# REPORT No. 121.

## THE MINIMUM INDUCED DRAG OF AEROFOILS.

By Max M. Munk.

### INTRODUCTION.

The following paper is a dissertation originally presented by the author to the University of Goettingen. It was intended principally for the use of mathematicians and physicists. The author is pleased to note that the paper has aroused interest in other circles, to the end that the National Advisory Committee for Aeronautics will make it available to a larger circle in America. The following introduction has been added in order to first acquaint the reader with the essence of the paper.

In the following development all results are obtained by integrating some simple expressions or relations. For our purposes it is sufficient, indeed, to prove the results for a pair of small elements. The qualities dealt with are integrable, since, under the assumptions we are allowed to make, they can not be affected by integrating. We have to consider only the relations between any two lifting elements and to add the effects. That is to say, in the process of integrating each element occurs twice—first, as an element producing an effect, and, second, as an element experiencing an effect. In consequence of this the symbols expressing the integration look somewhat confusing, and they require so much space in the mathematical expression that they are apt to divert the reader's attention from their real meaning. We have to proceed up to three dimensional problems. Each element has to be denoted twice (by a Latin letter and by a Greek letter), occurring twice in a different connection. The integral, therefore, is sixfold, six symbols of integration standing together and, accordingly, six differentials (always the same) standing at the end of the expression, requiring almost the fourth part of the line. The meaning of this voluminous group of symbols, however, is not more complicated and not less elementary than a single integral or even than a simple addition.

In section 1 we consider one aerofoil shaped like a straight line and ask how all lifting elements, which we assume to be of equal intensity, must be arranged on this line in order to offer the least drag.

If the distribution is the best one, the drag can not be decreased or increased by transferring one lifting element from its old position ($a$) to some new position ($b$). For then either the resulting distribution would be improved by this transfer, and therefore was not best before, or the transfer of an element from ($b$) to ($a$) would have this effect. Now, the share of one element in the drag is composed of two parts. It takes share in producing a downwash in the neighborhood of the other lifting elements and, in consequence, a change in their drag. It has itself a drag, being situated in the downwash produced by the other elements.


Fig 1

Considering only two elements, Fig. 1 shows that in the case of the lifting straight line the two downwashes, each produced by one element in the neighborhood of the other, are equal. For this reason the two drags of the two elements each produced by the other are equal, too, and hence the two parts of the entire drag of the wings due to one element. The entire drag produced by one element has twice the value as the drag of that element resulting from the downwash in its environs. Hence, the entire drag due to one element is unchanged when the element is transferred from one situation to a new one of the same downwash, and the distribution is the best only if the downwash is constant over the whole wing.

In sections 2 to 6 it is shown that the two parts of the drag change by the same value in all other cases, too. If the elements are situated in the same transverse plane, the two parts are equal. A glance at Fig. 2 shows that the downwash produced by (1) at (2), (3), (4), and (5)

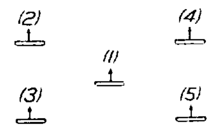


Fig. 2

is equal. But then it also equals the downwash due to (4), say, produced at (1). This holds true even for the component of the downwash in the direction of the lift if the elements are normal to each other (Fig. 3.); for this component is proportional $x.y/r^3$, according to the symbols

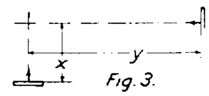


Fig. 3.

of the figure. Hence, it is proved for lift of any inclination, horizontal and vertical elements being able, by combination, to produce lift in any direction.

There remains only the question whether the two parts of the drag are also equal if the elements are situated one behind the other—that is to say, in different longitudinal positions. They are not; but their sum is independent of the longitudinal distance apart. To prove this, add in Fig. 4 to the lifting element (2) a second inverse lifting element (3) with inverse

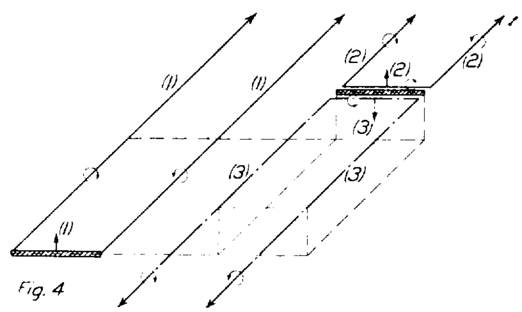


Fig. 4

linear longitudinal vortices in the inverse direction. The reader observes that the transverse vortices (2) and (3) neutralize each other; the longitudinal linear vortices, however, have the same sign, and all four vortices form a pair of vortices running from infinity to infinity. The drag, produced by the combination of (1) and this pair, is obviously independent of the longitudinal positions of (1) and (2). But the added element (3) has not changed the drag, for (1) and (3) are situated symmetrically and produce the same mutual downwash. The direction of the lift, however, is inverse, and therefore the two drags have the inverse sign, and their sum is zero.

If the two lifting elements are perpendicular to each other (chapter 5), a similar proof can be given.

Sections 6 and 7 contain the conclusions. The condition for a minimum drag does not depend upon the longitudinal coordinates, and in order to obtain it the downwash must be assumed to be constant at all points in a transverse plane of a corresponding system of aerofoils. This is not surprising; the wings act like two dimensional objects accelerating the air passing in an infinite transverse plane at a particular moment. Therefore the calculation leads to the consideration of the two dimensional flow about the projection of the wings on a transverse plane.

Section 8 gives the connection between the theory in perfect fluids and the phenomenon in true air. It is this connection that allows the application of the results to practical questions.

## I. THE LIFTING STRAIGHT LINE.

A system of aerofoils moving in an incompressible and frictionless fluid has a drag (in the direction of its motion) if there is any lift (perpendicular to the direction of its motion). The magnitude of this drag depends upon the distribution of the lift over the surface of the aerofoils. Although the dimensions of the given system of aerofoils may remain unchanged, the distribution of the lift can be radically altered by changes in details, such as the aerofoil section or the angle of attack. The purpose of the investigation which is given in the following pages is to determine (*a*) the distribution of lift which produces the least drag, and (*b*) the magnitude of this minimum drag.

Let us first consider a single aerofoil of such dimensions that it may be referred to with sufficient exactness as a lifting straight line, which is at right angles to the direction of its flight. The length or span of this line may be denoted by $l$. Let the line coincide with the horizontal, or $x$ axis of a rectangular system of coordinates having its origin at the center of the aerofoil. The density of the lift

$$A' = \frac{dA}{dx} \tag{1}$$

where $A$, the entire lift from the left end of the wing up to the point $x$, is generally a function of $x$ and may be denoted by $f(x)$. Let the velocity of flight be $v_0$.

The modern theory of flight[1] allows the entire drag to be expressed as a definite double integral, if certain simplifying assumptions are made. In order to find this integral, it is necessary to determine the intensity of the longitudinal vortices which run from any lifting element to infinity in a direction opposite to the direction of flight. These vortices are generally distributed continuously along the whole aerofoil, and their intensity per unit length of the aerofoil is

$$\Gamma' = \frac{1}{v_0 \cdot \rho} \cdot \frac{dA'}{dx} \tag{2}$$

where $\rho$ is the density of the fluid. Now, for each lifting element $dx$, we shall calculate the downwash $w$, which, in accordance with the law of Biot-Savart, is produced at it by all the longitudinal vortices. A single vortex, beginning at the point $x$, produces at the point $x=\xi$ the downwash

$$dw = \frac{1}{4\pi\rho v_0} \cdot dA' \cdot \frac{1}{\xi - x} \tag{3}$$

Therefore the entire downwash at the point $\xi$ is

$$w = \frac{1}{4\pi\rho v_0} \int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{dA'}{dx} \cdot \frac{1}{\xi - x} dx \tag{4}$$

The integration is to be performed along the aerofoil; and the principal value of the integral is to be taken at the point $x=\xi$. This rule also applies to all of the following integrals. Hence it follows that the drag according to the equation

$$\frac{dW}{dx} = W' = \frac{w}{v_0} \cdot A' \tag{5}$$

is

$$W = \frac{1}{4\pi v_0^2 \rho} \int_{-\frac{l}{2}}^{+\frac{l}{2}} \left\{ \int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{dA'}{dx} \cdot \frac{1}{\xi - x} dx \right\} \cdot A' d\xi \tag{6'}$$

[1] See L. Prandtl, Tragflügeltheorie, I. Mitteilung. Nachrichten der Ges. d. Wiss. zu Göttingen, 1918.

or, otherwise expressed,

$$W=\frac{1}{4\pi v_0^2\rho}\int_{-\frac{l}{2}}^{+\frac{l}{2}}\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f'(x)\cdot f(\xi)}{\xi-x}dxd\xi \tag{6}$$

$f'$ here signifies the derivative of $f$ with respect to $x$ or $\xi$. The entire lift is represented by

$$A=\int_{-\frac{l}{2}}^{+\frac{l}{2}}f(x)\,dx \tag{7}$$

Hence the solution of the problem to determine the best distribution of lift depends upon the determination of the function $f$ so that the double integral

$$J_1=\int_{-\frac{l}{2}}^{+\frac{l}{2}}\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f'(x)\cdot f(\xi)}{\xi-x}\,dxd\xi \tag{8}$$

shall have a value as small as possible; while at the same time the value of the simple integral

$$J_2=\int_{-\frac{l}{2}}^{+\frac{l}{2}}f(x)dx=\text{const.} \tag{9}$$

is fixed.

The first step towards the solution of this problem is to form the first variation of $J_1$

$$\delta J_1=\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left\{\delta f(x)\,dx\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f'(\xi)}{\xi-x}d\xi\right\}+\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left\{\delta f'(\xi)d\xi\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f(x)}{\xi-x}d\xi\right\} \tag{10}$$

The second integral on the right side of (10) can be reduced to the first. By exchanging the symbols $x$ and $\xi$ and by partial integration with respect to $x$, considering $f'(\xi)$ as the integrable factor, there is obtained

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left\{\delta f'(\xi)d\xi\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f(x)}{\xi-x}dx\right\}=-\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left\{\delta f(x)dx\cdot\frac{d}{dx}\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f(\xi)}{x-\xi}d\xi\right\} \tag{11}$$

The second member disappears since $f=0$ at the limits of integration.[2] Further, the right hand part of (11)

$$\frac{d}{dx}\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f(x)}{x-\xi}d\xi$$

upon substitution of the new variables $x$ and $t=x-\xi$ for $x$ and $\xi$, is transformed into

$$\frac{d}{dx}\int_{x+\frac{l}{2}}^{x-\frac{l}{2}}\frac{f(x-t)}{t}\,dt$$

[2] If this were not true, there would be infinite velocities at these points.

Now

$$\frac{d}{dx}\int_{x+\frac{l}{2}}^{x-\frac{l}{2}}\frac{f(x-t)}{t}\,dt=\frac{f\left(\frac{l}{2}\right)}{x-\frac{l}{2}}-\frac{f\left(-\frac{l}{2}\right)}{x+\frac{l}{2}}+\int_{x+\frac{l}{2}}^{x-\frac{l}{2}}\frac{f'(x-t)}{t}\,dt$$

or, since $f$ disappears at the limits of integration,

$$\frac{d}{dx}\int_{x+\frac{l}{2}}^{x-\frac{l}{2}}\frac{f(x-t)}{t}\,dt=\int_{x+\frac{l}{2}}^{x-\frac{l}{2}}\frac{f'(x-t)}{t}\,dt$$

which, upon the replacement of the original variables, becomes

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f'(\xi)}{x-\xi}\,d\xi$$

so that, finally,

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left\{\delta f'(\xi)d\xi\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f(x)}{\xi-x}dx\right\}=\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left\{\delta f(x)dx\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f'(\xi)}{\xi-x}d\xi\right\}\qquad(12)$$

Substituting this in (10) there finally results

$$\delta J_1=2\int_{-\frac{l}{2}}^{+\frac{l}{2}}\delta f(x)dx\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f'(\xi)}{\xi-x}d\xi\qquad(13)$$

From which the condition for the minimum amount of drag, taking into consideration the second condition (9), is

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}}\frac{f'(\xi)}{\xi-x}\,d\xi+\lambda=0\qquad(14)$$

or, when equation (4) is taken into consideration

$$w=\text{const.}=w_0\qquad(15)$$

*The necessary conditión for the minimum of drag for a lifting straight line is that the downwash produced by the longitudinal vortices be constant along the entire line.*

That this necessary consideration is also sufficient results from the obvious meaning of the second variation, which represents the infinitesimal drag produced by the variation of the lift if it alone is acting, and therefore it is always greater than zero.

## 2. PARALLEL LIFTING ELEMENTS LYING IN A TRANSVERSE PLANE.

The method just developed may be applied at once to problems of a more general nature. If, instead of a single aerofoil, there are several aerofoils in the same straight line perpendicular to the direction of flight, only the limits of integration are changed in the development. The integration in such cases is to be performed along all of the aerofoils. However, this is nonessential for all of the equations and therefore the condition for the minimum drag (equation 15) applies to this entire system of aerofoils.

Let us now discard the condition that all of the lifting lines are lying in the same straight line, but retain, however, the condition that they are parallel to each other, perpendicular to the line of flight as before, and that they are all lying in a plane perpendicular to the line of flight. Let the height of any lifting line be designated by $z$ or $\zeta$. Equation (3) transforms into a similar one which gives the downwash produced at the point $x$, $z$ by the longitudinal vortex beginning on the lifting element at the point $\zeta$:

$$dw = \frac{1}{4\pi\rho v_0} dA' \frac{\xi - x}{(\xi - x)^2 + (\zeta - z)^2} \tag{3a}$$

The expression, which must now be a minimum, is

$$J_1 = \int\int\left[\frac{d}{dx} f(x, z)\right] \cdot f(\xi, \zeta) \frac{\xi - x}{(\xi - x)^2 + (\zeta - z)^2} d\xi\, dx \tag{8a}$$

with the unchanged secondary condition

$$J_2 = \int f(x, z)\, dx = \text{const.} \tag{9a}$$

These integrals are to be taken over all of the aerofoils.

This new problem may be treated in the same manner as the first.

$$\frac{\xi - x}{(\xi - x)^2 + (\zeta - z)^2}$$

is always to be substituted for $\frac{1}{\xi - x}$. It may be shown that this substitution does not affect the correctness of equations (10) to (15). Therefore

$$w = \text{const.} = w_0 \tag{15a}$$

is again obtained as the necessary condition for the minimum of the entire drag.

Finally, this also holds true for the limiting case in which, over a limited portion of the transverse plane, the individual aerofoils, like venetian blinds, lie so closely together that they may be considered as a continuous lifting part of a plane. Including all cases which have been considered so far, the condition for a minimum of drag can be stated:

*Let the dimensions of a system of aerofoils be given, those in the direction of flight being small in comparison with those in other directions. Let the lift be everywhere directed vertically. Under these conditions, the downwash produced by the longitudinal vortices must be uniform at all points on the aerofoils in order that there may be a minimum of drag for a given total lift.*

### 3. THREE DIMENSIONAL PARALLEL LIFTING ELEMENTS.

The three-dimensional problem may be based upon the two-dimensional one. Let now the dimensions in the direction of flight be considerable and let the lifting elements be distributed in space in any manner. Let $y$ or $\eta$ be the coordinates of any point in the direction of flight. For the time being, all lifting forces are assumed to be vertical.

The calculation of the density of drag for this case is somewhat more complicated than in the preceding cases. Consideration must be given not only to the longitudinal vortices, which are treated as before, but also to the transverse vortices which run perpendicular to the lift at any point and to the direction of flight. Their intensity at any point where there is a lifting element is

$$\Gamma = A' \cdot \frac{1}{v_0 \cdot \rho} = f(x, y, z) \cdot \frac{1}{v_0 \rho} \cdot$$

The density of drag, $W^1$ now has two components, $W_1$ and $W_2$, the first being due to the transverse vortices and the second to the longitudinal vortices.

For the solution of the present problem only the total drag of all lifting elements

$$W=\int W'dx$$

is to be considered. In the first place it will be shown that the integral of those parts of the density resulting from the transverse vortices

$$W_1=\int W_1'dx$$

does not contribute to the total drag. A small element of one transverse vortex of the length $dx$ at the point $(x, y, z)$ produces at the point $(\xi, \eta, \zeta)$ the downwash

$$dw=\frac{1}{4\pi\rho v_0}\frac{\eta-y}{r^3}f(x, y, z)\cdot dx \tag{16}$$

where

$$r^2=(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2.$$

Therefore

$$W_1=\frac{1}{4\pi\rho v_0^2}\int\int f(x, y, z)\cdot f(\xi, \eta, \zeta)\frac{\eta-y}{r^3}dxd\xi. \tag{17}$$

This integration is to be extended over all the aerofoils. It is possible to write this expression in such a manner that it holds for a continuous distribution of lift over parts of surfaces or in space. This is true, moreover, for most of the expressions in this paper. Now, exchanging the variables $x, y, z$, for $\xi, \eta, \zeta$, in equation (17) does not change the value of the integral, since the symbols for the variables have no influence on the value of a definite integral. On the other hand, the factor $(\eta-y)$, and therefore the integral also, changes its sign. Hence

$$W_1=-W_1=0 \tag{18}$$

and, as stated,

$$W=W_2. \tag{19}$$

*Therefore the entire drag may be calculated without taking into consideration the transverse vortices.*

The method of calculating the effect of the longitudinal vortices can be greatly simplified. At the point $(\xi, \eta, \zeta)$ that part of the density of drag resulting from a longitudinal vortex beginning at the point $(x, y, z)$ is

$$W_2'=\frac{1}{v_0^2\rho}\int f(\xi, \eta, \zeta)f'(x, y, z)\cdot\psi dx \tag{20}$$

where

$$f'=\frac{d}{dx}f,\text{ resp. }\frac{d}{d\xi}f$$

and

$$\psi=\frac{1}{4\pi}\int_y^\infty\frac{\xi-x}{t^3}ds;\ t^2=(\xi-x)^2+(\eta-s)^2+(\zeta-z)^2. \tag{21}$$

The entire drag is

$$W=\int W_2'dx=\frac{1}{v_0^2\rho}\int\int f(\xi, \eta, \zeta)f'(x, y, z)\psi d\xi dx. \tag{22}$$

Now, in the double integral (22) the variables $x, y, z$ may be exchanged with $\xi, \eta, \zeta$, as before, without affecting the value of the definite integral. Partial integration may then be performed twice, first with respect to $\xi$ and then with respect to $x$. The substitution results in

$$W=\frac{1}{v_0^2\rho}\int\int f(x, y, z)f'(\xi, \eta, \zeta)\bar{\psi}dxd\xi \tag{23}$$

$\bar{\psi}$ is obtained from $\psi$ upon the exchange of variables. Its value is therefore

$$\bar{\psi}=\frac{1}{4\pi}\int_{y}^{\infty}\frac{x-\xi}{\bar{t}^3}\,ds;\ \bar{t}^2=(\xi-x)^2+(s-y)^2+(\zeta-z)^2. \tag{24}$$

When partially integrating with respect to $d\xi$, the integrable factor is $f'$ $(\xi, \eta, \zeta)$

$$W=-\frac{1}{v_0^2\rho}\int\int f(\xi,\eta,\zeta)\frac{d}{d\xi}\bar{\psi}\cdot f(x,y,z)\,dxd\xi \tag{25}$$

In the subsequent partial integration with respect to $dx$, the integrable factor is $\frac{d}{d\xi}\bar{\psi}=-\frac{d}{dx}\bar{\psi}\cdot$

$$W=-\frac{1}{v_0^2\rho}\int\int f(\xi,\eta,\zeta)\cdot f'(x,y,z)\,\bar{\psi}dxd\xi. \tag{26}$$

Finally, by addition of (22) and (26), there is obtained

$$2\,W=\frac{1}{v_0^2\rho}\int\int f(\xi,\eta,\zeta)\,f'(x,y,z)\,(\psi-\bar{\psi})\,dxd\xi. \tag{27}$$

$\bar{s}=y+\eta-s$ may now be substituted in (24) for the variable of integration $s$. Then $t$ changes to $t$, and with the exception of the sign the integrand in (21) agrees with the resulting one in (22)

$$\psi=-\frac{1}{4\pi}\int_{y}^{-\infty}\frac{x-\xi}{t^3}d\bar{s} \tag{28}$$

Subtracting (28) from (21) there results finally

$$\psi-\psi=\frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{\xi-x}{t^3}ds. \tag{29}$$

Hence, $\psi-\psi$ and therefore the entire right side of equation (22) is seen to be independent of the longitudinal coordinates $y$ of the lifting elements.

*Therefore the entire resistance of a three-dimensional system of aerofoils with parallel lifting elements does not depend upon the longitudinal positions of the lifting elements.*

### 4. LIFTING ELEMENTS ARRANGED IN ANY DIRECTIONS IN A TRANSVERSE PLANE.

The problem considered in section 2 can also be generalized in another way. For the present the condition that all lifting elements be in one transverse plane may remain. However, they need no longer be parallel, and the lift may be due to not only a great number of infinitesimal lifts $dA$ but also to similar transverse forces $dB$. In the first place let the direction of all lifting elements be arbitrary, but such that there is a minimum drag, and let this direction be an unknown quantity to be determined.

In the present problem it is desirable to consider a continuous distribution of lift over given areas instead of lines. The last case can be deduced from the first at any time by passing to the limit.

Let $A'=f(x,z)$ be the density of the vertical lift per unit area, and $B'=F(x,z)$ the density of the lateral force per unit area. The lateral force is considered positive when acting in the positive direction of the $X$-axis. Then the density of the transverse vortices has the components $\frac{1}{v_o p}\cdot A'$ and $-\frac{1}{v_o p}B'$. The density of the longitudinal vortex is the divergence of the density of the transversal vortex, or $\frac{1}{v_o p}\left(\frac{dA'}{dx}-\frac{dB'}{dz}\right)$. The longitudinal vortices beginning at the point $(x, z)$ therefore produce at the point $(\xi,\zeta)$ the downwash and the transverse velocity

$$dw = \frac{1}{4\pi v_0 \rho}\left(\frac{dA'}{dx} - \frac{dB'}{dz}\right) dxdz \frac{\xi - x}{r^2} \tag{3b}$$

$$du = \frac{1}{4\pi v_0 \rho}\left(\frac{dA'}{dx} - \frac{dB'}{dz}\right) dxdz \frac{z - \zeta}{r^2} \tag{3c}$$

According to the above, the density of the drag is

$$d\,W = A'\frac{w}{v_0} dxdz + B'\frac{u}{v_0} dxdz \tag{5b}$$

With these symbols there results for the total drag the expression

$$W = \frac{1}{4\pi v^2 \rho}\left\{\iiiint f'(x,z)\, f(\xi,\zeta)\, \frac{\xi - x}{r^2}\, dxdzd\xi d\zeta - \iiiint F'(x,z)\; F(\xi,\zeta) \frac{z-\zeta}{r^2} dxdzd\xi d\zeta + \right. \tag{30}$$
$$\left. \iiiint f'(x,z)\; F(\xi,\zeta)\, \frac{z-\zeta}{r^2}\, dxdzd\xi d\zeta - \iiiint F'(x,z)\, f(\xi,\zeta)\, \frac{\xi - x}{r^2}\, dxdzd\xi d\zeta \right\}$$

All of these integrals are to be taken over all of the lifting surfaces. Now the first two integrals have forms corresponding to the integral in (8), and therefore there is a possibility of substituting (12) for these. A similar relation also holds for the last two integrals. For example, the variation of the third integral is

$$\delta \iiiint f'(x,z)\;F(\xi,\zeta)\,\frac{z-\zeta}{r^2}\,dxdzd\xi d\zeta =$$
$$\iiiint \left[\delta f'(x,z)\cdot F(\xi,\zeta)\,\frac{z-\zeta}{r^2} + f'(x,z)\;\delta F(\xi,\zeta)\,\frac{z-\zeta}{r^2}\right] dxdzd\xi d\zeta \tag{31}$$

Now in the first term on the right-hand side the variables $x$ and $z$ may be exchanged with $\xi$ and $\zeta$. It may then be partially integrated with respect to $d\xi$, the integrable factor being $df^1(\xi,\zeta)$. This gives

$$\iiiint \delta f'(x,z)\cdot F(\xi,\zeta)\,\frac{z-\zeta}{r^2}\,dxdzd\xi d\zeta = -\iiiint \delta f(\xi,\zeta)\cdot \frac{d}{d\xi}\,F(x,z)\,\frac{\zeta - z}{r^2}\,dxdzd\xi d\zeta \tag{32}$$

This may be partially integrated with respect to $dz$, the integrable factor being

$$\frac{d}{d\zeta}\frac{\zeta - z}{r^2} = -\frac{d}{dz}\frac{\xi - x}{r^2}$$

$$\iiiint \delta f'(x,z)\;F(\xi,\zeta)\,\frac{z-\zeta}{r^2}\,dxdzd\xi d\zeta = -\iiiint \delta f(\xi,\zeta)\cdot F'(x,z)\,\frac{\xi - x}{r^2}\,dxdzd\xi d\zeta \tag{33}$$

Hence the first term of the variation of the third integral of (30) can be transformed into the second term of the variation of the fourth integral of this equation. In a similar manner the two other terms may be transformed into each other. It is therefore demonstrated that the variation of the entire drag may be written

$$\delta\,W = 2\iint \delta f\cdot w\cdot dxdz + 2\iint \delta F\cdot u\cdot dxdz \tag{13b}$$

Two problems of variation can now be stated. In the first place limited parts of the surfaces may be at our disposal, over which the vertical lift $A$ and the horizontal transversal force $B$ may have any distribution. Only the total lift

$$A = \iint f\,(x,z)\,dxdz = \text{const.} \tag{9b}$$

will be given in this case.

Then

$$w = \text{const.} = w_0;\;\; u = o \tag{15b}$$

is the condition for the least drag.

If, however, the lifting parts are similar to lines, there is generally one other condition to fulfill. It is then required that the lift disappear everywhere along the direction of the aerofoils. That is to say,

$$f \sin \beta - F \cos \beta = o \tag{34}$$

where $\beta$ is the angle of inclination of the aerofoil to the horizontal $X$-axis. In order to add the new requirement (34) a second Lagrange constant $\mu$ is introduced. The condition for the least drag is now

$$w + \lambda + \frac{\mu}{\cos \beta} = o,\;\; u - \frac{\mu}{\sin \beta} = o \tag{34a}$$

and after the elimination of $\mu$

$$w \cos \beta + u \sin \beta = w_o \cos \beta \tag{15c}$$

the constant 2 $\lambda$ being replaced by $-w_0$, as before. In words:

*If all lifting elements are in one transverse plane, the component of the velocity perpendicular to the wings, produced by the longitudinal vortices, must be proportional, at all lifting elements, to the cosine of the angle of lateral inclination.*

#### 5. LIFT DISTRIBUTED AND DIRECTED IN ANY MANNER.

The results obtained previously can be generalized not only for lifting elements distributed in a transverse plane but also for lifting elements distributed in any manner in space. That part of the total drag resulting from the transverse vortices is, in the general case

$$\begin{aligned} W_1 = \frac{1}{4\pi\rho v^2}\Big[ &\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int f\,(x,y,z)\, f\,(\xi\,\eta,\zeta)\,\frac{n-y}{r^3}\, dxdydzd\xi d\eta d\zeta \\ &+ \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int F\,(x,y,z)\; F\,(\xi,\eta,\zeta)\,\frac{n-y}{r^3}\, dxdydzd\zeta d\eta d\zeta \Big] \end{aligned} \tag{17a}$$

Both terms have the same form as the integral in (17). The demonstration for (17) therefore applies to both. In the general case also the total drag can be calculated from the longitudinal vortices without taking into consideration the transverse vortices.

$$\begin{aligned} W = \frac{1}{4\pi\rho v^2}\Big[ &\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int f(x,y,z)\, f\,(\xi,\eta,\zeta)\,\psi_1\, dxdydzd\xi d\eta d\zeta \\ &+ \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int F\,(x,y,z)\; F\,(\xi,\eta,\zeta)\,\psi_2\, dxdydzd\xi d\eta d\zeta \\ &- \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int f\,(x,y,z)\; F'\,(\xi,\eta,\zeta)\,\psi_1\, dxdydzd\xi d\eta d\zeta \\ &- \int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int F\,(x,y,z)\, f'\,(\xi,\eta,\zeta)\,\psi_2\, dxdydzd\xi d\eta d\zeta \Big] \end{aligned} \tag{22a}$$

In this as in (20),

$$\psi_1 = \frac{1}{4\pi}\int_y^\infty \frac{\xi - x}{t^3}\,ds;\;\; t^2 = (\xi - x)^2 + (\eta - s)^2 + (\zeta - z)^2$$

$$\psi_2 = \frac{1}{4\pi}\int_y^\infty \frac{\zeta - z}{t^3}\,ds$$

The first two terms in (22a) have the same form as the right-hand side of (22), and the same conclusions are therefore valid for each. It can be proved directly for (22a) as for (22) that each of the two double integrals is independent of the longitudinal coordinates of the lifting elements. This proof can now be extended over the last two integrals of equation (22a).

The third integral, after changing the variables, becomes

$$\iiiiiint f(x, y, z)\ F'(\xi, \eta, \zeta)\ \psi_1\ dxdydzd\xi d\eta d\zeta = \iiiiiint f(\xi, \eta, \zeta)\ F'(x, y, z)\ \psi_1\ d\xi d\eta d\zeta dxdydz \tag{35}$$

where

$$\bar{\psi}_1 = -\frac{1}{4\pi}\int_y^{\infty}\frac{\xi - x}{t^3}\,ds;\ t^2 = (\xi - x)^2 + (s - y)^2 + (\zeta - z)^2$$

Now, let $F'$ be chosen as the integrable factor and be partially integrated with respect to $z$.

$$\iiiiiint f(x, y, z)\ F'(\xi, \eta, \zeta)\ \psi_1 dxdydzd\xi d\eta d\zeta = \iiiiiint f(\xi, \eta, \zeta)\ F(x, y, z)\ \frac{d}{dz}\bar{\psi}_1 d\xi d\eta d\zeta dxdydz. \tag{36}$$

As in the previous cases, the second integral to be expected vanishes since $f$ as well as $F$ disappear at the limits of the integration. Next $\frac{d}{dz}\bar{\psi}_1 = -\frac{d}{d\xi}\bar{\psi}_2$ is chosen as the integrable factor and partially integrated with respect to $x$. By $\bar{\psi}_2$, by analogy, is meant

$$\bar{\psi}_2 = -\frac{1}{4\pi}\int_y^{\infty}\frac{\zeta - z}{t^3}\,ds$$

$$\iiiiiint f(x, y, z)\ F'(\xi, \eta, \zeta)\ \psi_1 dxdydzd\xi d\eta d\zeta = \iiiiiint f'(\xi, \eta, \zeta)\ F(x, y, z)\ \bar{\psi}_2 d\xi d\eta d\zeta dxdydz. \tag{37}$$

Now $\bar{\psi}_2$ may be transformed, the variable $s$ in the defining equation being replaced by $\eta + y - s$. The result is that

$$\bar{\psi}_2 = \frac{1}{4\pi}\int_y^{-\infty}\frac{\xi - z}{t^3}\,ds;\ t^2 = (\xi - x)^2 + (\eta - s)^2 + (\zeta - z)^2.$$

It is seen that the integrand agrees with that of the defining integral $\psi_2$. Therefore, and since the right-hand side of (37) contains the same function under the double integral as the fourth turn in (22a), this fourth term can be combined with the transformed third member. This gives

$$\begin{aligned}&\iiiiiint f(x, y, z)\ F'(\xi, \eta, \zeta)\ \psi_1 dxdydzd\xi d\eta d\zeta + \\ &\iiiiiint F(x, y, z)\, f'(\xi, \eta, \zeta)\cdot\psi_2 dxdydzd\xi d\eta d\zeta = \\ &\iiiiiint F(x, y, z)\, f'(\xi, \eta, \zeta)\ (\psi_2 - \bar{\psi}_2)\ dxdydzd\xi d\eta d\zeta\end{aligned} \tag{38}$$

where

$$\psi_2 - \bar{\psi}_2 = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{z - \xi}{t^3}\,ds;$$

$\psi_2 - \bar{\psi}_2$ and therefore the two sides of (38) are independent of $y$. This is therefore demonstrated for the whole right-hand side of (22a).

20167—23——25

In general it can therefore be said:

*The total resistance is always independent of the longitudinal coordinates of the lifting elements.*

And further:

*The most favorable distribution of the lift, with reference to the total drag, occurs when this is also the case for the projection of the lifting elements on a transverse plane.*

That is to say, all of the lifting elements are projected on a plane perpendicular to the direction of flight, and any element so obtained has a lift equal to the sum of the lifts of all lifting elements projected onto it.

### 6. DETERMINATION OF THE SOLUTIONS.

The previous demonstrations show that the investigation for the distribution of lift which causes the least drag is reduced to the solution of the problem for systems of aerofoils which are situated in a plane perpendicular to the direction of flight. In addition, the condition for least drag (15c), which becomes the condition of uniform downwash (15) if the lift is vertical, leads to a problem which has often been investigated in the theory of two-dimensional flow with a logarithmic potential. The flow produced within the lifting transverse plane by the longitudinal vortices originating in it is, indeed, of this type. Each such vortex produces a distribution of velocity such as is produced by a two-dimensional vortex of half its intensity, and the whole distribution of velocity is obtained by adding the distributions produced by the longitudinal vortices. The potential flow sought is determined by the condition of (15c). Let it be combined with the flow of constant vertical upward motion $\omega = -\omega_0$. The resulting flow satisfies the condition at the boundaries

$$\omega \cos \beta + \mu \sin \beta = 0 \tag{39}$$

and there results, for the case of lifting lines:

*The two dimensional potential flow is of the type that encircles the lifting lines, and at a great distance the velocity is directed upwards and has the value* $w = -w_0$.

Within lifting-surfaces the velocity is zero according to the condition (15b), and the fluid therefore flows around the contour.

The intensity of the longitudinal vortices at any point is twice the rotation of the two dimensional flow. In the case of the lifting lines, therefore, the density of the longitudinal vortices is double the discontinuity of velocity from one side to the other. The intensity of the transversal vortices is determined by integrating the longitudinal vortices along the aerofoils and therefore equals twice the difference of the velocity-integral produced on the two sides of the aerofoil. Now the integral of the velocity produced is identical with the potential and hence it appears:

*The density of the lift perpendicular to the lifting line is proportional to the discontinuity of potential* $\varphi_2 - \varphi_1$, *and has the value*

$$\sqrt{A'^2 + B'^2} = 2v_0\rho(\varphi_2 - \varphi_1) \tag{40}$$

Hence the total lift obtained by integrating over all aerofoils is

$$A = 2v_0\rho\int(\varphi_2 - \varphi_1)dx \tag{41}$$

Sometimes a transformation of this equation is useful. In order to obtain it, suppose that all of the lifting lines are divided into small parts. Then, on the two ends of each lifting element there begin two inverse longitudinal vortices, the effect of which on a distant point is that of a double vortex. Their velocity-potential $\varphi$ and their stream function $\psi$ may be combined in the complex function $\psi + i\varphi$, and, not considering the existence of a parallel flow, which is without any importance in the calculation, this complex function has the form for a lifting line,

$$d(\psi + i\varphi) = \frac{dA + idB}{z - z_0} \tag{42}$$

where $z$ represents $x+iy$ and $z_0=x_0+iy_0$, $x_0$ and $y_0$ being the coordinates of the lifting elements of the line. For a lift distributed over areas a similar equation can be formed. The integration of (42) gives

$$\psi+i\varphi=\int\frac{dA+idB}{z-z_0} \tag{42a}$$

Now the residuum of the integrand at infinity is $dA+idB$ and therefore the residuum of the integral is $A+iB$. Therefore the expression can be written.

$$A=2v_0\rho R\,[\text{Res}(\psi+i\varphi)] \tag{41a}$$

where the last part means the real part of the residuum of $\psi+i\varphi$ at infinity. In the most important case of horizontal aerofoils the residuum itself is real and can be used directly to calculate the lift. The density of drag at any point is proportional to the perpendical component of the density of lift and is $W'=\frac{w_0}{v_0}\cdot A'$, from which results $W=\frac{w_0}{v_0}\cdot A$. Making use of (41) one obtains

$$W=A^2\frac{1}{2v_0^2\rho}\frac{1}{\int\left(\frac{\varphi_2-\varphi_1}{w_0}\right)dx} \tag{43}$$

$$W=A^2\frac{1}{2v_0^2\rho}\frac{w_0}{R\,[\text{Res}\;(\psi+i\varphi)]} \tag{43a}$$

The integral in the denominator of (43) represents an area characteristic of the system of aerofoils investigated. Frequently the easiest method of calculation is to assume from the beginning the velocity $w_0$ at infinity to be unity.

The case of the lift continuously distributed over single parts of areas is derived from the preceding one by passing to the limit. Since the vertical velocity $w$ disappears at all points in the lifting surfaces, the velocity is zero at all points and the rotation vanishes.

*Therefore, in the case of the most favorable distribution of lift, all of the longitudinal vortices from the continuously lifting areas begin at the boundaries of the areas.*

Equations (43) and (43a) remain. The distribution of lift is indeterminate to a certain extent. On the other hand, it is possible to connect the points of the contour having the same potential $\varphi$ by strips of any form, and it is only necessary that the lift be always perpendicular to the strip and its density have a constant value along the whole strip. According to equation (40) this equals the difference of the potential at the contour between the two borders of the strip. Worthy of note is the special case in which all of the strips run along the contour, thus coming again to the case of lifting lines. It appears that:

*Closed lines have the same mimimum of drag as the enclosed areas when continuously loaded.*

Especially important are those symmetrical contours which are cut by horizontal lines in only two points. With such the limitation to vertical lift does not involve an increase of the minimum drag. For this case it appears that:

*The density of the vertical lift per unit area must be proportional to the vertical component of the velocity of the two-dimensional flow at the point of the contour of the same height z. It is*

$$\frac{dA}{dF}=2v_0\rho\frac{d\varphi}{dz} \tag{44}$$

The corresponding density of drag is

$$\frac{dW}{dF}=2w_0\rho\frac{d\varphi}{dz} \tag{45}$$

## 7. EXAMPLES OF CALCULATIONS.

Examples of calculation of the previous demonstrations can be based on any calculated two-dimensional potential flow around parts of lines or areas. The simplest flow of the first kind is that around a single horizontal line. It leads to the problem investigated at the beginning of this paper.

In this case the potential is the real part of $\sqrt{p^2-1}$, where $p$ denotes $x+iz$. The lifting line joins the two points $z=0,\ x=-1$ and $z=0,\ x=+1$, and has the length 2. The velocity at infinity is $w=1$. The discontinuity of potential along the lifting line is $\varphi_2-\varphi_1=2\sqrt{1-x^2}$. The density of lift is distributed according to the same law, therefore if plotted over the span the density of lift would be represented by the half of an ellipse.

The minimum drag is

$$W=A^2\,\frac{1}{v_0^2\rho/_2}\,\frac{1}{4\pi} \tag{46}$$

If, instead of the value 2, the span had the general value $b$, the minimum drag would be

$$W=A^2\cdot\frac{1}{v^2\rho/_2}\,\frac{1}{\pi b^2} \tag{47}$$

This same result has been obtained by Prof. Prandtl by another method.[3]

The simplest example for a lifting vertical area is the circle. Let its center coincide with the origin of the system of coordinates. Then the potential of the flow around this circle is

$$\varphi=\frac{z}{r^2}+z \tag{48}$$

where $r=\sqrt{x^2+z^2}$. At infinity $W_0=1$. Under the condition of and according to equation (40) the density of lift is

$$A'=2v_0\rho\,\frac{d}{dz}\left(\frac{z}{r^2}+z\right)_{r=1} \tag{49}$$

This results in a constant density of lift of $A'=2$. Therefore the drag is

$$W=A^2\cdot\frac{1}{2v_0^2\rho}\cdot\frac{1}{\int\int 2dxdz}=A^2\,\frac{1}{v_0^2\rho/_2\cdot 8\pi} \tag{50}$$

The double integral is to be taken over the circle. If the general case for the diameter equal to $D$ be considered, then the least drag is

$$W=A^2\cdot\frac{1}{v^2\rho/_2}\cdot\frac{1}{D^2 2\pi} \tag{51}$$

Hence in respect to the minimum drag the circle is equivalent to a lifting line having a length $\sqrt{2}$ times the diameter.

A lifting circular line would have the same minimum drag as the circular area.

This result was also obtained by Prof. Prandtl by another method.[4] A reduction of the original problem of variation to the two-dimensional flow sometimes enables a survey of the result to be made without calculation. For instance, let a third aerofoil be added between the two aerofoils of a biplane having a small gap. (The gap may be about one-sixth of the span.) Then, in order to find the most favorable distribution of lift, the double line about which the flow occurs is to be replaced by three lifting lines. Now, in the region of the middle lifting line the velocity is small, even before this line is introduced. Therefore the discontinuity of the potential along the middle line is very much smaller than that along the others. Hence it results that the middle aerofoil of a triplane should lift less than the other two.

[3] First communication concerning t[illegible]is in Zeitschrift für Flugtechnik und Motorl. 1914. S. 239, in a note by Betz.

[4] See Technische Berichte der Flugzeugmeisterei Bd. II Heft 3.

## 8. PROCEDURE FOR THE CASE OF FLUIDS WITH SMALL VISCOSITY.

The preceding results do not apply so much to the calculation of the most favorable distribution of lift as to the calculation of the least drag. For it appears, and the results are checked by calculation, that even considerable variations from the condition of most favorable distribution of lift do not increase the drag to any great extent. Usually the minimum drag can be considered as the real drag of the system of aerofoils and in order to allow for the effect of friction of the air it is sufficient to make an addition. This addition depends chiefly upon the aerofoil section; it also depends, omitting the Reynolds Number, only upon the area of the wings and on the dynamical pressure. It is independent of the dimensions of the system of wings themselves. It may be useful to have a name for that part of the density of drag, independent of the friction of the air, which results from the theory developed in this paper. It is called the "induced drag." Generally it is not the drag itself but an absolute coefficient which is considered. This coefficient is defined by

$$c_{wi}=\frac{W_i}{q\cdot F} \qquad (52)$$

where $W_i$ is the drag previously denoted by $W$, $q$ is the dynamical pressure $v_0^2.\rho/2$, and $F$ is the total area of the wings. Equation (43) can now be written

$$c_{wi}=\frac{c_a^2\cdot F}{\pi(k\cdot b)^2} \qquad (53)$$

where $c_a$ is the lift coefficient $\frac{A}{F.q}$ corresponding to $c_w$. The greatest horizontal span $b$ of the system of wings perpendicular to the direction of flight is arbitrarily chosen as a length characteristic of the proportions of the system, $k$ is a factor characteristic of the system of aerofoils and has, according to the preceding, the value.

$$k=\sqrt{\frac{4}{\pi}\cdot\frac{1}{b^2}\int\frac{\varphi_2-\varphi_1}{w_0}dx} \qquad (54)$$

It has a special physical significance.

*Under the same conditions a single aerofoil with a span of $k$ times the maximum span of a system of aerofoils has the same induced minimum resistance as the system.*

## 9. REFINEMENT OF THE THEORY.

The demonstrations given rest on the assumption that the velocities produced by the vortices are small in comparison with the velocity of flight. The next assumption, more accurate, would be that only powers higher than the first power could be neglected.

In this case the solutions just found for lifting elements in a transverse plane can be considered as the first step towards the calculation of more exact solutions. The following steps must be taken: The exact density of drag is $W'=A'\frac{w}{v_0+v}$ where $v$ is the horizontal velocity produced at the lifting elements by the transverse vortices. It can be calculated exactly enough from the first approximation. Now, the condition of least drag is

$$w\cdot\cos\beta+\mu\sin\beta=w_0\cos\beta\left(1+\frac{v}{v_0}\right) \qquad (15d)$$

and the flow of potential, according to this condition at the boundary, is to be found. Compared with the first approximation the density below is in general somewhat increased and the density above is somewhat decreased. The minimum drag changes only by quantities of the second order.

If the lifting elements are distributed in three dimensions a similar refinement can easily be found. In this case there is to be taken into consideration a second factor which always comes in if the differences of the longitudinal coordinates of the lifting elements are considerable. The direction of the longitudinal vortices do not agree exactly with the direction of flight, but they coincide with the direction of the velocity of the fluid around the aerofoil. They are therefore somewhat inclined downwards. A better approximation is obtained by projecting the lifting elements not in the direction of flight but in a direction slightly inclined upwards from the rear to the front. This inclination is about $\frac{2w_0}{v_0}$. Except for this, the method of calculation remains unchanged.