(NASA-CR-162499) INVESTIGATIONS ON THE HIERARCHY OF REFERENCE FRAMES IN GEODESY AND GEODYNAMICS (Ohio State Univ. Research Foundation) 127 p HC A07/MF A01 CSCL 08G

N80-13731

Unclas
G3/46 46195

Reports of the Department of Geodetic Science

Report No. 289

INVESTIGATIONS ON THE HIERARCHY OF REFERENCE FRAMES IN GEODESY AND GEODYNAMICS

by

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Prepared for

National Aeronautics and Space Administration
Goddard Space Flight Center
Greenbelt, Maryland 20770

Grant No. NSG 5265
OSURF Project 711055

The Ohio State University
Research Foundation
Columbus, Ohio 43212

August, 1979

PREFACE

This research project was partially sponsored by NASA/GSFC under Grant No. NSG 5265, Ivan I. Mueller, supervisor. The grant is under the technical direction of David E. Smith, NASA/GSFC, Greenbelt, Md.

The authors of this report are affiliated with the following organizations: Grafarend and Richter with the Institute of Astronomical and Physical Geodesy, University FAF at Munich, FRG; Papo with the Department of Civil Engineering, Technion, Haifa, Israel; and Mueller with OSU. The research described was performed while Mueller spent a year in Munich under the auspices of the Alexander von Humboldt Foundation and later when Papo was at OSU as a Postdoctoral Fellow. Grafarend also visited OSU during this time period, while he was a Visiting Scientist at the National Geodetic Survey under NAS/NRC sponsorship.

ABSTRACT

Modern high accuracy measurements of the non-rigid earth are to be referred to four-dimensional, i.e., time- and space-dependent, reference frames. Geodynamic phenomena derived from these measurements are to be described in a terrestrial reference frame in which both space- and time-like variations can be monitored. Existing conventional terrestrial reference frames (e.g. CIO, BIH) are no longer suitable for such purposes.

The ultimate goal of this study is the establishment of a reference frame, moving with the earth in some average sense, in which the geometric and dynamic behavior of the earth can be monitored, and whose motion with respect to inertial space can also be determined.

The study is conducted in three parts. In the first part problems related to reference directions are investigated, the second part deals with the reference origins and the third part with problems related to scale.

The approach is based on the fact that reference directions at an observation point on the earth surface are defined by fundamental vectors (gravity, earth rotation, etc.), both space and time variant. These reference directions are interrelated by angular parameters, also derived from the fundamental vectors. The interrelationships between these space- and time-variant angular parameters are illustrated in

hierarchic structures or towers, which make the derivations of the various relationships convenient. In order to determine the above parameters from observations using least squares techniques, model towers of triads are also presented to allow the formation of linear observation equations. Although the model towers are also space and time variant, their variations are described by adopted parameters representing our current knowledge of the earth.

After the translational and rotational degrees of freedom (origin and orientation) have been discussed, the notion of a length, scale degrees of freedom are introduced and studied under spacelike/timelike variations.

According to the notion of scale parallelism, originated by H. Weyl, scale factors with respect to a unit length are given. Three-dimensional geodesy is constructed from the set of three base vectors (gravity, earth-rotation and the ecliptic normal vector). Space and time variations are given with respect to a polar and singular value decomposition or in terms of changes in translation, rotation, deformation (shear, dilatation or angular and scale distortions).

TABLE OF CONTENTS

INVESTIGATIONS ON THE HIERARCHY OF REFERENCE FRAMES IN GEODESY AND GEODYNAMICS

PART I: SYSTEM OF REFERENCE DIRECTIONS: THE E-TOWER

by

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Introduction

In order to take full advantage of high quality geodetic observational systems, such as lunar and satellite laser ranging and radio interferometry to quasars, an appropriate terrestrial reference frame is needed in which geodynamic phenomena can be detected and monitored. The importance of the definition, determination and subsequent maintenance of such a terrestrial reference frame has been recognized by many, although, so far, no satisfactory and comprehensive proposals for its realization have been put forward [Kolaczek and Weiffenbach, 1975; IAU, in press].

The ultimate goal of this study is the establishment of such a reference frame, moving with the earth in some average sense, and whose motion with respect to inertial space can also be determined.

In attempting a solution to the problem, a "zero base" approach is taken. Being fully aware of the large body of accumulated knowledge in the relevant disciplines of geodesy, astronomy and geophysics, we conduct a step-by-step analysis of known concepts and relationships with the purpose of establishing an unbiased and systematic foundation. In many cases all we do is redefine and reformulate familiar concepts and quantities as necessary. The earth and its environment are considered in their full complexity. Only at a much later stage do we intend to make approximations and only after a quantitative analysis of their effects. This paper which deals with the directional aspects of the problem will be followed by subsequent ones which will treat the problems of reference origins and scale, and also the question of how the reference frame can be established and maintained in practice.

1. Fundamental Natural Vectors

Natural vectors are defined as such by their property of being dependent only on some natural phenomena and consequently independent of any artifacts such as coordinate systems, reference models, etc. Consider a point P on the surface of the earth and another point Q which serves as a target being observed at some epoch T from the point P. For the epoch T we define a number of natural vectors at the point P, designated as the fundamental vectors.

$\overline{Q}$ - the Observational Vector. The light ray which travels from Q to P (or vice versa) is generally a space curve due to the refraction by the atmosphere. What we actually observe is the direction of the tangent to that space curve at the point P. This tangent line is defined as the observational fundamental vector and is denoted by $\overline{Q}$.

$-\overline{\Gamma}$ - the Local Vertical Vector. The gravity vector at the point P is denoted by $\overline{\Gamma}$. Its magnitude is the value of gravity at P. We define the second fundamental vector $-\overline{\Gamma}$, opposite in direction to $\overline{\Gamma}$, to be referred to as the local vertical vector.

$\overline{\Omega}$ - the Rotation Vector. Rotation is change of orientation of a body or mass element with respect to some inertial system. It can be found by studying the space-like change of the velocity vector of mass points with respect to inertial space. For example, if the space-like change is zero, that is, constant velocity at all points, there is no rotation, but only a translation. Let $\overline{V}$ be the velocity vector with respect to inertial space, then $\overline{\Omega} = \text{rot}\ \overline{V}$ is by definition the rotation vector, also called the vorticity vector. Its magnitude is the instantaneous rotation velocity.

The definition separates reasonably rotation and deformation since the earth is not rigid. rot $\overline{V}$ just contains the antisymmetric part of the tensor grad $\overline{V}$, whereas the symmetric part describes deformation. The earth rotation vector changes with respect to time due to precession, nutation and polar motion and with respect to space due to the deformability.

$\overline{X}$ - the Ecliptic Normal. The ecliptic is the osculating plane of the space curve which the earth-moon barycenter is moving along. It is referred to a heliocentric system with inertial orientation. The vector $\overline{X}$ is the binormal vector of this curve. An approximation is the normal vector of the plane being spanned by the heliocenter (considered as fixed) and the earth-moon barycenter.

Basic Angular Parameters

Project the four fundamental vectors $\overline{Q}$, $-\overline{\Gamma}$, $\overline{\Omega}$, and $\overline{X}$ onto a unit sphere centered at point P (Fig. 1). At any instant the four points are related by five basic angular parameters as follows:

B altitude (observable)

A azimuth (astronomically observable)

Φ latitude (astronomically observable)

H hour angle of vernal equinox

E obliquity of the eclitic

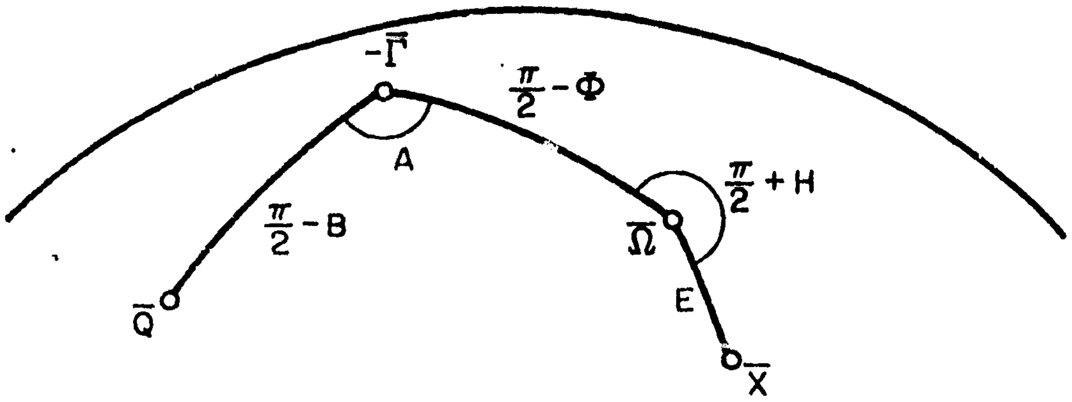

Fig. 1

The vernal equinox Υ is defined by $\overline{\Upsilon} = \overline{\Omega} \times \overline{X}$. The five angular parameters depend on the positions of the four fundamental vectors. As the vectors were defined in general to be space and time variant, it follows that the basic angular parameters are also space and time variant.

2. Reference Model

Analysis of a natural phenomenon is usually conducted through the introduction of an approximation, a so-called reference model. Using current knowledge of the phenomenon, a relatively simple model may be defined so that a reasonably good prediction of the phenomenon can be made for given space and time coordinates. In this section we define a reference model for the earth, the fundamental vectors, and the basic angular parameters defined in Section 1.

Reference Model of the Earth. The model earth is defined dynamically (from the points of view of its gravity field and rotation) as a rotationally symmetric level ellipsoid with major semiaxis a and eccentricity e. The ellipsoid rotates versus inertial space with uniform velocity ω about an axis which is slightly inclined to its minor (figure) axis in accordance with a specified polar motion model. The mass of the ellipsoid m is equal to the mass of the earth, and the parameters a, e, and ω are selected so that the normal (model) gravity potential on its surface is constant and is equal to the gravity potential on its surface is constant and is equal to the gravity potential on the geoid. The normal gravity potential at a given point, external to the ellipsoid, can be calculated from Gm, a, e, ω, and the coordinates of the point where G is the Newtonian gravitational constant [Heiskanen and Moritz, 1967, pp. 64-67].

The orientation of the rotational axis versus inertial space for a given epoch is calculated by the currently adopted models and parameters of general precession and astronomic nutation.

Geometrically, the model earth has a rigid irregular surface: the telluroid at a specified fundamental epoch [ibid., pp. 291-204]. Thus distances and angles between model surface points are assumed to be time invariant.

The Fundamental Model Vectors. We define the fundamental vectors of the model in a similar manner as for the natural case:

$\bar{q}$ the observational vector is the straight line from the observing point P to the target point Q as affected by aberration and parallax

$-\overline{\gamma}$ the local vertical vector is opposite in direction to the vertical gradient of the normal gravity field at P

$\overline{\omega}$ the model rotational vector at point P

$\overline{x}$ the vector normal to the mean ecliptic plane

The model fundamental vectors at a given epoch are related through model angular parameters similar to the natural ones as follows:

β model altitude

α model azimuth

ϕ model latitude

h model hour angle of the vernal equinox

ε obliquity of the mean ecliptic

At a given epoch we can project on the unit sphere the natural and the model fundamental vectors as shown in Fig. 2. The four differences $\delta\overline{q}$, $\delta\overline{\gamma}$, $\delta\overline{\omega}$, $\delta\overline{x}$ are called disturbance vectors. The disturbances in the basic angular parameters are

$$\delta\beta = B - \beta$$

$$\delta\alpha = A - \alpha$$

$$\delta\phi = \Phi - \phi$$

$$\delta h = H - h$$

$$\delta\varepsilon = E - \varepsilon$$

The mathematical relationships between the disturbance vectors and the disturbances in the angular parameters are given in Section 7.

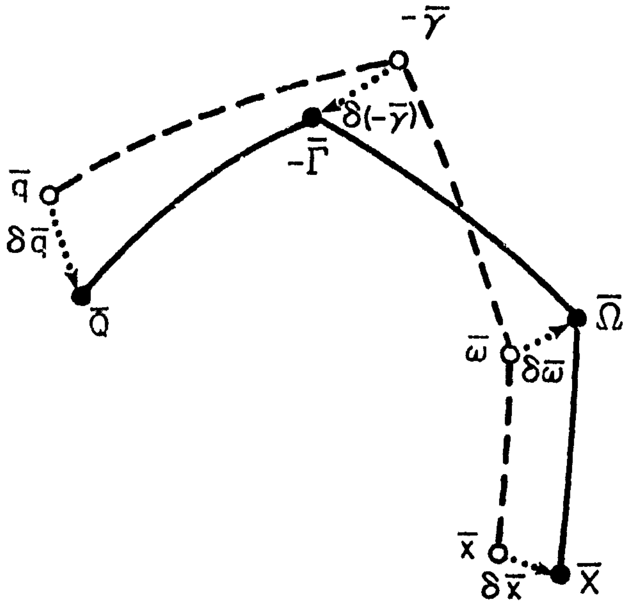

Fig. 2

3. Space- and Time-Like Variations of the Fundamental Vectors

The fundamental vectors defined for the natural case and for the reference model vary in space and in time. The space-like variation of $\overline{V}$ is the difference between $\overline{V} + d\overline{V}$ at a second point P + dP, in the neighborhood of P, and $\overline{V}$ at the same epoch. We can express the space-like variation of $\overline{V}$ as its partial derivative versus the space variable: $\partial\overline{V}/\partial S$.

In a way similar to the space-like variation, we define the time-like variation of $\overline{V}$ at point P and epoch T as its partial derivative versus the time variable: $\partial\overline{V}/\partial T$. The interpretation of the time-like variation is complicated by the necessity of defining the inertial frame as the common reference for the two states of the vector, i.e., $\overline{V}_{(T)}$ and $\overline{V}_{(T + dT)}$. To simplify our treatment of variations, the fundamental vectors are placed in a hierarchy beginning with $\overline{Q}$ through

$-\overline{\Gamma}$, $\overline{\Omega}$, $\overline{X}$, up to $\overline{i}$ which is considered as any inertial vector. Thus, in order to obtain the absolute derivative of a fundamental vector $\overline{V}$, $\frac{\partial \overline{V}}{\partial T}$, we have to differentiate both the coordinates of $\overline{V}$ and the base vectors defined by the fundamental vector on the next higher stage. Therefore we have to connect these base vectors with the inertial frame by means of time-variable rotation matrices. These systems of base vectors and rotation matrices will be introduced in detail in the next chapter. In Table 1 we have listed certain phenomena causing the variations and disturbances of the four fundamental vectors. A point to be kept in

Table 1

Sources of Variations of the Fundamental Vectors

Fundamental Vector	Space-Like Variations		Time-Like Variations	
	Model	Disturbances	Model	Disturbances
$\overline{Q}$	parallax, aberration	refraction	relative motion of target	perturbations in motion of target
$-\overline{\Gamma}$	positional difference	deflections of the vertical	constant spin rate, model polar motion	correction to polar motion, spin rate variations, tides, mass redistributions
$\overline{\Omega}$	--	local rotations	luni-solar precession + nutation	correction to luni-solar precession + nutation
$\overline{X}$	--	--	planetary precession	correction to planetary precession, ecliptic wobble
$\overline{i}$	--	--		

mind is that certain phenomena associated with space-like variations of a vector are not necessarily time invariant and vice versa, as, for example, refraction or deflections of the vertical.

4. Natural and Model Triads

The fundamental vectors defined in the preceding sections can be used to define orthonormal vector bases, or triads. According to the vectors used there will be natural and model triads.

Observational Triad - E1. The three axes of the triad E1 at the point P and epoch T are defined by the vectors $\overline{Q}$ and $-\overline{\Gamma}$ as

$$\overline{E1}^3 = \text{norm}\ \overline{Q}$$

$$\overline{E1}^2 = \text{norm}\ [\overline{Q} \times (-\overline{\Gamma})]$$

$$\overline{E1}^1 = \overline{E1}^3 \times \overline{E1}^2$$

Local Horizon Triad - E2. The axes of E2 are defined by the vectors $-\overline{\Gamma}$ and $\overline{\Omega}$ as follows:

$$\overline{E2}^3 = \text{norm}\ -\overline{\Gamma}$$

$$\overline{E2}^2 = \text{norm}\ [\overline{\Omega} \times (-\overline{\Gamma})]$$

$$\overline{E2}^1 = \overline{E2}^3 \times \overline{E2}^2$$

Equatorial Triad - E3. The axes of E3 are defined by the vectors $\overline{\Omega}$ and $\overline{X}$ as follows:

$$\overline{E3}^3 = \text{norm}\ \overline{\Omega}$$

$$\overline{E3}^1 = \text{norm}\ (\overline{\Omega} \times \overline{X})$$

$$\overline{E3}^2 = \overline{E3}^3 \times \overline{E3}^1$$

Ecliptic Triad - E4. The axes of E4 are defined by the vectors $\overline{X}$ and $\overline{i}$. At this stage we introduce the inertial triad e which is a space- and time-invariant orthonormal vector base. Its specific orientation is not important at the moment and will be left undefined. Vector $\overline{i}$ is parallel to axis $\overline{e}^3$ of the inertial triad e. The definition of E4 is as follows:

$$\overline{E4}^3 = \text{norm } \overline{X}$$
$$\overline{E4}^1 = \text{norm } (\overline{i} \times \overline{X})$$
$$\overline{E4}^2 = \overline{E4}^3 \times \overline{E4}^1$$

Note that axis 1 of E4 does not necessarily point towards the vernal equinox as is the case with the ecliptic system used in astronomy.

The triads of the reference model are defined similarly, the only difference being the substitution of the model fundamental vectors $\overline{q}$, $-\overline{\gamma}$, $\overline{\omega}$, $\overline{x}$ for the natural ones. The model triads are denoted by lower case letters e1, e2, etc.).

The above definitions result in left-handed systems in E1 and E2, and in angular parameters (altitude, azimuth, etc.) in accordance with geodetic conventions (see [Mueller, 1969, pp. 32-42]). The triads, based on the same fundamental vectors, could also be defined more systematically (i.e., all right-handed), but in that case the angular parameters would not comply with presently accepted conventions.

Transformation Between the Triads. We derive the orthogonal (rotational) transformations between the sequence of triads by introducing three additional angular patameters, ψ_1, ψ_2, ψ_3 (see Fig. 3), which together with the basic angular parameters α, β, ϕ, h, and ε

serve as parameters in the transformations. The sequence of transformations is as follows:

$$e4 = R_1(\psi_2)\ R_3(-\psi_3)\ e$$

$$e3 = R_1(-\epsilon)\ R_3(\psi_1)\ e4$$

$$e2 = P_1 R_2(\pi/2 - \phi)\ R_3(h) e3$$

$$e1 = R_2(\pi/2 - \beta)\ R_3(\alpha)\ e2$$

where $R_j(\mu)$ is a conventional rotational matrix around the j axis by an angle μ ($j = 1, 2, 3$) [Mueller, 1969, pp. 43-44],

P_k is a permutation matrix of the axis k ($k = 1, 2, 3$)

ei stands for the triad $\begin{bmatrix} \overline{ei}^1 \\ \overline{ei}^2 \\ \overline{ei}^3 \end{bmatrix}$

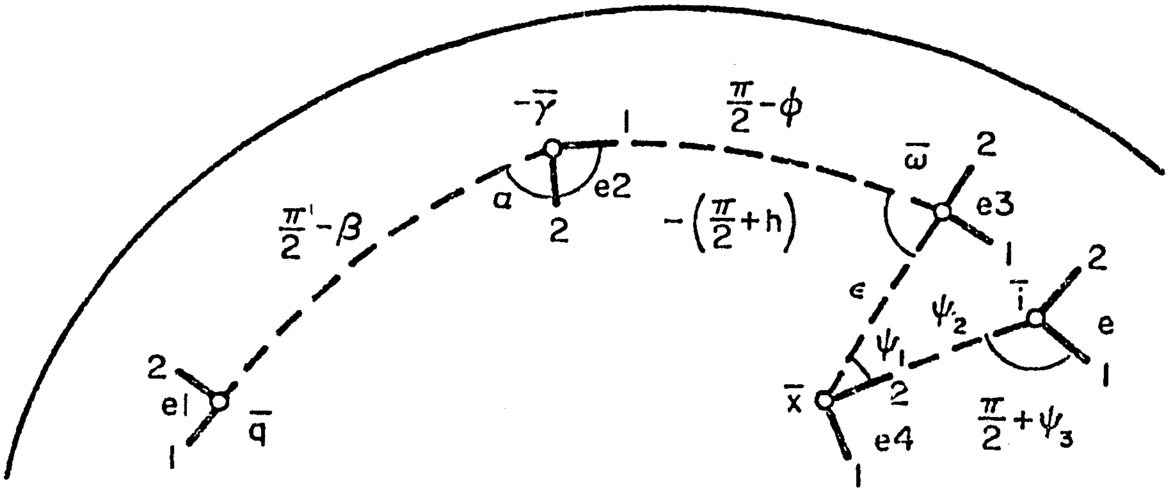

Fig. 3

The transformations are orthogonal so the inverse relations can be obtained in general by reversing the order of the rotational matrices and also the sign of the rotational angle. The transformations between the sequence of the natural triads Ei (i = 1, 2, 3, 4) and e (the inertial triad) are the same, except that instead of the model angles one must use the natural parameters A, B, Φ, H, E and also Ψ_1, Ψ_2, Ψ_3 (the latter group for the transformation between the inertial triad and E4).

5. Variations of a Triad

Since the triads are defined by the fundamental vectors, it is obvious that their directional variations will involve a rotation of the triad. Such variations are possible in three dimensions: 1) in space, 2) in time, 3) by the transition from the natural to the model fundamental vectors or vice versa.

Instead of analyzing separately the effects of these variations and disturbances, we shall study in a general way the influence of the variation of the fundamental vectors on the triads defined by them. It should be relatively easy, once the general formulae are available, to specify the kind of variation and the specific triad to which it applies. The same holds true for the disturbances.

Let $\overline{Z}$ and $\overline{D}$ be two fundamental vectors ($\overline{Z}$ the "lower" and $\overline{D}$ the "upper" one). The triad of which $\overline{D}/|\overline{D}|$ (= norm D) is the 3-vector is called $\overline{E}^{\cdot} = [\overline{E}^{1\cdot}, \overline{E}^{2\cdot}, \overline{E}^{3\cdot}]^T$. The triad of which norm $\overline{Z}$ is the 3-vector is called $\overline{E}^{*}$ being defined as

$$\overline{E}^{3*} = \text{norm } \overline{Z}$$

$$\overline{E}^{2*} = \text{norm } (\overline{D} \times \overline{Z})$$

$$\overline{E}^{1*} = \overline{E}^{2*} \times \overline{E}^{3*}$$

The representation of $\overline{D}$ in both systems is:

$$\overline{D} = [D^{1\cdot}, D^{2\cdot}, D^{3\cdot}]\ \overline{E}^{\cdot}$$

with the coordinates $D^{1\cdot} = D^{2\cdot} = 0$, $D^{3\cdot} = |\overline{D}|$, and, since the relation between $E^{\cdot}$ and E^{*} is

$$\overline{E}^{*} = R_2(\Pi/2 - \Phi)\ R_3(\Lambda)\ \overline{E}^{\cdot}\ ,$$
$$\overline{D} = [D^{1*},\ D^{2*},\ D^{3*}]\ \overline{E}^{*}$$

where $[D^{1*},\ D^{2*},\ D^{3*}]^T = R_2(\Pi/2 - \Phi)\ R_3(\Lambda)\ [D^{1\cdot},\ D^{2\cdot},\ D^{3\cdot}]^T$
$= [-|\overline{D}|\ \cos\Phi,\ 0,\ |\overline{D}|\ \sin\Phi]$

or $\quad \overline{D} = -|\overline{D}|\ \cos\Phi\ \overline{E}^{1*} + |\overline{D}|\ \sin\Phi\ \overline{E}^{3*}\ .$

The fundamental vector $\overline{Z}$ is in the $\overline{E}^{*}$-system:

$$\overline{Z} = [Z^{1*},\ Z^{2*},\ Z^{3*}]\ \overline{E}^{3*}$$

with the coordinates $Z^{1*} = Z^{2*} = 0,\ Z^{3*} = |\overline{Z}|,$

or $\quad \overline{Z} = |\overline{Z}|\ \overline{E}^{3*}$

The variations of $\overline{D}$ and $\overline{Z}$ are

$$d\overline{D} = [dD^{1\cdot},\ dD^{2\cdot},\ dD^{3\cdot}]\ \overline{E}^{\cdot}$$
$$= [dD^{1*},\ dD^{2*},\ dD^{3*}]\ \overline{E}^{*}$$

with $\begin{bmatrix} dD^{1*} \\ dD^{2*} \\ dD^{3*} \end{bmatrix} = \begin{bmatrix} \cos\Lambda\sin\Phi\, dD^{1\cdot} + \sin\Lambda\sin\Phi\, dD^{2\cdot} - \cos\Phi\, dD^{3\cdot} \\ -\sin\Lambda\, dD^{1\cdot} + \cos\Lambda\, dD^{2\cdot} \\ \cos\Lambda\cos\Phi\, dD^{2\cdot} + \sin\Lambda\cos\Phi\, dD^{2\cdot} + \sin\Phi\, dD^{3\cdot} \end{bmatrix}$

$$d\overline{Z} = [dZ^{1*},\ dZ^{2*},\ dZ^{3*}]\ \overline{E}^{*}$$

Now let us construct the new base vectors $\overline{E}^{*} + d\overline{E}^{*}$ after a small variation of the fundamental vectors $\overline{D}$ and $\overline{Z} \Rightarrow \overline{D} + d\overline{D}$ and $\overline{Z} + d\overline{Z}$.

$$\begin{aligned}\bar{E}^{3*} + d\bar{E}^{3*} &= \text{norm } (\bar{Z} + d\bar{Z}) \\ &= \text{norm } \{[dZ^{1*}, dZ^{2*}, Z^{3*} + dZ^{3*}] \bar{E}^{*}\} \\ &= [\frac{dZ^{1*}}{Z^{3*}}, \frac{dZ^{2*}}{Z^{3*}}, 1] \bar{E}^{*}\end{aligned}$$

$$\begin{aligned}\bar{E}^{2*} + d\bar{E}^{2*} &= \text{norm } [(\bar{D} + d\bar{D}) \times (\bar{Z} + d\bar{Z})] \\ &= \text{norm } \{[dD^{2*}(Z^{3*} + dZ^{3*}) - (D^{3*} + dD^{3*})dZ^{2*}] \bar{E}^{1*} + \\ &\qquad + [(D^{3*}+dD^{3*})dZ^{1*} - (D^{1*}+dD^{1*})(Z^{3*}+dZ^{3*})]\bar{E}^{2*} + \\ &\qquad + [(D^{1*}+dD^{1*})dZ^{2*} - dD^{2*}dZ^{1*}] \bar{E}^{3*}\} \\ &= [-\frac{dD^{2*}}{D^{1*}} + \frac{D^{3*}dZ^{2*}}{D^{1*}Z^{3*}}, 1, -\frac{dZ^{2*}}{Z^{3*}}] \bar{E}^{*}\end{aligned}$$

$$\begin{aligned}\bar{E}^{1*} + d\bar{E}^{1*} &= (\bar{E}^{2*} + d\bar{E}^{2*}) \times (\bar{E}^{3*} + d\bar{E}^{3*}) \\ &= [1, \frac{dD^{2*}}{D^{1*}} - \frac{D^{3*}dZ^{2*}}{D^{1*}Z^{3*}}, -\frac{dZ^{1*}}{Z^{3*}}] \bar{E}^{*}\end{aligned}$$

Collecting the new base vectors in one column matrix, we obtain

$$\begin{bmatrix} \bar{E}^{1*} + d\bar{E}^{1*} \\ \bar{E}^{2*} + d\bar{E}^{2*} \\ \bar{E}^{3*} + d\bar{E}^{3*} \end{bmatrix} = \bar{E}^{*} + d\bar{E}^{*} = (I + \Omega) \bar{E}^{*}$$

where the antisymmetric matrix

$$\Omega = \begin{bmatrix} 0 & \frac{dD^{2*}}{D^{1*}} - \frac{D^{3*}dZ^{2*}}{D^{1*}Z^{3*}} & -\frac{dZ^{1*}}{Z^{3*}} \\ -\frac{dD^{2*}}{D^{1*}} + \frac{D^{3*}dZ^{2*}}{D^{1*}Z^{3*}} & 0 & -\frac{dZ^{2*}}{Z^{3*}} \\ \frac{dZ^{1*}}{Z^{3*}} & \frac{dZ^{2*}}{Z^{3*}} & 0 \end{bmatrix}$$

is the Cartan matrix Ω [Grafarend, 1977, pp.159-160]. Expressing the elements of the $\overline{D}$-vector in terms of the $\overline{E}'$-frame we get

$$\Omega = \begin{bmatrix} 0 & \begin{matrix} \sin\Lambda\sec\Phi\frac{dD^{1\cdot}}{|\overline{D}|} \\ -\cos\Lambda\sec\Phi\frac{dD^{2\cdot}}{|\overline{D}|} + \tan\Phi\frac{dZ^{2*}}{|\overline{Z}|} \end{matrix} & -\frac{dZ^{1*}}{|\overline{Z}|} \\ \begin{matrix} -\sin\Lambda\sec\Phi\frac{dD^{1\cdot}}{|\overline{D}|} \\ +\cos\Lambda\sec\Phi\frac{dD^{2\cdot}}{|\overline{D}|} - \tan\Phi\frac{dZ^{2*}}{|\overline{Z}|} \end{matrix} & 0 & -\frac{dZ^{2*}}{|\overline{Z}|} \\ \frac{dZ^{1*}}{|\overline{Z}|} & \frac{dZ^{2*}}{|\overline{Z}|} & 0 \end{bmatrix}$$

Now apply the general expressions derived above to any of the geodetic triads. For example, by identification of $\overline{D}$ with $\overline{\Omega}$ and $\overline{Z}$ with $-\overline{\Gamma}$, of $d\overline{D}$ with polar motion and $d\overline{Z}$ with a change of the vertical direction, we find the influence of polar motion and of a change of the vertical onto the orientation of the horizontal system. Λ and Φ are then longitude and latitude, $\frac{dD^{1\cdot}}{|\overline{D}|} = x$ and $\frac{dD^{2\cdot}}{|\overline{D}|} = -y$ the components of polar motion, $\frac{dZ^{1*}}{|\overline{Z}|} = k_1$ and $\frac{dZ^{2*}}{|\overline{Z}|} = k_2$ the angles of vertical change in north-south and east-west directions respectively. In order to get the horizontal system north-oriented, some signs have to be changed. Thus we finally obtain

$$d\bar{E}2^1 = (-\sin\Lambda\sec\Phi\, x - \cos\Lambda\sec\Phi\, y + \tan\Phi\, k_2)\, \bar{E}2^2 + k_1\, \bar{E}2^3$$

$$d\bar{E}2^2 = (\sin\Lambda\sec\Phi\, x + \cos\Lambda\sec\Phi\, y - \tan\Phi\, k_2)\, \bar{E}2^1 + k_2\, \bar{E}2^3$$

$$d\bar{E}2^3 = -k_1\bar{E}2^1 - k_2\, \bar{E}2^2$$

There are many similar applications of the general formula, for instance the influence of a change of the vertical and a motion of the target on the observational triad, or the dependence of the equatorial system on planetary precession, luni-solar precession and nutation. While in the first two examples the motion is relative to an earth-fixed observer, it is described relative to an inertial system in the latter one. The general formula is valid for both cases.

We can interpret the Cartan matrix as a rotation matrix of three differential Cardan angles about the three E^*-axes, the first angle being $-\frac{dZ^{2*}}{|\bar{Z}|}$, the second $\frac{dZ^{1*}}{|\bar{Z}|}$ and the third $\sin\Lambda\sec\Phi\frac{dD^{1\cdot}}{|\bar{D}|}$ - $-\cos\Lambda\sec\Phi\frac{dD^{2\cdot}}{|\bar{D}|} + \tan\Phi\frac{dZ^{2*}}{|\bar{Z}|}$. Later on we shall call them τ^i in the first level (observational triad), ν^i in the second level (horizontal triad), ξ^i in the third level (equatorial triad) and μ^i in the fourth level (ecliptic triad), i = 1,2,3.

6. The Commutative Diagram of Triads

To obtain a better insight into the interrelations between the various triads, we will construct three-dimensional structures to be referred to as the E(e) Towers or the Commutative Diagram of Triads (see Fig. 4). Each point in the diagram represents a certain triad

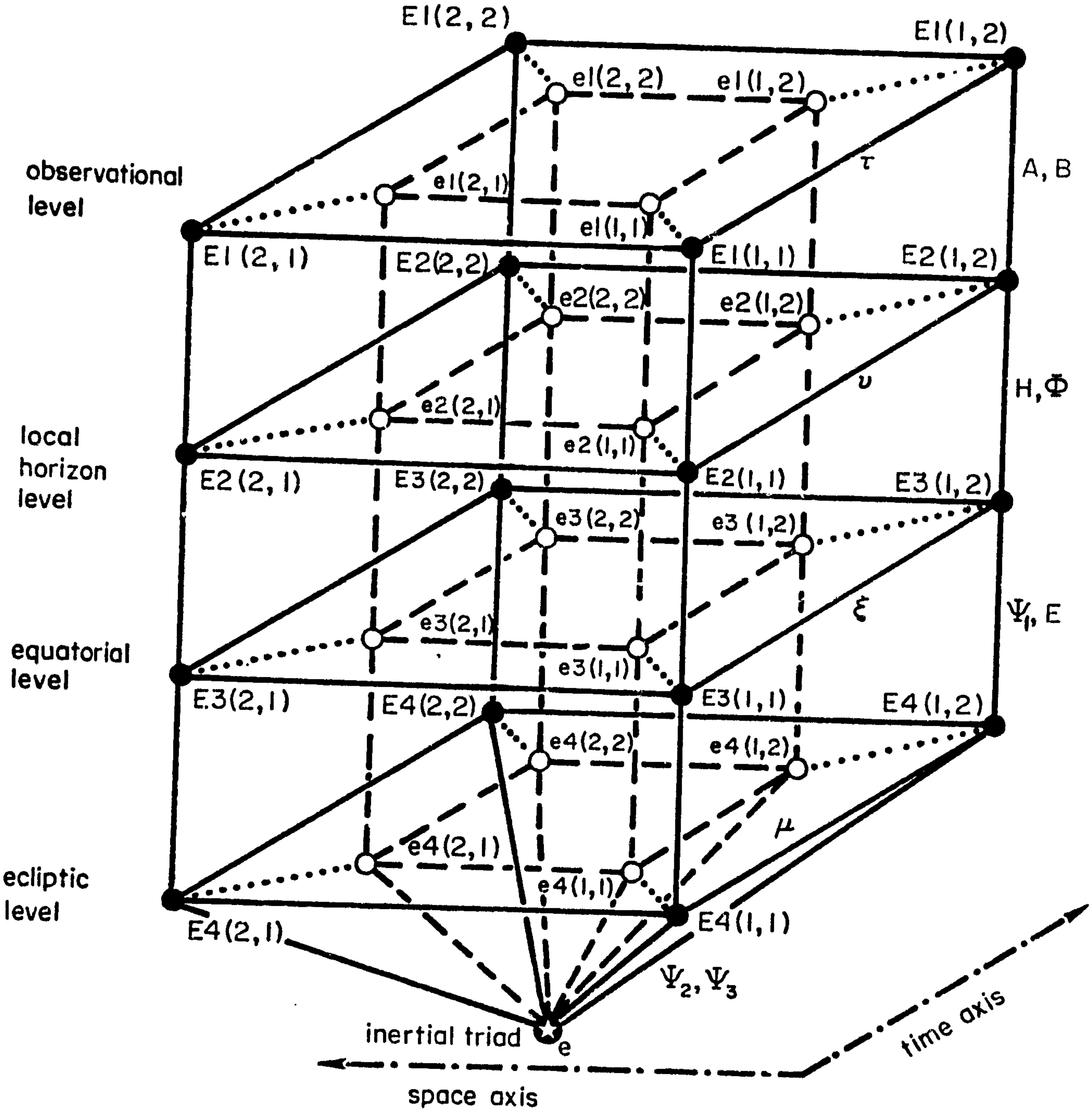
E1(2,2)
E1(1,2)
e1(2,2)
e1(1,2)
τ
A, B
observational level
e1(2,1)
e1(1,1)
E1(2,1)
E2(2,2)
E1(1,1)
E2(1,2)
e2(2,2)
e2(1,2)
υ
H, Φ
e2(2,1)
local horizon level
e2(1,1)
E2(2,1)
E3(2,2)
E2(1,1)
E3(1,2)
e3(2,2)
e3(1,2)
ξ
Ψ_1, E
e3(2,1)
equatorial level
e3(1,1)
E3(2,1)
E4(2,2)
E3(1,1)
E4(1,2)
e4(2,2)
e4(1,2)
μ
e4(2,1)
ecliptic level
e4(1,1)
E4(2,1)
E4(1,1)
Ψ_2, Ψ_3
inertial triad
e
time axis
space axis

Fig. 4

according to the label attached to it. The straight lines between the points represent orthogonal (rotational) transformations between the respective triads. The overall organization of the diagram is as follows:

E-tower	tower of the natural triads (solid lines)
e-tower	tower of the model triads (dashed lines)
levels	1, 2, 3, 4 - according to the type of triads, i.e., observational, horizontal, etc.
space-like variations	the lines parallel to the space axis represent space-like variations of the triads
time-like variations	the lines parallel to the time axis represent time-like variations of the triads
disturbances	the diagonal (dotted) lines which run on level i between an Ei triad and the corresponding ei triad. These are the only connections between the E-tower and the e-tower and represent the disturbances explained in Section 2.

The diagram thus represents all triads, their space- and time-like variations, and model disturbances at a single point P. In order to identify space- and time-like variations, we introduce two indices (j, k) which follow the symbol Ei or ei of the triad. Both j and k can be 1 or 2, where index j = 1 stands for triads at P and j = 2 at P + dP. In a similar manner k = 1 stands for triads at epoch T while k = 2 at T + dT. Thus the index (1, 1) indicates the situation at P at epoch T, (2, 1) is after a space-like, (1, 2) after a time-like variation; (2, 2) represents the situation when both space- and time-like variations affected the trial (1, 1).

Interlevel Transformations. In Section 4 we derived the interlevel transformations along a typical sequence Ei(j, k), i = 1,2,3,4. Fig. 4 shows the pairs of parameters involved in a transformation between two adjacent triads along a column of the tower: A, B; H, Φ; etc. As these interlevel parameters are space and time variant, it is obvious that they carry j and k indices matching the column of triads. We have identified the various parameters and the respective triad columns of the tower structure where they apply in Table 2. For compactness of representation, denote by σ the vector of model angular parameters as follows:

$$\sigma^T = [\alpha, \beta, h, \phi, \varepsilon, \psi_1, \psi_2, \psi_3]$$

Following the notation introduced in Section 3, we denote space variations by $\partial\sigma/\partial S$, time variations by $\partial\sigma/\partial T$, disturbances (natural minus model) by $\delta\sigma$, space variations of disturbances by $\partial(\delta\sigma)/\partial S$, and time variations of disturbances by $\partial(\delta\sigma)/\partial T$.

Table 2

Interlevel Transformations

σ_s	**Transformation Parameters**	**Tower Column**
σ_1	σ	**ei (1,1)**
σ_2	$\sigma + \frac{\partial \sigma}{\partial S} dS$	**ei (2,1)**
σ_3	$\sigma \qquad + \frac{\partial \sigma}{\partial T} dT$	**ei (1,2)**
σ_4	$\sigma + \frac{\partial \sigma}{\partial S} dS + \frac{\partial S}{\partial T} dT$	**ei (2,2)**
σ_5	$\delta\sigma + \sigma$	**Ei (1,1)**
σ_6	$\delta\sigma + \sigma + \left[\frac{\partial \sigma}{\partial S} + \frac{\partial (\delta\sigma)}{\partial S}\right] dS$	**Ei (2,1)**
σ_7	$\delta\sigma + \sigma \qquad + \left[\frac{\partial \sigma}{\partial T} + \frac{\partial (\delta\sigma)}{\partial T}\right] dT$	**Ei (1,2)**
σ_8	$\delta\sigma + \sigma + \left[\frac{\partial \sigma}{\partial S} + \frac{\partial (\delta\sigma)}{\partial S}\right] \delta S + \left[\frac{\partial \sigma}{\partial T} + \frac{\partial (\delta\sigma)}{\partial T}\right] dT$	**Ei (2,2)**

Inlevel Transformations. We have defined in Section 5 the differential inlevel transformation vectors. In Fig. 5 we can see a total of 12 such vectors for level one. As the changes in the space and time variables dS and dT are differential and the diagram is commutative, there are seven independent conditions to be fulfilled:

$$\bar{\tau}_6 = \bar{\tau}_1 \qquad \bar{\tau}_{10} = -\bar{\tau}_2 + \bar{\tau}_3 + \bar{\tau}_5$$
$$\bar{\tau}_7 = \bar{\tau}_2 \qquad \bar{\tau}_{11} = -\bar{\tau}_6 + \bar{\tau}_{10} + \bar{\tau}_8$$
$$\bar{\tau}_8 = \bar{\tau}_4 \qquad \bar{\tau}_{12} = \bar{\tau}_7 + \bar{\tau}_{11} + \bar{\tau}_9$$
$$\bar{\tau}_9 = \bar{\tau}_5$$

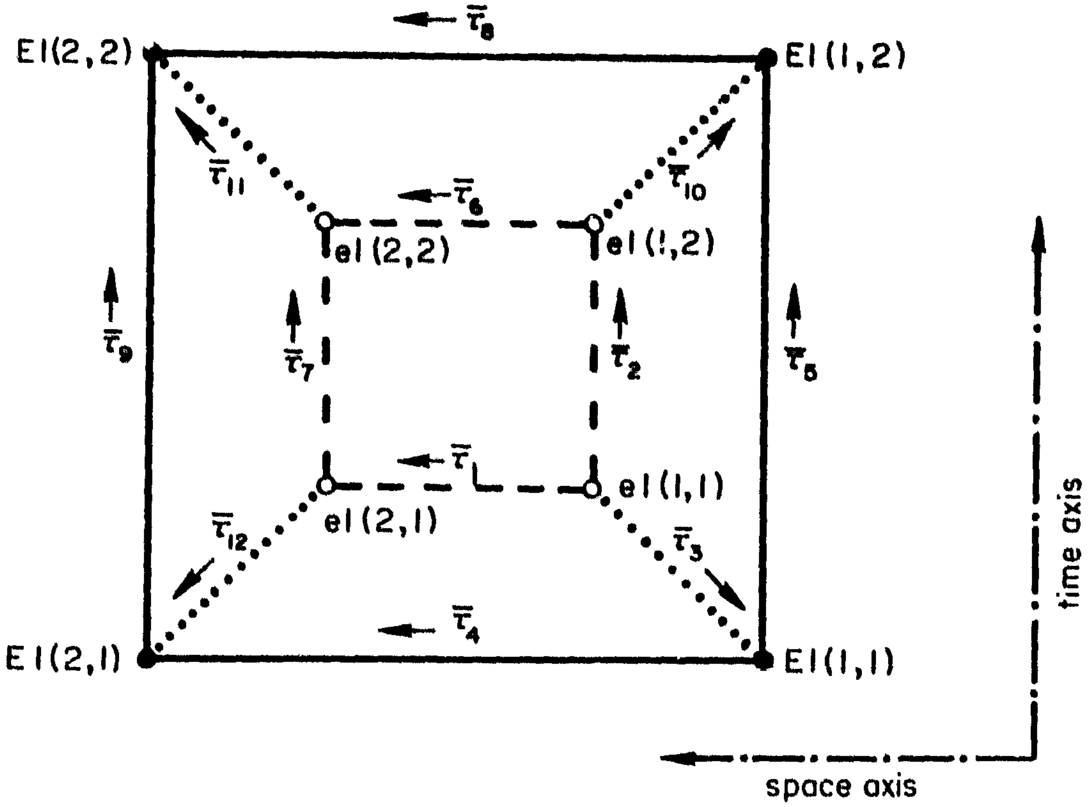

Fig. 5

and therefore only five independent $\bar{\tau}_n$ vectors left. These represent the following variations in the triads of level one:

$\bar{\tau}_1$	space-like	variations of model triads
$\bar{\tau}_2$	time-like	
$\bar{\tau}_3$	disturbances	
$\bar{\tau}_4$	space-like	variations of natural triads
$\bar{\tau}_5$	time-like	

The various inlevel transformation parameters ($\bar{\tau}$, $\bar{\nu}$, etc.) can be expressed as functions of the relevant space- and time-like variations of the fundamental vectors as shown in Section 5.

7. Variations in the Basic Angular Parameters as a Function of Variations in the Fundamental Vectors

We are faced with a large number of transformation parameters required to relate the various triads in the towers. We have already taken a step towards reducing their number in the inlevel transformations. We will complete the reduction process by expressing the variations of the basic angular parameters (interlevel transformations) as a function of variations of the fundamental vectors and show that the transformation between any two triads in the towers is dependent on the variations of the four fundamental vectors only.

Following ideas in [Grafarend, 1977, pp. 207-212], in Fig. 6 we have four triads which together form a closed loop of a commutative diagram. This loop is used as a typical example, and therefore the subscripts of $\bar{\tau}$ and $\bar{\nu}$ (which are $\bar{\tau}_3$, $\bar{\nu}_3$, i.e., disturbances) are not indicated. From previous sections we have

$$e2(1,1) = R_E(\alpha,\beta)\ e1(1,1)$$

$$E1(1,1) = R_C(\bar{\tau})\ e1(1,1)$$

$$E2(1,1) = R_C(\bar{\nu})\ e2(1,1)$$

$$E2(1,1) = R_E(\alpha + \delta\alpha,\ \beta + \delta\beta)\ E1(1,1)$$

where $R_E(\alpha,\beta) = R_3(-\alpha)\ R_2(\beta - \pi/2)$, Eulerian rotation matrix

$R_C(\bar{\tau}) = R_3(\tau_3)\ R_2(\tau_2)\ R_1(\tau_1)$ and

$R_C(\bar{\nu}) = R_3(\nu_3)\ R_2(\nu_2)\ R_1(\nu_1)$, Cardanian rotation matrices

Thus

$$R_C(\bar{\nu})\ e2(1,1) = R_E(\alpha + \delta\alpha,\ \beta + \delta\beta)\ E1(1,1), \text{ or}$$

$$R_C(\bar{\nu})\ R_E(\alpha,\beta) = R_E(\alpha + \delta\alpha,\ \beta + \delta\beta)\ R_C(\bar{\tau})$$

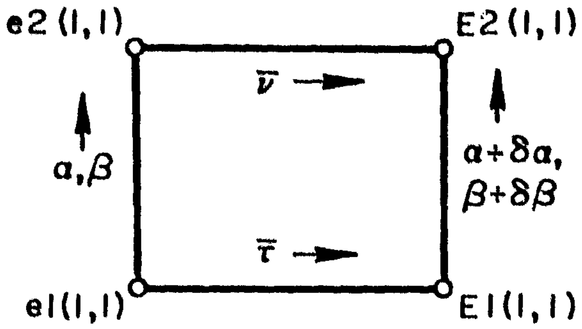

Fig. 6

From here we can arrive, by patient algebra, at the following expressions:

$$\begin{bmatrix}\delta\alpha\\ \delta\beta\end{bmatrix} = \begin{bmatrix}-\cos\beta & 0 & \sin\beta\\ 0 & -1 & 0\end{bmatrix}\bar{\tau} + \begin{bmatrix}0 & 0 & -1\\ -\sin\alpha & \cos\alpha & 0\end{bmatrix}\bar{\nu}$$

Treating in a similar way the other loops of the columns Ei(1,1), ei(1,1), we get

$$\begin{bmatrix}\delta h\\ \delta\phi\end{bmatrix} = \begin{bmatrix}-\cos\phi & 0 & -\sin\phi\\ 0 & 1 & 0\end{bmatrix}\bar{\nu} + \begin{bmatrix}0 & 0 & -1\\ -\sin h & \cos h & 0\end{bmatrix}\bar{\xi}$$

$$\begin{bmatrix}\delta\psi_1\\ \delta\varepsilon\end{bmatrix} = \begin{bmatrix}0 & -\sin\varepsilon & \cos\varepsilon\\ -1 & 0 & 0\end{bmatrix}\bar{\xi} + \begin{bmatrix}0 & 0 & -1\\ \cos\psi_1 & \sin\psi_1 & 0\end{bmatrix}\bar{\mu}$$

Substitute into $\bar{\tau}$, $\bar{\nu}$, $\bar{\xi}$, $\bar{\mu}$ their equivalents, perform the multiplications, and rearrange. The results are summarized in Table 3. The matrix presented is actually the matrix of partial derivatives of the basic angular parameters vs. variations of the fundamental vectors. It should be obvious that the matrix would not change if we considered space-like variations or time-like variations instead of the

Table 3

Angular Parameter Variations As a Function of Variations in the Fundamental Vectors

	δq_1	δq_2	$\delta(-\gamma)_1$	$\delta(-\gamma)_2$	$\delta\omega_1$	$\delta\omega_2$	δx_1	δx_3
$\delta\alpha$	0	$\sec\beta$	$\tan\beta\sin\alpha$	$\tan\varphi$ $-\tan\beta\cos\alpha$	$\sec\varphi\sin h$	$-\sec\varphi\cos h$	0	0
$\delta\beta$	-1	0	$\cos\alpha$	$\sin\alpha$	0	0	0	0
δh	0	0	0	$\sec\varphi$	$\cot\epsilon$ $+\tan\varphi\sin h$	$-\tan\varphi\cos h$	$-\operatorname{cosec}\epsilon\cdot\cos\psi_1$	$-\operatorname{cosec}\epsilon\cdot\sin\psi_1$
$\delta\varphi$	0	0	1	0	$\cos h$	$\sin h$	0	0
$\delta\epsilon$	0	0	0	0	0	1	$\sin\psi_1$	$-\cos\psi_1$

Note: For phenomena associated with variations of the fundamental vectors, see Table 1.

disturbances in the derivation as long as Fig. 4 is a commutative diagram. Table 3 represents the situation in the model. For the natural parameters, the relationships are, of course, identical.

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Appendix A

Differentials of a Compound Rotation Matrix

Preliminaries

Analytical expression for the differentials of an orthogonal matrix R which represents a sequence of elementary rotations is the subject of this Appendix. Rotation matrices $R_i(\theta)$ are used in orthogonal coordinate transformations as shown in [Mueller, 1969, p. 43] where θ is the angle of rotation and i is the axis about which the rotation is performed.

The differentiation of a rotation matrix $R_i(\theta)$ with respect to the angle θ is obtained by pre- or post-multiplying the $R_i(\theta)$ matrix by a skew symmetric L_i matrix

$$\frac{\partial R_i(\theta)}{\partial \theta} = L_i R_i(\theta) = R_i(\theta) L_i$$

The L_i matrix is defined as the i layer of the skew-symmetric e_{ijk} system as shown in [Lucas, 1963]. The rotation and Lucas' matrices are

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad L_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad L_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A rotation matrix $R_i(\theta)$ or a product of two or more rotation matrices are orthogonal 3 x 3 matrices with the following two properties:

(i) The determinant is equal to one.

(ii) The inverse is equal to the transpose.

The above properties of a 3 x 3 orthogonal matrix A can be utilized for deriving the elements of the adjoint matrix of A. As is well known the adjoint of a nonsingular matrix (the transposed matrix of its cofactors) divided by its determinant is equivalent to its inverse

$$\frac{\text{adj. } A}{|A|} = A^{-1}$$

According to the properties of A as stated above, i.e.,

$$|A| = 1 \quad \text{and} \quad A^{-1} = A^T$$

one has

$$\text{adj. } A = A^T$$

or explicitly

$$\text{adj } A = \begin{bmatrix} (a_{22}a_{33}-a_{32}a_{23}) & -(a_{12}a_{33}-a_{13}a_{32}) & (a_{12}a_{23}-a_{22}a_{13}) \\ -(a_{21}a_{33}-a_{31}a_{23}) & (a_{11}a_{33}-a_{31}a_{13}) & -(a_{11}a_{23}-a_{12}a_{13}) \\ (a_{21}a_{32}-a_{22}a_{31}) & -(a_{11}a_{32}-a_{12}a_{31}) & (a_{11}a_{22}-a_{12}a_{21}) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Use the above result in deriving an expression for the matrix product S

$$S = ABA^T$$

where A is a 3 x 3 orthogonal matrix and B is a skew symmetric matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad B = \begin{bmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{bmatrix}$$

Perform the multiplication, regroup to obtain

$$S = \begin{matrix} 0 & \begin{array}{l} (a_{12}a_{23}-a_{22}a_{13})b_1 \\ +\ (a_{21}a_{13}-a_{23}a_{11})b_2 \\ +\ (a_{22}a_{11}-a_{21}a_{12})b_3 \end{array} & \begin{array}{l} (a_{33}a_{12}-a_{32}a_{13})b_1 \\ +\ (a_{31}a_{13}-a_{33}a_{11})b_2 \\ +\ (a_{32}a_{11}-a_{31}a_{12})b_3 \end{array} \\ & 0 & \begin{array}{l} (a_{33}a_{22}-a_{32}a_{23})b_1 \\ +\ (a_{31}a_{23}-a_{33}a_{21})b_2 \\ +\ (a_{32}a_{21}-a_{31}a_{22})b_3 \end{array} \\ \text{skew-symmetric} & & 0 \end{matrix}$$

Using the property of the adjoint of an orthogonal matrix (A),

$$S = \begin{bmatrix} 0 & (a_{31}b_1+a_{32}b_2+a_{33}b_3) & -\ (a_{21}b_1+a_{22}b_2+a_{23}b_3) \\ & 0 & (a_{11}b_1+a_{12}b_2+a_{13}b_3) \\ \text{skew-symmetric} & & 0 \end{bmatrix}$$

Differentials of a Sequence of Rotations

The compound rotation matrix R which represents a sequence of elementary rotations $R_{i_j}(\theta_j)$ is defined as their product

$$R = R_{i_n}(\theta_n) \ldots R_{i_2}(\theta_2)R_{i_1}(\theta_1)$$

Derive an expression for the partial derivative of R with respect to one of the angles θ_j where $j = 1,2,\ldots,n$ and in a form which is convenient for programming on a computer.

Partition R into three parts

$$R = A\, R_{i_j}(\theta_j) B$$

where A, $R_{i_j}(\theta_j)$ and B are orthogonal.

$$\frac{\partial R}{\partial \theta_j} = A\, R_{i_j}(\theta_j) L_i B\ .$$

$L_i B$ can be represented as BQ_i where Q_i is a skew symmetric matrix with elements which are a function of B.

$$Q_i = B^T L_i B$$

Using the expression for S as developed earlier for each of the three cases i = 1,2,3 one gets

$$Q_1 = \begin{bmatrix} 0 & b_{13} & -b_{12} \\ -b_{13} & 0 & b_{11} \\ b_{12} & -b_{11} & 0 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 0 & b_{23} & -b_{22} \\ -b_{23} & 0 & b_{21} \\ b_{22} & -b_{21} & 0 \end{bmatrix} \quad Q_3 = \begin{bmatrix} 0 & b_{33} & -b_{32} \\ -b_{33} & 0 & b_{31} \\ b_{32} & -b_{31} & 0 \end{bmatrix}$$

The resulting partial derivative of R is thus

$$\frac{\partial R}{\partial \theta_j} = A\, R_{i_j}(\theta_j) BQ_i = RQ_i$$

The variation of the R matrix as a function of variations of the θ_j angles is obtained now easily from the above results

$$\delta R = \frac{\partial R}{\partial \theta_n} \delta\theta_n + \dots \frac{\partial R}{\partial \theta_j} \delta\theta_j + \dots \frac{\partial R}{\partial \theta_1} \delta\theta_1$$

$$\delta R = R \cdot \sum_{j=1}^{n} Q_{i_j} \delta\theta_j = R \cdot \Omega$$

where Q_{i_j} is a function of the i row of the product of the j - 1 elementary rotation matrices to the right of $R_{i_j}(\theta_j)$.

Differentials of Cardanian and Eulerian Rotation Matrices

There are two special types of compound rotation matrices which have been used extensively in deriving the various relationships in the E-tower:

Cardanian rotation matrix

$$R_C(\alpha,\beta,\gamma) = R_3(\gamma)R_2(\beta)R_1(\alpha)$$

and

Eulerian rotation matrix

$$R_E(\alpha,\beta,\gamma) = R_3(\gamma)R_2(\pi/2-\beta)R_3(\alpha)$$

Using notation and formulae developed in the preceding section one obtains for a Cardanian matrix,

$$R_C = \begin{bmatrix} \cos\beta\cos\gamma & \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma & \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma \\ -\cos\beta\sin\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{bmatrix}$$

$$\frac{\partial R_C}{\partial\alpha} = R_C L_1 \; ; \; \frac{\partial R_C}{\partial\beta} = R_3(\gamma)R_2(\beta)L_2R_1(\alpha) \; ; \; \frac{\partial R_C}{\partial\gamma} = L_3R_C$$

$$Q_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad Q_\beta = \begin{bmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & 0 \\ \cos\alpha & 0 & 0 \end{bmatrix} \quad Q_\gamma = \begin{bmatrix} 0 & \cos\alpha\cos\beta & \sin\alpha\cos\beta \\ -\cos\alpha\cos\beta & 0 & \sin\beta \\ -\sin\alpha\cos\beta & -\sin\beta & 0 \end{bmatrix}$$

$$\delta R_C = R_C \cdot [Q_\alpha\delta_\alpha + Q_\beta\delta_\beta + Q_\gamma\delta_\gamma] = R_C\Omega_C$$

$$\Omega_C = \begin{bmatrix} 0 & \sin\alpha\delta\beta + \cos\alpha\cos\beta\delta\gamma & -\cos\alpha\delta\beta + \sin\alpha\cos\beta\delta\gamma \\ & 0 & \delta\alpha + \sin\beta\delta\gamma \\ \text{skew symmetric} & & 0 \end{bmatrix}$$

The elements of the Ω_C matrix are differentially small, thus

$$R_C + \delta R_C = R_C \cdot (I+\Omega_C)$$

$$= R_C(\alpha,\beta,\gamma)R_C(\delta\alpha+\sin\beta\delta\gamma, \cos\alpha\delta\beta-\sin\alpha\cos\beta\delta\gamma, \sin\alpha\delta\beta+\cos\alpha\cos\beta\delta\gamma)$$

The derivation of a variational equation for the Eulerian matrix is as follows:

$$R_E = \begin{bmatrix} \cos\alpha\cos\gamma\sin\beta - \sin\alpha\sin\gamma & \sin\alpha\cos\gamma\sin\beta + \cos\alpha\sin\gamma & -\cos\beta\cos\gamma \\ -\cos\alpha\sin\gamma\sin\beta - \sin\alpha\cos\gamma & -\sin\alpha\sin\gamma\sin\beta + \cos\alpha\cos\gamma & \cos\beta\sin\gamma \\ \cos\alpha\cos\beta & \sin\alpha\cos\beta & \sin\beta \end{bmatrix}$$

$$\frac{\partial R_E}{\partial\alpha} = R_E L_3 \; ; \; \frac{\partial R_E}{\partial\beta} = R_3(\gamma)R_2(\pi/2-\beta)L_2^T R_3(\alpha) \; ; \; \frac{\partial R_E}{\partial\gamma} = L_3 R_E$$

$$Q_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad Q_\beta = \begin{bmatrix} 0 & 0 & \cos\alpha \\ 0 & 0 & \sin\alpha \\ -\cos\alpha & -\sin\alpha & 0 \end{bmatrix} \quad Q_\gamma = \begin{bmatrix} 0 & \sin\beta & -\sin\alpha\cos\beta \\ -\sin\beta & 0 & \cos\alpha\cos\beta \\ \sin\alpha\cos\beta & -\cos\alpha\cos\beta & 0 \end{bmatrix}$$

$$\delta R_E = R_E \cdot [Q_\alpha\delta_\alpha + Q_\beta\delta_\beta + Q_\gamma\delta_\gamma] = R_E \cdot \Omega_E$$

$$\Omega_E = \begin{bmatrix} 0 & \delta\alpha + \sin\beta\delta\gamma & \cos\alpha\delta\beta - \sin\alpha\cos\beta\delta\gamma \\ & 0 & \sin\alpha\delta\beta + \cos\alpha\cos\beta\delta\gamma \\ \text{skew symmetric} & & 0 \end{bmatrix}$$

$$R_E + \delta R_E = R_E(I+\Omega_E)$$

$$= R_E(\alpha,\beta,\gamma)\cdot R_C(\sin\alpha\delta\beta+\cos\alpha\cos\beta\delta\gamma,-\cos\alpha\delta\beta+\sin\alpha\cos\beta\delta\gamma,\delta\alpha+\sin\beta\delta\gamma)$$

Note the similarities between the Ω_C and Ω_E matrices:

$$\Omega_{C_{12}} = \Omega_{E_{23}} \qquad \Omega_{C_{23}} = \Omega_{E_{12}} \qquad \Omega_{C_{13}} = -\Omega_{E_{13}}$$

Appendix B

Differential Relationships Between Model and Natural Triads, Vectors and Angular Parameters

Derivation of the differential relationships between model and natural quantities as presented in their final form in the main text are the subject of this Appendix. The results obtained in the last section of Appendix A are used extensively. For the sake of completeness, certain formulae given in the main text are repeated.

Levels 1 and 2

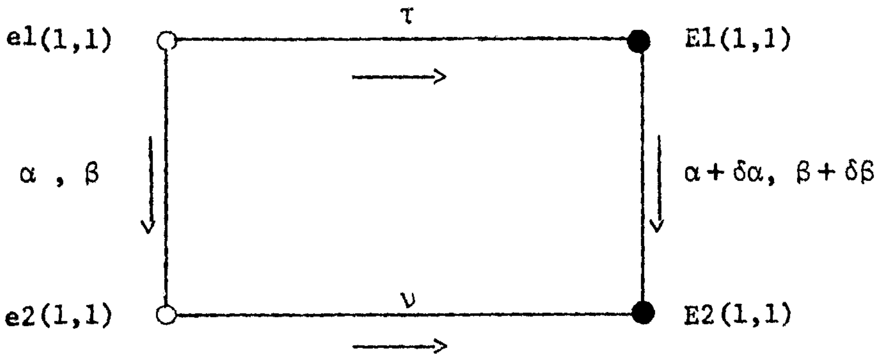

Fig. B.1

The disturbances $(\delta\alpha, \delta\beta)$ of model azimuth and altitude respectively as well as the components of the two rotation vectors $\tau^T = [\tau_1\ \tau_2\ \tau_3]$, $\nu^T = [\nu_1\ \nu_2\ \nu_3]$ are regarded as differentially small angles so that the Cardanian rotation matrices $R_c(\tau)$ and $R_c(\nu)$ can be written as follows:

$$R_c(\tau) = \begin{bmatrix} 1 & \tau_3 & -\tau_2 \\ -\tau_3 & 1 & \tau_1 \\ \tau_2 & -\tau_1 & 1 \end{bmatrix} \qquad R_c(\nu) = \begin{bmatrix} 1 & \nu_3 & -\nu_2 \\ -\nu_3 & 1 & \nu_1 \\ \nu_2 & -\nu_1 & 1 \end{bmatrix}$$

From the commutative diagram in Fig. B-1,

$$E2(1,1) = R_3[-(\alpha+\delta\alpha)]R_2[(\beta+\delta\beta)-\pi/2]E1(1,1)$$

$$E1(1,1) = R_c(\tau)\ e1(1,1)$$

$$E2(1,1) = R_c(\nu)\ e2(1,1)$$

$$e2(1,1) = R_3(-\alpha)/R_3(\beta-\pi/2)\ e1(1,1) = R_{12}\ e1(1,1)$$

where $R_{12} = \begin{bmatrix} \sin\beta\cos\alpha & -\sin\alpha & \cos\beta\cos\alpha \\ \sin\beta\sin\alpha & \cos\alpha & \cos\beta\sin\alpha \\ -\cos\beta & 0 & \sin\beta \end{bmatrix}$

Using the formula for variation in R_{12} as derived in Appendix A,

$$E2(1,1) = R_{12}(I+\Omega_{12})E1(1,1)$$

where

$$\Omega_{12} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\delta\beta + \begin{bmatrix} 0 & -\sin\beta & 0 \\ \sin\beta & 0 & \cos\beta \\ 0 & -\cos\beta & 0 \end{bmatrix}\delta\alpha = \begin{bmatrix} 0 & -\sin\beta\delta\alpha & -\delta\beta \\ \sin\beta\delta\alpha & 0 & \cos\beta\delta\alpha \\ \delta\beta & -\cos\beta\delta\alpha & 0 \end{bmatrix}$$

From the four equations above and substituting the expressions for δR_{12},

$$R_c(\nu) = R_{12}(I+\Omega_{12})R_c(\tau)\ R_{12}^T$$

$$= R_{12}\begin{bmatrix} 1 & \tau_3 - \sin\beta\delta\alpha & -\tau_2 - \delta\beta \\ -\tau_3 + \sin\beta\delta\alpha & 1 & \tau_1 + \cos\beta\delta\alpha \\ \tau_2 + \delta\beta & -\tau_1 - \cos\beta\delta\alpha & 1 \end{bmatrix} R_{12}^T$$

R_{12} is an orthogonal matrix and so the development for S in Appendix A can be applied:

$$\begin{bmatrix} 1 & \nu_3 & -\nu_2 \\ -\nu_3 & 1 & \nu_1 \\ \nu_2 & -\nu_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \begin{matrix} -\cos\beta(\tau_1+\cos\beta\delta\alpha) \\ +\sin\beta(\tau_3-\sin\beta\delta\alpha) \end{matrix} & \begin{matrix} -\sin\beta\sin\alpha(\tau_1+\cos\beta\delta\alpha)-\cos\alpha(\tau_2+\delta\beta) \\ -\cos\beta\sin\alpha(\tau_3-\sin\beta\delta\alpha) \end{matrix} \\ & 1 & \begin{matrix} \sin\beta\cos\alpha(\tau_1+\cos\beta\delta\alpha)-\sin\alpha(\tau_2+\delta\beta) \\ +\cos\beta\cos\alpha(\tau_3-\sin\beta\delta\alpha) \end{matrix} \\ \text{skew symmetric} & & 1 \end{bmatrix}$$

from which it follows after regrouping:

$$\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = \begin{bmatrix} 0 & -\sin\alpha \\ 0 & \cos\alpha \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix} + \begin{bmatrix} \sin\beta\cos\alpha & -\sin\alpha & \cos\beta\cos\alpha \\ \sin\beta\sin\alpha & \cos\alpha & \cos\beta\sin\alpha \\ -\cos\beta & 0 & \sin\beta \end{bmatrix} \cdot \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

The last expression in a compact notation is

$$\nu = A_{12} \begin{bmatrix} \delta a \\ \delta b \end{bmatrix} + R_{12}\tau$$

Noting that $A_{12}^T \cdot A_{12} = I$ and also R_{12} being orthogonal, the last expression premultiplied by A_{12}^T and R_{12}^T respectively yields:

$$\begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix} = A_{12}^T \nu - A_{12}^T R_{12} \tau$$

$$\tau = R_{12}^T \nu - R_{12}^T A_{12} \begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix}$$

or explicitly

$$\begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -\sin\alpha & \cos\alpha & 0 \end{bmatrix} \cdot \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} + \begin{bmatrix} -\cos\beta & 0 & \sin\beta \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} -\cos\beta & 0 \\ 0 & -1 \\ \sin\beta & 0 \end{bmatrix} \begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix} + \begin{bmatrix} \sin\beta\cos\alpha & \sin\beta\sin\alpha & -\cos\beta \\ -\sin\alpha & \cos\alpha & 0 \\ \cos\beta\cos\alpha & \cos\beta\sin\alpha & \sin\beta \end{bmatrix} \cdot \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}$$

Levels 2 and 3

The same approach is followed in the derivation of differential expressions for levels 2-3 using the commutative diagram in Fig. B-2. The disturbances δh, $\delta\phi$ of the respective model hour angle of vernal equinox and latitude are regarded as differentially small angles as are the components of $\xi^T = [\xi_1 \ \xi_2 \ \xi_3]$. The derivations are given without further comments.

$$\begin{tikzcd}
e2(1,1) \arrow[r, "\nu"] \arrow[d, "{h,\phi}"'] & E2(1,1) \arrow[d, dash, "{h+\delta h,\phi+\delta\phi}"] \\
e3(1,1) \arrow[r, "\xi"] & E3(1,1)
\end{tikzcd}$$

Fig. B-2

$$E3(1,1) = P_1R_3(h+\delta h)R_2(\pi/2-\phi-\delta\phi)E2(1,1) = R_{23}(I+\Omega_{23})E2(1,1)$$

$$E2(1,1) = R_c(\nu) \ e2(1,1)$$

$$E3(1,1) = R_c(\xi) \ e3(1,1)$$

$$e3(1,1) = P_1R_3(h)R_2(\pi/2-\phi) \ e2(1,1) = R_{23}e2(1,1)$$

where P_1 is the permutation matrix for the reversal of the first axis [see Mueller, 1969, p. 43],

$$R_{23} = \begin{bmatrix} -\sin\phi\cos h & -\sin h & \cos\phi\cos h \\ -\sin\phi\sin h & \cos h & \cos\phi\sin h \\ \cos\phi & 0 & \sin\phi \end{bmatrix}$$

and

$$\Omega_{23} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\delta\phi + \begin{bmatrix} 0 & \sin\phi & 0 \\ -\sin\phi & 0 & \cos\phi \\ 0 & -\cos\phi & 0 \end{bmatrix}\delta h = \begin{bmatrix} 0 & \sin\phi\delta h & \delta\phi \\ -\sin\phi\delta h & 0 & \cos\phi\delta h \\ -\delta\phi & -\cos\phi\delta h & 0 \end{bmatrix}$$

From the set of four equations above

$$R_c(\xi) = I + R_{23}\begin{bmatrix} 0 & \nu_3 + \sin\phi\delta h & -\nu_2 + \delta\phi \\ & 0 & \nu_1 + \cos\phi\delta h \\ \text{skew symmetric} & & 0 \end{bmatrix} R_{23}^T$$

Due to the permutation matrix P_1 the determinant of the R_{23} matrix is -1. Accordingly Adj. $R_{23} = -R_{23}^T$.

Skipping a few obvious steps the following is obtained:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 0 & -\sin h \\ 0 & \cos h \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \delta h \\ \delta\phi \end{bmatrix} + \begin{bmatrix} \sin\phi\cos h & \sin h & -\cos\phi\cos h \\ \sin\phi\sin h & -\cos h & -\cos\phi\sin h \\ -\cos\phi & 0 & -\sin\phi \end{bmatrix}\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}$$

$$\xi = A_{23} \cdot \begin{bmatrix} \delta h \\ \delta\phi \end{bmatrix} - R_{23} \cdot \nu$$

Premultiplying as in levels 1 and 2 and regrouping

$$\begin{bmatrix}\delta h\\ \delta\phi\end{bmatrix} = \begin{bmatrix}0 & 0 & -1\\ -\sin h & \cos h & 0\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} + \begin{bmatrix}-\cos\phi & 0 & -\sin\phi\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\nu_1\\ \nu_2\\ \nu_3\end{bmatrix}$$

$$\begin{bmatrix}\nu_1\\ \nu_2\\ \nu_3\end{bmatrix} = \begin{bmatrix}-\cos\phi & 0\\ 0 & 1\\ -\sin\phi & 0\end{bmatrix}\begin{bmatrix}\delta h\\ \delta\phi\end{bmatrix} + \begin{bmatrix}\sin\phi\cos h & \sin\phi\sin h & -\cos\phi\\ \sin h & -\cos h & 0\\ -\cos\phi\cos h & -\cos\phi\sin h & -\sin\phi\end{bmatrix}\cdot\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}$$

Levels 3 and 4

As in the upper levels the disturbances $\delta\psi_1$ and $\delta\varepsilon$ are differentially small angles as are the components of the μ rotation vector. The derivations are presented without comments.

$$\begin{tikzcd}
e3(1,1) \arrow[r, "\xi"] \arrow[d, "{\psi,\varepsilon}"'] & E3(1,1) \arrow[d, "{\psi_1+\delta\psi_1,\ \varepsilon+\delta\varepsilon}"] \\
e4(1,1) \arrow[r, "\mu"] & E4(1,1)
\end{tikzcd}$$

Fig. B-3

$$E4(1,1) = R_3(-\psi_1-\delta\psi_1)R_1(\varepsilon+\delta\varepsilon)E3(1,1) = R_{34}(I+\Omega_{34})E3(1,1)$$

$$E4(1,1) = R_c(\mu)\ e4(1,1)$$

$$E3(1,1) = R_c(\xi)\ e3(1,1)$$

$$e4(1,1) = R_3(-\psi_1)R_1(\varepsilon)\ e3(1,1) = R_{34}\ e3(1,1)$$

where

$$R_{34} = \begin{bmatrix} \cos\psi_1 & -\sin\psi_1\cos\varepsilon & -\sin\psi_1\sin\varepsilon \\ \sin\psi_1 & \cos\psi_1\cos\varepsilon & \cos\psi_1\sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{bmatrix}$$

and

$$\Omega_{34} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\delta\varepsilon + \begin{bmatrix} 0 & -\cos\varepsilon & -\sin\varepsilon \\ \cos\varepsilon & 0 & 0 \\ \sin\varepsilon & 0 & 0 \end{bmatrix}\delta\psi_1 = \begin{bmatrix} 0 & -\cos\varepsilon\delta\psi_1 & -\sin\varepsilon\delta\psi_1 \\ \cos\varepsilon\delta\psi_1 & 0 & \delta\varepsilon \\ \sin\varepsilon\delta\psi_1 & -\delta\varepsilon & 0 \end{bmatrix}$$

From the set of four equations above,

$$R_c(\mu) = I + R_{34} \begin{bmatrix} 0 & \xi_3 - \cos\varepsilon\delta\psi_1 & -\xi_2 - \sin\varepsilon\delta\psi_1 \\ & 0 & \xi_1 + \delta\varepsilon \\ \text{skew symmetric} & & 0 \end{bmatrix} R_{34}^T$$

The resulting three matrix equations are:

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 0 & \cos\psi_1 \\ 0 & \sin\psi_1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \delta\psi_1 \\ \delta\varepsilon \end{bmatrix} + \begin{bmatrix} \cos\psi_1 & -\sin\psi_1\cos\varepsilon & -\sin\psi_1\sin\varepsilon \\ \sin\psi_1 & \cos\psi_1\cos\varepsilon & \cos\psi_1\sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$$\mu = A_{34} \begin{bmatrix} \delta\psi_1 \\ \delta\varepsilon \end{bmatrix} + R_{34} \cdot \xi$$

$$\begin{bmatrix} \delta\psi_1 \\ \delta\varepsilon \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ \cos\psi_1 & \sin\psi_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} + \begin{bmatrix} 0 & -\sin\varepsilon & \cos\varepsilon \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$

$$\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} = \begin{bmatrix}0 & 1\\ \sin\varepsilon & 0\\ -\cos\varepsilon & 0\end{bmatrix}\begin{bmatrix}\delta\psi_1\\ \delta\varepsilon\end{bmatrix} + \begin{bmatrix}\cos\psi_1 & \sin\psi_1 & 0\\ -\sin\psi_1\cos\varepsilon & \cos\psi_1\cos\varepsilon & -\sin\varepsilon\\ -\sin\psi_1\sin\varepsilon & \cos\psi_1\sin\varepsilon & \cos\varepsilon\end{bmatrix}\cdot\begin{bmatrix}\mu_1\\ \mu_2\\ \mu_3\end{bmatrix}$$

Combination and Summary of Differential Relationships

In this section the formulae derived in the first three sections of Appendix B are combined with the results presented earlier.

For compactness adopt the following notation:

$$\delta\eta = \begin{bmatrix}\delta\alpha\\ \delta\beta\end{bmatrix} ; \quad \delta k = \begin{bmatrix}\delta h\\ \delta\phi\end{bmatrix} ; \quad \delta\chi = \begin{bmatrix}\delta\psi_1\\ \delta\varepsilon\end{bmatrix}$$

$$\delta q = \begin{bmatrix}\delta q_1\\ \delta q_2\end{bmatrix}; \quad \delta(-\gamma) = \begin{bmatrix}\delta(-\gamma)_1\\ \delta(-\gamma)_2\end{bmatrix} ; \quad \delta\omega = \begin{bmatrix}\delta\omega_1\\ \delta\omega_2\end{bmatrix} ; \quad \delta x = \begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix}$$

where $\overline{q}$, $-\overline{\gamma}$, $\overline{\omega}$, $\overline{x}$ are the model fundamental vectors.

The matrix D^i_j stands for the partial derivative of vector i with respect to vector j. Earlier we have derived the following differential expressions:

$$\begin{array}{l|l|l}
\delta\eta = D^{\eta}_{\nu}\cdot\nu + D^{\eta}_{\tau}\cdot\tau & \nu = D^{\nu}_{\eta}\cdot\delta\eta + D^{\nu}_{\tau}\cdot\tau & \tau = D^{\tau}_{\eta}\cdot\delta\eta + D^{\tau}_{\nu}\cdot\nu\\
\delta k = D^{k}_{\xi}\cdot\xi + D^{k}_{\nu}\cdot\nu & \xi = D^{\xi}_{k}\cdot\delta k + D^{\xi}_{\nu}\cdot\nu & \nu = D^{\nu}_{k}\cdot\delta k + D^{\nu}_{\xi}\cdot\xi\\
\delta\chi = D^{\chi}_{\mu}\cdot\mu + D^{\chi}_{\xi}\cdot\xi & \mu = D^{\mu}_{\chi}\cdot\delta\chi + \Gamma^{\mu}_{\xi}\cdot\xi & \xi = D^{\xi}_{\chi}\cdot\delta\chi + D^{\xi}_{\mu}\cdot\mu
\end{array}$$

and the following:

$$\tau = D^{\tau}_{q}\ \delta q + D^{\tau}_{-\gamma}\delta(-\gamma)$$

$$\nu = D^{\nu}_{-\gamma}\delta(-\gamma) + D^{\nu}_{\omega}\ \delta\omega$$

$$\xi = D^{\xi}_{\omega}\ \delta\omega + D^{\xi}_{x}\ \delta x$$

$$\mu = D^{\mu}_{x}\ \delta x$$

Substituting the second into the first group of equations

$$\delta\eta = D^{\eta}_{\tau}D^{\tau}_{q}\delta q + (D^{\eta}_{\tau}D^{\tau}_{-\gamma} + D^{\eta}_{\nu}D^{\nu}_{-\gamma})\delta(-\gamma) + D^{\eta}_{\nu}D^{\nu}_{\omega}\ \delta\omega$$

$$\delta k = D^{k}_{\nu}D^{\nu}_{-\gamma}\delta(-\gamma) + (D^{k}_{\nu}D^{\nu}_{\omega} + D^{k}_{\xi}D^{\xi}_{\omega})\delta\omega + D^{k}_{\xi}D^{\xi}_{x}\ \delta x$$

$$\delta\chi = D^{\chi}_{\xi}D^{\xi}_{\omega}\ \delta\omega + (D^{\chi}_{\xi}D^{\xi}_{x} + D^{\chi}_{\mu}D^{\mu}_{x})\delta x$$

Multiplication of the D^{i}_{j} matrices followed by rearrangement of terms results finally in Table 3 in the text.

Appendix C

Applications of the Differential Relationships

Here examples of the application of some of the relationships presented in Table 3 are given. The examples have been selected from two important areas of analysis, i.e., space-like and time-like variations of the fundamental vectors and their effect on variations of the basic angular parameters.

Space-like variations. Fig. C-1 is the disturbance column of the E-tower over the first three levels. The disturbances, i.e., the differences between the natural and the model basic angular parameters (α, β, h, ϕ) and fundamental vectors ($\overline{q}$, $-\overline{\gamma}$, $\overline{\omega}$), are differentially small angles. The pairs of orthogonal components of $\delta\overline{q}$, $\delta(-\overline{\gamma})$ and $\delta\overline{\omega}$ have the following interpretations (see Table 1):

δq_1	refraction in altitude
δq_2	refraction in azimuth
$\delta(-\gamma)_1$	meridional component of the deflection of the vertical
$\delta(-\gamma)_2$	prime vertical component of the deflection of the vertical
$\delta\omega_1$	nonparallelity of the rotation axis in (ecliptic) longitude
$\delta\omega_2$	nonparallelity of the rotation axis in obliquity

From the first two rows of Table 3,

$$\begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix} = \begin{bmatrix} 0 & \sec\beta \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \delta q_1 \\ \delta q_2 \end{bmatrix} + \begin{bmatrix} \sin\alpha\tan\beta & \tan\phi - \cos\alpha\tan\beta \\ \cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \delta(-\gamma)_1 \\ \delta(-\gamma)_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} \sin h \sec\phi & -\cos h \sec\phi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\omega_1 \\ \delta\omega_2 \end{bmatrix}$$

Thus the above equation relates possible errors (corrections) in refraction, deflection of the vertical, and parallelity of the model (ellipsoidal) rotation axis versus natural rotation axis to those in azimuth and altitude. Assuming $\delta\overline{q}$ to be zero and with a slightly different notation, we have the generalized Laplace conditions as shown in [Grafarend and Richter, 1977].

The above set could also be utilized directly as linearized observational equations where $\delta\alpha$, $\delta\beta$ are the respective (observed minus model) azimuth and altitude and $\delta\overline{q}$, $\delta(-\overline{\gamma})$ and $\delta\overline{\omega}$ are the unknowns.

Time-like variations. Fig. C-2 shows the time-like variational column of the F-tower at the second, third, and fourth levels. In this case we are considering natural angular parameters (H, Φ, Ψ_1 E) and their values δT later, denoted by H', Φ', Ψ', and E'. In accordance with Table 2, the expressions for the time-like variations of, e.g., the parameter H, are

$$\begin{aligned} H' &= H + \frac{\partial H}{\partial T}\,\delta T \\ &= h + \delta h + \frac{\partial h}{\partial T}\,\delta T + \frac{\partial(\delta h)}{\partial T}\,\delta T \end{aligned}$$

Subtract the corresponding model quantities and rearrange to obtain

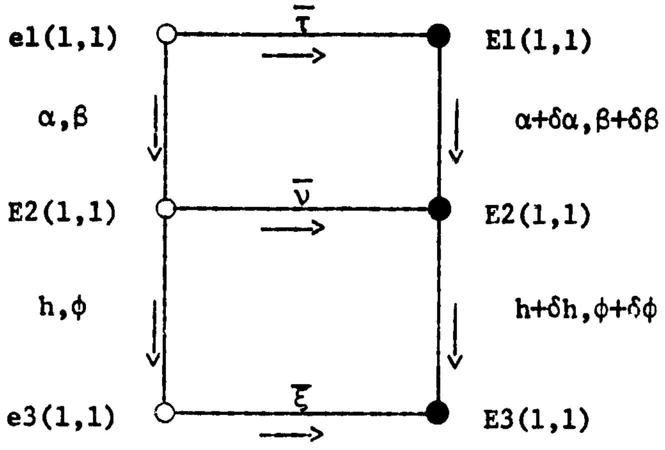
e1(1,1)
$\bar{\tau}$
E1(1,1)
α,β
α+δα,β+δβ
E2(1,1)
$\bar{\nu}$
E2(1,1)
h,φ
h+δh,φ+δφ
e3(1,1)
$\bar{\xi}$
E3(1,1)

Fig. C-1

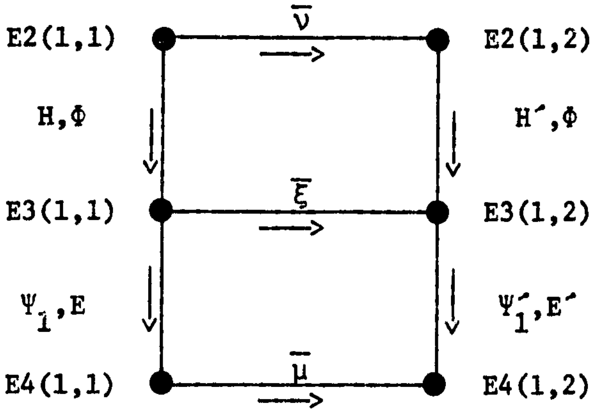
E2(1,1)
$\bar{\nu}$
E2(1,2)
H,Φ
H´,Φ
E3(1,1)
$\bar{\xi}$
E3(1,2)
Ψ_1,E
Ψ_1',E´
E4(1,1)
$\bar{\mu}$
E4(1,2)

Fig. C-2

$$(H' - H) - (h' - h) = \frac{\partial(\delta h)}{\partial T}\,\delta T \equiv \dot{\delta}h$$

Similar notation is adopted for Φ, $-\gamma$, ω, and x. Now apply the third and fourth rows of Table 3 and write the following expressions:

$$\begin{bmatrix} \dot{\delta}h \\ \dot{\delta}\phi \end{bmatrix} = \begin{bmatrix} 0 & \sec\phi \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\delta}(-\gamma)_1 \\ \dot{\delta}(-\gamma)_2 \end{bmatrix} + \begin{bmatrix} \cot\varepsilon + \tan\phi\sin h & -\tan\phi\cos h \\ \cos h & \sin h \end{bmatrix} \begin{bmatrix} \dot{\delta}\omega_1 \\ \dot{\delta}\omega_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} -\mathrm{cosec}\,\varepsilon\cos\psi_1 & -\mathrm{cosec}\,\varepsilon\sin\psi_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\delta}x_1 \\ \dot{\delta}x_2 \end{bmatrix}$$

where

$\dot{\delta}h$	variation in the disturbance (natural minus model) of the hour angle of vernal equinox
$\dot{\delta}\phi$	variation in the disturbance of the latitude
$\dot{\delta}(-\bar{\gamma})$	effect of local (plate) motions plus differences (natural minus model) in polar motion and spin rate
$\dot{\delta}\bar{\omega}$	difference (natural minus model) in luni-solar precession and nutation
$\dot{\delta}\bar{x}$	difference in planetary precession

The above equation thus relates errors (corrections) in earth rotation (precession, nutation, polar motion, spin rate) to those in latitude and hour angle (longitude + Greenwich sidereal time). It could also be utilized as linearized observational equations where $\dot{\delta}h$, $\dot{\delta}\phi$ are the observables and $\dot{\delta}(-\bar{\gamma})$, $\dot{\delta}\bar{\omega}$, and $\dot{\delta}\bar{x}$ are the unknowns.

References

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INVESTIGATIONS ON THE HIERARCHY OF REFERENCE FRAMES IN GEODESY AND GEODYNAMICS

PART II: SYSTEM OF ORIGINS: THE P-TOWER

by

Haim B. Papo

1. Introduction

The fundamental vectors at a point on earth, their reference model, and the commutative tower of triads--the E tower--were introduced and studied in [Grafarend et al., 1979] as a first step in our investigation for reference frames in geodesy and geodynamics. Time was defined as the fourth independent coordinate, and an attempt was made to distinguish between natural--observable--quantities and their models corresponding to present day knowledge. The concepts presented there (vectors, triads, parameters, transformations, variations, etc.) were only directional. Distances, coordinates, linear velocities, scale and deformations were not considered. Consequently, no metric data of any kind could be analyzed with the help of the E-tower alone.

In the following, we introduce the tower of origins (P-tower) which complements the directional E-tower in defining concepts, identifying parameters, and analyzing interrelationships and variations of positions and distances between points in space and time. The approach follows closely the one employed in [Grafarend et al., 1979]. The distances, coordinates, linear velocities of the various points, regarded as natural (real) quantities are paralleled by a set of models of the same in a one-to-one correspondence. As in the E-tower we are interested in the difference between the real and the model quantities to be represented subsequently as functions of a selected set of parameters. The two towers are closely related in sharing certain concepts and parameters and, in fact, it would have been impossible to present the P-tower without repeated reference to the E-tower.

The tower of origins (P-tower) presented and studied in the following chapters should not be regarded as a problem-solution-procedure type of report. It is rather an attempt to provide a method of analysis, to lay down foundations, to create a consistent and logical language and nomenclature for a subsequent study and solution of specific problems. Many of the results in terms of concepts, relationships and variations may seem trivial and not necessarily new. But this is exactly the purpose of this report, i.e., to redefine, reorganize and systematize certain aspects of our present knowledge and understanding of the geometry, kinematics and dynamics of the earth without resorting to too many basic assumptions and hypotheses. We have tried to identify and clarify parameters and phenomena which apply to directions (E-tower) as well as to positions and distances (P-tower) between points in space and time. The creation of this common basis is essential for our future treatment of specific problems where we should be able to use as necessary a combination of concepts and formulae derived and associated with either of the two towers.

The ultimate goal of our studies of reference frames for geodesy and geodynamics is the establishment of a conventional terrestrial coordinate system (CTCS) through the combined analysis of a selected set of high quality observations (laser ranging, radio interferometry. etc.). The CTCS should represent in some average but nonetheless well-defined manner the space-time behavior of the earth vs. inertial space. Dynamic or geometric variations of the earth in space and time would be referred to the inertial frame of reference through the CTCS. The P-tower presented in this report is another step toward the achievement of the above goal.

2. The Tower of Origins

The relative positions and motions of points in space and time which are characterized as the origins of various reference frames for geodesy and geodynamics are presented and studied in a diagrammatic structure to be referred to as the tower of origins or the P-tower.

The overall appearance, organization, and notation of the P-tower (see Fig. 1) are similar to those of the E-tower. The points in the diagram symbolize certain physically meaningful points at a given epoch. Capital Pi denote natural-real origins, while their models are denoted by pi. The integer i signifies the level of the origin and assumes the values of 1, 2, 3, or 4, for the topocenter, bodycenter, barycenters, respectively.

On a given level the points are organized along two axes: the space axis and the time axis. The integers within the parentheses (j, k) are the space and time indices of the point to be interpreted as follows:

j = 1 point related to the observer
j = 2 point related to the target
k = 1 epoch T
k = 2 "next" epoch T + dT where dT is a differentially small time interval.

One should note the different interpretation given here to the j index as compared to the corresponding j index in the E-tower: In the P-tower Pi(1, k) and Pi(2, k) are two distinctly different (not adjacent) points, which, in general, have different velocities in space. The level of a point depends on its nature and on its function which is

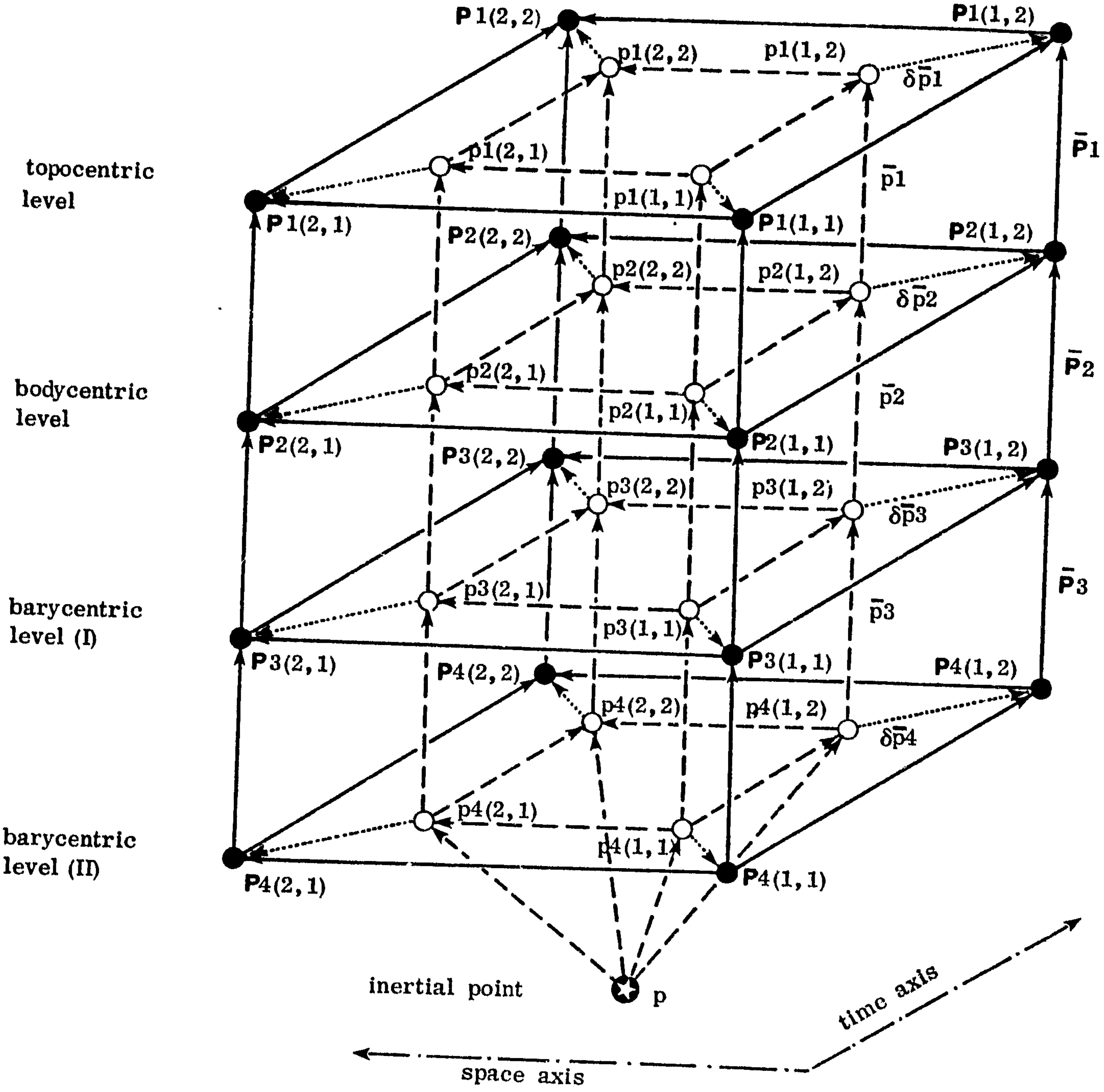


Fig. 1 The P-Tower

associated in general with the measurement of distances, directions or gravity.

Topocentric (observational) level P1

A point is considered at the topocentric level if it serves either as an observing point or as a target. Stars and quasars are not considered as target points since their three-dimensional coordinates are not known with equal precision. The principal point of a telescope, an EDM instrument, or of a radiotelescope are a few examples of observing points. The principal point of an artificial satellite's transponder or laser retro-reflector are a few examples of target points.

Bodycentric level P2

The center of mass of a body serves as origin on the bodycentric level. A body is defined here as a conglomerate of mass points which are connected to each other fairly rigidly so that variations in relative positions between the mass points (deformations) are small as compared to the overall size of the body. The earth and the moon are typical examples of such bodies and their respective centers of mass are points of the P2 level. We see that a P2 point has a definite physical meaning although it cannot be directly reached by observations. A point of the P1 level is normally located on the surface of a body and as such is associated with a certain P2 point which is the same body's mass center. Exception to this rule is a close satellite of a planet (the earth or the moon) which is defined as a P1 point while its P2 point is the mass center of the planet.

Barycentric levels P3, P4

A point is considered at the barycentric level if it is at the center of mass of a set of bodies. The selection of the set is more or less arbitrary and thus identity of the P3 point depends on the composition of the set (its elements). There may be several barycentric levels according to some hierarchy. A good example for a P3 barycentric level (I) origin is the earth-moon barycenter. As the earth and the moon are a subset of the solar system set (the sun and the planets) we can define a P4 origin at the barycenter of the solar system (barycentric level (II)). It should be obvious that one could continue with P5 at the barycenter of our galaxy, etc.

Inertial level p

The inertial origin p is defined as a point which is fixed or moving with uniform velocity in inertial space [see Goldstein, 1965]. The positions and motions of all the points in the P-tower are referred to this p point in accordance with the laws of Newtonian mechanics.

The points in the diagram are marked either as full (black) circles or as hollow (white) circles depending on whether they represent a natural point or its model. Thus, in Fig. 1 we can distinguish between the natural P-tower (the black points) and the model p-tower (the hollow circles).

The lines between two points in the double tower represent vectors in inertial space. The interpretation of these vectors depends on the axis to which the vectors are parallel and also on the nature of the points being connected by it.

We will examine first vectors at the topocentric (observational) level. For example, let Pl(1, 1) be an observing point on the earth surface and Pl(2, 1) represent a target on the lunar surface. As the k index (in the parentheses) is 1 for both points, the epoch T at which both points are defined is the same.

The vector $\overline{Pl(1, 1)\ Pl(2, 1)}$ (see Fig. 2) represents the natural geometric distance and direction between the two points. Analogously the vector $\overline{Pl(1, 2)\ Pl(2, 2)}$ represents the natural distance and direction between the same two points only at a "later" epoch T + dT.

The vector which connects the positions of the same point at two different epochs (T and T + dT) is defined as the linear velocity vector of that point vs. inertial space. For example,

$\overline{Pl(1, 1)\ Pl(1, 2)}$	velocity of Pl(1, 1) at T
$\overline{Pl(2, 1)\ Pl(2, 2)}$	velocity of Pl(2, 1) at T

The interpretation of the vectors connecting the points pl(1, 1), pl(2, 1), pl(1, 2), pl(2, 2) is the same as above but for the model. The differences between the instantaneous positions of the natural points and their models are represented by the following vectors (see Fig. 2):

$\overline{pl(1, 1)\ Pl(1, 1)} \equiv \overline{\delta pl}(1, 1)$	positional disturbance vector at epoch T for point Pl(1, 1)
$\overline{pl(2, 1)\ Pl(2, 1)} \equiv \overline{\delta pl}(2, 1)$	positional disturbance vector at epoch T for point Pl(2, 1)

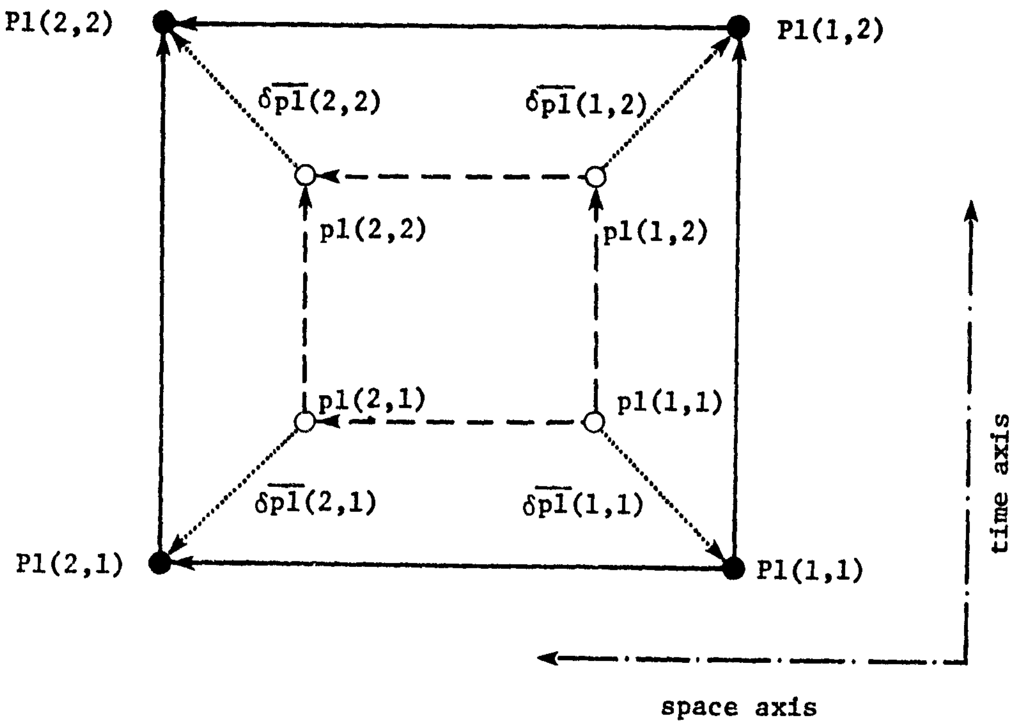


Fig. 2 The topocentric level.

$\overline{pl(1,\ 2)\ Pl}(1,\ 2) \equiv \delta\overline{pl}(1,\ 2)$ positional disturbance vector at epoch T + dT for point Pl(1, 2)

$\overline{pl(2,\ 2)\ Pl}(2,\ 2) \equiv \delta\overline{pl}(2,\ 2)$ positional disturbance vector at epoch T + dT for point Pl(2, 2)

The vectors between the inertial point p and any of the natural or model points in the P-tower symbolize their position vectors in an inertial frame of reference with origin at p. By virtue of the above definition the P-tower is a commutative diagram of the vectors in inertial space, i.e., the sum of the vectors forming a closed loop is identically zero. Using the commutative property at the topocentric level (see Fig. 2) we derive the following relationships:

$$P\overline{1(1,2)P1}(2,2) - P\overline{1(1,1)P1}(2,1) = P\overline{1(2,1)P1}(2,2) - P\overline{1(1,1)P1}(1,2)$$

$$P\overline{1(1,1)P1}(1,2) - p\overline{1(1,1)p1}(1,2) = \delta p\overline{1(1},2) - \delta p\overline{1(1},1)$$

An important property of a vector commutative diagram is that the vector relationships are independent of the coordinate system chosen to represent those vectors. The components of the various vectors may change from one coordinate system to another; however, their magnitude as well as their relative orientation remains invariant.

We will examine next a vertical wall of the P-tower (see Fig. 3). The k indices of all the points being 2 means that the wall represents a situation at epoch T + dT. In Fig. 3 we have used a shortened notation for the vectors along the vertical lines as follows:

$$P\overline{1(1},2) \equiv P\overline{2(1,2)\ P1}(1,2)$$

$$P\overline{2(1},2) \equiv P\overline{3(1,2)\ P2}(1,2)$$

etc.

The interpretation of these vectors follows from the identity of the end points:

$P\overline{1(1},2)$	is geocentric (mass center) position vector of the observer at T + dT
$P\overline{2(1},2)$	is earth-moon barycentric position vector of the geocenter at T + dT

etc.

The vectors $p\overline{1(1},2)$ and $p\overline{2(1},2)$ are analogous to the above but for the model.

The vectors which connect the model points with the corresponding natural points are defined as positional disturbances. For example,

$$\begin{tikzcd}
p1(1,2) \arrow[r, dotted, "\delta\overline{p1}(1,2)"] & P1(1,2) \\
p2(1,2) \arrow[u, dashed, "\overline{p1}(1,2)"] \arrow[r, dotted, "\delta\overline{p2}(1,2)"] & P2(1,2) \arrow[u, "\overline{P1}(1,2)"'] \\
p3(1,2) \arrow[u, dashed, "\overline{p2}(1,2)"] \arrow[r, dotted, "\delta\overline{p3}(1,2)"] & P3(1,2) \arrow[u, "\overline{P2}(1,2)"'] \\
p4(1,2) \arrow[u, dashed, "\overline{p3}(1,2)"] \arrow[r, dotted, "\delta\overline{p4}(1,2)"] & P4(1,2) \arrow[u, "\overline{P3}(1,2)"'] \\
p \arrow[u, dashed, "\overline{p4}(1,2)"] \arrow[ur, "\overline{P4}(1,2)"'] &
\end{tikzcd}$$

Fig. 3 A column layer of the P-tower

$\delta p\overline{1(1},2)$ is the positional disturbance of the observer at $T+dT$

$\delta p\overline{2(1},2)$ is the positional disturbance of the geocenter at $T+dT$

etc.

The diagram in Fig. 3 being part of the P-tower is also commutative. Using the commutative property, we can write, for example,

$$\overline{P1(1},2) - p\overline{1(1},2) = \delta\overline{p1(1},2) - \delta\overline{p2(1},2)$$

$$[P\overline{1(1},2) + P\overline{2(1},2)] - [p\overline{1(1},2) + p\overline{2(1},2)] = \delta\overline{p1(1},2) - \delta\overline{p3(1},2)$$

etc.

We will complete the examination of the structure and significance of the P tower by studying the interrelations between points on one of the time-variation walls as shown in Fig. 4. The meaning of the vectors connecting points along a column has been discussed above. The two vectors $\dot{\overline{P1(1}},1)$ and $\dot{\overline{P2(1}},1)$ connecting the geocentric position vectors $\overline{P1(1},1)$ and $\overline{P1(1},2)$ are the respective linear velocity vectors vs. inertial space of the P1(1,1) and P2(1,1) points at epoch T. Using the property of commutativity, we can write the following:

$$\overline{P1(1},2) - \overline{P1(1},1) = \dot{\overline{P1(1}},1) - \dot{\overline{P2(1}},1)$$

$$\overline{P2(1},2) - \overline{P2(1},1) = \dot{\overline{P2(1}},1) - \dot{\overline{P3(1}},1)$$

etc.

The expressions on the right-hand side represent the relative linear velocities of the observer vs. the geocenter and of the geocenter vs. the earth-moon barycenter, respectively. An interesting corollary is the following inequality:

$$\frac{\partial}{\partial T}\,\overline{P1(1},1) \neq \dot{\overline{P1(1}},1) \ .$$

Summarizing our discussion of the P-tower structure and the interpretation of the various points and vectors in it, we see that it can serve as a convenient means for representing and studying the whole range of positional and velocity information related to points in the natural world as well as in its model.

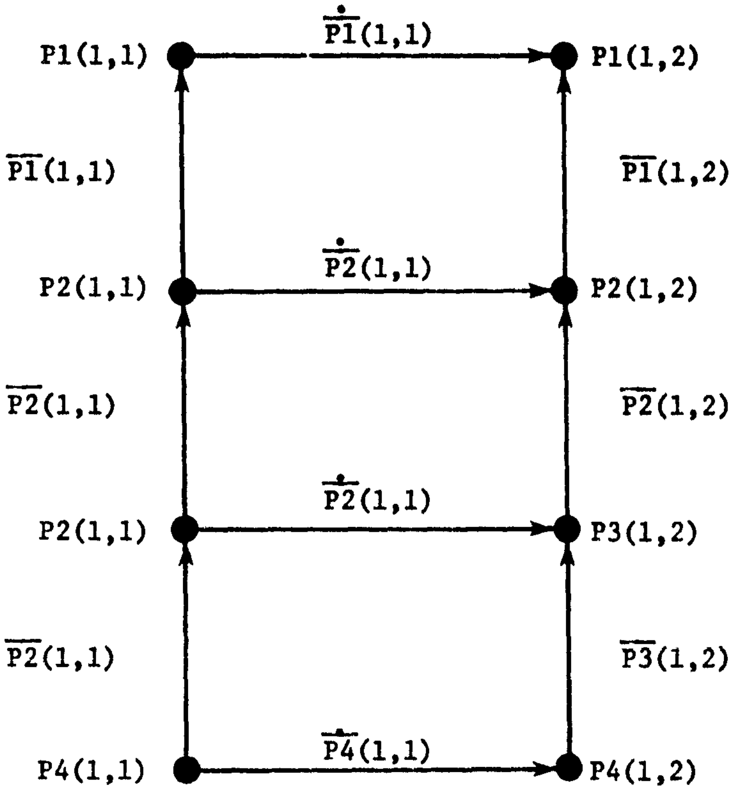


Fig. 4 A vertical "time" wall of the P-tower

It should be kept in mind that certain vectors in the P-tower can be null vectors due to the two end points being coincident. For example, if P1(1,1) and P1(2,1) are both points on the earth surface, the points P2(1,1) and P2(2,1) represent the same point, i.e., the geocenter, and therefore the vector $\overline{P2(1,1)\ P2}(2,1)$ is a null vector. For this case, we can easily deduce the following identities:

$P\overline{2(1},1) \equiv P\overline{2(2},1)$

$p\overline{2(1},1) \equiv p\overline{2(2},1)$

$\delta\overline{p2(}1,1) \equiv \delta\overline{p2(}2,1)$

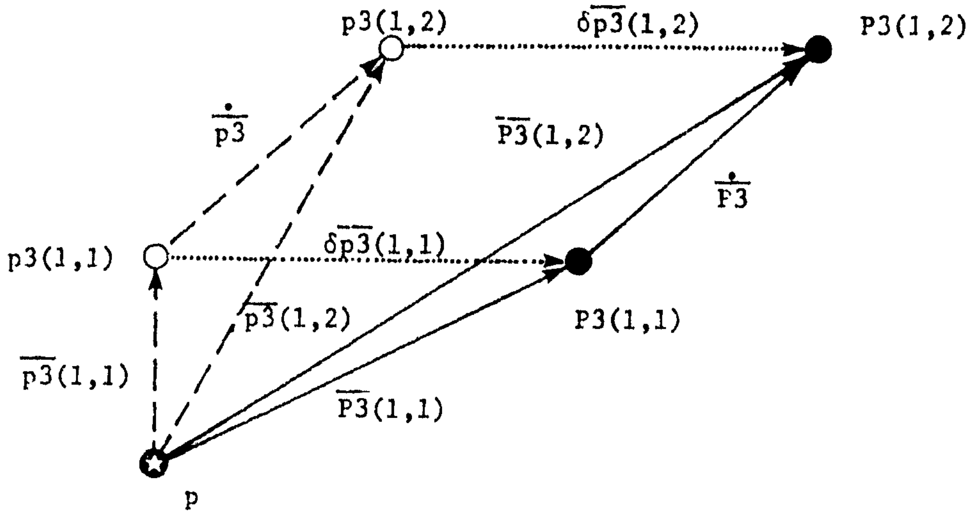
etc.

3. Barycentric and Bodycentric Levels

In Chapter 3 the nature and interrelationships of origins at the barycentric and bodycentric levels are studied. The major objective is to identify the positional disturbances and their time-like variations at these levels with inadequacies in current theories and respective constants. In particular we study the problem of possible dependence of second- and third-level disturbances on the rotation of the earth and its mass distribution.

In Fig. 5 we have reproduced part of Level 3 of the P-tower relating it directly to the p point. Bypassing Level 4 in the above figure is the equivalent to the assumption that the solar system barycenter P4 and its model p4 are coincident and are taken as the inertial point. The vectors $\overline{P3(1,1)}$, $\overline{P3(1,2)}$ and their difference $\frac{\partial}{\partial T}\overline{P3}$ (or, equivalently, in this case $\dot{\overline{P3}}$) represent the motion of the earth-moon barycenter P3 with respect to the barycenter of the solar system. As the $p\overline{3(1,1)}$, $p\overline{3(1,2)}$, and $\dot{\overline{p3}}$ represent the model of the same, computable with current theory, it should be obvious that inadequacies in that theory will be represented by the respective disturbances. Accordingly, $\delta p\overline{3(1,1)}$, $\delta p\overline{3(1,2)}$ and their difference $\frac{\partial}{\partial T}\delta\overline{p3}$ are all non-zero vectors.

Little as we know at present about the $\overline{\delta p3}$ vector and its time-like variation $\frac{\partial}{\partial T}\overline{\delta p3}$, we can at least state the following: The theory of motion of P3 about the barycenter of the solar system is a function of the combined masses of the earth and the moon, the masses of the sun and the other planets in addition to constants of integration (or zero epoch state vectors). Accordingly, phenomena such as (i) the motion of the mass centers of the earth and the moon vs. their barycenter P3,



P3	earth-moon system barycenter
p3	model earth-moon system barycenter
p	inertial point (identified with the solar system barycenter)

Fig. 5 The barycentric level

(ii) the mass distribution within the earth or the moon vs. their respective mass centers, (iii) the rotational motion in space of the earth or the moon, are not parts of the disturbances in the motion of the earth-moon barycenter. Another way of stating the above would be that measurements within the earth-moon system would not be sensitive to the $\overline{\delta p3}$ disturbance or to its time-like variations.

In Fig. 6 we have added Level 2 to the previous case. $\overline{P2}$ represents the earth-moon barycentric position vector of the geocenter (or selenocenter) P2. Using the commutative property of the loop formed by the four natural points, we can derive an expression for the vector $\overline{P2(1,1)P2}(1,2)$ denoted in the diagram as $\dot{\overline{P2}}$

$$\dot{\overline{P2}} = \dot{\overline{P3}} + \overline{P2}(1,2) - \overline{P2}(1,1)$$

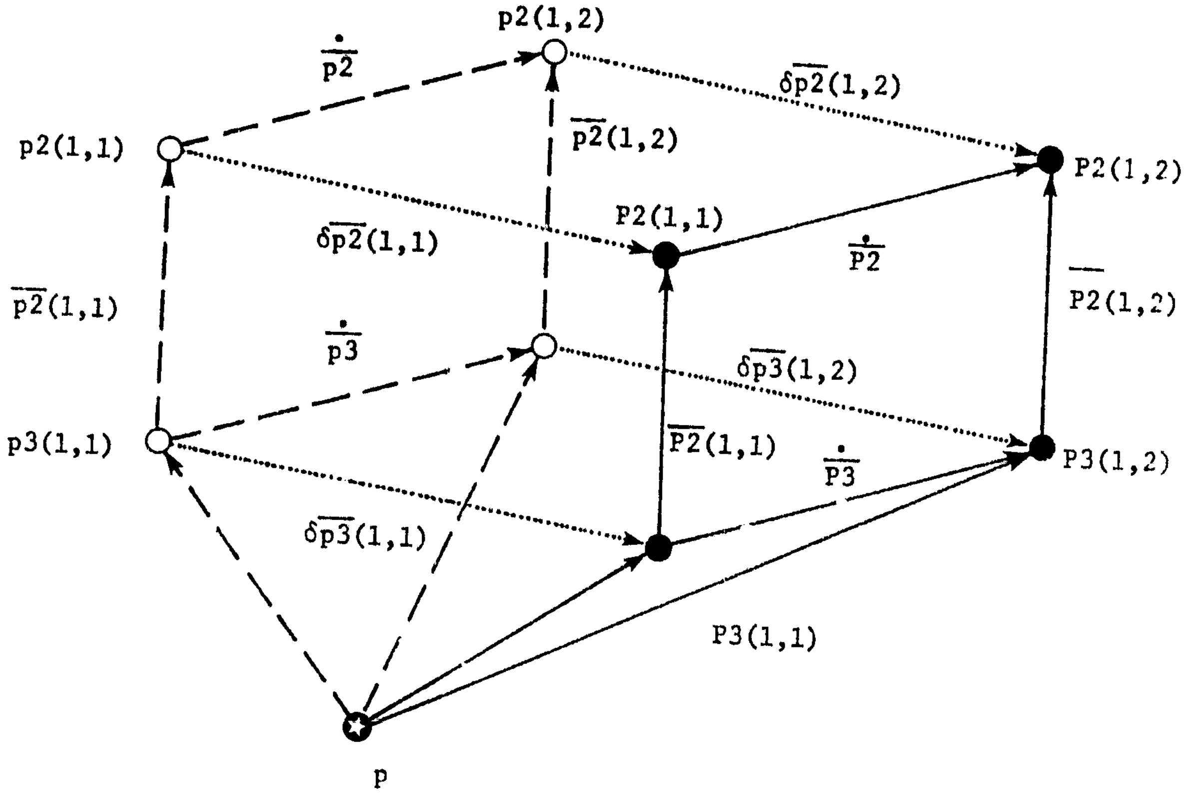


Fig. 6 The bodycentric and barycentric levels

but at Level 3 we had

$$\dot{\overline{P3}} = \frac{\partial}{\partial T}\overline{P3}$$

and so it follows

$$\dot{\overline{P2}} = \frac{\partial\overline{P3}}{\partial T} + \frac{\partial\overline{P2}}{\partial T} .$$

Using the loop at Level 2 and the above results, we can write

$$\delta\overline{p2(1,2)} - \delta\overline{p2(1,1)} = \frac{\partial\overline{P2}}{\partial T} + \frac{\partial\overline{P3}}{\partial T} - \frac{\partial\overline{p2}}{\partial T} - \frac{\partial\overline{p3}}{\partial T}$$

$$\frac{\partial}{\partial T}\,\delta\overline{p2} = \frac{\partial}{\partial T}\,(\overline{P2} - \overline{p2}) + \frac{\partial}{\partial T}\,(\overline{P3} - \overline{p3})$$

$$\frac{\partial}{\partial T}\,(\delta\overline{p2} - \delta\overline{p3}) = \frac{\partial}{\partial T}\,(\overline{P2} - \overline{p2})$$

$(\overline{P2} - \overline{p2})$ and its time derivative $\frac{\partial}{\partial T}(\overline{P2} - \overline{p2})$ represent the difference between the real and the model motions of the geocenter about the earth-moon barycenter. We denote by c the ratio between the masses of the earth and the moon

$$c = \frac{M_e}{M_m} \sim 81.3$$

and identify

$\overline{P2}(1,1)$ as the position vector of the geocenter with respect to the earth moon barycenter P3 and

$\overline{P2}(2,1)$ as the position vector of the selenocenter with respect to the same origin P3.

From the definition of the barycenter of a two-body system we have the following (see Fig. 1):

(i) $\overline{P2}(1,1)$, $\overline{P2}(2,1)$ and $\overline{P2(1,1)P2(2,1)}$ are collinear

(ii) $|\overline{P2}(1,1)| + |\overline{P2}(2,1)| = |\overline{P2(1,1)P2(2,1)}|$

(iii) $|\overline{P2}(2,1)| \,/\, |\overline{P2}(1,1)| = c.$

The above equations demonstrate the simple relationship between $\overline{P2}(1,1)$ and the lunar theory which in principle gives the components of $\overline{P2(1,1)P2(2,1)}$ and its time derivatives.

$\overline{p2}$ and $\frac{\partial}{\partial T}\overline{p2}$ can be computed from the current dynamic theory of earth-moon system (essentially the lunar theory), while $(\overline{P2} - \overline{p2})$ and its time derivatives represent corrections to that theory.

From the above we can draw two conclusions which complement each other:

(a) The positional disturbance of the geocenter $\delta\overline{p2}$ and its time derivatives $\frac{\partial}{\partial T}\delta\overline{p2}$ consist of two components which represent corrections to the theories of motion of the barycenter of the earth-moon system and that of the geocenter (selenocenter) with respect to the barycenter of the solar system and to each other respectively.

(b) $\delta\overline{p2}$ and $\frac{\partial}{\partial T}\,\delta\overline{p2}$ do not depend on the rotation of the earth (or the moon) or on variations in its mass distribution.

Summarizing this chapter and extending its conclusions to $\delta\overline{p1}$ we can state:

(a) Unaccounted perturbations in the theory of motion of the earth-moon barycenter with respect to the solar system barycenter dominate the $\delta\overline{p3}$ disturbances and their time derivatives.

(b) Unaccounted perturbations in the lunar theory dominate the $\delta\overline{p2} - \delta\overline{p3}$ disturbances and their time derivatives.

(c) The Level 1 disturbances $\delta\overline{p1}$ or actually $(\delta\overline{p1} - \delta\overline{p2})$ and their time derivatives are dominated by the rotation of the earth (or the moon) and by mass redistributions.

It is important to realize that according to (b) above $\delta\overline{p2}$ does not have a diurnal motion and it is independent of inadequacies in the adopted gravity model of the earth (or the moon).

4. The Topocentric (Observer-Target) Level

In this chapter we study problems which are related to the $\overline{P1}$ bodycentric position vector and its time-like variations. Our discussions are limited to Level 1 origins which are located on the surface of the earth (or the moon).

For a point on the earth (moon) surface there are three issues of fundamental importance which have to be carefully studied in order to understand the nature and significance of Level 1 positional disturbances and their time-like variations:

(i) The rotational motion of $\overline{P1}$ with respect to an inertial frame of reference centered at P2.

(ii) The relationship between $\overline{P1}$, $\frac{\partial}{\partial T}\overline{P1}$ and the variable gravitational potential field of the earth (or the moon).

(iii) The explicit definition (and realization) of p1--the model topocentric origin.

The proper order of introducing and studying the above three topics is not arbitrary as they are interdependent. Accordingly we begin with (i) proceed through (ii) and finally end up with (iii).

4.1. The Rotational Vector $\bar{\Omega}$, Its Model and Disturbances

The time-like variations of the geocentric position vectors and their disturbances are strongly dependent on the rotational motion of the earth (moon) vs. its mass center. We will devote this sub-chapter to sharpening the concepts associated with the rotation vector $\bar{\Omega}$ for the earth and the rotational motion of $\overline{P1}$ around it.

The rotational vector $\bar{\Omega}$ describes by its direction and magnitude the rotational motion of a point on the earth surface P1 vs. the geocenter P2 and with respect to inertial space. The time-like variation in position (linear velocity) of P1 with respect to P2 is obtained by the well-known vector equation

$$\frac{\partial}{\partial T}\overline{P1} = \bar{\Omega} \times \overline{P1}$$

which is rigorous for a rigid body. The motion of P1 on the non-rigid earth vs. the geocenter P2 can be partitioned into two parts as follows:

$$\frac{\partial}{\partial T}\overline{P1} = \bar{\Omega} \times \overline{P1} + \left(\frac{\partial|\overline{P1}|}{\partial T}\right)\frac{\overline{P1}}{|\overline{P1}|}$$

where the first term represents variation in the direction of $\overline{P1}$ vs. inertial space and the second term is the variation in its magnitude. We will discuss in a subsequent sub-chapter the second term and its association with variations in the gravitational potential at P1. In the present sub-chapter we will be concerned only with the first term which is equivalent in form with the rigid rotation of $\overline{P1}$ about the mass center P2 (see Fig. 7). The absolute rotational motion of $\overline{P1}$ with respect to a non-rotating, inertial triad, represented in Fig. 7, by the inertial vector $\bar{i}$, is quite complicated but can be partitioned into a sequence of simpler relative motions. For that purpose we define a sequence of rotational vectors - axes between $\bar{i}$ and $\overline{P1}$ ranked in the following order:

$$\bar{i},\ \bar{\Omega}_S,\ \bar{\Omega}_E,\ \bar{\Omega},\ \overline{P1}$$

where a higher rank is associated with the nearness to $\bar{i}$ (Fig. 8). The three rotational vectors are defined as follows:

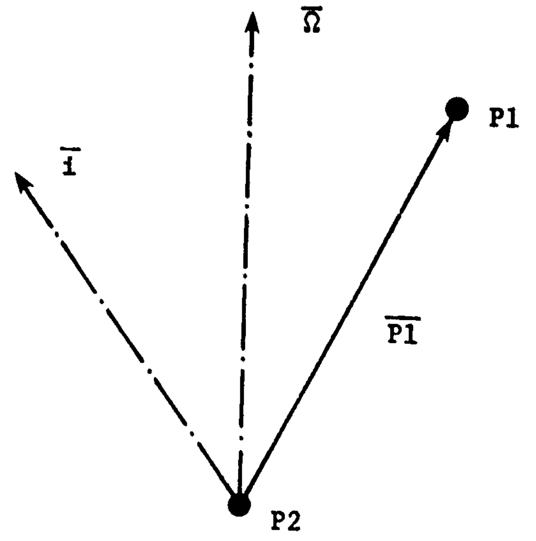

$\overline{\Omega}$
P1
$\overline{i}$
$\overline{P1}$
P2

Fig. 7

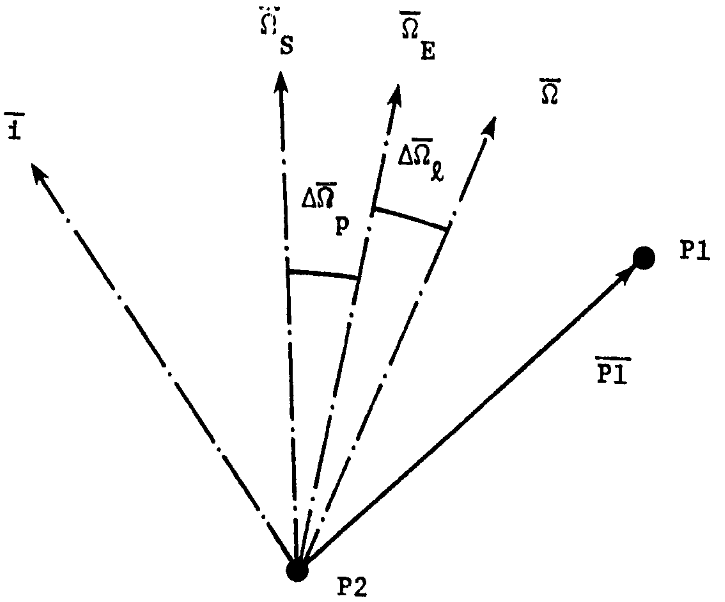

$\overline{\Omega}_S$
$\overline{\Omega}_E$
$\overline{\Omega}$
$\overline{i}$
$\Delta\overline{\Omega}_\ell$
$\Delta\overline{\Omega}_p$
P1
$\overline{P1}$
P2

Fig. 8

$\bar{\Omega}_S$ - is the spin vector. Its orientation vs. $\bar{i}$ is defined by the general (planetary + lunisolar) precession and by the forced terms of nutation. It does not contain terms of diurnal or higher frequencies. Its magnitude is changing in time with unpredictable variations to be determined by means of observations.

$\bar{\Omega}_E$ - is the Eulerian vector which rotates around $\bar{\Omega}_S$ with a nearly diurnal frequency and with a small angular amplitude $|\Delta\bar{\Omega}_p|/|\bar{\Omega}_S|$ where $\Delta\bar{\Omega}_p = \bar{\Omega}_E - \bar{\Omega}_S$. Both frequency and amplitude of $\Delta\bar{\Omega}_p$ are unpredictable and can be determined only through observations. The $\Delta\bar{\Omega}_p$ vector in magnitude and in orientation represents the polar motion phenomenon. The $\bar{\Omega}_E$ axis and its motion vs. $\bar{i}$ represents the complete solution of the differential equations of rotational motion of the earth.

Note: We should point out that both $\bar{\Omega}_S$ and $\bar{\Omega}_E$ are space invariant, i.e., they are the same in orientation and in magnitude for any point Pl. Thus Ω_E can be regarded as the instantaneous global rotational axis of the earth.

$\bar{\Omega}$ - is the instantaneous rotational vector at Pl. It has a nearly diurnal rotational motion around $\bar{\Omega}_S$ and its angular distance from $\bar{\Omega}_E$ is extremely small of the order of 10^{-6} seconds of arc. The $\Delta\bar{\Omega}_\ell$ vector represents local motions of Pl.

In spite of the fact that $\bar{\Omega}$ or approximately $\bar{\Omega}_E$ are the true instantaneous axes of rotation of $\bar{P}$l, it has been demonstrated by Atkinson [1975] and also by Leick [1978] that the axis that can be detected directly by observations is the $\bar{\Omega}_S$ axis, while the $\bar{\Omega}_E$ and the $\bar{\Omega}$ axes are unobservable. Intuitively, the above statement could be explained by the fact that both $\overline{\Omega}_E$ and $\overline{\Omega}$ have a nearly diurnal rotational motion around $\overline{\Omega}_S$ just like $\overline{P1}$. An observer at P1 cannot detect on a short time basis (one day or less) the motion of P1 vs. $\overline{\Omega}_E$ and $\overline{\Omega}$. Only on a much longer time basis, i.e., days for $\overline{\Omega}_E$ and years for $\overline{\Omega}$ can one detect the accumulated effect of the small perturbations $\Delta\overline{\Omega}_p$ and $\Delta\overline{\Omega}_\ell$ on the orientation of $\overline{P1}$ vs. $\overline{\Omega}_S$ and through it vs. $\overline{i}$.

To recapitulate, we can state that the time-like variations of $\overline{P1}$ are defined by the following vector equation which is identical to the one written at the very beginning of this chapter.

$$\frac{\partial}{\partial T}\overline{P1} = (\overline{\Omega}_S + \Delta\overline{\Omega}_p + \Delta\overline{\Omega}_\ell) \times \overline{P1} + \frac{\partial}{\partial T}|\overline{P1}| \cdot \frac{\overline{P1}}{|\overline{P1}|}$$

The above equation means, for example, that the angle between $\overline{P1}$ and $\overline{\Omega}_S$, the instantaneous geocentric colatitude, varies in time due to the combination of polar motion and local motions. In the second part of this sub-chapter we will define the model of the rotational vector $\overline{\Omega}$.

The dynamical model of the earth is defined as in [Grafarend et al., 1979] as a rotational level ellipsoid rotating with a constant spin rate. The orientation of its spin axis $\overline{\omega}_S$ with respect to an inertial frame is given by general precession and forced terms of nutation as presently adopted. With respect to the axis of figure $\overline{z}$ (minor axis of the ellipsoid) the spin axis describes a cone with an amplitude 0".15 and a period of 1.1828 years. The sense of the model polar motion, thus defined, is counterclockwise as seen from the north. The two constants (0".15, 1.1828 years) correspond to the average amplitude and period of polar motion between 1970 and 1976 [Markowitz, 1976].

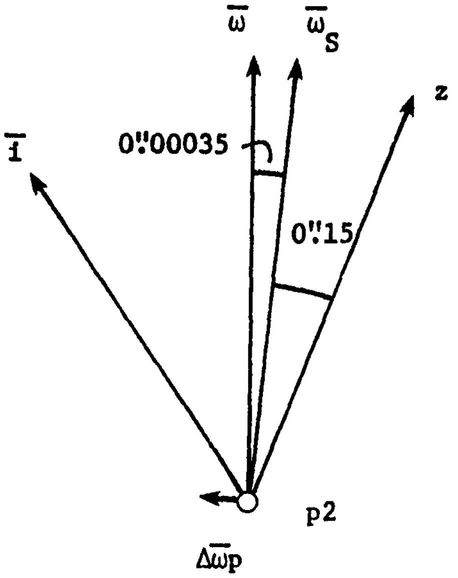


Fig. 9

The total rotational motion of the model geocentric position vector $\overline{p1}$ is given by the following equation

$$\frac{\partial}{\partial T}\,\overline{p1} = (\bar{\omega}_S + \Delta\bar{\omega}_p) \times \overline{p1} = \bar{\omega} \times \overline{p1}$$

where $\Delta\bar{\omega}_p$ is the model polar motion vector. Its magnitude is

$$|\Delta\bar{\omega}_p| = \frac{2\pi}{1.1828}\,\frac{0''.15}{\rho''} = 3.863085\ 10^{-6}\ \text{rad/year},$$

it is normal to $\bar{\omega}_S$, coplanar with $\bar{\omega}_S$ and $\bar{z}$ (axis of figure of the ellipsoid) and points in a direction such that $\bar{\omega}_S$ is between $\Delta\bar{\omega}_p$ and $\bar{z}$ (see Fig. 9). $\Delta\bar{\omega}_p$ rotates around $\bar{\omega}_S$ with an angular velocity slightly higher than $|\bar{\omega}_S|$

$$|\bar{\omega}_S| + \frac{2\pi}{1.1828} = 2306.4797\ \text{rad/year}$$

The angle between the Chandlerian axis $\bar{\omega}$ and the spin axis $\bar{\omega}_S$ is

$$\frac{|\bar{\omega}_p|}{|\bar{\omega}_S|} \simeq 0''.0003463$$

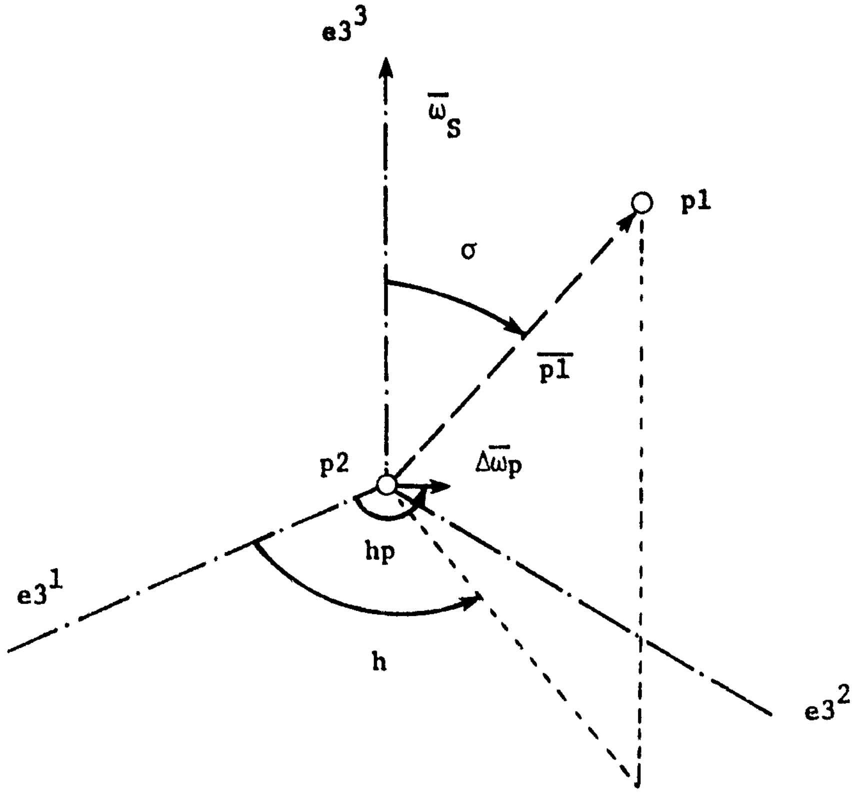


Fig. 10

Because of polar motion in the model the instantaneous colatitude and the hour angle of the vernal equinox h of the position vector $\overline{p1}$ vary in time (see Fig. 10). These variations can be computed as follows:

$$\dot{\sigma} = -|\Delta\overline{\omega}_p| \cdot \sin (h - h_p)$$

$$\dot{h} = |\overline{\omega}_S| - |\Delta\overline{\omega}_p|\cos (h - h_p) \cdot \cot \sigma$$

As the model of the earth is rigid the coordinates of p1 in the x,y,z geocentric reference frame, fixed to the ellipsoid with z as the minor axis, are also invariant. After one polar motion cycle, i.e., 1.1828 years, the σ and h coordinates of $\overline{p1}$ will be back at their initial values.

In order to illustrate the feasibility of this type of parametrization of polar motion we evaluated the effect of the model polar motion on the geocentric equatorial coordinates of five stations. Over a period of 440 sidereal days which is slightly longer than the period of model polar motion (1.1828 years), we numerically integrated the time rates of the σ_i and h_i coordinates assuming spin and polar motion to be the only causes of their variation. The constants used were as follows:

ω_s = 2301.1676 rad/year spin rate of the earth

$\dot{h}_p$ = 2306.4797 rad/year spin rate of the polar motion vector.

$\Delta\omega_p$ = $3.863085 \cdot 10^{-6}$ rad/year polar motion magnitude

The initial coordinates of the five stations are given in the following table:

Table 1. Initial Equatorial Coordinates

Station	h	σ
1	10°	50°
2	82°	50°
3	154°	50°
4	226°	50°
5	298°	50°

The initial value of h_p was set at 180°. Table 2 shows the time-like variations of the h, σ coordinates of the five stations in arc seconds computed at 20 sidereal day intervals over a period of 440 days.

The values in the table are computed by subtracting from the numerically integrated h_i, σ_i as affected by polar motion the equivalent h_i, σ_i values without polar motion.

In Fig. 11 we have plotted in addition to the varying σ, h coordinates of the five stations also the varying position of a reference pole vs. the spin axis and the vernal equinox. The reference pole is defined in a way similar to that of the CIO pole, i.e., the angular distances to the five stations are invariant.

We summarize this sub-chapter by writing up the equations for the disturbance vector in $\bar{\omega}$, i.e., the difference between the real and the model instantaneous rotation vectors (at P1 and p1 respectively) as follows:

$$\delta\bar{\omega} = \bar{\Omega} - \bar{\omega} = (\bar{\Omega}_S - \bar{\omega}_S) + (\Delta\bar{\Omega}_p - \Delta\bar{\omega}_p) + \Delta\bar{\Omega}_\ell = \delta\bar{\omega}_S + \delta\bar{\omega}_p + \delta\bar{\omega}_\ell$$

where

$\delta\bar{\omega}_S$ - is the disturbance in the spin vector of the earth, its first and second components being due to inadequacies of current theory and constants of precession and nutation and its third component representing spin rate variations.

$\delta\bar{\omega}_p$ - is the polar motion disturbance vector; it is normal to $\bar{\omega}_S$ and represents the difference between real and model polar motions.

$\delta\bar{\omega}_\ell \equiv \Delta\bar{\Omega}_\ell$ - is the local component (space variant) of the $\delta\bar{\omega}$ vector and is associated with local motions of P1.

We repeat that $\delta\bar{\omega}_S$ and $\delta\bar{\omega}_p$ are global in nature, i.e., they are space invariant while $\delta\omega_\ell$ is different, in general, for different points.

TABLE 2 . VARIATIONS IN EQUATORIAL COORDINATES DUE TO POLAR MOTION

EPOCH	H1	SIG1	H2	SIG2	H3	SIG3	H4	SIG4	H5	SIG5
0.	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
20.	0.04	0.0	0.01	0.04	-0.03	0.02	-0.03	-0.03	0.01	-0.04
40.	0.07	-0.01	0.03	0.08	-0.05	0.06	-0.06	-0.04	0.01	-0.08
60.	0.10	-0.03	0.06	0.11	-0.07	0.10	-0.10	-0.05	0.01	-0.13
80.	0.13	-0.06	0.09	0.12	-0.07	0.14	-0.13	-0.04	-0.01	-0.16
100.	0.14	-0.10	0.13	0.13	-0.06	0.18	-0.17	-0.02	-0.04	-0.19
120.	0.15	-0.15	0.16	0.12	-0.05	0.22	-0.19	0.02	-0.07	-0.21
140.	0.14	-0.19	0.20	0.10	-0.02	0.25	-0.21	0.05	-0.11	-0.22
160.	0.13	-0.23	0.22	0.07	0.01	0.27	-0.22	0.10	-0.14	-0.22
180.	0.10	-0.26	0.24	0.04	0.05	0.28	-0.21	0.14	-0.18	-0.20
200.	0.07	-0.28	0.25	-0.01	0.08	0.28	-0.20	0.18	-0.20	-0.17
220.	0.04	-0.30	0.25	-0.05	0.12	0.27	-0.18	0.21	-0.22	-0.13
240.	0.0	-0.30	0.24	-0.09	0.15	0.24	-0.15	0.24	-0.23	-0.09
260.	-0.03	-0.28	0.22	-0.13	0.17	0.21	-0.11	0.25	-0.23	-0.05
280.	-0.06	-0.26	0.19	-0.15	0.18	0.16	-0.07	0.25	-0.22	-0.01
300.	-0.09	-0.22	0.15	-0.17	0.18	0.12	-0.04	0.24	-0.20	0.03
320.	-0.10	-0.18	0.12	-0.17	0.17	0.08	-0.01	0.22	-0.18	0.06
340.	-0.10	-0.14	0.08	-0.16	0.16	0.04	0.02	0.19	-0.14	0.07
360.	-0.10	-0.10	0.05	-0.14	0.13	0.01	0.03	0.15	-0.11	0.08
380.	-0.08	-0.06	0.02	-0.11	0.10	-0.01	0.04	0.10	-0.07	0.07
400.	-0.05	-0.03	0.01	-0.07	0.06	-0.02	0.03	0.06	-0.04	0.05
420.	-0.02	-0.01	0.0	-0.03	0.02	-0.01	0.02	0.02	-0.01	0.02
440.	0.01	0.0	0.0	0.01	-0.01	0.01	-0.01	-0.01	0.01	-0.01

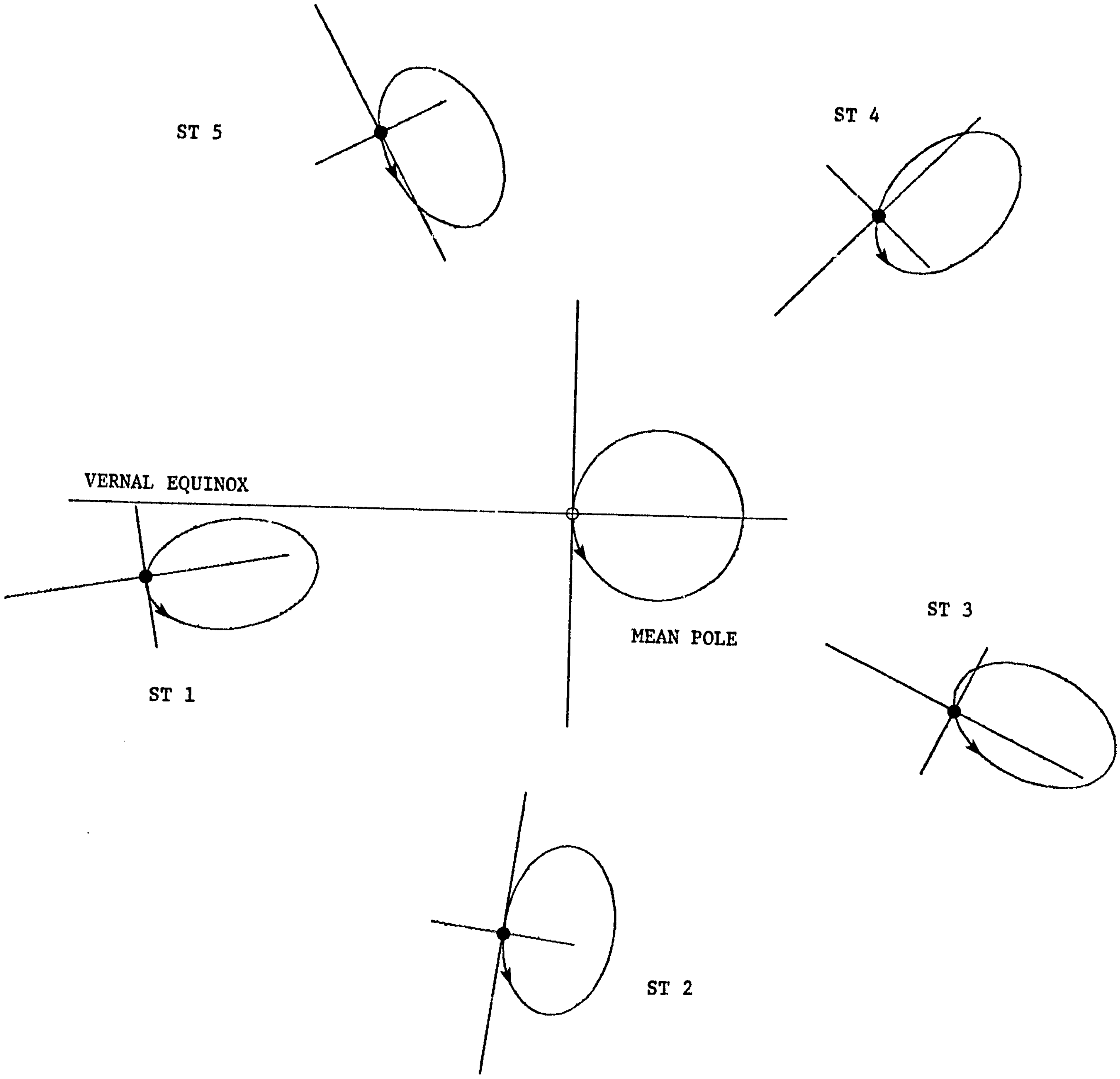


Fig. 11

4.2 Gravity and the Position Vector

The relationships between the geocentric position vector $\overline{P1}$ and its time-like variations on the one hand and the potential of gravity and its derivatives at P1 on the other are studied in this sub-chapter.

The time-like variations in $\overline{P1}$ were partitioned (see 4.1) into directional and magnitudinal components as follows:

$$\frac{\partial}{\partial T} \overline{P1} = \overline{\Omega} \times \overline{P1} + \frac{\partial}{\partial T} |\overline{P1}| \cdot \frac{\overline{P1}}{|\overline{P1}|}$$

We will study first the various causes for variations in the potential W at P1 and their relationship to variations in the magnitude of the geocentric position vector.

The gravity potential of the earth at P1 (which is a point on the earth surface) is evaluated by the following well-known formula (see Fig. 12):

$$W = G \int_V \frac{\rho}{\ell} dv + \frac{1}{2} (\overline{\Omega} \times \overline{P1}) \cdot (\overline{\Omega} \times \overline{P1})$$

where dv is an element of volume,

ρ, ℓ are the density of dv and its distance from P1 respectively,

G is the gravitational constant.

The integration is extended over V which includes in our case the solid earth, the oceans and the atmosphere. W thus obtained would be the measured value of W from which the potential of extraterrestrial masses, the tidal potential has been subtracted. Considering the total mass within V to be invariant

$$\int_V \rho dv = \text{const} = m$$

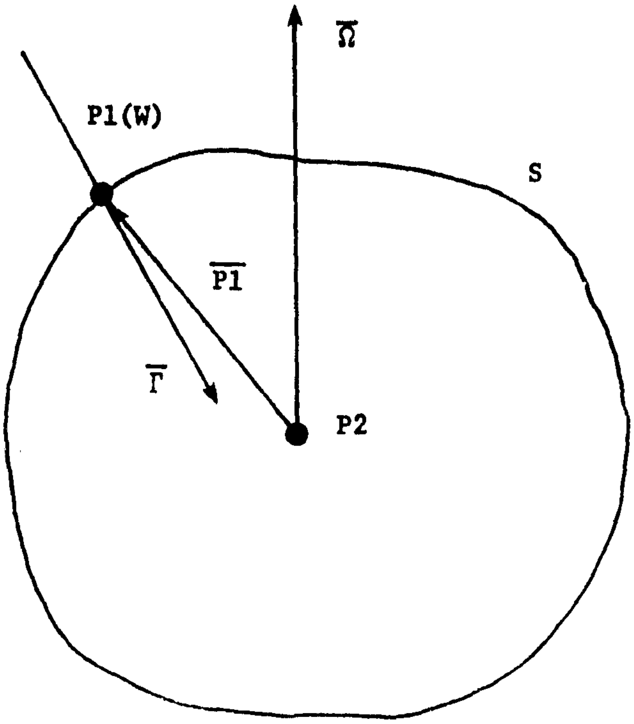


Fig. 12

or $\frac{\partial m}{\partial T} = 0$

we can differentiate W with respect to time to obtain the time-like variations in W

$$\frac{\partial W}{\partial T} = G \int_V \left(\frac{\partial \rho}{\partial T} \cdot \frac{1}{\ell} - \frac{\partial \ell}{\partial T} \cdot \frac{\rho}{\ell^2}\right) dv + \left(\frac{\partial W}{\partial T}\right)_\Omega$$

where $\left(\frac{\partial W}{\partial T}\right)_\Omega$ denotes variations in rotational potential.

The first term in the integral is not associated with any changes in the relative distances between P1 and other material points including possible target points P1(2,K) or other observing points P1(1,K). In other words, time-like variation in gravitational potential due to density redistribution within the earth are not accompanied by time-like variations in <u>relative position</u>. However, the position of the mass center P2 does change (mass center shift) within the surface S and consequently the geocentric vector $\overline{P1}$ also changes. The variations in the position vector $\overline{P1}$ due to mass center shift are space invariant, i.e., they are the same for all the points on the earth.

The second term in $\frac{\partial W}{\partial T}$ is explicitly associated with variations in relative distances between P1 and the totality of mass points which constitute the earth. The phenomenon which dominates this term is the local differential motion of P1. One out of several causes for local motions is the elastic response of the earth to variations in the tidal potential.

By definition, the "horizontal" component of motion of $\overline{P1}$ is normal to $-\overline{\Gamma}$ (the direction of the local vertical) and produces zero variation in the potential. Thus, the only component of $\frac{\partial \overline{P1}}{\partial T}$ which is related to $\frac{\partial W}{\partial T}$ is the vertical component, i.e.,

$$\frac{\partial \overline{P1}}{\partial T} \cdot (-\overline{\Gamma}) \simeq \frac{\partial \overline{P1}}{\partial T} \cdot \frac{\overline{P1}}{|\overline{P1}|} = \frac{\partial}{\partial T} |\overline{P1}|$$

where $-\overline{\Gamma}$ is the unit vector along the local vertical (see Fig. 13) and the approximation is permissible due to the small angle between $\overline{P1}$ and $-\overline{\Gamma}$.

The time-like variations <u>in magnitude</u> of the geocentric position vector $\overline{P1}$ are related to variations in the potential W by a modification of the well-known formula [Heiskanen and Moritz, 1967] which relates potential and height differences:

$$\frac{\partial W}{\partial T} \simeq -g \cdot \frac{\partial}{\partial T} |\overline{P1}|$$

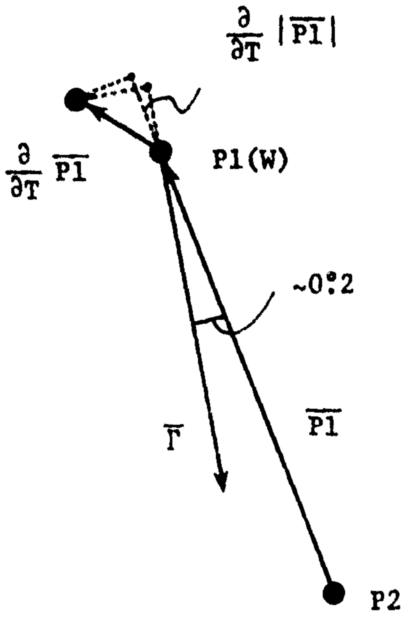


Fig. 13

from which it follows directly

$$\frac{\partial}{\partial T}|\overline{P1}| = -\frac{1}{g} \cdot \frac{\partial W}{\partial T}$$

In the second part of this sub-chapter we study the relationship between the local gravity vector $-\overline{\Gamma}$ at P1 and the geocentric position vector $\overline{P1}$, their models and the corresponding disturbances.

Fig. 14 shows a schematic spatial diagram of the geocentric position vector $\overline{P1}$; the $-\overline{\Gamma}$ local vertical vector and their respective models $\overline{p1}$ and $-\overline{\gamma}$. The disturbances $\delta\overline{p2}$ and $\delta\overline{p1}$ as well as $\delta(-\overline{\gamma})$ are also shown. $\overline{z}$ is the axis of figure of the reference ellipsoid. We derive first an expression for the angle between $-\overline{\gamma}$ and $\overline{p1}$.

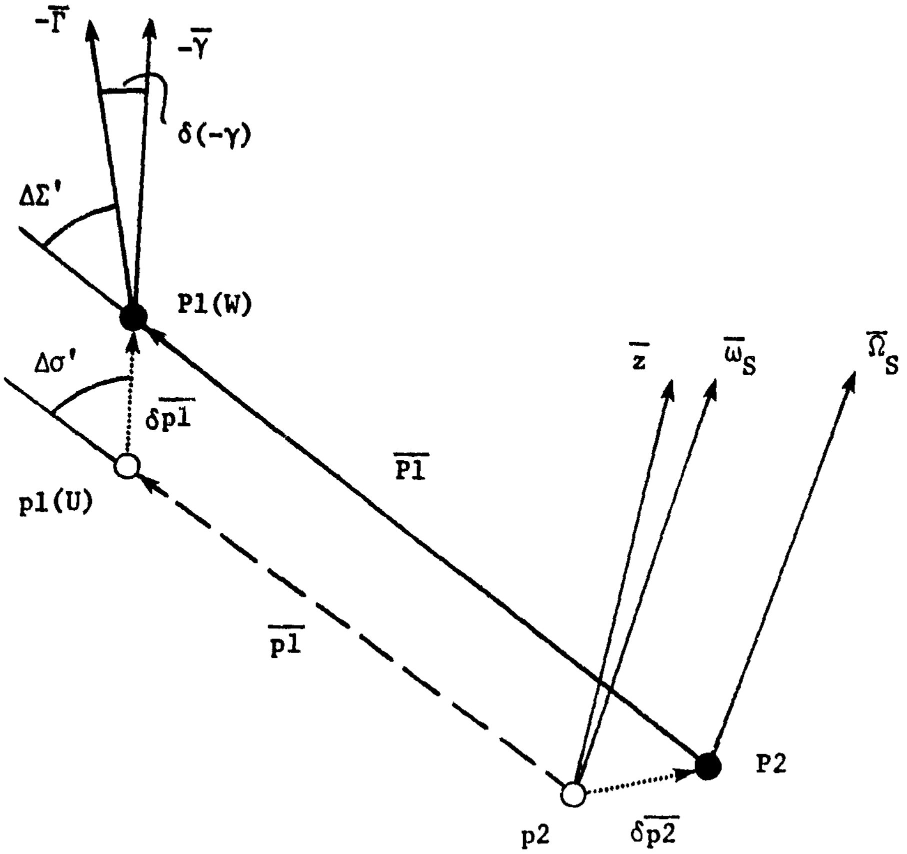


Fig. 14

A right-handed Cartesian coordinate system x, y, z is defined (see Fig. 15) which has its origin at p2 and which is fixed to the reference ellipsoid. The ellipsoid, or equivalently, the x,y,z system, rotates vs. inertial space around the spin axis $\overline{\omega}_S$ which is inclined by 0".15 vs. $\overline{z}$. The point p1 is defined in the x,y,z system by its three Cartesian coordinates p1(x,y,z) or by the three geocentric spherical coordinates ρ, σ', λ which are the geocentric radial distance, colatitude and longitude respectively. Since the model is assumed to be rigid the three coordinates are constant.

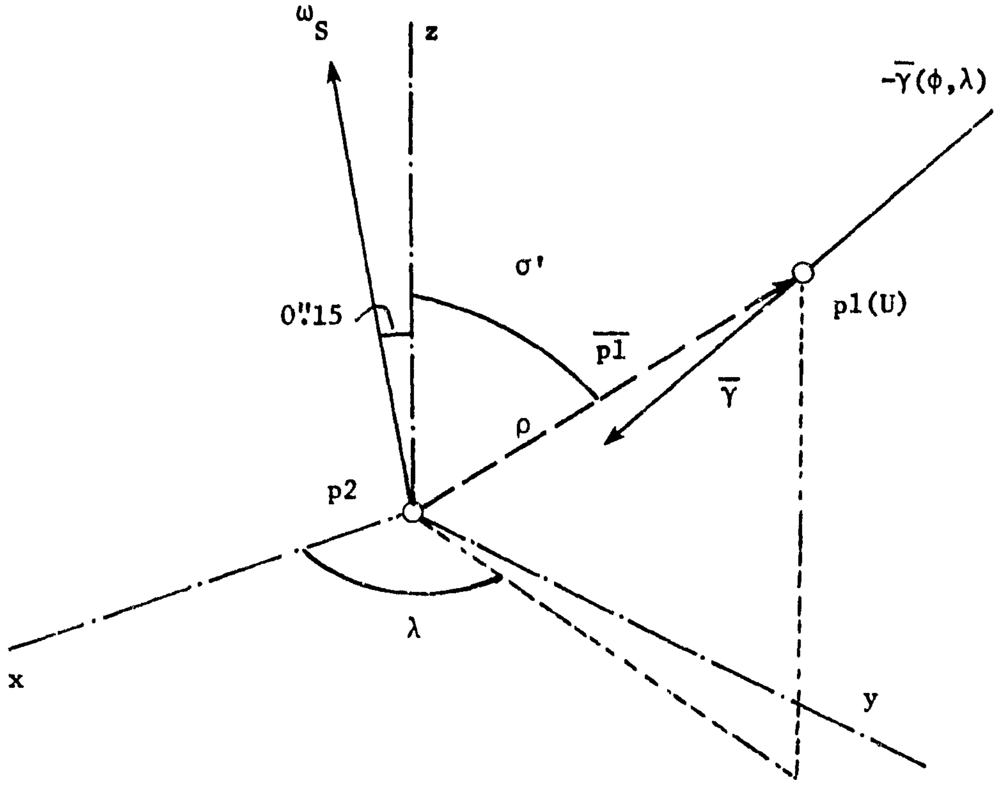


Fig. 15

We will use also ellipsoidal coordinates of p1 in the x,y,z coordinate system, namely u, β, λ which are convenient in that the gravity (normal) potential U of the model ellipsoid and its derivatives can be represented in closed formulae.

For the computation of U and the components of its gradient $\overline{\gamma}$ we will assume that the component ω of $\overline{\omega}_S$ along z is equal in magnitude to $|\overline{\omega}_S|$. The difference between ω and $|\overline{\omega}_S|$ divided by ω is negligible - of the order of 10^{-12}. The value of the normal potential U at the point p1 is computed as a function of the four parameters of the level ellipsoid a, e, m, ω and the components of the position vector $\overline{p1}$. According to [Heiskanen and Moritz, 1967] and utilizing ellipsoidal coordinates u, β, λ, the following expressions hold:

$$u = \sqrt{\frac{\rho^2 - a^2e^2}{2} + \sqrt{a^2e^2\rho^2\cos^2\sigma' + \frac{(\rho^2 - a^2e^2)^2}{4}}}$$

$$\beta = \text{arc sin } \frac{\rho \cos \sigma'}{u}$$

the inverse relationship being

$$\rho = \sqrt{u^2 + a^2e^2\cos^2\beta}$$

$$\sigma' = \text{arc tan } \sqrt{1 + \left(\frac{ae}{u}\right)^2} \cot \beta$$

Longitude is the same in spherical or ellipsoidal coordinates. The potential at $pl(u,\beta,\lambda)$ is computed by the following formula (see ibid.) where the effect of the small equatorial (x-y plane) component of ω_S has been neglected:

$$U(u,\beta) = \frac{Gm}{ae} \text{ arc tan } \frac{ae}{u} + \frac{1}{2}\, \omega^2 a^2 \, \frac{q}{qo}\, (\sin^2\beta - \tfrac{1}{3}) + \frac{1}{2}\, \omega^2 (u^2 + a^2e^2)\cos^2\beta$$

where

$$q = \frac{1}{2}\left[\left(1 + 3\, \frac{u^2}{a^2e^2}\right) \text{arc tan } \frac{ae}{u} - 3\, \frac{u}{ae}\right]$$

$$qo = \frac{1}{2}\left[\left(1 + 3\, \frac{(1-e^2)}{e^2}\right) \text{arc tan } \frac{e}{\sqrt{1-e^2}} - 3\, \frac{\sqrt{1-e^2}}{e}\right]$$

The vector along the gradient of U at pl is $\overline{\gamma}$ the normal gravity vector. Its components along the ellipsoidal coordinates are:

$$\frac{\partial U}{\partial u} = \frac{-Gm}{(u^2 + a^2e^2)} + \omega^2 u \cos^2\beta + \frac{1}{2}\, \frac{\omega^2 a^2}{qo} \left(\sin^2\beta - \frac{1}{3}\right)\left[\frac{3u}{a^2e^2} \text{ arc tan } \frac{ae}{u} - \frac{3u^2 + 2a^2e^2}{ae(u^2 + a^2e^2)}\right]$$

$$\frac{\partial U}{\partial \beta} = \omega^2 \sin\beta \cos\beta \left[a^2 \frac{q}{qo} - (u^2 + a^2 e^2)\right]$$

$$\frac{\partial U}{\partial \lambda} = 0$$

In order to obtain the components of $\overline{\gamma}$ in the x,y,z system we need the Jacobian of transformation from u, β, λ into x,y,z coordinates.

The independence of U from λ implies that the three vectors $\overline{z}$, $\overline{pl}$ and $\overline{\gamma}$ are coplanar. Accordingly, instead of transforming from u, β, λ into x,y,z, we transform from u, β into r, z where r, z, λ are the cylindrical coordinates of pl and $r = \sqrt{x^2 + y^2}$ is the distance from the z axis. The transformation equations are simple

$$r = \sqrt{u^2 + a^2 e^2} \cos\beta$$

$$z = u \sin\beta$$

The components of $\overline{\gamma}$ in the r,z system would be computed then in a row vector form

$$\left[\frac{\partial U}{\partial r} \; \frac{\partial U}{\partial z}\right] = \left[\frac{\partial U}{\partial u} \; \frac{\partial U}{\partial \beta}\right] \cdot J$$

where

$$J = \frac{\partial \begin{pmatrix} u \\ \beta \end{pmatrix}}{\partial \begin{pmatrix} r \\ z \end{pmatrix}} = \frac{\sqrt{u^2 + a^2 e^2}}{u^2 + a^2 e^2 \sin^2\beta} \begin{bmatrix} u \cos\beta & \sqrt{u^2 + a^2 e^2} \sin\beta \\ - \sin\beta & \dfrac{u \cos\beta}{\sqrt{u^2 + a^2 e^2}} \end{bmatrix}$$

The angle between $\overline{\gamma}$ and the z axis is thus given by

$$\sigma'_{\gamma} = \text{arc tan} \left[\frac{\frac{\partial U}{\partial r}}{\frac{\partial U}{\partial z}} \right]$$

while the corresponding angle of the $\overline{p1}$ vector is (see Fig. 16):

$$\sigma'_{p1} = \text{arc tan}\,\frac{r}{z} = \text{arc tan}\left[\frac{\sqrt{u^2 + a^2e^2}}{u}\cot\beta\right]$$

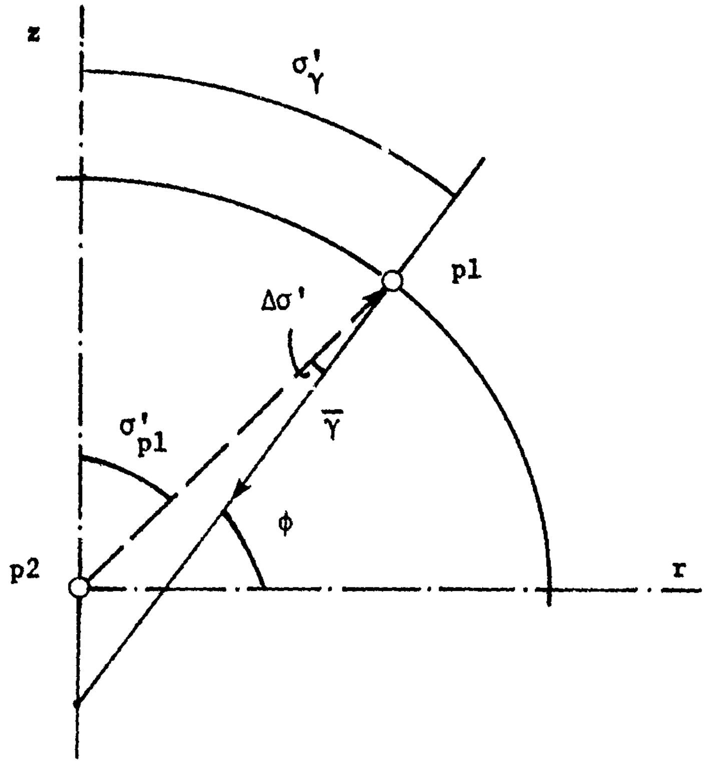


Fig. 16

We point out that ϕ the complement of σ'_γ is different from the conventional geodetic latitude which is computed for a point on the ellipsoidal surface. We shall see in a subsequent sub-chapter that the model of P1 is not on the ellipsoid. The angle between $-\overline{\gamma}$ and $\overline{p1}$ is thus

$$\Delta\sigma' = \sigma'_\gamma - \sigma'_{p1}$$

and for mid latitudes and a few kilometers height above the ellipsoid it is of the order of 10'.

We will derive now an approximate expression for the angle between $\overline{P1}$ and the $-\overline{\Gamma}$ vector as a function of $\Delta\sigma'$, the positional $[\delta\overline{p2}, \delta\overline{p1}]$ and the angular $[\delta(-\overline{\gamma})]$ disturbances. The angle between $\overline{P1}$ and $\overline{p1}$ denoted by Δ is evaluated from the dot product:

$$\frac{\overline{P1} \cdot \overline{p1}}{|\overline{P1}|\ |\overline{p1}|} = \cos\Delta$$

From Fig. 14 we have

$$\overline{P1} = \overline{p1} + (\delta\overline{p1} - \delta\overline{p2})$$

$$\cos\Delta = \frac{\overline{p1} \cdot \overline{p1} + (\sigma\overline{p1} - \sigma\overline{p2}) \cdot \overline{p1}}{|\overline{P1}|\ |\overline{p1}|}$$

$$= \frac{1}{|\overline{P1}|}\left[|\overline{p1}| + (\delta\overline{p1} - \delta\overline{p2}) \cdot \frac{\overline{p1}}{|\overline{p1}|}\right]$$

We expand $\cos\Delta$ (Δ is a small angle) and regroup

$$\Delta \approx \sqrt{2}\sqrt{1 - \left[\frac{|\overline{p1}|}{|\overline{P1}|} + \frac{(\delta\overline{p1} - \delta\overline{p2})}{|\overline{P1}|} \cdot \frac{\overline{p1}}{|\overline{p1}|}\right]}$$

If $(\delta\overline{p1} - \delta\overline{p2})$ is collinear with $\overline{p1}$, Δ will be zero. Δ will be maximum for $|\overline{p1}| = |\overline{P1}|$ from which we derive finally (see Fig. 17)

$$\Delta \leq \frac{|\delta\overline{p1} - \delta\overline{p2}|}{|\overline{p1}|}$$

If we define $\overline{p1}$ and $\overline{p2}$ so that the magnitude of the difference $(\delta\overline{p1} - \delta\overline{p2})$ is of the order of a few kilometers, Δ will be a small angle of the order of tens of seconds of arc.

Considering now that $(-\overline{\Gamma})$, $(-\overline{\gamma})$, $\overline{P1}$ and $\overline{p1}$ are not necessarily coplanar we can see according to Fig. 18 that the angle between $\overline{P1}$ and $-\overline{\Gamma}$ can be approximated by $\Delta\sigma'$ the error being smaller than the sum

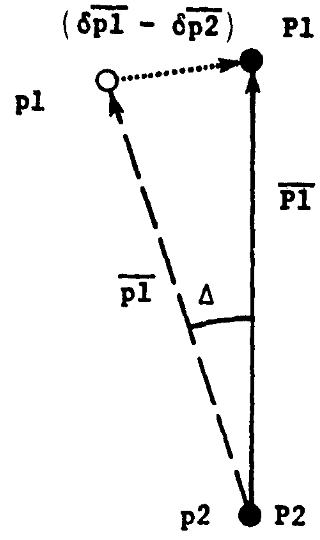
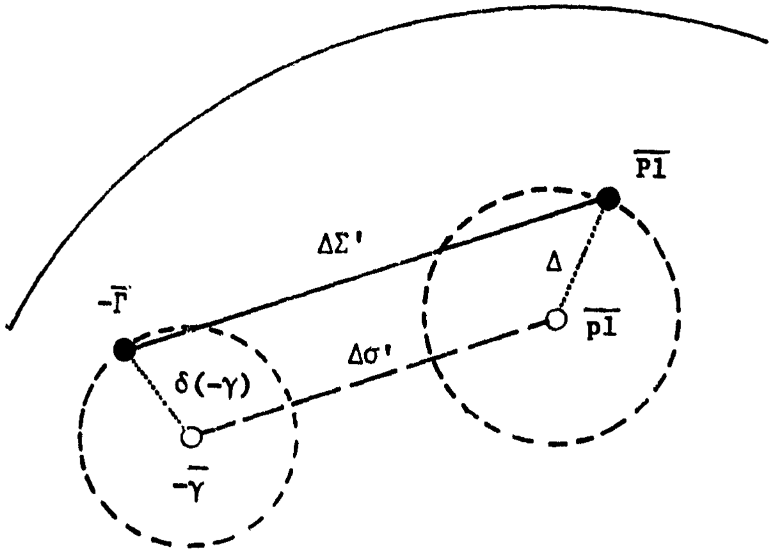


Fig. 17



The unit sphere

Fig. 18

$[\Delta + \delta(-\gamma)]$. The difference between the real $\Delta\Sigma'$ and its model $\Delta\sigma'$, i.e., the disturbance in $\Delta\sigma'$ depends on the deflections of the vertical $\delta(-\gamma)$ and on the positional disturbance difference $(\delta\overline{p1} - \delta\overline{p2})$.

4.3. Time-Like Variations in Level 1 Positional Disturbances

In Chapter 3 of this report we studied the general nature of the second- and third-level positional disturbances and their variations with time. In the case of Level 1 positional disturbances and their time-like variations we will be more specific. In this sub-chapter we derive the differential equations of Level 1 positional disturbances in terms of disturbances of the rotational vector $\overline{\Omega}$ and also in terms of variations in the magnitude of the geocentric vector $\overline{P1}$. Assume that the real geocenter P2 and its model p2 are coincident, i.e., the second level positional disturbance $\delta\overline{p2}$ is identically a zero vector.

Later we will relax this condition and will show the resulting implications. The time-like variations of the geocentric vector $\overline{P1}$, $\frac{\partial\overline{P1}}{\partial T}$ or $\dot{\overline{P1}}$ was partitioned above (see 4.1, 4.2) into a variation in direction and a variation in magnitude (see Fig. 19). For completeness we rederive the expression for $\dot{\overline{P1}}$ in a slightly different form:

$$\frac{\partial\overline{P1}}{\partial T} = \frac{\partial}{\partial T}\left[\frac{\overline{P1}}{|\overline{P1}|} \cdot |\overline{P1}|\right] = \frac{\partial}{\partial T}\ \frac{\overline{P1}}{|\overline{P1}|} \cdot |\overline{P1}| + \frac{\overline{P1}}{|\overline{P1}|} \cdot \frac{\partial}{\partial T}|\overline{P1}|$$

$$\frac{\partial}{\partial T}\frac{\overline{P1}}{|\overline{P1}|} \cdot |P1| = \tilde{\Omega} \mathrm{x} \overline{P1} = (\bar{\omega} \mathrm{x}\ \delta\bar{\omega})\ \mathrm{x}\ (\overline{p1} + \delta\overline{p1}) =$$

$$\bar{\omega} \mathrm{x}\ \overline{p1} + \bar{\omega} \mathrm{x}\ \delta\overline{p1} + \delta\bar{\omega} \mathrm{x}\ \overline{p1} + \delta\bar{\omega} \mathrm{x}\ \delta\overline{p1}$$

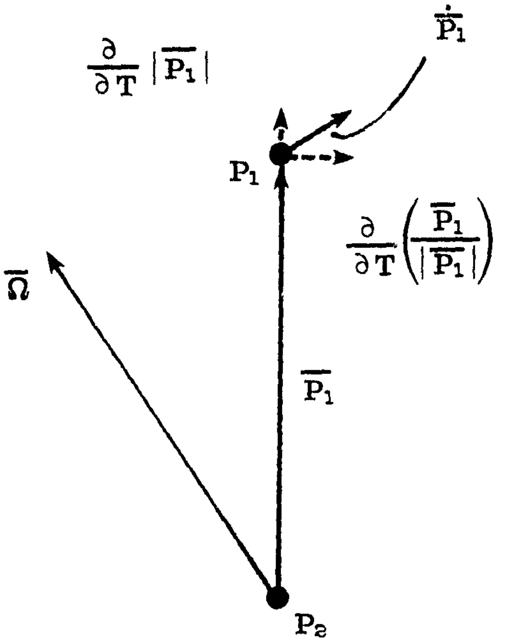


Fig. 19

The time-like variation of the $\overline{p1}$ vector by definition consists of a directional component only as the magnitude of $\overline{p1}$ is invariant (The model of the earth is assumed to be rigid.)

$$\frac{\partial}{\partial T}\,\overline{p1} = \overline{\omega} \times \overline{p1}$$

where $\overline{\omega}$ is the sum of the diurnal rotational (spin) vector $\overline{\omega}_s$ and $\overline{\Delta\omega}_p$ the model polar motion rotation vector.

The expression for the Level 1 positional disturbance simplified by the assumption $\delta\overline{p2} = 0$ is

$$\delta\overline{p1} = \overline{P1} - \overline{p1}$$

The time-like variation of $\delta\overline{p1}$ is obtained by differentiation as follows:

$$\frac{\partial}{\partial T}\,\delta\overline{p1} = \frac{\partial}{\partial T}\,\overline{P1} - \frac{\partial}{\partial T}\,\overline{p1}$$

We substitute expressions derived above, neglect two terms of the second order $(\overline{\Delta\omega}p \times \delta\overline{p1})$, $(\delta\overline{\omega} \times \delta\overline{p1})$, and obtain

$$\frac{\partial}{\partial T}\,\delta\overline{p1} \simeq \frac{\overline{P1}}{|\overline{P1}|} \cdot \frac{\partial}{\partial T}\,|\overline{P1}| + \delta\overline{\omega} \times \overline{p1} + \overline{\omega}_s \times \delta\overline{p1}$$

Rearranging and substituting for $\overline{P1}$ its equivalent we obtain the final form of $\frac{\partial}{\partial T}\,\delta\overline{p1}$

$$\frac{\partial}{\partial T}\,\delta\overline{p1} \simeq \overline{\omega}_s \times \delta\overline{p1} + \delta\overline{\omega} \times \overline{p1} + \frac{\partial}{\partial T}\,|\overline{p1} + \delta\overline{p1}|\,\frac{(\overline{p1} + \delta\overline{p1})}{|\overline{p1} + \delta\overline{p1}|}$$

This is a set of three first-order differential equations of the positional disturbance $\delta\overline{p1}$ with $\delta\overline{\omega}$ as an independent parameter. It demonstrates the relationship between the Level 1 positional disturbances and those of the rotational vector. If the positional disturbance $\delta\overline{p1}$ at some initial epoch is known, we can integrate numerically the differential equations of $\delta\overline{p1}$ using rotational vector disturbances and variations in the magnitude of $\overline{P1}$ (or equivalently variations in the potential W), which have been determined from observations.

We will show now that the above equation holds also for the case where $\delta\overline{p2}$ is not zero. Using a portion of the P tower as in Fig. 20 we can use the commutative property to derive the following (Level 4 is excluded without loss of generality):

$$\frac{\partial}{\partial T}\,\delta\overline{p3} = \delta\overline{p3}(1,2) - \delta\overline{p3}(1,1) = \frac{\partial\overline{P3}}{\partial T} - \frac{\partial\overline{p3}}{\partial T}$$

$$\frac{\partial}{\partial T}\,\delta\overline{p2} = \delta\overline{p2}(1,2) - \delta\overline{p2}(1,1) = \frac{\partial}{\partial T}\,\delta\overline{p3} + \frac{\partial\overline{P2}}{\partial T} - \frac{\partial\overline{p2}}{\partial T}$$

$$\frac{\partial}{\partial T}\,\delta\overline{p1} = \delta\overline{p1}(1,2) - \delta\overline{p1}(1,1) = \frac{\partial}{\partial T}\,\delta\overline{p2} + \overline{\Omega} \text{ x } \overline{P1} + \frac{\partial|\overline{P1}|}{\partial T} \cdot \frac{\overline{P1}}{|\overline{P1}|}$$

$$- \overline{\omega} \text{ x } \overline{p1} = \frac{\partial}{\partial T}\,\delta\overline{p2} + \overline{\omega} \text{ x } \overline{P1} + \delta\overline{\omega} \text{ x } \overline{P1} - \overline{\omega} \text{ x } \overline{p1}$$

$$+ \frac{\partial|\overline{P1}|}{\partial T} \cdot \frac{\overline{P1}}{|\overline{P1}|}$$

Neglecting second-order terms, substituting $\overline{\omega}_s$ for $\overline{\omega}$ and regrouping we have:

$$\frac{\partial}{\partial T}(\delta\overline{p1} - \delta\overline{p2}) \approx \overline{\omega}_s \text{ x } (\delta\overline{p1} - \delta\overline{p2}) + \delta\overline{\omega} \text{ x } \overline{p1} + \frac{\overline{p1}}{|\overline{p1}|} \cdot \frac{\partial}{\partial T}|\overline{p1} + (\delta\overline{p1} - \delta\overline{p2})|$$

The resulting vector differential equation of the geocentric positional disturbance $(\delta\overline{p1} - \delta\overline{p2})$ is similar to the one obtained earlier for $\delta\overline{p2} = 0$.

In the second part of this sub-chapter we will develop a specific model of the Level 1 origins for the earth. The models of all the P1 points of the earth constitute thus the <u>geometrical</u> model of the earth surface. The rotational level ellipsoid discussed earlier in this report is the <u>dynamical</u> model of the earth. By making the distinction

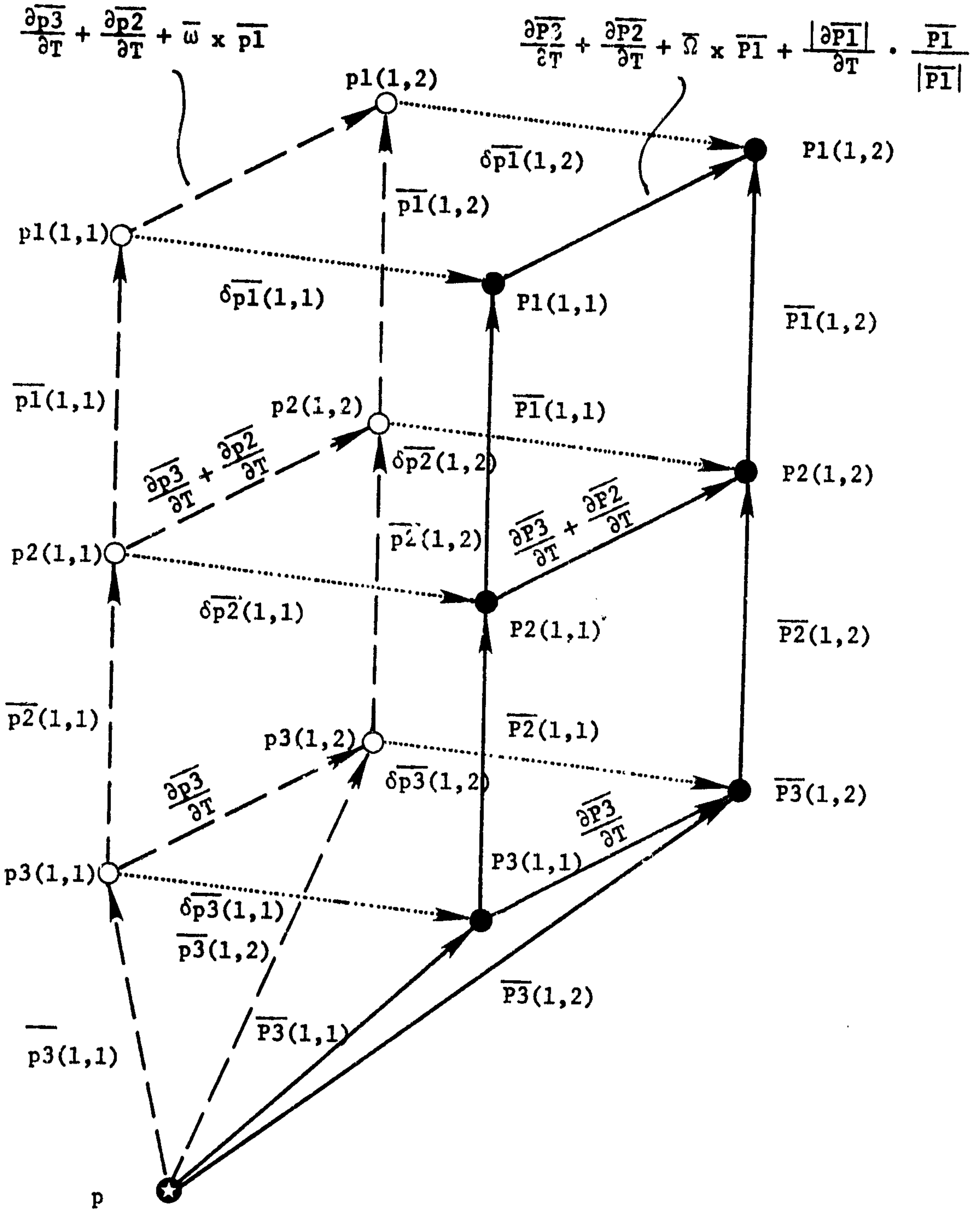

$\frac{\partial\overline{p3}}{\partial T} + \frac{\partial\overline{p2}}{\partial T} + \overline{\omega} \times \overline{p1}$
$\frac{\partial\overline{P3}}{\partial T} + \frac{\partial\overline{P2}}{\partial T} + \overline{\Omega} \times \overline{P1} + \frac{|\partial\overline{P1}|}{\partial T} \cdot \frac{\overline{P1}}{|\overline{P1}|}$
p1(1,2)
P1(1,2)
$\delta\overline{p1}(1,2)$
$\overline{p1}(1,2)$
p1(1,1)
$\delta\overline{p1}(1,1)$
P1(1,1)
$\overline{P1}(1,2)$
$\overline{p1}(1,1)$
p2(1,2)
$\overline{P1}(1,1)$
$\delta\overline{p2}(1,2)$
P2(1,2)
$\frac{\partial\overline{p3}}{\partial T} + \frac{\partial\overline{p2}}{\partial T}$
$\frac{\partial\overline{P3}}{\partial T} + \frac{\partial\overline{P2}}{\partial T}$
$\overline{p2}(1,2)$
p2(1,1)
$\delta\overline{p2}(1,1)$
P2(1,1)
$\overline{P2}(1,2)$
$\overline{p2}(1,1)$
$\overline{P2}(1,1)$
p3(1,2)
$\delta\overline{p3}(1,2)$
$\overline{P3}(1,2)$
$\frac{\partial\overline{p3}}{\partial T}$
$\frac{\partial\overline{P3}}{\partial T}$
P3(1,1)
p3(1,1)
$\delta\overline{p3}(1,1)$
$\overline{p3}(1,2)$
$\overline{P3}(1,2)$
$\overline{P3}(1,1)$
$\overline{p3}(1,1)$
p

Fig. 20

between geometrical and dynamical models of the earth we actually define pl as a point which is not located on the surface of the ellipsoid. As we shall see, pl is located on the telluroid as defined by Hirvonen [1960] and as described also in [Heiskanen and Moritz, 1967].

The selection of the telluroid as the geometrical model of the earth is essential for establishing a clear and unambiguous relationship between the potential of model (normal) gravity at pl and its derivatives on the one hand and the geocentric model position vector $\overline{p1}$ and model gravity vector $\overline{\gamma}$ at pl on the other.

In the P tower we have denoted a point on the earth surface as either P1(1,k) or P1(2,k) where indices 1 or 2 indicate an observing station or a target at the epoch T_k $(k = 1,2)$. In the following discussions we will drop the indices for convenience, as it will become clear that the particular values of the two indices within brackets are irrelevant. The fundamental vectors, discussed in [Grafarend et al., 1979], $-\overline{\Gamma}$, $\overline{\Omega}$ are specifically referred to the P1 point, where in particular $\overline{\Gamma}$ is the direction of gravity at P1 and $\overline{\Omega}$ is parallel to the axis around which the geocentric vector $\overline{P1}$ rotates with respect to inertial space (see sub-chapter 4.1).

We denoted the model of P1 as pl and denoted the vector difference $\overline{p1P1}$ as $\delta\overline{p1}$ the positional disturbance of pl. Just as for P1 above, the models of the fundamental vectors $-\overline{\gamma}$, $\overline{\omega}$ refer to the pl point. In particular the $\overline{\gamma}$ vector is defined as the direction of model (normal) gravity at pl and $\overline{\omega}$ is parallel to the axis around which the model geocentric vector $\overline{p1} \equiv \overline{p2p1}$ rotates in inertial space. In order

2 - 2

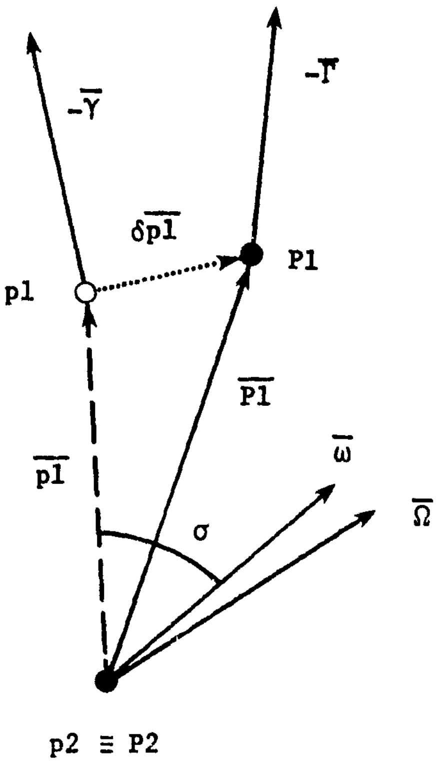


Fig. 21

to focus on $\overline{p1}$, $\overline{P1}$ and $\delta\overline{p1}$ alone we will make the assumption that p2 and P2 coincide, i.e., $\delta\overline{p2} = 0$ (see Fig. 21).

In principle $\delta\overline{p1}$ cannot <u>be and remain</u> a zero vector due to the essential difference in the rotational motion of the two vectors $\overline{p1}$ and $\overline{P1}$. $\overline{\Omega}$ and $\overline{\omega}$ are different in direction and in magnitude; $|\overline{p1}|$ is constant by definition (rigidity) while $|\overline{P1}|$ varies in time due to various causes like tides, mass redistributions, regional uplifts, etc.

We will define now the relationship between $\overline{P1}$ and $\overline{p1}$ (see Fig. 22) through the concepts of the height anomaly ζ and the telluroid as described in [Heiskanen and Moritz, 1967]. In addition to the basic angular parameters Φ, H which define the orientation of $-\overline{\Gamma}$ versus $\overline{\Omega}$ we introduce the gravity potential W at P1 or actually the potential

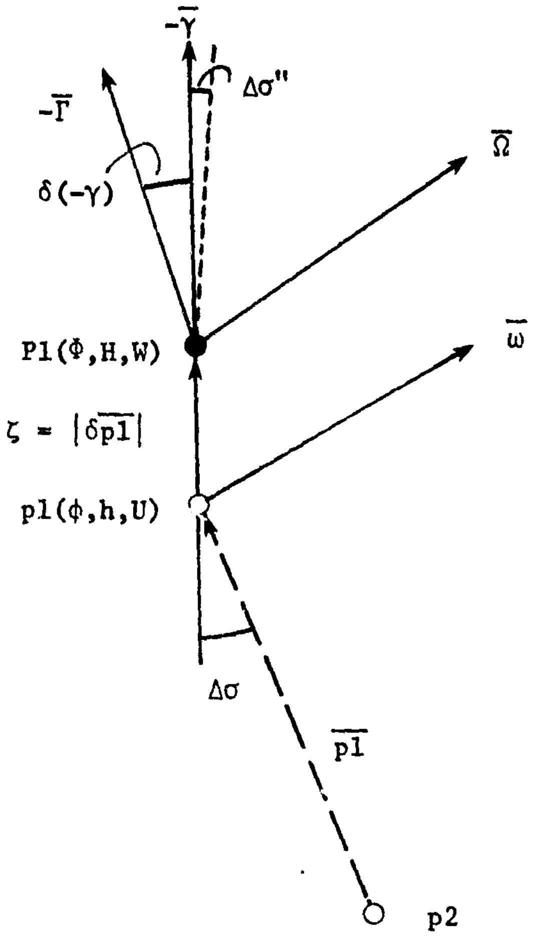


Fig. 22

difference $W_{oo} - W$ between the geoid and P1. The parameters of the reference ellipsoid (a, e, ω, m) are chosen so that the model (normal) potential on its surface U_{oo} is equal to W_{oo}. In Fig. 22 the quantities Φ, H and W define the position of P1 in space and the direction of $-\overline{\Gamma}$ there. Apply to $-\overline{\Gamma}$ the disturbance $\delta(-\overline{\gamma})$ with an opposite sign to obtain (except for a small correction Δσ") the $-\overline{\gamma}$ vector.

Beginning from P1 we measure the height anomaly ζ along the $-\overline{\gamma}$ vector and obtain the p1 point, i.e., the model of the P1 point. The positional disturbance vector $\delta\overline{p1}$ thus is defined in magnitude by ζ and by $-\overline{\gamma}$ in direction. The point p1 so defined is on the telluroid. According to the definitions of ζ and the telluroid the normal gravity potential at p1 is equal to the natural potential W at P1 or equivalently the respective potential differences versus the ellipsoid and the geoid are equal

$$U_{oo} - U_{p1} = W_{oo} - W_{p1}$$

Now apply to $\overline{\Omega}$ the disturbance $\delta\overline{\omega}$ with opposite sign and obtain the direction of $\overline{\omega}$ in space. Using a rigorous transformation from $-\overline{\gamma}$ to $\overline{p1}$ (see 4.2) obtain the direction of $\overline{p1}$ in space.

The magnitude of $\overline{p1}$ is obtained from U at p1, the σ angle between $\overline{p1}$ and z (σ' after being corrected for model polar motion) and the parameters of the ellipsoid. Thus, we arrive finally at the p2 point, the mass center of the ellipsoid.

We can summarize in concept the above relationships as follows:

(i) Three quantities (x, y, z) or (ϕ, λ, U), are needed to define the $\overline{p1}$ and $-\overline{\gamma}$ vectors.

(ii) Two disturbances ($\delta(-\overline{\gamma})$ and ζ) are needed to transform from $\overline{p1}$ and $-\overline{\gamma}$ into $\overline{P1}$ and $-\overline{\Gamma}$. The two disturbances are represented by three numbers: two for $\delta(-\overline{\gamma})$, the deflections of the vertical, and one for ζ, the height anomaly which is close in value to the undulations of the geoid.

The disturbances $\delta(-\overline{\gamma})$ and ζ as defined above correspond to the quantities which would be evaluated through well-known techniques of physical geodesy [Heiskanen and Moritz, 1967].

There are two basic difficulties involved in the above definition of the positional disturbance $\delta\overline{p1}$:

(i) The gravity potential W at P1 varies in time and correspondingly the model potential U at p1 which is equal to W should vary also. This, however, would require a non-rigid telluroid which contradicts our definition of a rigid model of the earth.

(ii) $\overline{\Omega}$ and $\overline{\omega}$ are different in magnitude and in direction. As the above vectors represent the rotational velocity vectors of P1 and p1 respectively, it is obvious that the two points will not remain aligned along the $-\overline{\gamma}$ vector, except at an initial epoch.

A possible solution which allows us to retain some of the obvious advantages of the telluroid as the geometrical model of the earth without sacrificing the rigidity principle is as follows:

The geometrical model of the earth is assumed to be rigid. It is defined as the telluroid at a specified zero epoch. From the zero epoch and on the positional disturbances vary according to the differential equation derived in the first part of this sub-chapter.

4.4 Time-Like Variations of the Distance Between Two Earth Surface Points

In this sub-chapter we study variations in the distance between points at the topocentric level in order to identify the global and local parameters which can be recovered. Consider the distance between two points on the earth surface, i.e., P1(1,1) and P1(2,1), the observing and the target points at the topocentric level of the P tower (see Fig. 23). As both points are defined on the earth surface their body-centric reference point P2 is the same (the geocenter) for both and so the vectors $\overline{P2(1,1)P2(2,1)}$, $\overline{P2(1,2)P2(2,2)}$, etc. are all null vectors.

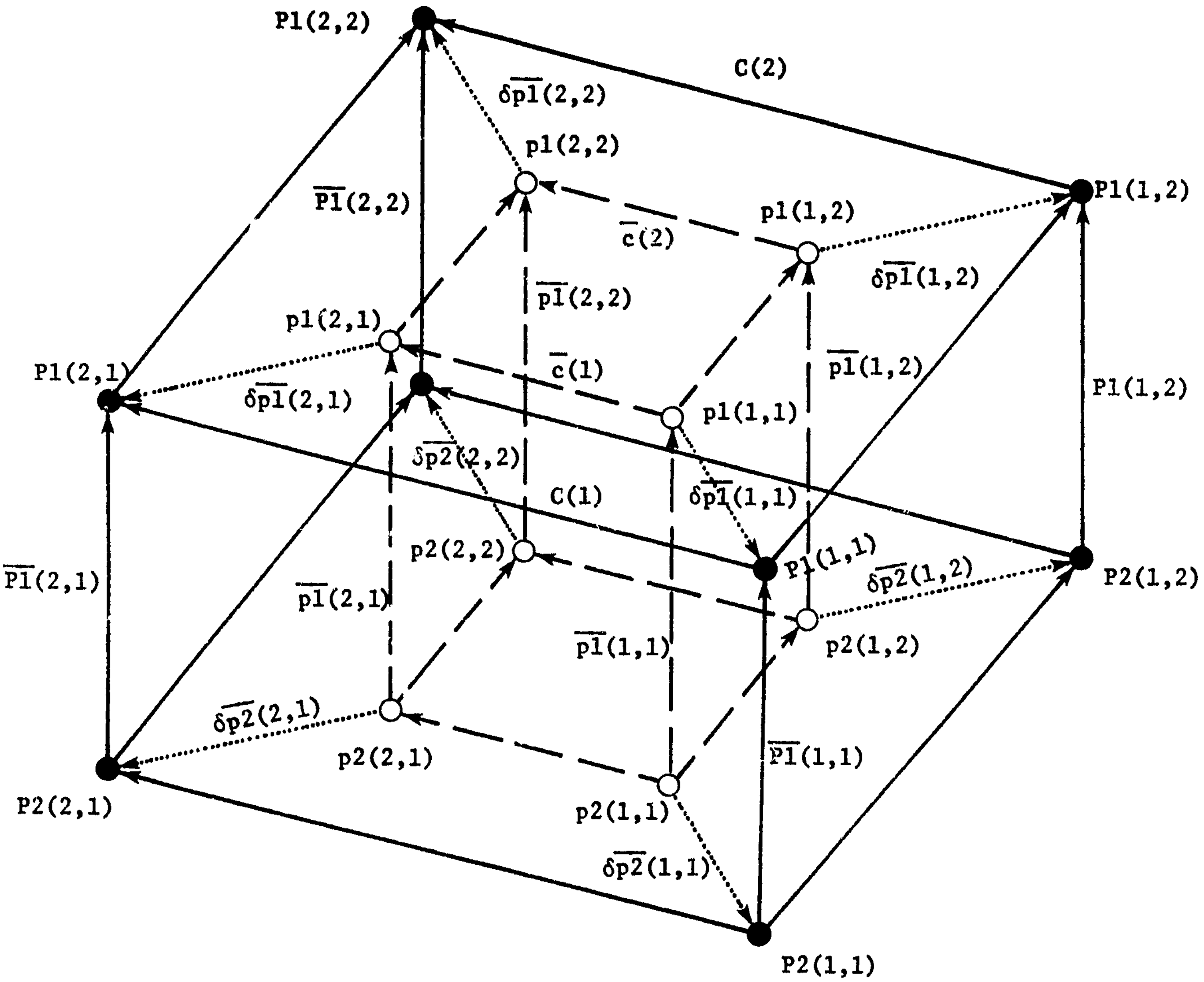


Fig. 23

We will simplify the notation in this sub-chapter and adopt the following:

$$\overline{C} = \overline{P1}(2,1) - \overline{P1}(1,1)$$

$$\overline{c} = \overline{p1}(2,1) - \overline{p1}(1,1)$$

$$\delta\overline{c} = \delta\overline{p1}(2,1) - \delta\overline{p1}(1,1)$$

where $\overline{C}$, $\overline{c}$, $\delta\overline{c}$ are the respective observer-target vector, its model and its disturbance. From the commutative properties of the P tower (topocentric level) we can easily derive the following (see Fig. 23):

$$\overline{C} = \overline{c} + \delta\overline{c}$$

The rates of change of the above vectors are reflected in the differences $\overline{C}(2) - \overline{C}(1)$, $\overline{c}(2) - \overline{c}(1)$ $\delta\overline{c}(2) - \delta\overline{c}(1)$ and can be obtained by formal differentiation vs. the time variable:

$$\dot{\overline{C}} = \dot{\overline{c}} + \delta\dot{\overline{c}}$$

By $\dot{C}$ we will denote the rate of change of the magnitude (length) of the vector $\overline{C}$. From sub-chapter 4.1 we have

$$\frac{\partial}{\partial T}\overline{p1} = \overline{\omega} \times \overline{p1}$$

which when applied to the difference $\overline{p1}(2,1) - \overline{p1}(1,1)$, and remembering that $\overline{\omega}$ is space invariant, results in:

$$\dot{\overline{c}} = \overline{\omega} \times \overline{c}$$

Note that $\dot{c} = 0$, i.e., the distance between any two model points is invariant according to the assumption of rigidity.

The disturbance in the rotation vector $\delta\overline{\omega}$ is presented in two components as follows (see sub-chapter 4.1):

$\delta\overline{\omega}_g = \delta\overline{\omega}_S + \delta\overline{\omega}_p$ global component

$\delta\overline{\omega}_\ell$ local component.

From sub-chapter 4.3 we have

$$\frac{\partial}{\partial T}(\delta\overline{p1} - \delta\overline{p2}) = \overline{\omega} \times (\delta\overline{p1} - \delta\overline{p2}) + \delta\overline{\omega} \times \overline{P1} + \frac{\partial}{\partial T}|\overline{P1}| \cdot \frac{\overline{p1}}{p1}$$

which when applied to the difference $\delta\overline{pl}(2,1) - \delta\overline{pl}(1,1)$ results in:

$$\dot{\delta\overline{c}} \simeq \overline{\omega} \times \delta\overline{c} + \delta\overline{\omega}_g \times \overline{C} + [\delta\overline{\omega}_{\ell 2} \times \overline{pl}(2,1) - \delta\overline{\omega}_{\ell 1} \times \overline{pl}(1,1)]$$

$$+ [\frac{\partial}{\partial T} |\overline{Pl}(2,1)| \cdot \frac{\overline{pl}(2,1)}{|\overline{pl}(2,1)|} - \frac{\partial}{\partial T} |\overline{Pl}(1,1)| \cdot \frac{\overline{pl}(1,1)}{|\overline{pl}(1,1)|}]$$

$$\dot{\delta\overline{c}} \simeq \overline{\omega} \times \delta\overline{c} + \delta\overline{\omega}_g \times \overline{C} + (\delta\overline{L} + \delta\overline{M})$$

With the above the rate of change of $|\overline{C}|$, i.e., $\dot{C}$ is

$$\dot{C} = \frac{\dot{\overline{C}} \cdot \overline{C}}{C} = \frac{(\dot{\overline{c}} + \dot{\delta\overline{c}}) \cdot (\overline{c} + \delta\overline{c})}{C}$$

$$= \frac{1}{C} [\overline{\omega} \times \overline{c} \cdot \overline{c} + \overline{\omega} \times \overline{c} \cdot \delta\overline{c} + \overline{\omega} \times \delta\overline{c} \cdot \overline{c} + \overline{\omega} \times \delta\overline{c} \cdot \delta\overline{c}$$
$$+ \delta\overline{\omega}_g \times \overline{C} \cdot \overline{C} + (\delta\overline{L} + \delta\overline{M}) \cdot \overline{C}]$$

The first, fourth and fifth terms in the square brackets are zero due to the fact that two of the three vectors in the mixed vector product are the same. The second and the third terms cancel being of the same magnitude and opposite sign. Thus finally, we have the following:

$$\dot{C} = \frac{\overline{C}}{C} \cdot (\delta\overline{L} + \delta\overline{M}) \simeq \frac{\overline{c}}{c} \cdot (\delta\overline{L} + \delta\overline{M})$$

Explicitly written the result is

$$\frac{\partial}{\partial T} |\overline{Pl}(2,1) - \overline{Pl}(1,1)| \simeq \frac{\overline{pl}(2,1) - \overline{pl}(1,1)}{|\overline{pl}(2,1) - \overline{pl}(1,1)|}$$

$$\cdot \left\{ \left[\delta\overline{\omega}_{\ell 2} \times \overline{pl}(2,1) - \delta\overline{\omega}_{\ell 1} \times \overline{pl}(1,1) \right] \right.$$
$$\left. + \left[\frac{\partial}{\partial T} |\overline{Pl}(2,1)| \cdot \frac{\overline{pl}(2,1)}{|\overline{pl}(2,1)|} - \frac{\partial}{\partial T} |\overline{Pl}(1,1)| \cdot \frac{\overline{pl}(1,1)}{|\overline{pl}(1,1)|} \right] \right\}$$

From inspection of the above equation we can state the following:

(a) The rate of change of the distance between two earth surface points is independent of global phenomena.

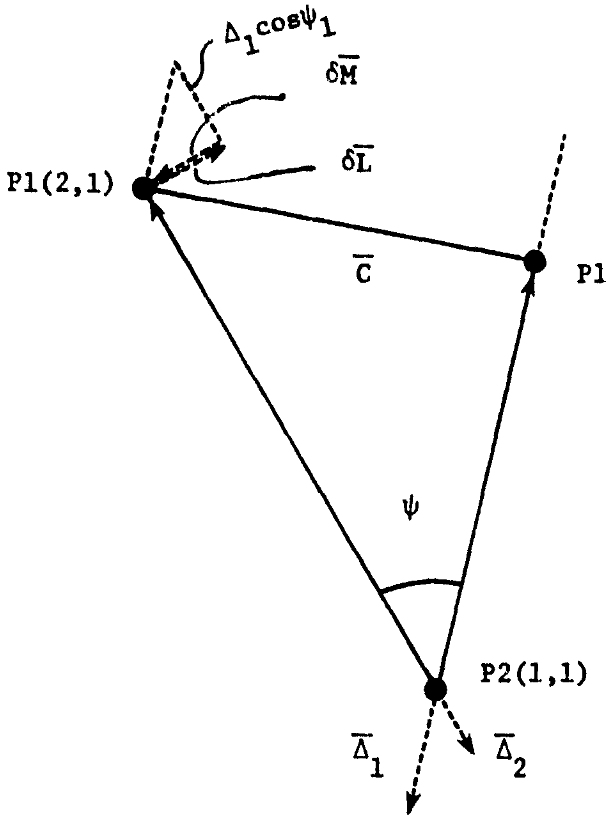


Fig. 24

(b) The vector sum ($\overline{\delta L} + \overline{\delta M}$) represents the difference in local horizontal and vertical motions (the relative motion) between the two points.

In the last part of this sub-chapter we will study the effect of a shift of the geocenter (due to mass redistributions) on the distance between two surface points.

Denote the shift of P2 vs. P1(1,1) and P1(2,1) by $\overline{\Delta}$ and decompose it into three vector components $\overline{\Delta}_1$, $\overline{\Delta}_2$, $\overline{\Delta}_3$ along the directions of $\overline{P1}(1,1)$, $\overline{P1}(2,1)$ and $\overline{P1}(2,1) \times \overline{P1}(1,1)$ respectively (see Fig. 24).

The component $\overline{\Delta}_3$ is normal to the plane defined by $\overline{P1}(1,1)$ and $\overline{P1}(2,1)$ and also to the vector $\overline{C}$. Accordingly its contribution to the sum$(\delta\overline{L} + \delta\overline{M})$ is also normal to $\overline{C}$ and so the dot product is zero:

$$\overline{C} \cdot (\delta\overline{L} + \delta\overline{M})_{\Delta_3} = 0$$

The effect of $\overline{\Delta}_1$ on the sum $(\delta\overline{L} + \delta\overline{M})$ can be represented by the equivalent parallel shifts of P1(1,1) and P1(2,1) in the opposite direction. The magnitudes of $\delta\overline{L}$ and $\delta\overline{M}$ due to $\overline{\Delta}_1$ are as follows $(\Delta_1 = |\overline{\Delta}_1|)$:

$$|\delta\overline{L}| = \Delta_1 \sin\psi - 0$$

$$\frac{\partial}{\partial T}|\overline{P1}(1,1)| = \Delta_1$$

$$\frac{\partial}{\partial T}|\overline{P1}(2,1)| = \Delta_1 \cos\psi$$

$$|\delta\overline{M}| = \sqrt{\Delta_1{}^2 - \Delta_1{}^2 \cos^2\psi} = \Delta_1 \sin\psi$$

The magnitudes of $\delta\overline{L}$ and $\delta\overline{M}$ being the same and by inspection of Fig. 23 we get finally

$$\delta\overline{L} + \delta\overline{M} = 0$$

A similar proof can be derived for $\overline{\Delta}_2$.

Thus we see that although the shift of the geocenter, $\overline{\Delta}$ causes local variations in the orientation and the magnitude of $\overline{P1}$ vectors, it has no effect on the distance between P1 (surface) points.

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INVESTIGATIONS ON THE HIERARCHY OF REFERENCE FRAMES IN GEODESY AND GEODYNAMICS

PART III: SCALE SYSTEMS: THE S-TOWER

by

Erik W. Grafarend

The third hierarchic structure in Euclid space: the tower of geodetic scale systems

0. Introduction

While the *hierarchic structures* which rule *orientation and origin* (rotational and translational degrees of freedom) have been presented with respect to space-time geodesy in E. Grafarend (1978 a, b) and E. Grafarend, I. Mueller, H. Papo and B. Richter (1979), the *third hierarchic structure* will be introduced here, namely *scale*. Any vector space is furnished with the topological notion of length, here the lengths of geodetic reference vectors like the length of the gravity vector, of the rotation vector, of the ecliptic normal, etc. Beside *directional parallelism scale parallelism* is needed, a notion introduced by H. Weyl (1952 p. 121-138).
Spacelike and timelike changes of fundamental geodetic length with respect to a fixed length or scale unit (unit length, unit time and others) will be studied, extending first results of refraction studies in E. Grafarend (1976) where Weyl-geometry was used. The variations will be finally *applied* to the *three base vector* system ($\underset{\sim}{\Gamma}$, $\underset{\sim}{\Omega}$, $\underset{\sim}{\Psi}$) which establishes three-dimensional geodesy. As a special technique polar and singular value decomposition are used in order to separate angular and dilatational distortions. The results can be embedded into the general theory of deformations introduced by C. Boucher (1978).

1. The local structure of the scale system

From the *differential point of view* two derivations of the basic scale structure in Euclid space are given. The relation to Weyl geometry is emphasized.

1.1

Here, let us introduce $\underset{\sim}{v}(x,y,z,t)$, a four-dimensional or space-time vector field which is a function of space-time coordinates $x^1=x$, $x^2=y$, $x^3=z$, $x^4=t$ in Euclid space. The vector $\underset{\sim}{v}$ is represented twofold, firstly with respect to an orthonormal triad $(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$ such that its coordinates are $(0,0,v)$ where v is the *length* of the vector, secondly, with respect to an orthonormal triad $(\underset{\sim}{e}_{1^\circ},\underset{\sim}{e}_{2^\circ},\underset{\sim}{e}_{3^\circ})$ which is *fixed* in space-time or *invariant with respect to a translation* in space-time. The base vectors are related by a rotation, $\underset{\sim}{e}_o \rightarrow \underset{\sim}{e} = R\underset{\sim}{e}_o$, where R is a threedimensional rotation matrix. Space - and/or timelike variations are studied by differentiation:

$$1(1)\quad \underset{\sim}{v} = (0,0,v) \begin{bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{bmatrix} = (0,0,sv_o)\, R \begin{bmatrix} \underset{\sim}{e}_{1^\circ} \\ \underset{\sim}{e}_{2^\circ} \\ \underset{\sim}{e}_{3^\circ} \end{bmatrix}$$

$$1(2)\quad d\underset{\sim}{v} = (0,0,dv) \begin{bmatrix} \underset{\sim}{e}_1 \\ \underset{\sim}{e}_2 \\ \underset{\sim}{e}_3 \end{bmatrix} + (0,0,v) \begin{bmatrix} d\underset{\sim}{e}_1 \\ d\underset{\sim}{e}_2 \\ d\underset{\sim}{e}_3 \end{bmatrix} =$$

$$d\{(0,0,sv_o)\}\, R \begin{bmatrix} \underset{\sim}{e}_{1^\circ} \\ \underset{\sim}{e}_{2^\circ} \\ \underset{\sim}{e}_{3^\circ} \end{bmatrix} + (0,0,sv_o)\, d\{R \begin{bmatrix} \underset{\sim}{e}_{1^\circ} \\ \underset{\sim}{e}_{2^\circ} \\ \underset{\sim}{e}_{3^\circ} \end{bmatrix}\} =$$

$$\{0,0,dsv_o + s\, dv_o)\}\, R \begin{bmatrix} \underset{\sim}{e}_{1^\circ} \\ \underset{\sim}{e}_{2^\circ} \\ \underset{\sim}{e}_{3^\circ} \end{bmatrix} + (0,0,sv_o)\, \{dR \begin{bmatrix} \underset{\sim}{e}_{1^\circ} \\ \underset{\sim}{e}_{2^\circ} \\ \underset{\sim}{e}_{3^\circ} \end{bmatrix} + R \begin{bmatrix} d\underset{\sim}{e}_1 \\ d\underset{\sim}{e}_2 \\ d\underset{\sim}{e}_3 \end{bmatrix} \quad \}$$

The length of the vector v has been expressed by the product of a *scale factor* s and a *fundamental length* v_o. For example, a length of 10 m is the product of the scale factor $s = 10$ and the fundamental length $v_o = 1$ m. In addition to the translational invariance of the *direction reference system* e_o we will assume that the fundamental length v_o is invariant with respect to translation, too. Thus beside the postulate of directional parallelism $de_o = 0$ we have the *postulate of scale parallelism* $dv_o = 0$. These postulates lead to a variation of the vector field v given by

$$1(3) \quad dv = (0,0,dv) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + (0,0,v) \begin{bmatrix} de_1 \\ de_2 \\ de_3 \end{bmatrix} =$$

$$\{(0,0,dsv_o)\}\, R \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + (0,0,sv_o)\{dR \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} =$$

$$\{(0,0,dss^{-1}v\} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + (0,0,v)\, dRR^{-1} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

or

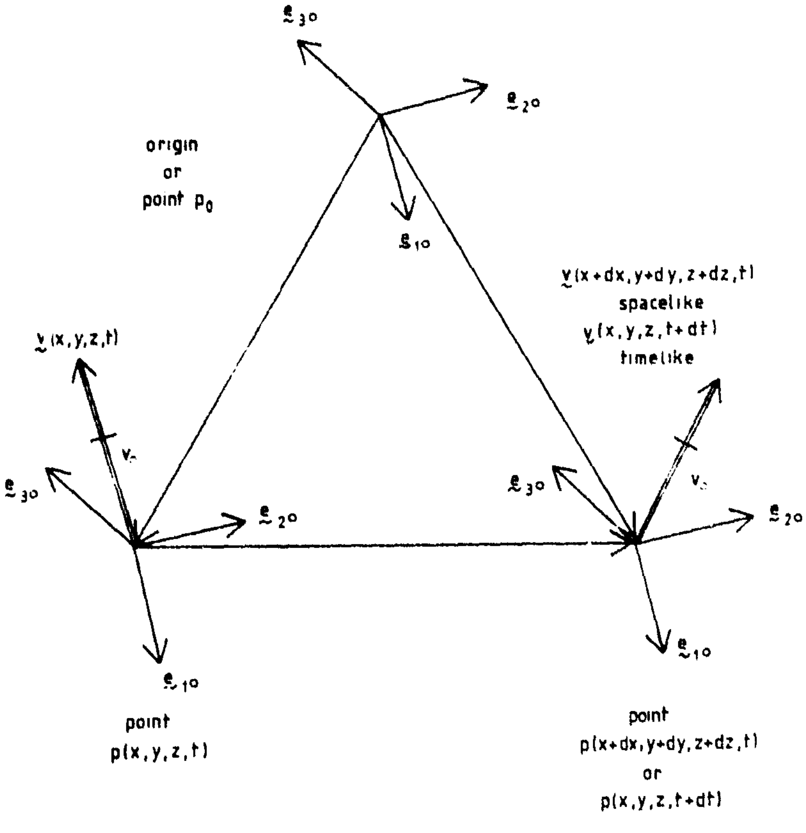
$$1(4) \quad dv = ds\, s^{-1} v$$

$$1(5) \quad de = dRR^{-1}\, e$$

Note that R is an orthogonal matrix, $|R| = +1$, or $R^{-1} = R'$. A verbal formulation of the fundamental result is this: The length of a vector v is changed under directional and scale parallelism proportional to the change of scale factor and the length itself, but inverse proportional to the scale factor. The orientation of the reference system e is changed under directional and scale parallelism proportional to the change of directional parameters within the rotation matrix R and the base vectors $\underset{\sim}{e}_i$ itself, but inverse proportional to the rotation matrix.

Fig. 1 illustrates the degrees of freedom of type translation, rotation, scale or origin, orientation, scale.

Fig. 1: Parallel transport of directions $(\underset{\sim}{e}_{1o}, \underset{\sim}{e}_{2o}, \underset{\sim}{e}_{3o})'$ and length unit v_o



1.2

Another derivation of the fundamental differential equations 1(4), 1(5) originates directly from the group of transformations. According to *Fig. 2* let us denote by $\underset{\sim}{v}(x_0,y_0,z_0,t_0)$ a vector at a space-time point x,y,z,t. Both vectors coincide if we change orientation and scale by

$$1(6) \quad \underset{\sim}{v}(x,y,z,t) = s\ R\ \underset{\sim}{v}(x_0,y_0,z_0,t_0)$$

$$1(7) \quad d\underset{\sim}{v} = dv\ \underset{\sim}{e}_3 + v\ d\underset{\sim}{e}_3 = (ds\ s^{-1} + dR\ R^{-1})\underset{\sim}{v}$$

or

$$dv = ds\ s^{-1}\ v$$

$$d\underset{\sim}{e} = dR\ R^{-1}\ \underset{\sim}{e}$$

Fig. 3 illustrates the different postulates of parallel transport of directions and scale.

Fig. 2: Degrees of freedom of type translation, rotation, scale

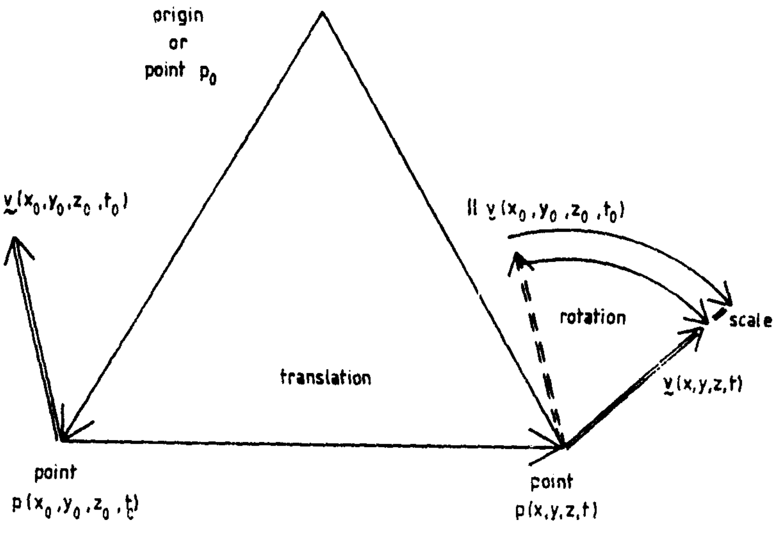
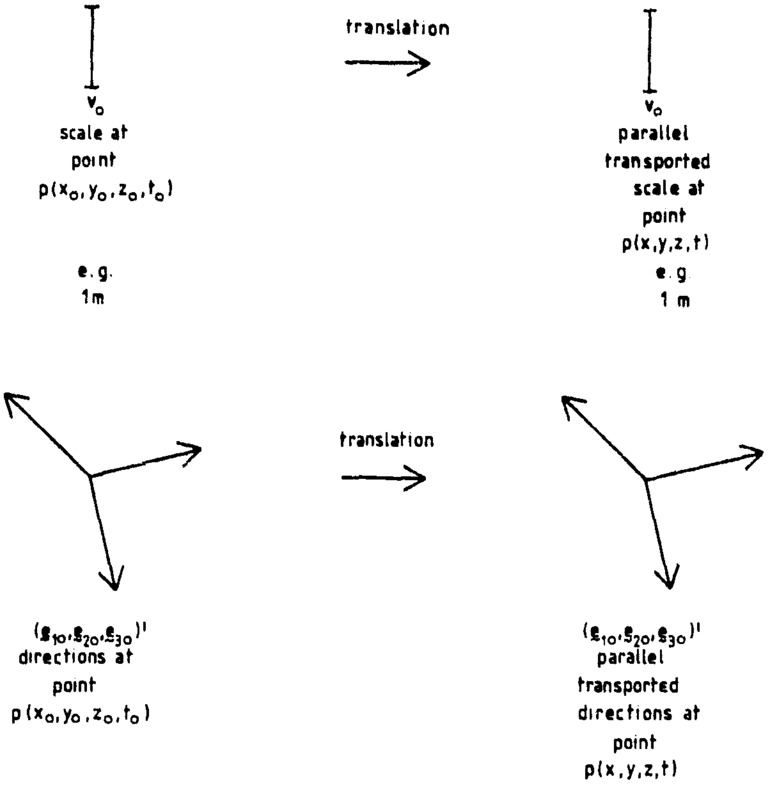


Fig. 3: Directions and scale under translational invariance: $d\xi_o = 0$, $dv_o = 0$



1.3

The classical treatment of length variation in differential geometry is based on the quadratic form $v^2 = ||v||^2$ of the vector v. dv^2 and dv are obviously related by

1(8) $dv^2 = 2v\ dv$

1(9) $dv = \frac{1}{2v}\ dv^2$

leading to

$$1(10) \quad \frac{1}{2} d \|v\|^2 = \|v\|^2 d \ln s = \|v\|^2 \frac{\partial \ln s}{\partial x^i} dx^i .$$

2. The global structure of scale systems

From the *integral point of view* a derivation of the basic scale structure in Euclid space is given. The invariance of observables under the group of transformations is emphasized.

2.1

Here let us introduce two vectors $v(x_0,y_0,z_0,t_0)$ and $v(x,y,z,t)$ at space-time points x_0,y_0,z_0,t_0 and x,y,z,t, respectively, which are parallel under a translation. Both vectors coincide if we change orientation and scale by

$$2(1) \quad v(x,y,z,t) = s\,R\,v(x_0,y_0,z_0,t_0)$$

$$2(2) \quad \delta_s v(x,y,z,t) = v(x_0+\delta x,y_0+\delta y,z_0+\delta z,t_0) - v(x_0,y_0,z_0,t_0)$$

$$2(3) \quad \delta_t v(x,y,z,t) = v(x_0,y_0,z_0,t_0+\delta t) \qquad - v(x_0,y_0,z_0,t_0)$$

$\delta_s v$ is called *spacelike variation*, $\delta_t v$ *timelike variation*. Let us introduce the rotation parameters by

$$2(4) \quad R = R_E(\Lambda,\Phi,0) = R_3(0)\, R_2(\tfrac{\pi}{2} -\Phi)\, R_3\,(\Lambda)$$

$$2(5) \quad R_E(\Lambda+\delta\Lambda,\Phi+\delta\Phi,0) = R_E(\Lambda,\Phi,0)$$

$$\begin{bmatrix} 1 & +\delta\Lambda & +\cos\Lambda\delta\Phi \\ -\delta\Lambda & 1 & +\sin\Lambda\delta\Phi \\ -\cos\Lambda\delta\Phi & -\sin\Lambda\delta\Phi & 1 \end{bmatrix} + \mathcal{O}_2$$

where $\mathcal{O}_2$ indicates terms of second order.

2(6) $\delta \underset{\sim}{v} = \delta s \; s^{-1} \underset{\sim}{v} + R_E(\Lambda,\Phi,0) \; \delta A \; R'_E(\Lambda,\Phi,0) \; \underset{\sim}{v}$

where the antisymmetric matrix δA can be represented by

2(7) $$\delta A = \begin{bmatrix} 0 & +\delta\Lambda & +\cos \Lambda\delta\Phi \\ -\delta\Lambda & 0 & +\sin \Lambda\delta\Phi \\ -\cos \Lambda\delta\Phi & -\sin \Lambda\delta\Phi & 0 \end{bmatrix}$$

2(8) $$\Omega = R_E(\Lambda,\Phi,0) \; \delta A \; R'_E(\Lambda,\Phi,0) =$$

$$\begin{bmatrix} 0 & +\delta\Lambda \sin \Phi & +\delta\Phi \\ -\delta\Lambda \sin \Phi & 0 & -\delta\Lambda \cos \Phi \\ -\delta\Phi & +\delta\Lambda \cos \Phi & 0 \end{bmatrix}$$

2(9) $\delta v = \delta s \; s^{-1} v$

2(10) $\delta \underset{\sim}{e} = \Omega \underset{\sim}{e}$

2.2

We will prove next that positional angles and lengths ratios are invariant with respect to the underlying similarity transformation

2(11) $\underset{\sim}{v} \rightarrow T\underset{\sim}{v} = s \; R \; \underset{\sim}{v} + \underset{\sim}{t}$

2(12) $$\frac{\langle T(\underset{\sim}{v}_2-\underset{\sim}{v}_1),\; T(\underset{\sim}{v}_3-\underset{\sim}{v}_1)\rangle}{\|T(\underset{\sim}{v}_2-\underset{\sim}{v}_1)\|\|T(\underset{\sim}{v}_3-\underset{\sim}{v}_1)\|} = \frac{s^2(\underset{\sim}{v}_2-\underset{\sim}{v}_1)'R'R(\underset{\sim}{v}_3-\underset{\sim}{v}_1)}{s^2\,\|\underset{\sim}{v}_2-\underset{\sim}{v}_1\|\;\|\underset{\sim}{v}_3-\underset{\sim}{v}_1\|} =$$

$$\frac{\langle v_2-v_1,\; v_3-v_1\rangle}{\|v_2-v_1\|\|v_3-v_1\|} \quad \text{q.e.d}$$

2(13) $$\frac{\|T(\underset{\sim}{v}_2-\underset{\sim}{v}_1)\|}{\|T(v_3-v_1)\|} = \frac{s\sqrt{(v_2-v_1)'\,R'R\,(v_2-v_1)}}{s\sqrt{(v_3-v_1)'\,R'R\,(v_3-v_1)}} = \frac{\|\underset{\sim}{v}_2-\underset{\sim}{v}_1\|}{\|\underset{\sim}{v}_3-\underset{\sim}{v}_1\|} \quad \text{q.e.d.}$$

Fig. 4 is an illustration of the invariance of positional angles and lengths ratios in a space-time triangle. Related commutative diagrams for translation, rotation and scale are given in *Fig. 5*

3. Examples

Threedimensional geodesy will be based on *three base vectors*, namely $[\Gamma, \underset{\sim}{\Omega}, \Psi]'$, located at the topocentre and referring to the vector fields of gravity, rotation and eliptic normal. The base vectors are neither orthogonal nor normalized. The gravity vector determines the local vertical. The rotation field is constructed from the inertial velocity vector $\underset{\sim}{v}$ of the topocentre by vorticity $\underset{\sim}{\Omega}$ = rot $\underset{\sim}{v}$ changing in space and time due to plate rotations and the dynamics of the planetary system. The eliptic normal is defined by the binormal vector of the curve of the mass centre of the earth in inertial space. The base vectors will be referred to a base vector system at initial epoch zero and space point zero, in detail by

3(1) $$\begin{bmatrix} \underset{\sim}{\Gamma} \\ \underset{\sim}{\Omega} \\ \underset{\sim}{\Psi} \end{bmatrix} = RU \begin{bmatrix} \underset{\sim}{\Gamma}_o \\ \underset{\sim}{\Omega}_o \\ \underset{\sim}{\Psi}_o \end{bmatrix} = VR \begin{bmatrix} \underset{\sim}{\Gamma}_o \\ \underset{\sim}{\Omega}_o \\ \underset{\sim}{\Psi}_o \end{bmatrix}$$

which corresponds to a systematic set-up of type

3(2) $\underset{\sim}{v} = RU\,\underset{\sim}{v}_o = VR\,\underset{\sim}{v}_o$.

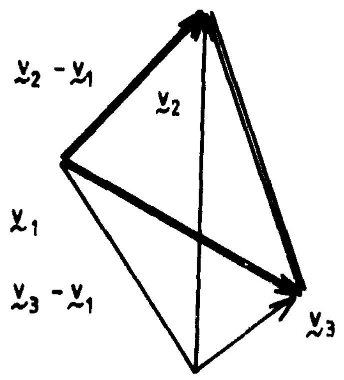


Fig. 4: Space-time triangle

$$\begin{tikzcd}[column sep=small]
 & \underset{\sim}{e}_0 \arrow[dl, "R_E(\Lambda,\Phi,O)"'] \arrow[dr, "R_E(\Lambda+\delta\Lambda,\Phi+\delta\Phi,O)"] & \\
\underset{\sim}{e}(x,y,z,t) \arrow[rr] & & \begin{array}{l} \underset{\sim}{e}(x,y,z,t+\delta t) \\ \text{or} \\ \underset{\sim}{e}(x+\delta x,y+\delta y,z+\delta z,t) \end{array}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & v_0 \arrow[dl, "s"'] \arrow[dr, "s+\delta s"] & \\
v(x,y,z,t) \arrow[rr] & & \begin{array}{l} v(x,y,z,t+\delta t) \\ \text{or} \\ v(x+\delta x,y+\delta y,z+\delta z,t) \end{array}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & p_0 \arrow[dl] \arrow[dr] & \\
p(x,y,z,t) \arrow[rr] & & \begin{array}{l} p(x,y,z,t+\delta t) \\ \text{or} \\ p(x+\delta x,y+\delta y,z+\delta z,t) \end{array}
\end{tikzcd}$$

Fig. 5: Commutative diagrams for degrees of freedom of type rotation, scale and translation

It includes the *polar decomposition* (Cauchy decomposition) where R is a *rotation matrix* ($|R| = +1$), U and V are *right and left stretch matrices* being symmetric. The matrices are related by

$$3(3) \quad V = RUR' \Longleftrightarrow RVR = U.$$

A *singular value decomposition* of the stretch matrices is

$$3(4) \quad V = R_V V^* R_V'$$

$$3(5) \quad U = R_u U^* R_u'$$

where

$$3(6) \quad V^* = \text{diag}\ (v_1,\ v_2,\ v_3)$$

$$3(7) \quad U^* = \text{diag}\ (u_1,\ u_2,\ u_3)$$

and v_1, v_2, v_3 and u_1, u_2, u_3 are eigen-values.

$$3(8) \quad \underset{\sim}{v} = RR_u U^* R_u' \underset{\sim}{v}_o = R_V V^* R_V' R \underset{\sim}{v}_o$$

leads to variations of type spacelike and/or timelike

$$3(9) \quad d\underset{\sim}{v} = (dRU + RdU)\underset{\sim}{v}_o = (dVR + VdR)\underset{\sim}{v}_o =$$

$$(dRR' + RdU\ U^{-1}\ R')\underset{\sim}{v} = (dVV^{-1} + VdRR'V^{-1})\underset{\sim}{v}$$

or

$$3(10) \quad d\underset{\sim}{v} = \{dRR' + R(dR_u U^* R_u' + R_u U^* dR_u' + R_u dU^* R_u') R_u U^{*-1} R_u' R\} \underset{\sim}{v}$$

$$= \{(dR_V V^* R_V' + R_V V^* dR_V' + R_V dV^* R_V') R_V V^{*-1} R_V' + VdRR'V^{-1}\} \underset{\sim}{v}$$

$$3(11)\ dU = dR_uU^*R'_u + R_uU^*dR'_u + R_udU^*R'_u$$

$$3(12)\ dV = dR_vV^*R'_v + R_vV^*dR'_v + R_vdVR'_v$$

$$3(13)\ d\underset{\sim}{v} = dR\ R'\ \underset{\sim}{v} + (RdR_uU^*R'_u + RR_uU^*dR'_u)R_u\ \mathrm{diag}(\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_3})R'_uR\underset{\sim}{v}$$

$$+ RR_u\ \mathrm{diag}(du_1, du_2, du_3)\ \mathrm{diag}\ (\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_3})R'_uR'\underset{\sim}{v}$$

$$3(14)\ d\underset{\sim}{v} = \{dR_v\ \mathrm{diag}(v_1, v_2, v_3)R'_v + R_v\ \mathrm{diag}\ (v_1, v_2, v_3)dR'_v\}R_v$$

$$\mathrm{diag}\ (\frac{1}{v_1}, \frac{1}{v_2}, \frac{1}{v_3})R'_v\ \underset{\sim}{v} + R_v\ \mathrm{diag}\ (dv_1, dv_2, dv_3)$$

$$\mathrm{diag}\ (\frac{1}{v_1}, \frac{1}{v_2}, \frac{1}{v_3})R'_v\underset{\sim}{v} + VdRR'V^{-1}\ \underset{\sim}{v}$$

The tensors

$$3(15)\ C = U^2,\ B = V^2$$

will be called *right and left deformation matrices* (Cauchy-Green matrices) which can be represented by

$$3(16)\ C = R_uU^{*2}R'_u = R_u\ \mathrm{diag}\ (u_1{}^2, u_2{}^2, u_3{}^2)R'_u$$

$$3(17)\ B = R_vV^{*2}R_v = R_v\ \mathrm{diag}\ (v_1{}^2, v_2{}^2, v_3{}^2)R'_v$$

What is the sense of all these strange computations?
At first we have *rotated* the three base vactors by a proper rotation matrix R. Secondly we have *stretched* the three base vectors by the matrices U and V, respectively. The singular value decomposition allows the *separation of angular and scale distortion*. By the matrices R_u and R_v, respectively, we have

rotated the matrices U and V, respectively, into their *principal directions*. Along the principal directions there is only a change in scale of the three base vectors $[\underset{\sim}{\Gamma}, \underset{\sim}{\Omega}, \underset{\sim}{\Psi}]'$. Thus we have found a decomposition into *shear and dilatation*, the off- and diagonal elements of the *deformation matrices* if we use this terminology. In general, the space-time change of geodetic base vectors can therefore be understood as a change in origin (translation), orientation (rotation) and scale. Fixed or translational invariant is always the base vector system $[\underset{\sim}{\Gamma}_0, \underset{\sim}{\Omega}_0, \underset{\sim}{\Psi}_0]$. In geodetic applications, the *nine* elements which describe the space-time change of a triplet of base vectors is parameterized in a slightly different way: The base vector $\underset{\sim}{\Omega}$ of rotation is projected onto the plane rectangular to the base vector $\underset{\sim}{\Gamma}$; the direction is called south. Orthogonal to south within this plane we direct east, equivalently by the vector product $\underset{\sim}{\Omega} \wedge -\underset{\sim}{\Gamma}$; the normalized triad as the final product is called the *horizontal* one. By a similar process applied to [illegible] and [illegible] we arrive at the *equatorial* triad. Angular parameters which connect these triads are always of type longitude and latitude. Totally there are *six angles* which connect the system of base vectors $[\underset{\sim}{\Gamma}, \underset{\sim}{\Omega}, \underset{\sim}{\Psi}]'$, which span the geodetic three-dimensional Euclid space locally, and the one $[\underset{\sim}{\Gamma}_0, \underset{\sim}{\Omega}_0, \underset{\sim}{\Psi}_0]$. In addition, there is a space-time change of lengths $\|\underset{\sim}{\Gamma}\|, \|\underset{\sim}{\Omega}\|, \|\underset{\sim}{\Psi}\|$ parameterized by *three scale factors* referring to a fixed length system $\|\underset{\sim}{\Gamma}_0\|, \|\underset{\sim}{\Omega}_0\|, \|\underset{\sim}{\Psi}_0\|$.

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