a_0(x),$$

$$D_y = b_q(y)\frac{d^q}{dy^q} + \ldots + b_1(y)\frac{d}{dy} + b_0(y),$$

defined on $C^p[I']$, $C^q[I'']$ respectively. We assume $a_i(x)$, $b_j(y)$ are real and continuous on their domains of definition, and $a_p(x) \neq 0$, $b_q(y) \neq 0$.

Denote by N_x and N_y the respective null spaces of the operators D_x and D_y. These are obviously subspaces of dimension p and q respectively.

Let ℓ'_i, $i = \overline{0,p-1}$ and m'_j, $j = \overline{0,q-1}$ be continuous linear functionals on $C^p[I']$, $C^q[I'']$, respectively, which form the bases for the dual null spaces. The functionals ℓ'_i, $i = 0,p-1$; m'_j, $j = \overline{0,q-1}$ are of the following form:

$$\ell'(f) = \sum_{k=0}^{p-1}\sum_{\ell=0}^{N_k-1} A_{k\ell}\, f^{(k)}(x_\ell) + \sum_{k=0}^{p-1}\int_{I'} \beta_k(x) f^{(k)}(x)dx,\ f \in C^p[I'],$$

$$m'(g) = \sum_{k'=0}^{q-1}\sum_{\ell'=0}^{M_{k'}-1} B_{k'\ell'}\, g^{(k')}(x_\ell) + \sum_{k'=0}^{q-1}\int_{I''} \alpha_{k'}(y) g^{(k')}(y)dy,\ g \in C^q[I''],$$

where $A_{k\ell}$, $B_{k'\ell}$ are constants, the functions $\beta_k(x)$ and $\alpha_{k'}(y)$ are piecewise continuous on I' and I'' respectively, x_ℓ and y_ℓ, are points in I' and I'' respectively.

Denote by ℓ_i, $i = \overline{0,p-1}$ and m_j, $j = \overline{0,q-1}$ the respective restrictions of the functionals ℓ'_i, $i = \overline{0,p-1}$ and m'_j, $j = \overline{0,q-1}$ to the subspaces N_x and N_y. We know that there exists a unique base $\phi_i(x)$, $i = \overline{0,p-1}$ for N_x and a unique base $\psi_j(y)$, $j = \overline{0,q-1}$ for N_y such that

$$\ell_i(\varphi_k) = \delta_{ik},\ i,\ k = \overline{0, p-1},$$

$$m_j(\psi_k) = \delta_{hl},\ k,\ j = \overline{0, q-1}.$$

Lemma 1: Every function $f(x) \in C^p[I']$ can be represented in the following form:

$$f(x) = \sum_{i=0}^{p-1} \ell_i(f)\,\phi_i(x) + \int_{I'} G_1(x,t)\,D_t(f)\,dt \qquad (1)$$

where G (x,t) is the Green's function of the differential problem

$$\begin{cases} D_x f = e \\ \ell_i(f) = 0 \quad i = \overline{0, p-1} \end{cases} \text{ where } \quad f \in C^p[I'] \xrightarrow{D_x} e \in L^2[I'].$$

(see [4], [10]). Similarly all $g(y) \in C^q[I'']$ can be written as

$$g(y) = \sum_{j=0}^{q-1} m_j(g)\,\psi_j(y) + \int_{I''} G_2(y,u)\,D_u(g)\,du \qquad (2)$$

$G_2(y,u)$ is the Green's function of the differential problem

$$\begin{cases} D_y g = h \\ m_j(g) = 0 \quad j = \overline{0, q-1} \end{cases} \qquad g \in C^q[I''] \xrightarrow{D_y} h \in L^2[I''].$$

Define the "partial" operators L_i, i = 0,p-1 and M_j, j = 0,q-1 on the space $C^{p,q}[R_u]$ such that

$$F(x,y)\ C^{p,q}[R_u],$$

$$L_i(F) = \ell_i[F(x,y^*)] \text{ for all } y^* \in I'',$$

$$M_j(F) = m_j[F(x^*,y)] \text{ for all } x^* \in I'.$$

Denote by D_x and D_y

$$\mathcal{D}_x = a_p(x)\,\frac{\partial^p}{\partial x^p} + \ldots + a_1(x)\,\frac{\partial}{\partial x} + a_0(x),$$

$$\mathcal{D}_y = b_q(y)\,\frac{\partial^q}{\partial y^q} + \ldots + b_1(y)\,\frac{\partial}{\partial y} + b_0(y),$$

the extensions of the operators D_x and D_y to the space $C^{p,q}[R_u]$.

Theorem: Every $F(x,y) \in C^{p,q}[R_u]$ can be written in the form

$$F(x,y) = \sum_{i=0}^{p-1} L_i(F)\phi_i(x) + \int_{I'} G_1(x,t) D_t [F(t,y)]\, dt = P_1(F) + R_1(F) \qquad (3)$$

and

$$F(x,y) = \sum_{j=0}^{q-1} M_j(F)\psi_j(y) + \int_{I''} G_2(y,u) D_u [F(x,u)]\, du = P_2(F) + R_2(F) \qquad (4)$$

(see Gordon [17], p. 17).

It is obvious that all functions $F(x,y) \in N_x \cap N_y$ can be written as

$$F(x,y) = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} L_i M_j(F)\phi_i(x)\psi_j(y) = P_1 P_2(F). \qquad (5)$$

Using the results given in [17], [7], [19] were able to assert that all $F(x,y) \in N_x' \cap N_y'$ can be represented as

$$F(x,y) = \iint_{R_u} G_1(x,t)\, G_2(y,u)\, D_t D_u F(t,u)\, dt\, du \qquad (6)$$

where N_x' and N_y' are the complementary spaces of N_x and N_y, respectively.

Suppose now we wish to obtain our function in the form

$$F = P(F) + R(F)$$

where $P(F)$ is the component of the function belonging to the subspace $N_x \cap N_y$ and satisfying the interpolation conditions:

$$L_i(F) = L_i(P), \quad i = 0, p-1,$$

$$M_j(F) = M_j(P), \quad j = 0, q-1,$$

and $R(F)$ if the corresponding error, i.e. the component of the function belonging to the complementary subspace of $N_x \cup N_y$.

Let us first make two remarks about the n^{th} order. These notes will be used constantly in this chapter.

1. Using Dieudonne's theorem (in [13], p. 123) we can obtain F as the sum of two components, one in the subspace and the other the corresponding error.

2. There exists an isomorphism between the lattice of the subspaces and the lattice of the corresponding representations. (See Gordon [19]).

Theorem 1: Every function $F(x,y) \in C^{p,q}[R_u]$ admits the following development:

$$F(x,y) = \mathcal{P}(F) + \mathcal{R}(F),$$

where

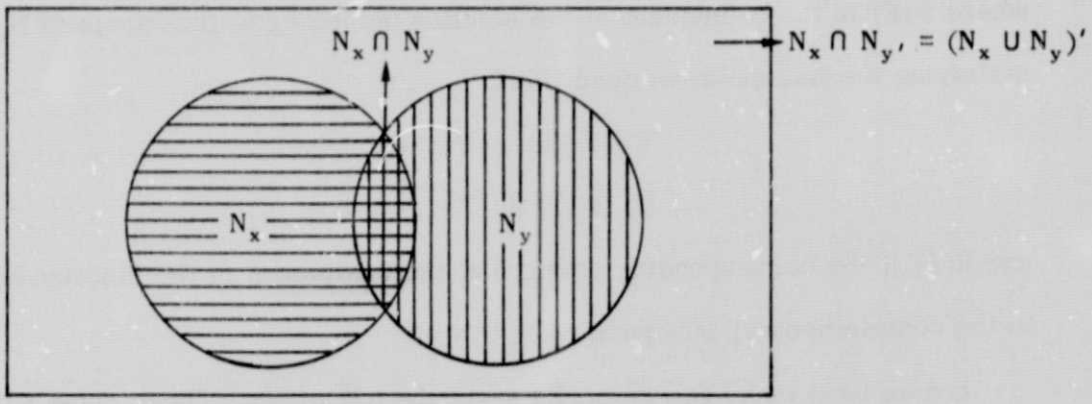
$$\mathcal{P}(F) = \sum_{i=0}^{p-1} L_i(F)\phi_i(x) + \sum_{j=0}^{q-1} M_j(F)\psi_j(y) - \sum_{i=0}^{p-1}\sum_{j=0}^{q-1} L_i M_j(F)\phi_i(x)\psi_j(y)$$

$$= P_1(F) + P_2(F) - P_1P_2(F),$$

$$\mathcal{R}(F) = \iint_{R_u} G_1(x,t)\, G_2(y,u)\, D_t D_u F(t,u)\, dt\, du.$$

By using the following diagram, remarks 1 and 2, and formulas (3), (4), (5) a and (6), we arrive at formula (7).

We can easily verify that P(F) satisfies all the necessary interpolation conditions.



This diagram suggests that we compose $P_1(F)$ and $P_2(F)$ to obtain P(F).

Denote this operation by $\oplus$, then

$$P(F) = (P_1 \oplus P_2)(F) = P_1(F) + P_2(F) - P_1 P_{\bar{2}}(F).$$

In the terminology introduced by Gordon ([19] p. 234) P(F) is called the maximal approximation.

We now look for F in the form

$$F(x,y) = p(F) + r(F),$$

where p(F) is the component of F in the subspace $N_x \cap N_y$ satisfying the interpolation conditions

$$L_i M_j(F) = L_i M_j [p(F)], \quad i = \overline{0, p-1}, \quad j = \overline{0, q-1}.$$

and p(F) is the corresponding error, i.e. the component of F in the complementary subspace of $N_x \cap N_y$.

Theorem 2: Any $F(x,y) \in C^{p,q} [R_u]$ can be written in the form

$$F = p(F) + r(F),$$

where

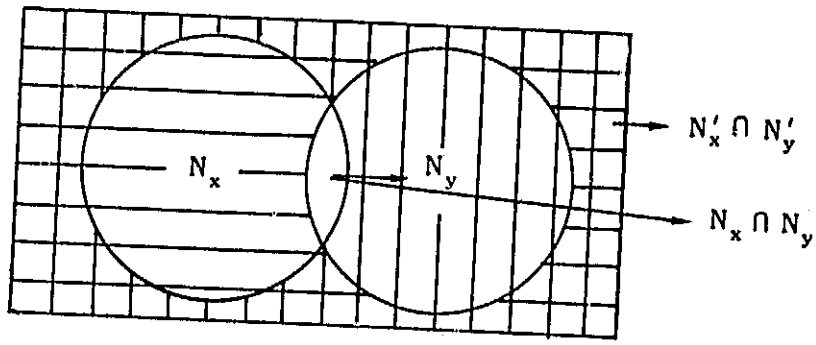
$$p(F) = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} L_i M_j(F) \phi_i(x) \psi_j(y) = P_1 P_2(F),$$

$$r(F) = \int_{I'} G_1(x,t) D_t F(t,y)\, dt + \int_{I''} G_2(y,u) D_u F(,u)\, du +$$

$$- \iint_{R_u} G_1(x,t)\, G_2(y,u)\, D_t D_u F(t,u)\, dt\, du =$$

$$= R_1(F) + R_2(F) - R_1 R_2(F) = (R_1 \oplus R_2)(F). \qquad (8)$$

An immediate justification of the formula is suggested by the following diagram:



Remarks:

1. Theorem 2 can be considered as a generalization of a development of Stancu (see [29]).

In fact, if one particularizes as follows,

$$\mathcal{D}_x = D_x^{(p,0)},\ \mathcal{D}_y = D_y^{(0,q)},\ I' = [0,1],\ I'' = [0,1],$$
$$L_i M_j(F) = DF^{(i,j)}(0,0),\ i = \overline{0,p-1},\ j = \overline{0,q-1},$$

one obtains the taylorian development of Stancu:

$$F(x,y) = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \frac{x^{i'}}{i!} \frac{y^j}{j!} DF^{(i,j)}(0,0) + \frac{1}{(p-1)!} \int_0^1 (x-t)_+^{p-1} DF^{(p,0)}(t,y)\,dt +$$

$$\frac{1}{(q-1)!} \int_0^1 (y-u)_+^{q-1} D^{(0,q)} F(x,u)\,du +$$

$$\int_0^1 \int_0^1 \frac{(x-t)_+^{p-1}}{(p-1)!} \frac{(y-u)_+^{q-1}}{(g-1)!} D^{(p,q)} F(t,u)\,dt\,du$$

2. Obviously p(F) satisfies the necessary interpolation conditions. Gordon calls p(f) the minimal approximation.

The diagram in this case gives us the errors $R_1(F)$ and $R_2(F)$ of the operation of composition, then r(F),

$$r(F) = (R_1 \oplus R_2)(F) = R_1(F) + R_2(F) - R_1 R_2(F).$$

We can obtain another expression for r(F) by making another diagram.

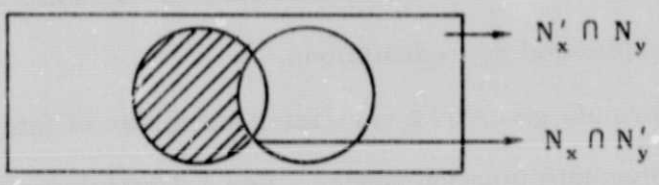
Theorem 3: Every function $F(x,y) \in C^{p,q}\ [R_u]$ can be written in the following form:

$$F(x,y) = P_1P_2(F) + P_1R_2(F) + P_2R_1(F) + R_1R_2(F) =$$

$$= \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} L_i M_j(F) \phi_i(x) \psi_j(y) + \sum_{i=0}^{p-1} \phi_i(x) \int_{I''} G_2(y,u) D_u [L_i(F)]\, du$$

$$+ \sum_{j=0}^{q-1} \psi_j(y) \int_{I'} G_1(x,t) D_t [M_j(F)]\, dt + \iint_{R_u} G_1(x,t)\, G_2(y,u) D_t D_u(F)\, dt\, du \quad (10)$$

We justify this formula in a manner identical to the preceding one but we begin with the following diagram:



For example, for the shaded part we have the following conditions: $F(x,y) \in N_x$ and $F(x,y) \in N'_y$. Then we have the representations

$$F(x,y) = \sum_{i=0}^{p-1} L_i(F) \phi_i(x) \quad (11)$$

$$F(x,y) = \int_{I''} G_2(y,u) D_u F(x,u)\, du \quad (12)$$

By substituting (11) into (12) we see that all functions $F(x,y) \in N_x \cap N'_y$ can be written in the form

$$F(x,y) = \sum_{i=0}^{p-1} \phi_i(x) \int_{I''} G_2(y,u) D_u [L_i(F)]\, du.$$

Similarly all $F(x,y) \in N_y \cap N'_x$ can be written as

$$F(x,y) = \sum_{j=0}^{q-1} \psi_j(y) \int_{I'} G_1(x,t) D_t [M_j(F)] \, dt.$$

In particular since we have already proposed Theorem 2 we obtain a taylorian development whose expression is different from the rest.

Formulas (7) and (10) were deduced by Gordon in a different way (see [18] p. 939, [17] p. 23, 2.5).

We have demonstrated a method in the very simple case of bivariate functions and presented for them the maximal and minimal approximations. This discussion is an introduction to the case of several variables and to the subsequently more complicated representations.

The above formula gives us a very interesting way of introducing a scalar product for the bivariate function spaces. For a particular simple choice of the operators and the functionals we obtain one of the Sard spaces [26] which is very often used in numerical analysis.

1.1.2 Generalizations of the Development of A. Sard

Let $R_u = I' \times I''$, $(a,b) \in R_u$ and p, q are non-negative integers such that p+q=n. A. Sard in [26] introduces the space $B_{p,q} = B_{p,q}(a,b) = B_{p,q}(R_u; a,b)$ as the collection of real-valued functions f(x,y) defined on R_u and having derivatives

$$D^{(p,q)} f(x,y) \quad (x,y) \in R_u$$

$$D^{(n-j,j)} f(x,b) \quad x \in I' \quad j < q,$$

$$D^{(i,n-j)} f(a,y) \quad y \in I'' \quad i < p,$$

which are continuous. Assume $p + q = 2m$, $a = 0$, $b = 0$.

Theorem 4: Every function $f(x,y) \in B_{p,q}$ can be written in the form:

$$f = A + B + C + D \tag{13}$$

where

$$A = \sum_{i+j<2m} \frac{x^i}{i!}\frac{y^i}{i!} Df^{(i,j)}(0,0),\quad B = \sum_{i<m} \frac{x^i}{i!}\int_0^1 \frac{(y-u)^{2m-i-1}}{(2m-i-1)!} Df^{(i,2m-1)}(0,u)du,$$

$$C = \sum_{j<m} \frac{y^j}{y!}\int_0^1 \frac{(x-t)_+^{2m-j-1}}{(2m-j-1)!} Df^{(2m-j,j)}(t,0)dt,\quad D = \iint_{R_u} \frac{(x-t)_+^{m-1}}{(m-1)!}\frac{(x-t)_+^{m-1}}{(m-1)!}\frac{(y-u)_+^{m-1}}{(m-1)!} Df^{(m,m)}(t,u)dtdu.$$

To begin we will write out in detail the first sum.

$$A = A_1 + \sum_{i=0}^{m-1} A_{2,i} + \sum_{j=0}^{m-1} A_{3,j}$$

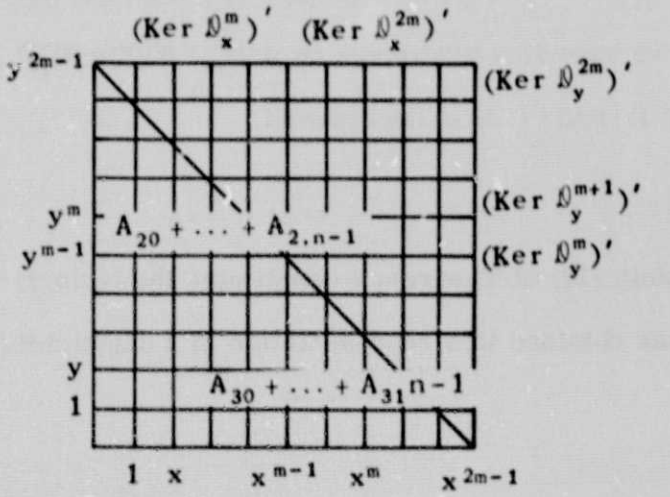
where

$$A_1 = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \frac{x^i}{i!}\frac{y^j}{j!} Df^{(i,j)}(0,0),$$

$$\sum_{i=0}^{m-1} A_{2,i} = \sum_{i=0}^{m-1}\sum_{j=m}^{2m-1-i} \frac{x^i}{i!}\frac{y^j}{j!} Df^{(i,j)}(0,0),$$

$$\sum_{j=0}^{m-1} A_{3,j} = \sum_{j=0}^{m-1}\sum_{j=m}^{2m-1-j} \frac{x^i}{i!}\frac{y^j}{j!} Df^{(i,j)}(0,0).$$

Construct the following diagram:



Denote by

$$D_x^i = \frac{\partial^i}{\partial x^i},\ D_y^i = \frac{\partial^i}{\partial y^i},\ i = \overline{0,2m}.$$

Also denote by $S[x]$, $S[y^m, \ldots, y^{2m-1}]$, the subspaces generated by x and y^m, $\ldots$, y^{2m-1}, respectively.

It is clear that all $f(x,y) \in \operatorname{Ker} D_x^m \cap \operatorname{Ker} D_y^m$ admits the representation A_1, all $f(x,y) \in S[y^j] \cap S[x^m, \ldots, x^{2m-1-j}]$, $j = \overline{0,m-1}$ the representation $A_{3,j}$, $j = \overline{0,m-1}$ and all functions $f(x,y) \in S[x^i] \cap S[y^m, \ldots, Y^{2m-1-i}]$ the representation $A_{2,i}$, $i = \overline{0,m-1}$.

Search the subspaces now for the one which has the B representation. Following the same diagram we prove that all $f(x,y)$ which are elements of the subspace $\{S[1] \cap (\operatorname{Ker} D_y^{2m})'\} \cup \{S[x] \cap (\operatorname{Ker} D_y^{2m-1})'\} \cup \ldots \cup \{S[x^m] \quad (\operatorname{Ker} D_y^{m+1})'\}$ can be written in the B form.

Actually for all $f(x,y) \in S[x^i] \cap (\operatorname{Ker} D_y^{2m-i})'$, $i = \overline{0,m-1}$, we have that

$$f(x,y) = \frac{x^i}{i!} \int_0^1 \frac{(y-u)_+^{2m-i-1}}{(2m-i-1)!} D^{(i,2m-1)} f(0,u)du, \quad i = \overline{0,m-1}.$$

Similarly we easily see the C representation corresponds to the subspace $\{S[1] \cap (\operatorname{Ker} D_x^{2m})'\} \cup \{S[y] \cap (\operatorname{Ker} D_X^{2m-1})'\} \cup \ldots \cup \{S[y^m] \cap (\operatorname{Ker} D_X^{m+1})'\}$.

Finally we know that all $f(x,y) \in (\operatorname{Ker} D_X^m)' \cap (\operatorname{Ker} D_Y^m)'$ admit the D representation and formula (13) is thus proved.

Remarks

1. Formula (13) of Theorem 4 constitutes the taylorian development of Sard. The author has obtained this representation in a completely different manner. (See [26]).

2. The method, the one by which we have just constructed the formula of Sard, permits us to obtain more general representations possessing similar characteristics. We have presented the method in the trivial case. We are able to obtain in an analogous manner more complex representations of the form (notation is obvious)

$$f(x,y) = \sum_{i+j<2m} \phi_i(x)\psi_j(y) L_i M_j(f) + \sum_{i<m} \phi_i(x) \int_0^1 G_{2m-i}(y,u) D^{(n,m)}f(t,u) \; [L_i(f)]\, du +$$

$$+ \sum_{j<m} \psi_j(y) \int_0^1 G_{2m-j}(x,t) D^{2m-j,0)} [M_j(f)]\, dt +$$

$$\int_0^1 \int_0^1 G_m(x,t) G_m(y,u) D^{(n,m)}f(t,u) dt\, du \qquad (14)$$

We have replaced the taylorian functionals by the continuous linear functionals $L_i M_j$, $i + j < 2m$.

This representation is considered as a generalization of the taylorian development of Sard.

2. Decomposition of Linear Differential Problems

In the second part of the report we will obtain representations of multivariable functions starting with appropriate decomposition of certain linear differential problems. We also find again certain developments obtained in the preceding part of the report. We organize the material in the following way.

Two differential problems, for the bivariate case, are treated in the first section and we obtain the maximal and minimal approximations and the corresponding errors. One case in particular permits us to concretely illustrate a result given by G. Birkhoff and W. J. Gordon [7].

In the second section we study two differential problems for the four variable case whose solutions can be interpreted as defining an interpolating function for a system of given curves and for a system of given surfaces, respectively. We also study the corresponding errors. As one last example, for the four variable case, we intend to illustrate a partial symmetric intermediate approximation.

Finally we consider the general case of n variables as providing the interpolating function of certain hypersurfaces in j variables, $j = \overline{1,n-1}$ and the corresponding errors as solutions of certain differential problems. For the n variable case we are able to obtain all the total symmetric intermediate approximations and the corresponding errors.

From the point of view of the theory of differential equations we obtained the solution to a certain class of differential problems by systematically using the same method based on the appropriate diagrams, suggesting the mentioned isomorphism between corresponding distributive lattices.

2.1 Bivariate Case

We adopt the same framework as found in part one.

Let the non-homogeneous differential problem be

$$\begin{cases} \partial_x \partial_y F(x,y) = e(x,y), \\ L_i(F) = f_i(y), \qquad i = \overline{0, p-1}, \\ M_j(F) = g_j(x), \qquad j = \overline{0, q-1}, \end{cases} \tag{17}$$

the symbolism is the same as in part one, $e(x,y) \in C^0(R_u)$, $f_i(y) \in C^q[I'']$, $g_j(x) \in C^p[I']$, $i = \overline{0,p-1}$, $j = \overline{0, q-1}$.

Suppose that the compatibility conditions

$$L_i(g_j) = M_j(f_i), \; I = \overline{0, p-1}, \; j = \overline{0, q-1}, \tag{18}$$

are satisfied. We can then assume a solution of the form

$$F(x,y) = F_1(x,y) + F_2(x,y), \; F \in C^{p,q}[R_u], \tag{19}$$

where $F_1(x,y)$ is the solution to the differential problem

$$\begin{cases} \partial_x \partial_y F_1(x,y) = 0 \\ L_i(F_1) = f_i(y), \quad i = \overline{0, p-1}, \\ M_j(F_1) = g_j(x), \quad j = \overline{0, q-1}. \end{cases} \tag{20}$$

and $F_2(x,y)$ is the solution of the problem

$$\begin{cases} \partial_x \partial_y F_2(x,y) = e(x,y) \\ L_i(F_2) = 0, \quad i = \overline{0, p-1}, \\ M_j(F_2) = 0, \quad j = \overline{0, q-1}, \end{cases} \tag{21}$$

We then have from the first part that F_2 is

$$F_2(x,y) = \iint_{R_u} G_1(x,t)\, G_2(y,u)\, e(t,u)\, dt du. \tag{22}$$

Following the diagram corresponding to Theorem 1, part one, and th relation

$$P_{\mathrm{Ker}\mathfrak{D}_x\mathfrak{D}_y} = P_{\mathrm{Ker}\mathfrak{D}_x} + P_{\mathrm{Ker}\mathfrak{D}_y} - P_{\mathrm{Ker}\mathfrak{D}_x\ \mathrm{Ker}\mathfrak{D}_y}, \tag{23}$$

where by $P_{\mathrm{Ker}\mathfrak{D}_x\mathfrak{D}_y}$ we mean the operator corresponding to all $F(x,y) \in C^{p,q}[R_n]$, the component in the subspace $\mathrm{Ker}\,\mathfrak{D}_x\mathfrak{D}_y$. We decompose the second problem into three differential problems:

$$\begin{cases} \mathfrak{D}_x F_{11} = 0, \\ L_i(F_{11}) = f_i(y), \quad i = \overline{0, p-1}, \end{cases}$$

$$\begin{cases} \mathfrak{D}_y F_{12} = 0, \\ M_j(F_{12}) = g_j(x), \quad j = \overline{0, q-1}, \end{cases}$$

$$\begin{cases} \mathfrak{D}_x F_{13} = 0, \\ \mathfrak{D}_y F_{13} = 0, \\ L_i M_j(F_{13}) = L_i(g_j) = \alpha_{ij}, \quad i = \overline{0, p-1}, \quad j = \overline{0, q-1}, \end{cases}$$

for which the solutions are:

$$F_{11} \equiv \sum_{i=0}^{p-1} f_i(y)\,\phi_i(x),$$

$$F_{12} = \sum_{i=0}^{q-1} g_j(x)\,\psi_j(y),$$

$$F_{13} = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} \alpha_{ij}\, f_i(y)\, g_j(x).$$

From (23) we conclude

$$F_1 = F_{11} + F_{12} - F_{13},$$

from which it follows that

$$F = F_{11} + F_{12} - F_{13} + F_2. \tag{24}$$

This is the result proposed by Theorem 1, part one. The uniqueness of the solution to problem (17) is due to the hypothesis made on the operators L_i and M_j, (see part one).

Remark:

For the particular case: $\mathcal{A}_x = D^{(2p,0)}$, $\mathcal{A}_y = D^{(0,2p)}$,

$$L_i M_j(F) = F^{(i,j)}(x_i, y_j),\ i = \overline{0,p-1},\ j = \overline{0,p-1},\ x_i,\ y_j = 0 \text{ or } 1,$$

G. Birkhoff and W. J. Gordon obtained the same result using an induction method (see [7]).

Particular Case

We are going to consider a particular case which appears in [7]. The solution given by the author on p. 204, formula (19) is inexact and will be corrected in what follows:

$$\begin{cases} F^{(4,4)}(x,y) = 0, \quad 0 \le x,y \le 1, \\ F(x,0) = 0, \ F(0,y) = 0 \\ F(x,1) = 0, \ F(1,y) = 0 \\ F_y(x,0) = g_0(x), \ F_x(0,y) = f_0(y), \\ F_y(x,1) = g_1(x), \ F_x(1,y) = f_1(y), \end{cases}$$

where $f_j(y), q_i(x) \in C^4[0,1]$, $i,j = 0,1$, and we have

$$F_{xy}(i,j) = \frac{d}{dx} f_j(i) = \frac{d}{dy} g_i(j), \ i,j = 0,1.$$

Then the unique solution in $C^{4,4}[R_{\mu}]$ is of the form

$$F(x,y) = \phi_0(y)\, f_0(x) + \phi_1(y)\, f_1(x) + \phi_0(x)\, g_0(y) + \phi_1(x)\, g_1(y) -$$

$$- \phi_0(y)\,\phi_0(x)\, F^{(1,1)}_{(0,0)} - \phi_0(x)\,\phi_0(x)\,\phi_1(y)\, F^{(1,1)}_{(0,1)} - \phi_1(x)\,\phi_0(y)\, F^{(1,1)}_{(1,0)} -$$

$$- \phi_1(x)\,\phi_1(y) F^{(1,1)}_{(1,1)} - \phi_2(x)\,\phi_0(y) F^{(1,0)}_{(0,0)} - \phi_2(x)\,\phi_1(y)\, F^{(1,0)}_{(1,0)} -$$

$$- \phi_3(x)\,\phi_0(y) F^{(1,0)}_{(0,1)} - \phi_3(x)\,\phi_1(y)\, F^{(1,0)}_{(1,1)} - \phi_2(y)\,\phi_0(x)\, F^{(0,1)}_{(0,0)} -$$

$$- \phi_2(y)\,\phi_1(x) F^{(0,1)}_{(0,0)} - \phi_3(y)\,\phi_0(x)\, F^{(0,1)}_{(1,0)} - \phi_3(y)\,\phi_1(x)\, F^{(0,1)}_{(1,1)},$$

where

$$\phi_0(x) = x(1-x)^2,$$

$$\phi_1(x) = x^2(1-x),$$

$$\phi_2(x) = (1+2x)(1-x)^2,$$

$$\phi_3(x) = (3-2x)x^2,$$

are the cardinal functions for the polynomial interpolation problem in the sense of Hermite (see Davis [12] p. 45).

The case where the four initial boundary conditions are non-zero has been considered by Gordon in [17] p. 38.

Let us now consider the following differential problem:

$$\begin{cases} \mathbb{D}_x F = f_1(x,y), \\ \mathbb{D}_y F = f_2(x,y), \\ L_i M_j(F) = \alpha_{ij}, \quad i = \overline{0,p-1}, \quad j = \overline{0,q-1}, \end{cases}$$

where $f_1(x,y) \in C^0[I'] \times C^q[I'']$, $f_2(x,y) \in C^p[I'] \times C^0[I'']$. The following condition

$$\mathbb{D}_x f_2(x,y) = \mathbb{D}_y f_1(x,y) \tag{26}$$

is assumed to be satisfied, then every solution F(x,y) is necessarily a sum

$$F(x,y) = F_1(x,y) + F_2(x,y)$$

where $F_1(x,y)$ is the solution of the problem

$$\begin{cases} \mathcal{D}_x F_1 = 0 \\ \mathcal{D}_y F_1 = 0 \\ L_i M_j (F_1) = a_{ij}, \quad i = \overline{0, p-1}, \quad j = \overline{0, q-1}, \end{cases} \tag{27}$$

and $F_2(x,y)$ is the solution of the problem

$$\begin{cases} \mathcal{D}_x F_2 = f_1(x,y), \\ \mathcal{D}_y F_2 = f_2(x,y), \\ L_i M_j (F_2) = 0, \quad i = \overline{0, p-1}, \quad j = \overline{0, q-1}. \end{cases} \tag{28}$$

The solution $F_1(x,y)$ is obviously given by

$$F_1(x,y) = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} a_{ij} \phi_i(x) \psi_j(y). \tag{29}$$

Looking for the expression of F_2 we return to the diagram corresponding to Theorem 2, part one. Let T be the operator considered in the problem. Then we deduce that

$$P_{(\mathrm{Ker}\, T)'} = P_{(\mathrm{Ker}\, \mathcal{D}_x)'} + P_{(\mathrm{Ker}\, \mathcal{D}_y)'} - P_{(\mathrm{Ker}\, \mathcal{D}_x)' \cap (\mathrm{Ker}\, \mathcal{D}_y)'} \tag{30}$$

by $(\mathrm{Ker}T)'$ we mean the complementary subspace of the null space of the operator T.

Then to find the expression for $F_2(x,y)$ we decompose the differential problem (28) into three differential problems:

$$\begin{cases} \mathbb{D}_x F_{21} = f_1(x.y), \\ L_i(F_{21}) = 0, \quad i = \overline{0, p-1}, \end{cases}$$

$$\begin{cases} \mathbb{D}_y F_{22} = f_2(x,y) \\ M_j(F_{22}) = 0, \quad j = \overline{0, q-1} \end{cases}$$

$$\begin{cases} \mathbb{D}_x \mathbb{D}_y F_{23} = \mathbb{D}_x f_2(x,y) \\ L_i(F_{23}) = 0, \quad i = \overline{0, p-1}, \\ M_j(F_{23}) = 0, \quad j = \overline{0, q-1}. \end{cases}$$

For the reasons given in part one, we then have

$$F_{21} = \int_0^1 G_1(x,t)\, f_1(t,y)\, dt,$$

$$F_{22} = \int_0^1 G_1(y,u)\, f_2(x,u)\, du,$$

$$F_{23} = \int_0^1 \int_0^1 G_1(x,t)\, G_2(y,u)\, \mathbb{D}_t \mathbb{D}_u f_2(t,u)\, dt\, du.$$

Having calculated the relation in (30) we obtain

$$F_2 = F_{21} + F_{22} - F_{23}. \tag{31}$$

Consider the same differential problem and work this time for the expression of F_2 from the diagram corresponding to Theorem 3, part one. Thus

$$F_2 = F_2^1 + F_2^2 + F_2^3 \tag{32}$$

where

(1) F_2^1 satisfies the differential problem

$$\begin{cases} \mathbb{D}_x F_2^1 = f_1(x,y), \\ L_i(F_2^1) = 0, \quad i = \overline{0, p-1}, \end{cases}$$

and corresponds to the null space of the operator $\mathcal{D}_y$,

$$F_2^1 = \sum_{j=0}^{q-1} M_j(F_2^1)\,\psi_j(y).$$

(2) F_2^2 satisfies the problem

$$\begin{cases} \mathcal{D}_y F_2^2 = f_2(x,y), \\ M_j(F_2^2) = 0, \quad j = \overline{0,q-1}, \end{cases}$$

and corresponds to the null space of the operator $\mathcal{D}_y$,

$$F_2^2 = \sum_{i=0}^{p-1} L_i(F_2^2)\,\phi_i(x)$$

(3) F_2^3, like above, is the solution to the problem

$$\begin{cases} \mathcal{D}_x \mathcal{D}_y F_2^3 = \mathcal{D}_x f_2(x,y), \\ L_i(F_2^3) = 0, \quad i = \overline{0,p-1}, \\ M_j(F_2^3) = 0, \quad j = \overline{0,q-1}. \end{cases}$$

It is a clear result that

$$F_2^1 = \sum_{j=0}^{q-1} \psi_j(y) \int_0^1 G_1(x,t)\, M_j[f_1(t,y)]\,dt,$$

$$F_2^2 = \sum_{i=0}^{p-1} \phi_i(x) \int_0^1 G_2(y,u)\, L_i[f_2(x,u)]\,du,$$

$$F_2^3 = \int_0^1 \int_0^1 G_1(x,t)\, G_2(y,u)\, \mathcal{D}_t f_2(t,u)\,dt\,du.$$

This is the result given by Theorem 3, part one.

In an analogous manner we can treat the case of three and more than three variables.

2.2 Spline Functions in Several Variables

This section is meant to briefly present in a unified manner certain types of spline functions in several variables. The major role in the construction of these spline functions is played by univariate spline functions.

The reasons behind placing this section at the end of this report are:

1. We use the methods of part one, sections one and two in order to derive the analytic expression of the spline function and the corresponding remainder.

2. The differential problems studied in part two of the report suggest to us the Euler-Lagrange equations of the minimization properties satisfied by the spline functions under consideration.

2.2.1 Recollections

Let $\pi'\colon 0 = x_0 < x_1 < \ldots < x_{p'} = 1$, $\pi''\colon 0 = y_0 < y_1 < \ldots < y_{q'} = 1$ and $\pi'''\colon 0 = z_0 < z_1 < \ldots < z_{r'} = 1$ be three partitions of the intervals $I' = [0,1]_x$, $I'' = [0,1]_y$, and $I''' = [0,1]_z$.

Let $E^{2p}_{\pi'}[I']$ be the collection of functions f defined on I' which are of class C^{2p} inside each subinterval $[x_i, x_{i+1}]$ of I' and which have simple jump discontinuities in the $(2m-2)^{th}$ derivative across the joints x_i, $1 < i < p-1$. Similarly define the space $E^{2p}_{\pi''}[I'']$ and $E^{2p}_{\pi'''}[I''']$.

Define $E^{2p,2p}_{\pi'\times\pi''}[I'\times I'']$ as the space of functions $f(x,y)$ defined on $I'\times I''$ and whose normal derivatives satisfy the conditions

$$(1^\circ)\quad f^{(\mu,0)}(x^*,y) \in E^{2p}_{\pi''}[I''],\ \forall\, x^* \in I',\ 0 \le \mu \le 2m-2,$$

$$(2^\circ)\quad f^{(0,\nu)}(x,y^*) \in E^{2p}_{\pi'}[I'],\ \forall\, y^* \in I'',\ 0 \le \nu \le 2m-2,$$

For the trivariate case we introduce, in an analogous fashion, the space $E^{2p,2p,2p}_{I'\times I''\times I'''}[R_u]$, when $R_u = I'\times I''\times I'''$. The partition $\pi'\times\pi''\times\pi'''$ of R_u yields parallelpipeds denoted R_{ijk}, $i = \overline{0,P'}$, $j = \overline{0,Q'}$, $k = \overline{0,R'}$,

Let the partial operators be

$$L_i(f) = \ell_i[f(x,y^*,z^*)],\ \text{for all } y^*_i z^* \in I''\times I''',$$

$$\forall\, f(x,y,z) \in E^{2p,2p,2p}_{I'\times I''\times I'''}[R_u],\ M_j(f) = m_j[f(x^*,y,z^*)],\ \text{for all } x^*_i z^* \in I'\times I''',$$

$$N_k(f) = n_k[f(x^*,y^*,z)]\ \text{for all } x^*,y^* \in I'\times I',$$

$i = \overline{1,P}$, $j = \overline{1,Q}$, $k = \overline{1,R}$.

The functionals ℓ_i, $1 \le i \le P$ are linear, continuous, defined on $E^{2p}_{\pi'}[I']$, and of the following form:

$$\forall\, f(x) \in E^{2p}_{\pi'}[I'],\ \ell_i[f(x)] \begin{cases} f(x_\ell), & \ell = \overline{0,p'} \\ f^{(\alpha)}_{(x_\ell)}, & \ell = 0,p',\ \alpha = \overline{1,p-1}. \end{cases}$$

Similarly on the spaces $E^{2p}_{\pi''}[I'']$, $E^{2p}_{\pi'''}[I''']$ we define the functionals m_j, $j = \overline{1,Q}$ and n_k, $k = \overline{1,R}$.

We introduce as in sections 1,2 and part one differential operators $\mathbb{D}_x$, $\mathbb{D}_y$, $\mathbb{D}_z$ of degree p, where P, Q, R > p.

Denote by S_x the collection of spline functions related to the differential operator $\mathbb{D}_x$ and the functionals f_i, $1 \le i \le p$. The dimension of S_x is P. Let $\{Q_i(x)\}_{i=1}^{p}$ be the cardinal base of S_x.

A spline function $s(x) \in S_x$ for the case of the differential operator D^p, has the following properties

(i) s(x) is a polynomial of degree less than or equal to 2p-1 in each interval $[x_i, x_{i+1}]$, $i = \overline{0,P'-1}$.

(ii) the first 2p-2 derivatives of s(x) are continuous at the joints x_i, $i = \overline{0,P'-1}$.

Starting with this unified definition we have for the trivariate case, we can generate the spline functions of Gordon and Ahlberg-Nilson-Walsh.

2.2.2 W. J. Gordon's Spline Functions

Suppose we wish to obtain an interpolation formula

$$f(x,y,z) = S(x,y,z) + r(x,y,z), \tag{49}$$

such that

$$\mathbb{D}_x^* \mathbb{D}_x \mathbb{D}_y^* \mathbb{D}_y \mathbb{D}_z^* \mathbb{D}_z \, S(x,y,z) = 0 \tag{50}$$

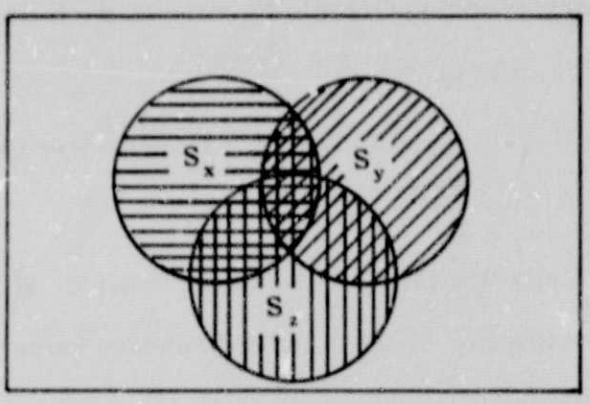
in the interior of the parallelpipeds R_{ijk}, $i = \overline{0,P'}$, $j = \overline{0,Q'}$, $k = \overline{0,R'}$, and that the following interpolation conditions are satisfied:

$$\begin{aligned} L_i(f) &= L_i(S), \quad i = \overline{1,P}, \\ M_j(f) &= M_j(S), \quad j = \overline{1,Q}, \\ N_k(f) &= N_k(S), \quad k = \overline{1,R}. \end{aligned} \tag{51}$$

We denote by $\mathbb{D}^*$ the adjoint form of the operator D.

The spline function s(x,y,z) was defined in a different way by Gordon [18].

To obtain the analytic expression of s(x,y,z) and r(x,y,z) we construct the following diagram:



Via analogous reasons as those of sections 1,2 and part 1 we can write the following formulas:

$$S(x,y,z) = \sum_{i=1}^{P} L_i(f)\phi_i(x) + \sum_{j=1}^{Q} M_j(f)\psi_j(y) + \sum_{k=1}^{R} N_k(f)\,X_k(z) -$$

$$- \sum_{i=1}^{P}\sum_{j=1}^{Q} L_i M_j(f)\phi_i(x)\psi_j(y) - \sum_{i=1}^{P}\sum_{k=1}^{R} L_i N_k(f)\phi_1(x)X_k(z) - \qquad (52)$$

$$- \sum_{j=1}^{Q}\sum_{k=1}^{R} M_j N_k(f)\psi_j(y)\,X_k(z) + \sum_{i=1}^{P}\sum_{j=1}^{Q}\sum_{k=1}^{R} L_i M_j N_k(f)\phi_i(x)\psi_j(y)X_k(z)$$

$$r(x,y,z) = \iiint_{R_u} G_1^{\circ}(x,t)\,G_2^{\circ}(y,u)\,G_3^{\circ}(z,v)\mathfrak{D}_t^{*}\mathfrak{D}_t\mathfrak{D}_u^{*}\mathfrak{D}_u\mathfrak{D}_v^{*}\mathfrak{D}_v f(t,u,v)dtdudv \qquad (53)$$

where $G_1^{\circ}(x,t)$ is the function which appears in the error formula for a univariate spline

$$f(x) = S(x) + r(x) \qquad (54)$$

$$S(x) = \sum_{i=1}^{P'-1} \ell_i(f)\phi_i(x) \qquad (55)$$

$$r(x) = \int_0^1 \left\{G_1(x,t) - \sum_{i=1}^{P'-1} \phi_i(x)\ell_{x^i}[G(x,t)]\right\}\mathfrak{D}_t^{*}\mathfrak{D}_t f(t)dt =$$

$$= \int_0^1 G_1^{\circ}(x,t)D_t^{*}D_t f(t)dt\,. \qquad (56)$$

$G_1(x,t)$ is the corresponding Green's function of the boundary value problem

$$\begin{cases} D_x^* D_x f(x) = e(x) \\ f^{(a)}(0) = 0, \qquad f(x) \in C^{2p}[I'] \xrightarrow{D_x^* D_x} e(x) \in L^2[I'] \\ f^{(a)}(1) = 0, \quad a = \overline{0, p-1}, \end{cases} \tag{57}$$

In the sum which appears in the integral, we consider all the functionals ℓ_i, $i = \overline{1,P}$, with the exception of those in the boundary value problem (57). (See Birkhoff and de Boor [6]).

The functions $G_2^o(y,u)$ and $G_3^o(z,u)$ are similarly defined.

The analytic expressions of the spline function $s(x,y,z)$ and the error $r(x,y,z)$ were obtained in a different way than Gordon's, in [18], for the bivariate case.

Minimization Property

Gordon's spline function minimizes the pseudo-norm

$$\iiint_{R_u} [\mathbb{D}_x \mathbb{D}_y \mathbb{D}_z f(x,y,z)]^2 \, dxdydz, \tag{58}$$

with respect to all functions which are in the space $C^{P,P,P}[R_u]$, and which satisfy the interpolation conditions (51). (See [18], for the example).

2.2.3 Multi-variable Spline Functions Introduced by Ahlberg, Nilson and Walsh

We are looking for an interpolation formula

$$f(x,y,z) = S_1(x,y,z) + r(x,y,z), \tag{59}$$

such that $S_1(x,y,z)$ satisfies the equations

$$\begin{aligned} \mathbb{D}_x^* \mathbb{D}_x \, S_1(x,y,z) &= 0, \\ \mathbb{D}_y^* \mathbb{D}_y \, S_1(x,y,z) &= 0, \\ \mathbb{D}_z^* \mathbb{D}_z \, S_1(x,y,z) &= 0, \end{aligned} \tag{60}$$

in each of the parallelpipeds R_{ijk}, with the following interpolation conditions

$$L_i M_j N_k(S_1) = L_i M_j N_k(f), \; = i = \overline{1,P}, \; j = \overline{1,Q}, \; k = \overline{1,R} \tag{61}$$

This definition is given in [1]. The authors have not studied the corresponding error (see also [9]). The diagram resembes that of part one section two but we replace N_x, N_y, N_z by S_x, S_y, S_z. We have the formulas:

$$S_1(x,y,z) = \sum_{i=1}^{P} \sum_{j=1}^{Q} \sum_{k=1}^{R} L_i M_j N_k(f) \phi_i(x) \psi_j(y) X_k(z), \tag{62}$$

$$r(x,y,z) = \sum_{i=1}^{P} \sum_{j=1}^{Q} \phi_i(x)\psi_j(y) \int_{I'''} G_3^\circ(z,v) \mathcal{D}_v^* \mathcal{D}_v [L_i M_j(f)] dv +$$

$$+ \sum_{j=1}^{Q} \sum_{k=1}^{R} \psi_j(y) X_k(z) \int_{I'} G_1^\circ(x,t) \mathcal{D}_t^* \mathcal{D}_t [M_j N_k(f)] dt + \sum_{i=1}^{P} \sum_{k=1}^{R} \phi_i(x) X_k(z) \cdot$$

$$\int_{I''} G_2^\circ(y,u) \mathcal{D}_u^* \mathcal{D}_u [L_i N_k(f)] du - \sum_{i=1}^{P} \phi_i(x) \iint_{I''\times I'''} G_2^\circ(y,u) G_3^\circ(z,v) \mathcal{D}_u^* \mathcal{D}_u \mathcal{D}_v^* \mathcal{D}_v [L_i(f)] dudv -$$

$$- \sum_{j=1}^{Q} \psi_j(y) \iint_{I'\times I'''} G_1^\circ(x,t) G_3^\circ(z,v) \mathcal{D}_t^* \mathcal{D}_t \mathcal{D}_v^* \mathcal{D}_v [M_j(f)] dtdv -$$

$$- \sum_{k=1}^{R} X_k(z) \iint_{I'\times I'''} G_1^\circ(x,t) G_2^\circ(y,u) \mathcal{D}_t^* \mathcal{D}_t \mathcal{D}_u^* \mathcal{D}_u [N_k(f)] dtdu$$

$$+ \iiint_{R_u} G_1^\circ(x,t) G_2^\circ(y,u) G_3^\circ(z,v) \mathcal{D}_t^* \mathcal{D}_t \mathcal{D}_u^* \mathcal{D}_u \mathcal{D}_v^* \mathcal{D}_v [f(t,u,v)] dtdudv. \tag{63}$$

Minimization Property

The spline function $s_1(x,y,z)$ minimizes the pseudo-norm of (58) with respect to all the functions in the space $C^{P,P,P}[R_u]$ which satisfy the interpolation conditions (61). (See [1]).

REMARKS

We can similarly introduce spline functions in n variables interpolating hypersurfaces in j variables $j = \overline{1,n-1}$. Using univariate spline functions we can

generate a lattice of projectors whose "minimal" projector will be the tensor product splines and whose "maximal" projector will be a spline function in n variables, interpolating a network of hypersurfaces in n - 1 variables. Of course of great interest will be all the spectrum of intermediate splines interpolating hypersurfaces in any number of variables from one to n - 1. It is unnecessary to mention the great importance to have at hand the lattice of corresponding remainders for numerous problems in numerical analysis.

Bibliography

1. Ahlberg, J. H., Nilson, E. N. and Walsh, J. L.: The Theory of Splines and Their Applications, Academic Press, 1967.

2. Anselone, P. M. and Laurent, P. J.: A general method for the construction of interpolating or smoothing spline functions, Num. Math., 12, 1968, pp. 68-82.

3. Atteia, M.: Existence et détérmination des fonctions spline a plusieurs variables, C.R.A.S. t.262, (1966), pp. 575-578.

4. Atteia, M.: Théorie et applications des fonctions-splines en analyse numérique, Thèse, Grenoble (1966).

5. Aubin, P. J.: Behavior of the error of the approximate solutions of the boundary value problems for linear elliptic operators by Galerkin's and finite difference methods, Annali della Sevola Normale di Pisa, Fasc. IV, (1967), pp. 599-639.

6. Birkhoff, G., de Boor, C.: Error bounds for spline interpolation, J. of Math. and Mech., vol. 13, 5, (1964), pp. 827-836.

7. Birkhoff, G., Gordon, W. J.: The draftsman's and related equations, J. of Approx. Theory, 1, (1968), pp. 199-208.

8. Birkhoff, G., Schultz, M. H. and Varga, R. S.: Piecewise Hermite interpolation in one and two variables with applications to partial differential equations, Num. Math., 11, (1968), pp. 232-256.

9. de Boor, C.: Bicubic spline interpolation, J. of Math. and Physics, 3, (1962), pp. 212-218.

10. de Boor, C., Lynch, R. E.: On splines and their minimum properties, J. of Math. and Mech., vol. 15, 6, (1966), pp. 953-969.

11. Cea, J.: Approximation variationnelle des problèmes aux limites, Ann. Inst. Fourier (Grenoble), 14 (1964), pp. 345-444.

12. Davis, P. J.: Interpolation and Approximation, Blaisdell Ed., New York, 1965.

13. Dieudonné, J.: Fondements de L'Analyse Moderne, Ed. Gauthier-Villars.

14. Edwards: Functional Analysis-Theory and Applications. Ed. by Holt, Reihart and Winston, (1965).

15. Goldberg, S.: Unbounded Linear Operators, London: McGraw-Hill, 1966.

16. Golomb, M., Weinberger, H. F.: Optimal approximation and error bounds. In: On Numerical Approximation, R. E. Langer (Ed.), Madison, Wisc.: University of Wisconsin Press 1959, pp. 117-190.

17. Gordon, W. J.: "Blending-functions" methods of bivariate and multivariate interpolation and approximation, General Motors Research Report, Oct. 1968.

18. Gordon, W. J.: Spline-blended surface interpolation through curve networks, J. of Math. and Mech., 18, (1969), pp. 931-952.

19. Gordon, W. J.: Distributive lattices and the approximation of multivariate functions. In: Approximation with Special Emphasis on Spline Functions (Ed. I. J. Schoenberg). London: Academic Press, (1969), pp. 223-277.

20. Lions, J. L.: Equations Différentielles Opérationnelles et Problèmes aux limites, Springer-Verlag, 1961.

21. Mansfield, L. E.: On the optimal approximation of linear functionals in spaces of bivariate functions, SIAM J. of Numer. Anal. 8, (1971), pp. 115-126.

22. Mansfield, L. E.: Optimal approximations and error bounds in spaces of bivariate functions, J. of Approx. Theory 5, (1972), pp. 77-96.

23. Nielson, G. M.: Surface approximations and data smoothing using generalized spline functions, Thesis, University of Utah, (June 1970).

24. Nielson, G. M.: Bivariate spline functions and the approximation of linear functionals, Num. Math., 21, (1973), pp. 138-140.

25. Ritter, D.: Two dimensional spline functions and best approximation of linear functionals, J. of Approx. Theory 3, (1970), pp. 352-368.

26. Sard, A.: Linear Approximation, Am. Math. Soc., 1963.

27. Schwarz, L.: Théorie des Distributions, Ed. Hermann.

28. Soboleff, S. L.: Applications of Functional Analysis in Mathematical Physics, Providence, Am. Math Soc., 1963.

29. Stancu, D. D.: The remainder of certain linear approximation formulas in two variables, SIAM J. Numer. Anal., 1, (1964), pp. 137-163.

30. Yosida, K.: Functional Analysis, Springer-Verlag, 1965.