View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by NASA Technical Reports Server

(1)

DOE/NASA/1028-79/25
NASA TM-79295

DO NOT DESTROY
RETURN TO LIBRARY

AEROELASTIC EQUATIONS OF MOTION OF A DARRIEUS VERTICAL-AXIS WIND-TURBINE BLADE

Krishna Rao V. Kaza
University of Toledo

and

Raymond G. Kvaternik
National Aeronautics and Space Administration
Langley Research Center

NASA-TM-79295

December 1979

24 MAR 1980
MCDONNELL DOUGLAS
RESEARCH & ENGINEERING LIBRARY
ST. LOUIS

Prepared for
U.S. DEPARTMENT OF ENERGY
Energy Technology
Distributed Solar Technology Division

M80-11689

NOTICE

This report was prepared to document work sponsored by the United States Government. Neither the United States nor its agent, the United States Department of Energy, nor any Federal employees, nor any of their contractors, subcontractors or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

DOE/NASA/1028-79/25
NASA TM-79295

AEROELASTIC EQUATIONS OF MOTION OF A DARRIEUS VERTICAL-AXIS WIND-TURBINE BLADE

Krishna Rao V. Kaza
University of Toledo
Toledo, Ohio 43606

and

Raymond G. Kvaternik
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

December 1979

Work performed for
U. S. DEPARTMENT OF ENERGY
Energy Technology
Distributed Solar Technology Division
Washington, D. C. 20545
Under Interagency Agreement E(49-26)-1028

PREFACE

Krishna Rao V. Kaza, who is associated with the University of Toledo, is presently at the NASA Lewis Research Center. He is working under grant NSG-3139 funded by the U.S. Department of Energy through the Lewis Wind Energy Project Office. The work is being conducted under the direction of Christos C. Chamis. Before coming to Lewis, Mr. Kaza was a Langley grantee doing similar work at the NASA Langley Research Center with Raymond G. Kvaternik, the coauthor of this report.

AEROELASTIC EQUATIONS OF MOTION OF A DARRIEUS VERTICAL-AXIS WIND-TURBINE BLADE

by

Krishna Rao V. Kaza
The University of Toledo
Toledo, Ohio 43606

and

Raymond G. Kvaternik
NASA Langley Research Center
Hampton, Virginia 23665

E-238

ABSTRACT

The second-degree nonlinear aeroelastic equations of motion for a slender, flexible, nonuniform, Darrieus vertical-axis wind-turbine blade which is undergoing combined flatwise bending, edgewise bending, torsion, and extension are developed using Hamilton's principle. The blade aerodynamic loading is obtained from strip theory based on a quasi-steady approximation of two-dimensional incompressible unsteady airfoil theory. The derivation of the equations has its basis in the geometric nonlinear theory of elasticity and the resulting equations are consistent with the small deformation approximation in which the elongations and shears (and hence strains) are negligible compared to unity. These equations are suitable for studying vibrations, both static and dynamic aeroelastic instabilities, and dynamic response. Several possible methods of solution of the equations, which have periodic coefficients, are discussed.

INTRODUCTION

Renewed interest in the wind as an alternative source of energy has resulted in a number of studies of various wind-turbine concepts. Presently, receiving considerable attention (see for example Refs. 1 and 2) is the vertical-axis wind turbine (VAWT), also known as the Darrieus rotor (Ref. 3). The VAWT rotor (Fig. 1) embodies long, slender, and flexible airfoil-shaped blades which are attached to a vertical rotating shaft at both ends. The curved shape of each blade is approximately that of a troposkien, i.e., the shape taken by a flexible cable of uniform density and cross section when it is spun at a constant angular velocity.

The efficient design, construction, and operation of large VAWT rotors require that the vibrating loads and stresses in the blades, as well as in the combined rotor-tower system be reduced to the lowest possible levels and that the system must be free from all types of instabilities. Thus, aeroelastic and structural dynamic considerations have a direct bearing on the manufacture, life, and operation of large VAWT systems. Although the basic dynamic phenomena associated with VAWT rotors are basically similar to those of helicopter rotors, proprotors, and horizontal-axis wind turbines, the structural configurations of VAWT systems are sufficiently different to necessitate comprehensive and independent analytical and experimental investigations of their aeroelastic stability and dynamic response characteristics.

Vertical-axis rotor systems can exhibit a variety of mechanical and aeroelastic instabilities such as resonance, ground resonance, whirl flutter, blade classical bending-torsion flutter, blade coupled bending-torsion-extension aeroelastic instabilities, blade stall flutter, and blade static divergence. Ground resonance and whirl flutter are associated with the entire VAWT system; the other instabilities are primarily associated with the individual blades of the system. A design requirement for VAWT systems is that each component as well as the entire system be free from all instabilities.

Experimental results obtained with wind-tunnel models (Refs. 4 and 5) have indicated that the blade aeroelastic instabilities involving coupling between flatwise bending, edgewise bending, and torsion are possible under certain conditions. To explain these instabilities, an analytical investigation was conducted in Ref. 6 using an approximate modal analysis. More recently, bending vibration equations of a rotating curved slender blade were derived and solved for special cases by using asymptotic methods in Ref. 7. An aeroelastic analysis of an existing 5m VAWT system with and without guying wires for the tower was conducted in Ref. 8. The analyses indicate the possibility of resonance, ground resonance, and aeroelastic-type instabilities. In Ref. 9, an analytical investigation of the aeroelastic stability of a different 5m VAWT system was performed using a finite element model. These results also show the possibility of several types of instabilities, depending on the system parameters.

Analyses based on finite element models are well-suited to accommodate the structural complexities of actual VAWT systems which may have blades with struts and towers with guying wires. However, the fundamental understanding of the basic mechanisms of aeroelastic instabilities and dynamic response phenomena and parametric studies associated with VAWT systems are better served by a continuum model which leads to a set of differential equations. A continuum model for a single blade can be viewed as a building block from which a continuum model for an entire VAWT system can be constructed.

The purpose of this report is to develop a set of second-degree nonlinear aeroelastic equations of motion of a Darrieus wind-turbine blade involving flatwise bending, edgewise bending, torsion, and extension. The nonlinear terms which will be retained in the present derivation are of the type which have been found to be important in aeroelastic stability of helicopter rotor blades.

The derivation of the nonlinear equations of motion herein follows the methodology of Refs. 10-12. The equations are derived using the geometric nonlinear theory of elasticity (Ref. 13) in which the elongations and shears (and hence strains) are negligible compared to unity. The generalized aerodynamic forces are obtained from strip theory based on a quasi-steady approximation of two-dimensional incompressible, unsteady airfoil theory. The equations of motion which are consistent with these approximations may be derived to any desired degree by retaining terms in the dependent variables to the appropriate degree throughout the development. The present development will be directed to the derivation of the second-degree nonlinear equations of motion in which one formally retains terms through second-degree in the dependent variables. Rigorous adherence to this retention scheme leads to an almost insurmountable amount of algebra. To circumvent this problem to some extent, an ordering scheme which is consistent with the assumption of a slender beam is imposed early in the development of the dynamic and elastic portions of the present equations. No ordering scheme is imposed in the development of the generalized aerodynamic forces herein because any ordering scheme which is imposed would depend on the order assigned to the nondimensional free-stream velocity and induced velocity both of which vary significantly in practice. Thus, to accommodate such general operating conditions with the present equations, the aerodynamic forces are left in general second-degree form. The aerodynamic forces acting on a blade element are functions of the blade azimuth angle and hence the final equations will contain periodic terms. For completeness, the gravitational forces are also included in the present development.

The equations developed herein are suitable for studying aeroelastic instabilities, aeroelastic response, and vibration characteristics of flexible, curved, and rotating blades. These equations form a building block from which a continuum model of an entire Darrieus-type VAWT system can be constructed. As these equations do not have closed form solutions, several possible approximate methods of solution are discussed.

SYMBOLS

a	airfoil lift-curve-slope
a_1, b_1, c_1	quantities defined in Eq. (A5)
A	cross-sectional area of blade

A_P	projected area of rotor in vertical plane
A_u, A_v, A_w	generalized aerodynamic forces per unit length in $\bar{e}_{x_{B3}}$, $\bar{e}_{y_{B3}}$, $\bar{e}_{z_{B3}}$ directions, respectively
A_ϕ	generalized aerodynamic moment per unit length about elastic axis
b	number of blades
B_V, B_T, B_G, B_A	boundary terms arising from strain energy, kinetic energy, work done by gravitational forces, and work done by aerodynamic forces, respectively
B_i (i = 1, 2, 3, 4)	sectional constants
c	blade chord
C_{d_o}	airfoil profile drag coefficient
C(k)	Theodorsen's circulation function
D	airfoil drag per unit length
e	chordwise offset of mass centroid from elastic axis (positive when in front of elastic axis)
e_A	chordwise offset of area centroid of cross section from elastic axis (positive when in front of elastic axis)
E	Young's modulus
$\bar{e}_{X_{B2}}$, $\bar{e}_{Y_{B2}}$, $\bar{e}_{Z_{B2}}$	unit vectors along X_{B2}, Y_{B2}, Z_{B2} axes
$\bar{e}_{X_{B3}}$, $\bar{e}_{Y_{B3}}$, $\bar{e}_{Z_{B3}}$	unit vectors along X_{B3}, Y_{B3}, Z_{B3} axes
$\bar{e}_{X_{B6}}$, $\bar{e}_{Y_{B6}}$, $\bar{e}_{Z_{B6}}$	unit vectors along X_{B6}, Y_{B6}, Z_{B6} axes
$\bar{e}_{X_I}$, $\bar{e}_{Y_I}$, $\bar{e}_{Z_I}$	unit vectors along X_I, Y_I, Z_I axes
$\bar{e}_{X_R}$, $\bar{e}_{Y_R}$, $\bar{e}_{Z_R}$	unit vectors along X_R, Y_R, Z_R axes
$\bar{F}_A$	aerodynamic force vector

$F_{x_{B3}}$, $F_{y_{B3}}$, $F_{z_{B3}}$	components of aerodynamic force vector $\bar{F}_A$ in the directions of $\bar{e}_{X_{B3}}$, $\bar{e}_{Y_{B3}}$, $\bar{e}_{Z_{B3}}$, respectively
$F_{x_{B6}}$, $F_{y_{B6}}$, $F_{z_{B6}}$	components of aerodynamic force vector $\bar{F}_A$ in the directions of $\bar{e}_{X_{B6}}$, $\bar{e}_{Y_{B6}}$, $\bar{e}_{Z_{B6}}$, respectively
$\bar{g}$	gravitational acceleration vector
G	shear modulus
G_u, G_v, G_w	gravitational forces per unit length in u, v, w directions, respectively
G_ϕ	generalized gravitational moment per unit length about elastic axis
H	height of wind turbine
$\dot{h}$	vertical velocity of two-dimensional section normal to free-stream
I_u, I_v, I_w	generalized inertia forces per unit length in $\bar{e}_{X_{B3}}$, $\bar{e}_{Y_{B3}}$, $\bar{e}_{Z_{B3}}$ directions, respectively
$I_{x_3x_3}$, $I_{y_3y_3}$	area moments of inertia about Y_3 and X_3 axes, respectively
I_ϕ	generalized inertia moment per unit length about elastic axis
J	torsional section constant
k	reduced frequency
k_A	polar radius of gyration of cross-sectional area about elastic axis
k_i (i = 1, 2,...,6)	notation used in writing the variation of the kinetic energy
k_m	polar radius of gyration of cross-sectional mass about elastic axis ($k_m^2 = k_{m_1}^2 + k_{m_2}^2$)
k_{m_1}, k_{m_2}	mass radii of gyration about Y_3 and X_3 axes, respectively
$k_{x_{B3}}$, $k_{y_{B3}}$, $k_{z_{B3}}$	components of curvature of elastic axis before deformation

$k_{x_{B6}}$, $k_{y_{B6}}$, $k_{z_{B6}}$	components of curvature of elastic axis after deformation
l_i, m_i, n_i, ($i = 1, 2, 3$)	direction cosines, Eq. (A1)
L	aerodynamic lift per unit length
m	mass of the blade per unit length
M, $M_{z_{B6}}$	aerodynamic moment about the deformed elastic axis per unit length
P	arbitrary point on the elastic axis before deformation; also origin of the blade-fixed axis system before deformation
P'	arbitrary point on the elastic axis after deformation
$\bar{r}_o$, $\bar{r}_1$	position vectors of a point in the cross section of the blade before and after deformation, respectively
$\bar{R}$, $\bar{R}_1$	position vectors of an arbitrary point on the elastic axis before and after deformation, respectively
s, s_1	running coordinates along the elastic axis before and after deformation, respectively
s_i (1, 2,...,9)	notation used in writing the variation of strain energy in a concise form
S	length of blade along undeformed elastic axis
S_u, S_v, S_w, S_ϕ	generalized elastic forces
t	time
T	kinetic energy; blade tension; rotor thrust
u, v, w	deformations of elastic axis in X_{B3}, Y_{B3}, and Z_{B3} directions, respectively
U	resultant of U_T and U_P
U_R, U_T, U_P	radial, tangential, and perpendicular components of the resultant velocity of a point on the elastic axis

v_i	induced velocity, positive in the negative X_I direction
V	strain energy
V_∞	free-stream velocity
$\overline{V}_a$	wind velocity vector
$\overline{V}_{X_{B3}Y_{B3}Z_{B3}}$, $\overline{V}_{X_{B6}Y_{B6}Z_{B6}}$	relative velocity of a point on the elastic axis expressed in $X_{B3}Y_{B3}Z_{B3}$ and $X_{B6}Y_{B6}Z_{B6}$ coordinate systems, respectively
W	sum of W_A and W_G
W_A	work done by aerodynamic forces
W_G	work done by gravitational forces
x_o, z_o	coordinates of a point on the undeformed elastic axis along X_I- and Z_I-axes, respectively
x_3, y_3, z_3	coordinates in $X_3Y_3Z_3$ coordinate system, the x_3 and y_3 axes are the minor and major principal axes of the cross section
$X_IY_IZ_I$	inertial axis system
$X_{B1}Y_{B1}Z_{B1}$	blade axis system, parallel to $X_RY_RZ_R$ coordinate system
$X_{B2}Y_{B2}Z_{B2}$	blade axis system obtained by rotating $X_{B1}Y_{B1}Z_{B1}$ system about the negative Y_{B1}-axis by an angle θ_o
$X_{B3}Y_{B3}Z_{B3}$	blade axis system obtained by rotating $X_{B2}Y_{B2}Z_{B2}$ system about the Z_{B2}-axis by an angle γ
$X_{B6}Y_{B6}Z_{B6}$	blade axis system in the deformed configuration obtained by translating and rotating the $X_{B3}Y_{B3}Z_{B3}$ system; the Z_{B6}-axis is tangent to the deformed elastic axis
$X_RY_RZ_R$	blade axis system obtained by rotating the $X_IY_IZ_I$ system about the Z_I-axis by an angle $\psi(=\Omega t)$
$[T]$	transformation matrix relating the angular orientation of the deformed and undeformed blade-fixed coordinate systems

$[\varepsilon_{ij}]$	Green's strain tensor
α	airfoil section angle of attack
α_x, α_y, α_z	quantities defined in Appendix A
β, ζ, θ	Eulerian-type rotation angles
γ	section total pitch angle (built-in twist plus pitch angle due to control inputs)
$\gamma_{x_3z_3}$, $\gamma_{y_3z_3}$, $\gamma_{z_3z_3}$	engineering strain components
$\delta( )$	variation of ( )
$\delta\theta_{z_{B6}}$	virtual rotation about the Z_{B6}-axis
ε	small parameter of the order of the bending slopes; airfoil section pitch angle with respect to free-stream velocity; also extensional component of Green's strain tensor along the elastic axis
$\varepsilon_{x_3z_3}$, $\varepsilon_{y_3z_3}$, $\varepsilon_{z_3z_3}$	strain components
θ_o	angle between blade local tangent and vertical axis, illustrated in Fig. 1
μ	nondimensional free-stream velocity, $V_\infty/\Omega R$
μ_i	nondimensional induced velocity, $v_i/\Omega R$
ρ	mass density of the blade; also mass density of air
$\sigma_{x_3z_3}$, $\sigma_{y_3z_3}$, $\sigma_{z_3z_3}$	engineering stresses
ϕ	angle of twisting deformation about the elastic axis
ψ	blade azimuth angle
$\bar{\omega}_{X_{B3}Y_{B3}Z_{B3}}$	curvature vector of the undeformed elastic axis
$\bar{\omega}_{X_{B6}Y_{B6}Z_{B6}}$	curvature vector of the deformed elastic axis
Ω	rotational speed of rotor

$( )_C$	circulatory aerodynamic term
$( )_{NC}$	noncirculatory aerodynamic term
$\dot{( )}$	time derivative $\frac{\partial}{\partial t}( )$
$( )'$	denotes differentiation with respect to s

MATHEMATICAL MODEL AND ATTENDANT ORDERING SCHEME

The mathematical model chosen in the present development is a continuum model. The presence of rotation introduces equilibrium centrifugal stresses which require the use of a geometric nonlinear theory of elasticity. There are several levels of approximation which may be considered in this theory (Refs. 10 and 13). The level of approximation used in the present development assumes that the elongations and shears (and hence strains) are negligible compared to unity.

The wind-turbine blade considered in the present development consists of a slender, curved, nonuniform blade which can undergo combined flatwise bending, edgewise bending, torsion, and extension (axial deformation). The elastic axis, mass axis, and tension axis are taken to be noncoincident. The elastic axis is assumed to be coincident with the quarter-chord of the blade. The generalized aerodynamic forces are obtained from strip theory based on a quasi-steady approximation of two-dimensional, incompressible and unsteady airfoil theory. Gravitational forces are included.

Blades presently being considered for VAWT applications have neither pretwist (built-in twist) nor control inputs for changing section pitch angle as do the blades for a horizontal-axis wind turbine. However, for completeness, a variable section pitch angle is included in deriving the second-degree expressions for the bending curvatures and twist rate and for the strains.

An ordering scheme consistent with the assumption of a slender beam is introduced here to provide a systematic procedure for discarding higher-order terms while deriving the second-degree nonlinear aeroelastic equations. A mathematical ordering scheme was introduced in Ref. 11 for deriving the nonlinear equations for a slender helicopter rotor blade. Considerations similar to those in Ref. 11 have been applied in the present report to establish an ordering scheme which is consistent with the slender curved blades of a VAWT rotor. In this scheme, a parameter ε which is taken to be of the same order as the nondimensional variables u/S, v/S, w/S, and ϕ is introduced. The order of the dependent variables and geometric quantities appearing in the equations of motion of this report are as follows:

$$
\begin{aligned}
u/S &= O(\varepsilon) & x_3/S &= O(\varepsilon) \\
v/S &= O(\varepsilon) & y_3/S &= O(\varepsilon) \\
w/S &= O(\varepsilon) & z_3/S &= s/S = O(1) \\
\phi &= O(\varepsilon) & \theta_o(s) &= O(1) \\
\alpha_x &= O(\varepsilon) & x_o(s) &= O(1) \\
\alpha_y &= O(\varepsilon) & y_o(s) &= O(1) \\
\alpha_z &= O(\varepsilon^2) & z_o(s) &= O(1)
\end{aligned} \qquad (1)
$$

By using the ordering assigned above, the order of the elastic and inertial terms which are retained in the second-degree nonlinear aeroelastic equations of motion of the curved blade considered herein are given in Table 1. The rationale for this scheme was discussed in Ref. 11. It should be noted that in the present development the extensional deformation (w/S) is $O(\varepsilon)$ instead of $O(\varepsilon^2)$ as in Ref. 11 because of the presence of the initial bending curvature.

Table 1 - Ordering scheme

Freedom	Elastic forces	Inertial forces
Extension	$O(\varepsilon^3)$	$O(\varepsilon^3)$
Bending	$O(\varepsilon^4)$	$O(\varepsilon^2)$
Torsion	$O(\varepsilon^5)$	$O(\varepsilon^3)$

COORDINATE SYSTEMS AND MOTION VARIABLES

Several orthogonal coordinate systems will be employed in the derivation of the equations of motion. Those which are common to both the dynamic and aerodynamic aspects are described in this section.

1. Inertial system (I-system) $X_I Y_I Z_I$ — The Z_I-axis of this system, as is shown in Fig. 1, coincides with the vertical axis of the shaft. The X_I-axis is aligned with the free-stream velocity V_∞.

2. Rotating system (R-system) $X_R Y_R Z_R$ — This system is obtained by rotating the I-system about the Z_I-axis by an angle $\psi = \Omega t$, as is shown in Fig. 1. The shaft rotational speed is given by Ω and is assumed constant. The coordinate transformation between the I- and R-systems is

$$\begin{Bmatrix} \bar{e}_{X_I} \\ \bar{e}_{Y_I} \\ \bar{e}_{Z_I} \end{Bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{e}_{X_R} \\ \bar{e}_{Y_R} \\ \bar{e}_{Z_R} \end{Bmatrix} \tag{2}$$

3. Blade system 1 (B1-system) $X_{B1}Y_{B1}Z_{B1}$ — This local blade-fixed coordinate system, as shown in Fig. 1, is fixed to an arbitrary point, P, on the elastic axis of the blade. This frame translates along the blade elastic axis and it is parallel to R-system.

4. Blade system 2 (B2-system) $X_{B2}Y_{B2}Z_{B2}$ — This system is obtained by rotating the B1-system about the negative Y_{B1}-axis by an angle θ_o, as shown in Fig. 1. The X_{B2}, Y_{B2}, and Z_{B2} axes are in the normal, binormal and tangential directions, respectively. The rotation angle θ_o can be obtained from the known geometry of the curved undeformed elastic axis in the X_IZ_I plane by the parametric equation

$$\bar{R} = x_o\bar{e}_{X_I} + z_o\bar{e}_{Z_I} \tag{3}$$

and is

$$\theta_o = \tan^{-1}(-x_o'/z_o') \tag{4}$$

The coordinate transformation between the B1- and B2-systems is

$$\begin{Bmatrix} \bar{e}_{X_{B1}} \\ \bar{e}_{Y_{B1}} \\ \bar{e}_{Z_{B1}} \end{Bmatrix} = \begin{bmatrix} \cos\theta_o & 0 & -\sin\theta_o \\ 0 & 1 & 0 \\ \sin\theta_o & 0 & \cos\theta_o \end{bmatrix} \begin{Bmatrix} \bar{e}_{X_{B2}} \\ \bar{e}_{Y_{B2}} \\ \bar{e}_{Z_{B2}} \end{Bmatrix} \tag{5}$$

5. Blade principal axis system (B3-system) X_{B3}, Y_{B3}, Z_{B3} — The X_{B3} and Y_{B3} axes are taken to be aligned with the minor and major principal axes of the blade cross section, respectively. The principal axes are obtained by rotating the normal and binormal axes by an angle γ as indicated in Fig. 2. The angle γ is the total section pitch angle, which is a combination of built-in twist (pretwist) and section pitch changes due to control inputs. The VAWT configurations presently considered in the literature do not have any section pitch angle, but it is included in developing the expressions for the curvatures and strains in order to indicate how one would include this effect in the analysis. The coordinate transformation between the B2- and B3-systems is

$$\begin{Bmatrix} \bar{e}_{X_{B2}} \\ \bar{e}_{Y_{B2}} \\ \bar{e}_{Z_{B2}} \end{Bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{e}_{X_{B3}} \\ \bar{e}_{Y_{B3}} \\ \bar{e}_{Z_{B3}} \end{Bmatrix} \tag{6}$$

The curvature vector of the undeformed blade is

$$\bar{\omega}_{X_{B3}Y_{B3}Z_{B3}} = -\theta_o'\bar{e}_{Y_{B2}} + \gamma'\bar{e}_{Z_{B3}} \tag{7}$$

This vector is also called the Darboux vector in the literature. Substituting $\bar{e}_{Y_{B2}}$ into Eq. (7) from Eq. (6), the curvature vector is given by

$$\bar{\omega}_{X_{B3}Y_{B3}Z_{B3}} = k_{x_{B3}}\bar{e}_{X_{B3}} + k_{y_{B3}}\bar{e}_{Y_{B3}} + k_{z_{B3}}\bar{e}_{Z_{B3}} \tag{8}$$

where

$$\begin{aligned} k_{x_{B3}} &= -\theta_o' \sin\gamma \\ k_{y_{B3}} &= -\theta_o' \cos\gamma \\ k_{z_{B3}} &= \gamma' \end{aligned} \tag{9}$$

6. Blade system 6 (B6-system) $X_{B6}Y_{B6}Z_{B6}$ — This system is shown in Fig. 3 and is obtained by translating and rotating the $X_{B3}Y_{B3}Z_{B3}$ system. The Z_{B6}-axis is tangent to the deformed elastic axis. The blade cross section itself is assumed rigid. The deformations of the elastic axis are denoted by u, v, w in the B3-system. The angular orientation of the B6-system with respect to the B3-system is given by three Eulerian-type angles β, ζ, and θ, which are, in turn, expressed in terms of the elastic deformations (u, v, w) and their derivatives (u', v', w'), and the twisting deformation (ϕ). The final relation between the B3- and B6-systems is developed in Appendix A and is given by

$$\begin{Bmatrix} \bar{e}_{X_{B6}} \\ \bar{e}_{Y_{B6}} \\ \bar{e}_{Z_{B6}} \end{Bmatrix} = \begin{bmatrix} 1 - \frac{\phi^2}{2} - \frac{\alpha_x^2}{2} & \phi & -\alpha_x - \phi\alpha_y \\ -\phi - \alpha_x\alpha_y & 1 - \frac{\phi^2}{2} - \frac{\alpha_y^2}{2} & \phi\alpha_x - \alpha_y \\ \alpha_x & \alpha_y & 1 - \frac{1}{2}(\alpha_x^2 + \alpha_y^2) \end{bmatrix} \begin{Bmatrix} \bar{e}_{X_{B3}} \\ \bar{e}_{Y_{B3}} \\ \bar{e}_{Z_{B3}} \end{Bmatrix} \tag{10}$$

where

$$\alpha_x = u' - vk_{z_{B3}} + wk_{y_{B3}}$$

$$\alpha_y = v' + uk_{z_{B3}} - wk_{x_{B3}}$$

$$\alpha_z = w' - uk_{y_{B3}} + vk_{x_{B3}} \tag{11}$$

HAMILTON'S PRINCIPLE

The governing equations of motion are derived using the extended Hamilton's principle (Ref. 14).

$$\int_{t_o}^{t_1} (\delta T - \delta V + \delta W)dt = 0 \tag{12}$$

where

$$\delta W = \delta W_G + \delta W_A \tag{13}$$

In Eq. (12), T is the kinetic energy, V is the strain energy, and W is the work done by gravitational and aerodynamic forces. For subsequent convenience, the variation of work is divided into two parts as indicated in Eq. (13): the first part, δW_G, is due to gravitational forces; and the second part, δW_A, is due to aerodynamic loading. In the following sections, explicit expressions for T, V, and W in terms of the dependent variables u, v, w, and ϕ and their derivatives and the blade sectional properties will be developed.

Strain Energy

The expression for the strain energy of the blade in terms of engineering strains and stresses is

$$V = \frac{1}{2} \int_0^S \iint_A \left(\sigma_{z_3z_3}\gamma_{z_3z_3} + \sigma_{z_3x_3}\gamma_{z_3x_3} + \sigma_{z_3y_3}\gamma_{z_3y_3}\right)dx_3dy_3dz_3^* \tag{14}$$

---

*The coordinates s and z_3 are used interchangeably.

where, assuming that the components of the engineering strains are equal to the corresponding components of the Lagrangian strain and using Hooke's law,

$$\sigma_{z_3z_3} = E\gamma_{z_3z_3} = E\epsilon_{z_3z_3}$$

$$\sigma_{z_3x_3} = G\gamma_{z_3x_3} = 2G\epsilon_{z_3x_3}$$

$$\sigma_{z_3y_3} = G\gamma_{z_3y_3} = 2G\epsilon_{z_3y_3} \tag{15}$$

Taking the first variation of V as given in Eq. (14), and using Eq. (15), yields

$$\delta V = \int_0^S E \iint_A \gamma_{z_3z_3}\delta\gamma_{z_3z_3}dx_3dy_3dz_3$$

$$+ \int_0^S G \iint_A (\gamma_{z_3x_3}\delta\gamma_{z_3x_3} + \gamma_{z_3y_3}\delta\gamma_{z_3y_3})dx_3dy_3dz_3 \tag{16}$$

The expressions for the required strain components are developed in Appendix B. For a slender curved blade with zero section pitch angle these expressions are given by Eqs. (B15), (B16), and (B17). Substituting these expressions into Eq. (16), taking the indicated variations, and integrating over the cross section of the blade leads to

$$\delta V = \int_0^S (s_1\delta u + s_2\delta u' + s_3\delta u'' + s_4\delta v'$$

$$+ s_5\delta v'' + s_6\delta w + s_7\delta w' + s_8\delta\phi + s_9\delta\phi')dz_3 \tag{17}$$

where, consistent with the ordering scheme discussed earlier,

$$s_1 = T\theta_o'$$

$$s_2 = T(u' - \theta_o' w)$$

$$s_3 = -\theta_o' \left[E\varepsilon I_{y_3y_3} - 2EB_3(v'' + \phi\theta_o')\right]$$

$$+ EI_{y_3y_3}\left[u'' - \theta_o' w' - \theta_o'' w - \frac{\theta_o'}{2}(v'^2 + \phi^2) + \phi v''\right]$$

$$+ T\phi e_A - E\phi I_{x_3x_3}(v'' + \phi\theta_o') + GJv'(\phi' - \theta_o' v')$$

$$s_4 = Tv' + EI_{y_3y_3}\theta_o' v'(u'' - \theta_o'' w - \theta_o' w')$$

$$+ GJ(\phi' - \theta_o' v')(u'' - w'\theta_o' - \theta_o'' w - \theta_o') - GJ\theta_o' v'(u'' - w'\theta_o' - \theta_o'' w)$$

$$s_5 = EI_{y_3y_3}\phi(u'' - \theta_o'' w - \theta_o' w') - Te_A + 2EB_3\theta_o'(u'' - \theta_o' w' - \theta_o'' w)$$

$$- EI_{x_3x_3}\left[-v'' + \phi(u'' - \theta_o'' w - \theta_o' w' - \theta_o')\right]$$

$$s_6 = Tw\theta_o'^2 - T\theta_o' u' - EI_{y_3y_3}\theta_o''(u'' - \theta_o'' w - \theta_o' w') + EI_{y_3y_3}\theta_o''\theta_o'\varepsilon$$

$$s_7 = T - EI_{y_3y_3}\theta_o'(u'' - \theta_o'' w - \theta_o' w')$$

$$s_8 = -\theta_o'(v'' + \theta_o'\phi)\left[EI_{y_3y_3}\varepsilon - EB_3(v'' + \phi\theta_o')\right]$$

$$+ EI_{y_3y_3}(v'' + \theta_o'\phi)(u'' - \theta_o'' w - \theta_o' w') + EB_3(v'' + \theta_o'\phi)^2\theta_o'$$

$$+ E(u'' - \theta_o' w' - \theta_o'' w)\left[A\varepsilon e_A - I_{x_3x_3}(v'' + \phi\theta_o')\right.$$

$$\left. - \theta_o' B_3(u'' - \theta_o' w' - \theta_o'' w)\right] - E\theta_o'\left\{Ae_A\varepsilon - B_3\left[\theta_o'(u'' - \theta_o'' w - \theta_o' w')\right.\right.$$

$$\left. - \theta_o'^2\frac{\phi^2}{2} - \theta_o'\phi v''\right] + \frac{B_4}{2}\theta_o'(\phi^2 + 2\phi v'') + I_{x_3x_3}\left[-v'' + \phi(u'' - \theta_o'' w\right.$$

$$\left.\left. - \theta_o' w') - \phi\theta_o'\right] + \frac{B_1}{2}\phi'(\phi' - 2\theta_o' v')\right\} - 2B_3E\theta_o'(u'' - \theta_o'' w - \theta_o' w')^2$$

(cont'd)

$$- \theta_o'^2 E \left\{ -B_3 \left[u'' - \theta_o' w' - \theta_o'' w + \phi v'' + \frac{1}{2} \theta_o' (\phi^2 + v'^2) \right] \right.$$

$$\left. - B_2 (\theta_o' v'' + \theta_o'^2 \phi) \right\} + E\theta_o' (v'' + \theta_o' \phi) \left[I_{x_3 x_3} \varepsilon - B_4 (v'' + \phi \theta_o') \right]$$

$$s_9 = (\phi' - \theta_o' v') \left[E(I_{y_3 y_3} + I_{x_3 x_3}) \varepsilon - E(B_3 + B_4)(v'' + \phi \theta_o') \right]$$

$$+ GJ \left[\phi' + v' (u'' - w' \theta_o' - \theta_o'' w - \theta_o') \right]$$

$$T = E \left\{ A\varepsilon - I_{y_3 y_3} \left[\theta_o' (u'' - \theta_o'' w - \theta_o' w') + \frac{\theta_o'^2 \phi^2}{2} + \theta_o' \phi v'' \right] \right.$$

$$+ \frac{I_{x_3 x_3}}{2} \theta_o' \phi (\theta_o' \phi + 2v'') + Ae_A \left[-v'' + \phi (u'' - \theta_o'' w - \theta_o' w' - \theta_o') \right]$$

$$\left. + \frac{Ak_A^2}{2} \phi' (\phi' - 2\theta_o' v') \right\} \tag{18}$$

The expression for the extensional strain ε on the elastic axis for the case in which the section pitch angle is zero is given by

$$\varepsilon = w' + \theta_o' u + \frac{1}{2} (u'^2 + \theta_o'^2 w^2 - 2\theta_o' u' w + v'^2) \tag{19}$$

Assuming that the cross section is symmetric about the Y_{B3}-axis, the sectional properties appearing in Eq. (18) are defined as follows:

$$A = \iint dx_3 dy_3 \qquad \iint x_3 dx_3 dy_3 = 0$$

$$Ae_A = \iint y_3 dx_3 dy_3 \qquad \iint x_3 y_3 dx_3 dy_3 = 0$$

$$I_{x_3 x_3} = \iint y_3^2 dx_3 dy_3 \qquad \iint x_3 (x_3^2 + y_3^2) dx_3 dy_3 = 0$$

(cont'd)

$$I_{y_3y_3} = \int\int x_3^2 dx_3 dy_3 \qquad \int\int y_3 x_3^3 dx_3 dy_3 = 0$$

$$Ak_A^2 = \int\int \left(x_3^2 + y_3^2\right) dx_3 dy_3 \qquad \int\int x_3 y_3^3 dx_3 dy_3 = 0$$

$$J = \int\int \left(x_3^2 + y_3^2\right) dx_3 dy_3 \qquad \int\int y_3^2 x_3 dx_3 dy_3 = 0$$

$$B_1 = \int\int y_3\left(x_3^2 + y_3^2\right) dx_3 dy_3 \qquad B_3 = \int\int x_3^2 y_3 dx_3 dy_3$$

$$B_2 = \int\int x_3^2 y_3^2 dx_3 dy_3 \qquad B_4 = \int\int y_3^3 dx_3 dy_3 \tag{20}$$

Integrating Eq. (17) by parts, the result can be put in the form

$$\delta V = \int_0^S (S_u \delta u + S_v \delta v + S_w \delta w + S_\phi \delta\phi) dz_3 + B_V \tag{21}$$

where the generalized elastic forces S_u, S_v, S_w, and S_ϕ are

$$\begin{aligned} S_u &= s_1 - s_2' + s_3'' \\ S_v &= -s_4' + s_5'' \\ S_w &= s_6 - s_7' \\ S_\phi &= s_8 - s_9' \end{aligned} \tag{22}$$

and the boundary term is given by

$$B_V = \left[(s_2 - s_3')\delta u + s_3 \delta u' + (s_4 - s_5')\delta v + s_5 \delta v' + s_7 \delta w + s_9 \delta\phi\right]\Big|_0^S \tag{23}$$

Kinetic Energy

The position vector of an arbitrary mass point of the blade is

$$\bar{r}_1 = \bar{R} + \Delta\bar{R} + x_3\bar{e}_{X_{B6}} + y_3\bar{e}_{Y_{B6}} \tag{24}$$

Using Eqs. (5), (10), and (A16) in Eq. (24), the expression for $\bar{r}_1$ with respect to the B3-system is

$$\bar{r}_1 = X_3\bar{e}_{X_{B3}} + Y_3\bar{e}_{Y_{B3}} + Z_3\bar{e}_{Z_{B3}} \tag{25}$$

where

$$\begin{aligned}
X_3 &= x_o \cos\theta_o + z_o \sin\theta_o + u + x_3\left(1 - \frac{\phi^2}{2} - \frac{\alpha_x^2}{2}\right) - y_3(\phi + \alpha_x\alpha_y) \\
Y_3 &= v + y_3\left(1 - \frac{\phi^2}{2} - \frac{\alpha_y^2}{2}\right) + x_3\phi \\
Z_3 &= -x_o \sin\theta_o + z_o \cos\theta_o + w - x_3(\alpha_x + \phi\alpha_y) + y_3(\phi\alpha_x - \alpha_y)
\end{aligned} \tag{26}$$

The angular velocity of the B3-system can be written as

$$\bar{\omega} = \Omega\bar{e}_{Z_I} = \Omega\bar{e}_{Z_R} = \Omega\bar{e}_{Z_{B1}} \tag{27}$$

Substituting for $\bar{e}_{Z_{B1}}$ from Eqs. (5) and (6), yields

$$\bar{\omega} = \Omega\left(\sin\theta_o \cos\gamma\bar{e}_{X_{B3}} - \sin\theta_o \sin\gamma\bar{e}_{Y_{B3}} + \cos\theta_o\,\bar{e}_{Z_{B3}}\right) \tag{28}$$

The section pitch angle γ is set to zero in the subsequent development. For this special case Eq. (28) simplifies to

$$\bar{\omega} = \Omega \sin\theta_o\bar{e}_{X_{B3}} + \Omega\cos\theta_o\bar{e}_{Z_{B3}} \tag{29}$$

The expression for the kinetic energy of the blade in terms of $\bar{r}_1$ and $\bar{\omega}$ is given by

$$T = \frac{1}{2} \int_0^S \iint_A \rho \frac{d\bar{r}_1}{dt} \cdot \frac{d\bar{r}_1}{dt} \, dx_3 dy_3 dz_3 \tag{30}$$

where

$$\frac{d\bar{r}_1}{dt} = \dot{\bar{r}}_1 + \bar{\omega} \times \bar{r}_1 \tag{31}$$

The variation of T, integrated between t_o and t_1, is given by

$$\int_{t_o}^{t_1} \delta T = \int_{t_o}^{t_1} \int_0^S \iint_A \rho \frac{d\bar{r}_1}{dt} \cdot \delta\left(\frac{d\bar{r}_1}{dt}\right) dx_3 dy_3 dz_3 dt \tag{32}$$

Substituting Eqs. (25) and (29) into Eq. (31) and the result into Eq. (32), integrating by parts over time where necessary, and then integrating over the cross section, the variation of T can be put into the form

$$\delta T = \int_0^S (k_1 \delta u + k_2 \delta u' + k_3 \delta v + k_4 \delta v' + k_5 \delta w + k_6 \delta \phi) dz_3 \tag{33}$$

where, consistent with the ordering scheme introduced earlier,

$$k_1 = - m(\ddot{u} - e\ddot{\phi}) + 2m\Omega\dot{v} \cos\theta_o + m\Omega^2 \cos^2\theta_o (x_o \cos\theta_o + u - e\phi)$$

$$- m\Omega^2 \sin\theta_o \cos\theta_o (-x_o \sin\theta_o + w - ev')$$

$$k_2 = - mv'ex_o\Omega^2 \cos\theta_o + mk_{m_1}^2\Omega^2 \sin\theta_o \cos\theta_o - m\Omega^2 e\phi x_o \sin\theta_o$$

$$k_3 = - m\ddot{v} + 2m\Omega \sin\theta_o(\dot{w} - e\dot{v}') - 2m\Omega \cos\theta_o(\dot{u} - e\dot{\phi}) + m\Omega^2(v + e)$$

$$k_4 = - me(u' - \theta_o' w)\Omega^2 x_o \cos\theta_o + me(\ddot{w} + 2\dot{v}\Omega \sin\theta_o) + m\Omega^2 ex_o \sin\theta_o$$

$$- m\Omega^2 ew \sin^2\theta_o + m\Omega^2 eu \sin\theta_o \cos\theta_o$$

(cont'd)

$$k_5 = m\Big\{k_{m_1}^2\theta_o'(u' - \theta_o'w)\Omega^2\cos^2\theta_o + v'\theta_o'\big[-e\ddot{u} + 2e\dot{v}\Omega\cos\theta_o$$

$$+ e\Omega^2\cos^2\theta_o(x_o\cos\theta_o + u) - e\Omega^2\sin\theta_o\cos\theta_o(-x_o\sin\theta_o + w)\big]$$

$$- \ddot{w} - e\ddot{\phi}(u' - \theta_o'w) - 2e\dot{\phi}(\dot{u}' - \theta_o'\dot{w}) + e\ddot{v}' - e\phi(\ddot{u}' - \theta_o'\ddot{w})$$

$$- 2\Omega\sin\theta_o(\dot{v} - e\phi\dot{\phi} - ev'\dot{v}') + \Omega^2\sin^2\theta_o\big[-x_o\sin\theta_o + w$$

$$+ e\phi(u' - \theta_o'w) - ev'\big] - \Omega^2\sin\theta_o\cos\theta_o\big[x_o\cos\theta_o + u$$

$$- ev'(u' - \theta_o'w) - e\phi\big] + \theta_o'\Big[k_{m_1}^2(\ddot{u}' - \theta_o'\ddot{w}) - 2\Omega k_{m_1}^2\dot{\phi}\sin\theta_o$$

$$- \Omega^2\sin^2\theta_o'k_{m_1}^2(u' - \theta_o'w) - \Omega^2\sin\theta_o\cos\theta_o k_{m_1}^2\Big]$$

$$- \phi\theta_o'\big] - e\ddot{w} - 2\Omega e\dot{v}\sin\theta_o + \Omega^2\sin^2\theta_o e(-x_o\sin\theta_o + w)$$

$$- \Omega^2\sin\theta_o\cos\theta_o e(x_o\cos\theta_o + u)\big]\Big\}$$

$$k_6 = me\ddot{u} - mk_m^2\ddot{\phi} - 2m\Omega e\dot{v}\cos\theta_o - m\Omega^2 ex_o\cos\theta_o - m\Omega^2 eu\cos^2\theta_o$$

$$+ m\Omega^2 ew\cos\theta_o\sin\theta_o + m\left(k_{m_2}^2 - k_{m_1}^2\right)\phi\Omega^2\cos^2\theta_o$$

$$- m\Omega^2\left(k_{m_2}^2 - k_{m_1}^2\right)v'\sin\theta_o\cos\theta_o - m\left(k_{m_2}^2 - k_{m_1}^2\right)\Omega^2\phi$$

$$- me\Omega^2 x_o(u' - \theta_o'w)\sin\theta_o - 2mk_{m_1}^2\Omega(\dot{u}' - \theta_o'\dot{w})\sin\theta_o$$

$$- me\Omega^2\phi v + 2me\Omega\phi\dot{u}\cos\theta_o + me\phi\ddot{v} - 2me\Omega\dot{w}\phi\sin\theta_o$$

$$- me\ddot{w}(u' - \theta_o'w) - 2me\Omega\dot{v}\sin\theta_o(u' - \theta_o'w) - me\Omega^2 u\sin\theta_o\cos\theta_o(u'$$

$$- \theta'w) + me\Omega^2 w\sin^2\theta_o(u' - \theta_o'w) \qquad (34)$$

The sectional properties appearing in Eq. (34) are defined as follows:

$$m = \iint \rho dx_3 dy_3 \qquad me = \iint \rho y_3 dx_3 dy_3$$

$$mk_{m_1}^2 = \iint \rho x_3^2 dx_3 dy_3 \qquad mk_{m_2}^2 = \iint \rho y_3^2 dx_3 dy_3$$

$$k_m^2 = k_{m_1}^2 + k_{m_2}^2 \qquad \iint \rho dx_3 dy_3 = 0$$

$$\iint \rho x_3 y_3 dx_3 dy_3 = 0 \tag{35}$$

Integrating Eq. (33) by parts, the resulting expression can be put in the form

$$\delta T = \int_0^S (I_u \delta u + I_v \delta v + I_w \delta w + I_\phi \delta\phi) dz_3 + B_T \tag{36}$$

where

$$I_u = k_1 - k_2'$$

$$I_v = k_3 - k_4'$$

$$I_w = k_5$$

$$I_\phi = k_6 \tag{37}$$

and the boundary term B_T is given by

$$B_T = (k_2 \delta u + k_4 \delta v)\Big|_0^S \tag{38}$$

Virtual Work of Gravity Forces

The virtual work due to gravity can be expressed in the form

$$\delta W_G = \int_0^S \iint_A \rho \bar{g} \cdot \delta \bar{r}_1 dx_3 dy_3 dz_3 \tag{39}$$

where $\bar{g}$ is the gravitational acceleration vector and is given by

$$\bar{g} = -g\bar{e}_{Z_I} \tag{40}$$

The vector $\bar{g}$ can be expressed with respect to the B3-system by substituting for $\bar{e}_{Z_I}$ from Eqs. (5) and (6). For the special case of zero section pitch angle considered in the present development, the vector $\bar{g}$ in the B3-system simplifies to

$$\bar{g} = -g\left(\bar{e}_{X_{B3}} \sin \theta_o + \bar{e}_{Z_{B3}} \cos \theta_o\right) \tag{41}$$

Taking the variation of $\bar{r}_1$ which is given in Eqs. (25) and (26), substituting the resultant expression together with Eq. (41) into Eq. (39), integrating over the cross section, and integrating the result by parts, Eq. (39) yields

$$\delta W_G = \int_0^S (G_u\delta u + G_v\delta v + G_w\delta w + G_\phi\delta\phi)dz_3 + B_G \tag{42}$$

where

$$\begin{aligned}
G_u &= -mg \sin \theta_o + (-mgev' \sin \theta_o + mge\phi \cos \theta_o)' \\
G_v &= \left[-mge(u' - \theta_o'w) \sin \theta_o' - mge \cos \theta_o\right]' \\
G_w &= -mgev'\theta_o' \sin \theta_o - mg \cos \theta_o + mge\theta_o'\phi \cos \theta_o \\
G_\phi &= mge \sin \theta_o - mge \cos \theta_o (u' - \theta_o'w)
\end{aligned} \tag{43}$$

and the boundary term B_G is given by

$$B_G = \left\{(mgev' \sin \theta_o - mge\phi \cos \theta_o)\,\delta u + \left[mge(u' - \theta_o'w) \sin \theta_o + mge \cos \theta_o\right]\delta v\right\}\Bigg|_0^S \tag{44}$$

Virtual Work of Aerodynamic Forces

The virtual work of the aerodynamic forces can be written as

$$\delta W_A = \int_0^S \left(\bar{F}_A \cdot \delta \bar{r}_1 + M_{z_{B6}} \delta\theta_{z_{B6}} \right) dz_3 \tag{45}$$

where $\delta \bar{r}_1$ is the virtual displacement of the position vector of an arbitrary point on the elastic axis, $\bar{F}_A$ is the aerodynamic force vector, $\delta\theta_{zB6}$ is the virtual rotation about the Z_{B6}-axis. Usually the aerodynamic force vector $\bar{F}_A$ is calculated in the B-6 blade axis system. Since the position vector $\bar{r}_1$ given by Eq. (25) is expressed with respect to the B3-system, the force vector $\bar{F}_A$ is transformed to the B3-system using the following relation

$$\{\bar{F}_A\}_{X_{B3}Y_{B3}Z_{B3}} = [T]^T \{\bar{F}_A\}_{X_{B6}Y_{B6}Z_{B6}} \tag{46}$$

The aerodynamic force in the Z_{B6} direction is F_{ZB6} and is a profile drag force. Following usual practice, this force component is assumed to be unimportant and is taken to be zero. Substituting Eq. (A39) into Eq. (46) and discarding terms which will lead to terms higher than second-degree in the final equations, one obtains

$$\begin{aligned} F_{x_{B3}} &= F_{x_{B6}} - F_{y_{B6}} \phi \\ F_{y_{B3}} &= F_{x_{B6}} \phi + F_{y_{B6}} \\ F_{z_{B3}} &= - F_{x_{B6}} (u' - \theta_o' w) - F_{y_{B6}} v' \end{aligned} \tag{47}$$

Taking the variation of the position vector $\bar{r}_1$ (Eq. (25)) on the elastic axis yields

$$\delta \bar{r}_1 = \delta u \bar{e}_{X_{B3}} + \delta v \bar{e}_{Y_{B3}} + \delta w \bar{e}_{Z_{B3}} \tag{48}$$

The virtual rotation $\delta\theta_{z_{B6}}$ is obtained from the expression for $k_{z_{B6}}$ given in Eq. (A40) by replacing θ_o' by $\delta\theta_o$, ϕ' by $\delta\phi$, and $(u' - \theta_o' w)'$ by $\delta(u' - \theta_o' w)$, and making use of the fact $\delta\theta_o$ equals zero, and is

$$\delta\theta_{z_{B6}} = \delta\phi + v'(\delta u' - \theta_o'\delta w) \tag{49}$$

Substituting Eqs. (47-49) into Eq. (45), the virtual work expression reduces to the form

$$\delta W_A = \int_0^S (A_u\delta u + A_v\delta v + A_w\delta w + A_\phi\delta\phi)dz_3 + B_A \tag{50}$$

where the generalized aerodynamic forces are

$$A_u = F_{x_{B6}} - F_{y_{B6}}\phi - (M_{z_{B6}}v')'$$

$$A_v = F_{y_{B6}} + F_{x_{B6}}\phi$$

$$A_w = -F_{x_{B6}}(u' - \theta_o'w) - F_{y_{B6}}v' - M_{z_{B6}}\theta_o'v'$$

$$A_\phi = M_{z_{B6}} \tag{51}$$

and the boundary term is

$$B_A = M_{z_{B6}}v'\delta u\Big|_0^S \tag{52}$$

There remains the task of expressing $F_{x_{B6}}$, $F_{y_{B6}}$, and $M_{z_{B6}}$ in terms of the dependent variables u, v, w, ϕ and the geometric angle θ_o. These expressions will be generated from two-dimensional, incompressible, quasi-steady, strip theory in which only the velocity components perpendicular to the span-wise axis (z_{B6}-axis) of the deformed blade are assumed to influence the aerodynamic loading. Account will be taken of the pulsating free-stream velocity V(t) associated with a rotating blade by employing Greenberg's extension of Theodorsen's unsteady theory (Ref. 15) for determining the aerodynamic lift and pitching moment acting on the blade. The resulting expressions are specialized to the case of quasi-steady flow by setting Theodorsen's circulation function to unity. Classical blade element momentum theory is used to calculate the steady flow induced by the rotor.

In the present application of Greenberg's theory, the airfoil is taken to be pivoted in pitch about the aerodynamic center at the quarter chord and to be executing harmonic motions in pitch ($\varepsilon(t)$) and plunge ($\dot{h}(t)$) while immersed in a pulsating airstream V(t), as shown in Fig. 4. The lift and moment acting on the elemental section

of the blade may be expressed in terms of the circulatory and non-circulatory components as

$$L = L_C + L_{NC}$$
$$M = M_C + M_{NC} \tag{53}$$

Since the blade elastic axis is assumed to be coincident with the aerodynamic center at the quarter chord, the individual components of Eq. (53) follow from Ref. (15) and can be written as

$$L_{NC} = \frac{1}{2}\rho a \frac{c^2}{4}\left(\ddot{h} + V\dot{\varepsilon} + \dot{V}\varepsilon + \frac{c}{4}\ddot{\varepsilon}\right)$$
$$L_C = \frac{1}{2}\rho acV\left(\dot{h} + V\varepsilon + \frac{c}{2}\dot{\varepsilon}\right)$$
$$M_{NC} = -\frac{1}{2}\rho ac\left(\frac{c}{4}\right)^2\left(\dot{V}\varepsilon + \ddot{h} + \frac{3c}{8}\ddot{\varepsilon}\right)$$
$$M_C = -\frac{1}{2}\rho ac\left(\frac{c}{4}\right)^2 2V\dot{\varepsilon} \tag{54}$$

In the course of arriving at the circulatory terms in Eq. (54), the quasi-steady approximation has been introduced by setting the reduced frequency to zero, in consequence of which Theodorsen's circulation function $C(k)$ assumes the value of unity.

The lifts and moments given in Eq. (54) must now be expressed in terms of U_R, U_T, and U_P, the radial, tangential, and perpendicular velocity components relative to a point on the elastic axis of the airfoil, Fig. 5. Now, the expression in parentheses for L_{NC} in Eq. (54) is the downward acceleration relative to the air of the mid-chord point of the airfoil, and the expression in parentheses for L_C is the downward velocity relative to the air of the three-quarter-chord point of the airfoil. Since U_P is the relative velocity component perpendicular to the quarter chord, the sectional lifts can also be written as

$$L_{NC} = \frac{1}{2}\rho a \frac{c^2}{4}\left(-\dot{U}_P + \frac{c}{4}\ddot{\varepsilon}\right)$$
$$L_C = \frac{1}{2}\rho acU\left(-U_P + \frac{c}{2}\dot{\varepsilon}\right) \tag{55}$$

where $V(t)$, appearing outside the parentheses of the expression for L_C in Eq. (54), has been approximated by the resultant of only the tangential and perpendicular velocity components and is given by

$$V \simeq U = \sqrt{U_P^2 + U_T^2} \tag{56}$$

As indicated in Fig. 6, the noncirculatory lift acts normal to the section chordline and the circulatory lift acts normal to the resultant velocity U. The profile drag force acts parallel to U and is given by

$$D = \frac{1}{2} \rho a c \frac{C_{d_o}}{a} U^2 \tag{57}$$

where C_{d_o} is the (constant) profile drag coefficient.

The components of the aerodynamic force in the directions of the X_{B6}, Y_{B6}, and Z_{B6} axes are given by

$$F_{x_{B6}} = L_{NC} + L_C \cos \alpha - D \sin \alpha$$

$$F_{y_{B6}} = - L_C \sin \alpha - D \cos \alpha$$

$$F_{z_{B6}} = - D_R \text{ (neglected)} \tag{58}$$

where

$$\sin \alpha = U_P/U$$

$$\cos \alpha = U_T/U \tag{59}$$

Substituting Eqs. (55), (57), and (59) into Eq. (58), leads to

$$F_{x_{B6}} = \frac{1}{2} \rho a c \left[-U_P U_T + \frac{c}{2} U_T \dot{\varepsilon} - \frac{c}{4} \dot{U}_P + \left(\frac{c}{4}\right)^2 \ddot{\varepsilon} - \frac{C_{d_o}}{a} U U_P \right]$$

$$F_{y_{B6}} = \frac{1}{2} \rho a c \left[U_P^2 - \frac{c}{2} U_P \dot{\varepsilon} - \frac{C_{d_o}}{a} U_T U \right] \tag{60}$$

The noncirculatory and circulatory moments in Eq. (54) can be written in terms of U_T, U_P, U, and ε and assume the form

$$M_{NC} = -\frac{1}{2}\rho ac\left(\frac{c}{4}\right)^2\left(-\dot{U}_P - U\dot{\varepsilon} + \frac{3c}{8}\ddot{\varepsilon}\right)$$

$$M_C = -\frac{1}{2}\rho ac\left(\frac{c}{4}\right)^2 2U\dot{\varepsilon} \tag{61}$$

from which the total pitching moment is given by

$$M = M_{z_{B6}} = -\frac{1}{2}\rho ac\left(\frac{c}{4}\right)^2\left(U\dot{\varepsilon} - \dot{U}_P + \frac{3}{8}c\ddot{\varepsilon}\right) \tag{62}$$

The necessary expressions for U_P, U_T and $\dot{\varepsilon}$ are developed in Appendix C, and are given by Eqs. (C14) and (C19). Substituting Eqs. (C14) and (C19) into Eqs. (60) and (62) and the resulting expressions into Eq. (51), one obtains the necessary expressions for generalized aerodynamic forces.

AEROELASTIC EQUATIONS

Expressions for δT, δV, and δW have been obtained in the previous sections. Substituting these expressions and their associated boundary terms into Eq. (12), there results an expression of the form

$$\int_{t_o}^{t_1}\left\{\int_0^S\left[(\,)\delta u + (\,)\delta v + (\,)\delta w + (\,)\delta\phi\right]dz_3 + B\right\}dt = 0 \tag{63}$$

For arbitrary admissible variations, δu, δv, δw, and $\delta\phi$, the four expressions in parentheses must vanish individually as must the assembly of boundary terms denoted by B. The first condition will yield the four governing nonlinear partial differential equations for u, v, w, and ϕ, and the second condition will give the associated boundary conditions at the ends of the blade. The governing equations of motion and boundary conditions are summarized below.

u equation:

$$m(\ddot{u} - e\ddot{\phi}) - 2m\Omega\cos\theta_o\dot{v} - m\Omega^2\cos\theta_o x_o - m\Omega^2\cos^2\theta_o(u - e\phi)$$

$$+ m\Omega^2\sin\theta_o\cos\theta_o(w - ev') + \Big[mk_{m_1}^2\Omega^2\sin\theta_o\cos\theta_o$$

$$- me\Omega^2 v' x_o\cos\theta_o - me\Omega^2 x_o\sin\theta_o\phi\Big]' + mg\sin\theta_o + (mg\,ev'\sin\theta_o$$

(cont'd)

$$- mge\ \phi\ \cos\ \theta_o)' + T\theta_o' - \left[T(u' - \theta_o'w)\right]' + \left\{-\theta_o'\left[EI_{y_3y_3}\varepsilon\right.\right.$$

$$\left.- 2EB_3(v'' + \phi\theta_o')\right] + EI_{y_3y_3}\left[u'' - \theta_o'w' - \theta_o''w - \frac{\theta_o'}{2}(v'^2 + \phi^2) + \phi v''\right]$$

$$\left. + T\phi e_A - E\phi I_{x_3x_3}(v'' + \phi\theta_o') + GJv'(\phi' - \theta_o'v')\right\}'' = A_u$$

v equation:

$$m\ddot{v} - m\Omega^2(v + e) - 2m\Omega\ \sin\ \theta_o(\dot{w} - e\dot{v}') + 2m\Omega\ \cos\ \theta_o(\dot{u} - e\dot{\phi})$$

$$+ \left[me\ddot{w} + 2me\dot{v}\Omega\ \sin\ \theta_o + m\Omega^2 ex_o\ \sin\ \theta_o - m\Omega^2 ew\ \sin^2\ \theta_o\right.$$

$$\left. + m\Omega^2 eu\ \sin\ \theta_o\ \cos\ \theta_o - me\Omega^2(u' - \theta_o'w)x_o\ \cos\ \theta_o\right]' + \left[mge\ \cos\ \theta_o\right.$$

$$\left. + mge(u' - \theta_o'w)\ \sin\ \theta_o\right]' - \left[Tv' + EI_{y_3y_3}\theta_o'v'(u'' - \theta_o''w - \theta_o'w')\right.$$

$$\left. + GJ(\phi' - \theta_o'v')(u'' - w'\theta_o' - \theta_o''w - \theta_o') - GJ\theta_o'v'(u'' - w'\theta_o' - \theta_o''w)\right]'$$

$$+ \left\{EI_{y_3y_3}\phi(u'' - \theta_o''w - \theta_o'w') - Te_A + 2EB_3\theta_o'(u'' - \theta_o'w' - \theta_o''w)\right.$$

$$\left. - EI_{x_3x_3}\left[-v'' + \phi(u'' - \theta_o''w - \theta_o'w' - \theta_o')\right]\right\}'' = A_v$$

w equation:

$$m\ddot{w} + me\ddot{\phi}(u' - \theta_o'w) + 2me\dot{\phi}(\dot{u}' - \theta_o'\dot{w}) - me\ddot{v}' + me\phi(\ddot{u}' - \theta_o'\ddot{w})$$

$$+ 2m\Omega\ \sin\ \theta_o(\dot{v} - e\phi\dot{\phi} - ev'\dot{v}') - m\Omega^2\ \sin^2\ \theta_o\left[-x_o\ \sin\ \theta_o + w\right.$$

$$\left. + e\phi(u' - \theta_o'w) - ev'\right] + m\Omega^2\ \sin\ \theta_o\ \cos\ \theta_o\left[x_o\ \cos\ \theta_o + u - ev'(u'\right.$$

$$\left. - \theta_o'w) - e\phi\right] - mk_{m_1}^2\theta_o'(u' - \theta_o'w)\Omega^2\ \cos^2\ \theta_o - mv'\theta_o'\left[-e\ddot{u} + 2e\dot{v}\Omega\ \cos\ \theta_o\right.$$

(cont'd)

$$+ e\Omega^2 x_o \cos\theta_o + e\Omega^2 u \cos^2\theta_o - e\Omega^2 w \sin\theta_o \cos\theta_o\Big]$$

$$+ m\theta_o' \left[-k_{m_1}^2 (\ddot{u}' - \theta_o'\ddot{w}) + 2\Omega k_{m_1}^2 \dot{\phi} \sin\theta_o + \Omega^2 \sin^2\theta_o' k_{m_1}^2 (u' - \theta_o' w)\right.$$

$$\left. + \Omega^2 k_{m_1}^2 \sin\theta_o \cos\theta_o\right] + m\phi\theta_o' \left[-e\ddot{w} - 2\Omega\dot{v}e \sin\theta_o - \Omega^2 e x_o \sin\theta_o\right.$$

$$\left. + \Omega^2 \sin^2\theta_o ew - \Omega^2 \sin\theta_o \cos\theta_o eu\right] + Tw\theta_o'^2 - T\theta_o' u'$$

$$- EI_{y_3y_3}\theta_o''(u'' - \theta_o'' w - \theta_o' w') + EI_{y_3y_3}\varepsilon\theta_o''\theta_o' - \Big[T - EI_{y_3y_3}\theta_o'(u'' - \theta_o'' w$$

$$\left. - \theta_o' w')\right]' + mg\cos\theta_o + mge \sin\theta_o v'\theta_o' - mge\theta_o'\phi \cos\theta_o = A_w$$

ϕ equation:

$$mk_m^2\ddot{\phi} - me\ddot{u} + 2me\dot{v} \cos\theta_o + me\Omega^2 x_o \cos\theta_o + me\Omega^2 u \cos^2\theta_o$$

$$- me\Omega^2 w \sin\theta_o \cos\theta_o - m\left(k_{m_2}^2 - k_{m_1}^2\right)\Omega^2\phi \cos^2\theta_o + m\Omega^2\left(k_{m_2}^2\right.$$

$$\left. - k_{m_1}^2\right)v' \sin\theta_o \cos\theta_o + m\left(k_{m_2}^2 - k_{m_1}^2\right)\Omega^2\phi + me\Omega^2 x_o \sin\theta_o(u' - \theta_o' w)$$

$$+ 2mk_{m_1}^2\Omega \sin\theta_o(\dot{u}' - \theta_o'\dot{w}) + me\Omega^2\phi v - 2me\Omega\phi\dot{u} \cos\theta_o - me\phi\ddot{v}$$

$$+ 2me\Omega\dot{w}\phi \sin\theta_o + me\ddot{w}(u' - \theta_o' w) + 2me\Omega\dot{v}(u' - \theta_o' w) \sin\theta_o$$

$$+ me\Omega^2 u(u' - \theta_o' w) \sin\theta_o \cos\theta_o - me\Omega^2 w(u' - \theta_o' w) \sin^2\theta_o$$

$$- \theta_o'(v'' + \theta_o'\phi)\left[EI_{y_3y_3}\varepsilon - EB_3(v'' + \phi\theta_o')\right] + EI_{y_3y_3}(v'' + \theta_o'\phi)\Big[u''$$

$$\left. - \theta_o'' w - \theta_o' w')\right] + EB_3(v'' + \theta_o'\phi)^2\theta_o' + E(u'' - \theta_o'' w - \theta_o' w')\left[Ae_A\varepsilon\right.$$

$$\left. - I_{x_3x_3}(v'' + \phi\theta_o') - \theta_o' B_3(u'' - \theta_o' w' - \theta_o'' w)\right] - E\theta_o'\Big\{Ae_A\varepsilon$$

(cont'd)

$$- B_3\left[\theta_o'(u'' - \theta_o''w - \theta_o'w') - \frac{\theta_o'^2\phi^2}{2} - \theta_o'\phi v''\right] + \frac{B_4\theta_o'}{2}(\phi^2 + 2\phi v'')$$

$$+ I_{x_3x_3}\left[-v'' + \phi(u'' - \theta_o''w - \theta_o'w') - \phi\theta_o'\right] + \frac{B_1}{2}\phi'(\phi' - 2\theta_o'v')\Big\}$$

$$- 2B_3E\theta_o'(u'' - \theta_o''w - \theta_o'w')^2 - E\theta_o'^2\left\{-B_3\left[(u'' - \theta_o'w' - \theta_o''w + \phi v''\right.\right.$$

$$\left.\left. + \frac{1}{2}\theta_o'(\phi^2 + v'^2)\right] - B_2\theta_o'(v'' + \theta_o'\phi)\right\} + E\theta_o'(v'' + \theta_o'\phi)\left[I_{x_3x_3}\varepsilon\right.$$

$$\left. - B_4(v'' + \theta_o'\phi)\right] - \left\{(\phi' - \theta_o'v')\left[E\left(I_{y_3y_3} + I_{x_3x_3}\right)\varepsilon - E(B_3 + B_4)(v''\right.\right.$$

$$\left.\left. + \phi\theta_o')\right] + GJ\left[\phi' + v'(u'' - w'\theta_o' - \theta_o''w - \theta_o')\right]\right\}' - mge \sin\theta_o$$

$$+ mge(u' - \theta_o'w)\cos\theta = A_\phi \tag{64}$$

where

$$\varepsilon = w' + \theta_o'u + \frac{1}{2}(u' - \theta_o'w)^2 + \frac{1}{2}v'^2$$

$$T = E\left\{A\varepsilon - I_{y_3y_3}\left[\theta_o'(u'' - \theta_o''w - \theta_o'w') + \theta_o'^2\frac{\phi^2}{2} + \theta_o'\phi v''\right]\right.$$

$$+ I_{x_3x_3}\frac{\theta_o'\phi}{2}(\theta_o'\phi + 2v'') + Ae_A\left[-v'' + \phi(u'' - \theta_o''w - \theta_o'w' - \theta_o'\right]$$

$$\left. + Ak_A^2\frac{\phi'}{2}(\phi' - 2\theta_o'v')\right\} \tag{65}$$

It should be noted that when Eq. (65) is substituted into Eq. (64) some third-degree terms in u, v, w, and ϕ and their derivatives result. Since only second-degree equations are of interest herein, the third-degree terms should be discarded. Similarly, when Eqs. (C14) and (C19) are substituted into Eqs. (60) and (62) and the resulting equations into Eq. (51) some terms higher than second-degree in u, v, w, and ϕ and their derivatives result. These terms should also be discarded in the final expressions for A_u, A_v, A_w, and A_ϕ.

The assembled collection of boundary terms denoted by B is given by

$$B = B_T - B_V + B_G + B_A \tag{66}$$

and the requirement of the vanishing of the individual variational components leads to the relations

$$\left[(s_1 - s_3') - k_2 - (mgev' \sin\theta_o - mge\phi\cos\theta_o) - M_{z_{B6}}v'\right]\delta u\Big|_0^S = 0$$

$$s_3\delta u'\Big|_0^S = 0$$

$$\left\{(s_4 - s_5') - k_4 - \left[mge(u' - \theta_o'w)\sin\theta_o + mge\cos\theta_o\right]\right\}\delta v\Big|_0^S = 0$$

$$s_5\delta v'\Big|_0^S = 0$$

$$s_7\delta w\Big|_0^S = 0$$

$$s_9\delta\phi\Big|_0^S = 0 \qquad (67)$$

METHODS OF SOLUTION

The governing equations of motion are coupled, nonlinear, partial differential equations with periodic coefficients in the dependent variables u, v, w, and ϕ. These equations have no closed-form solution and must be solved using approximate methods. Usual practice in solving these equations is to, first, eliminate the spatial dependence. This results in a set of coupled nonlinear ordinary differential equations with periodic coefficients. Various techniques can then be employed to solve these equations to determine either aeroelastic stability or response. Some of these techniques are briefly summarized below. The reader interested in detailed considerations should refer to the references cited.

The spatial dependence is usually eliminated by employing a modal approach with either assumed or calculated mode shapes (Ref. 14). An alternative procedure for eliminating the spatial dependence is by use of an integrating matrix approach (Refs. 16-18).

The nonlinear ordinary differential equations with periodic coefficients also have no closed-form solution and must be solved by approximate methods. A common practice has been to numerically integrate these equations in time to determine time histories of u, v, w, and ϕ from which aeroelastic stability and response of the system can be determined. The assessment of stability can be facilitated if fast Fourier transforms are performed on the time histories (Ref. 19).

Blade moments and shears can also be calculated using the time histories. Another method for solving the nonlinear ordinary differential equations has been the classical perturbation method for determining stability whereby the nonlinear equations are perturbed about a steady-state equilibrium position. This leads to two sets of equations: a set of nonlinear algebraic equations for the steady-state quantities and a set of nonlinear ordinary differential equations with periodic coefficients in the perturbation quantities. The nonlinear algebraic equations are solved by standard iterative techniques. Usual practice is to linearize the perturbation equations by discarding all the perturbation terms of second-degree or higher in the perturbation variables. These linearized equations are then solved using Floquet-Liapunov theory (Refs. 20-22) from which aeroelastic stability and response can be determined. Another method sometimes employed for solving the linear perturbation equations is an approximate solution based on time-averaged coefficients in conjunction with a so-called multi-blade coordinate transformation (Ref. 22). In this approach, the linear perturbation equations are first transformed into a non-rotating coordinate system by means of a multi-blade coordinate transformation where some of the periodicity in the coefficients is transformed into constant terms. A constant coefficient approximation is then made by time averaging the remaining periodic coefficients in the differential equations. Standard eigensolution techniques can then be used to determine aeroelastic stability and response.

CONCLUDING REMARKS

The second-degree nonlinear aeroelastic equations of motion for a slender, flexible, curved, and nonuniform Darrieus vertical-axis wind-turbine blade undergoing combined flatwise bending, edgewise bending, torsion, and extension have been derived using the extended Hamilton's principle. The blade aerodynamic loading is obtained from strip theory based on a quasi-steady approximation of two-dimensional, incompressible, unsteady airfoil theory. The derivation of the equations has its basis in the geometric nonlinear theory of elasticity and the resulting equations are consistent with the small deformation approximation in which the elongations and shears (and hence strains) are negligible compared to unity. A mathematical ordering scheme which is consistent with the assumption of a slender beam was adopted for the purpose of systematically discarding higher-order terms in the elastic and dynamic forces in the final equations of motion. The expressions for the generalized aerodynamic forces were left in general second-degree form from which one can obtain the aerodynamic loading to the order appropriate to any case of interest. The final equations, which have periodic coefficients, are suitable for studying vibrations, both linear and nonlinear aeroelastic stability and response. As these equations do not have closed form solutions, several approximate methods of solution have been discussed.

APPENDIX A

DERIVATION OF EXPRESSIONS FOR COORDINATE TRANSFORMATION MATRIX AND CURVATURES

The development of the second-degree nonlinear aeroelastic equations of a rotating curved blade requires second-degree nonlinear expressions for the rotational transformation matrix between the B3- and B6-systems (Fig. 3) and for the components of curvature. Since these expressions are independent of rotational speed, Ω, only a non-rotating blade is considered.

The elastic deformations translate and rotate the B3-system to the B6-system. Let the rotational transformation matrix between these two systems be $[T]$ such that

$$\begin{Bmatrix} \bar{e}_{X_{B6}} \\ \bar{e}_{Y_{B6}} \\ \bar{e}_{Z_{B6}} \end{Bmatrix} = [T] \begin{Bmatrix} \bar{e}_{X_{B3}} \\ \bar{e}_{Y_{B3}} \\ \bar{e}_{Z_{B3}} \end{Bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} \bar{e}_{X_{B3}} \\ \bar{e}_{Y_{B3}} \\ \bar{e}_{Z_{B3}} \end{Bmatrix} \tag{A1}$$

Let the expression for the curvature vector of the deformed elastic axis be

$$\bar{\omega}_{X_{B6}Y_{B6}Z_{B6}} = k_{x_{B6}} \bar{e}_{X_{B6}} + k_{y_{B6}} \bar{e}_{Y_{B6}} + k_{z_{B6}} \bar{e}_{Z_{B6}} \tag{A2}$$

The next step is to find the expressions for the components of the curvature vector in terms of the direction cosines ℓ_1, m_1, n_1, ... n_3, and the components of initial curvature. From Eq. (A1), one can write

$$\bar{e}_{Z_{B6}} = \ell_3 \bar{e}_{X_{B3}} + m_3 \bar{e}_{Y_{B3}} + n_3 \bar{e}_{Z_{B3}} \tag{A3}$$

Differentiating Eq. (A3) with respect to s leads to

$$\bar{e}'_{Z_{B6}} = a_1 \bar{e}_{X_{B3}} + b_1 \bar{e}_{Y_{B3}} + c_1 \bar{e}_{Z_{B3}} \tag{A4}$$

where

$$a_1 = \ell_3' - m_3 k_{z_{B3}} + n_3 k_{y_{B3}}$$

$$b_1 = m_3' + \ell_3 k_{z_{B3}} - n_3 k_{x_{B3}}$$

$$c_1 = n_3' + m_3 k_{x_{B3}} - \ell_3 k_{y_{B3}} \tag{A5}$$

Invoking the small strain assumption, one can write

$$\frac{d\bar{e}_{Z_{B6}}}{ds_1} = (1 + 2\varepsilon)^{-\frac{1}{2}} \frac{d\bar{e}_{Z_{B6}}}{ds} \simeq \bar{e}'_{Z_{B6}} \tag{A6}$$

The identity

$$\bar{e}'_{Z_{B6}} = \bar{\omega}_{X_{B6}Y_{B6}Z_{B6}} \times \bar{e}_{Z_{B6}} \tag{A7}$$

leads to

$$\bar{e}'_{Z_{B6}} = k_{y_{B6}} \bar{e}_{X_{B6}} - k_{x_{B6}} \bar{e}_{Y_{B6}} \tag{A8}$$

Substituting Eq. (A1) into Eq. (A4), one can also write $\bar{e}'_{Z_{B6}}$ in the form

$$\begin{aligned} \bar{e}'_{Z_{B6}} = {} & (a_1\ell_1 + b_1m_1 + c_1n_1)\bar{e}_{X_{B6}} \\ & + (a_1\ell_2 + b_1m_2 + c_1n_2)\bar{e}_{Y_{B6}} \\ & + (a_1\ell_3 + b_1m_3 + c_1n_3)\bar{e}_{Z_{B6}} \end{aligned} \tag{A9}$$

From Eqs. (A8) and (A9), the expressions for $k_{x_{B6}}$ and $k_{y_{B6}}$ are

$$k_{x_{B6}} = -(a_1\ell_2 + b_1m_2 + c_1n_2) \tag{A10}$$

$$k_{y_{B6}} = a_1\ell_1 + b_1m_1 + c_1n_1 \tag{A11}$$

The identities

$$\bar{e}'_{X_{B6}} = \bar{\omega}_{X_{B6}Y_{B6}Z_{B6}} \times \bar{e}_{X_{B6}} = - k_{y_{B6}} \bar{e}_{z_{B6}} + k_{z_{B6}} \bar{e}_{Y_{B6}}$$

$$k_{z_{B6}} = \bar{e}'_{X_{B6}} \cdot \bar{e}_{Y_{B6}} \tag{A12}$$

and the substitution of Eq. (A1) into Eq. (A12) lead to

$$k_{z_{B6}} = \left(\ell_1 \bar{e}_{X_{B3}} + m_1 \bar{e}_{Y_{B3}} + n_1 \bar{e}_{Z_{B3}}\right)' \cdot \left(\ell_2 \bar{e}_{X_{B3}} + m_2 \bar{e}_{Y_{B3}} + n_2 \bar{e}_{Z_{B3}}\right) \tag{A13}$$

Expanding Eq. (A13), using Eq. (8), and making use of the fact that each element of the orthogonal matrix $[T]$ is equal to its cofactor, Eq. (A13) simplifies to

$$k_{z_{B6}} = \ell_3 k_{x_{B3}} + m_3 k_{y_{B3}} + n_3 k_{z_{B3}} + \ell_2 \ell'_1 + m_2 m'_1 + n_2 n'_1 \tag{A14}$$

Thus far, the expressions for the components of curvature have been developed in terms of the direction cosines and the components of the initial curvature of the undeformed elastic axis. The next task is to express the direction cosines in terms of u, v, w, and ϕ. To this end, the direction cosines are first expressed in terms of the Eulerian-type angles, β, ζ, and θ which are defined as follows:

1. A positive rotation β about the Y_{B3}-axis resulting in $X_{B4}Y_{B4}Z_{B4}$.
2. A positive rotation ζ about the negative X_{B4}-axis resulting in $X_{B5}Y_{B5}Z_{B5}$.
3. A positive rotation θ about the Z_{B5}-axis resulting in $X_{B6}Y_{B6}Z_{B6}$.

The explicit form of the transformation matrix $[T]$ in terms of the Eulerian-type angles is

$$[T] = \left[\begin{array}{c|c|c} \cos\theta\cos\beta & \sin\theta\cos\zeta & -\cos\theta\sin\beta \\ -\sin\theta\sin\zeta\sin\beta & & -\sin\theta\sin\zeta\cos\beta \\ \hline -\sin\theta\cos\beta & \cos\theta\cos\zeta & \sin\theta\sin\beta \\ -\cos\theta\sin\zeta\sin\beta & & -\cos\theta\sin\zeta\cos\beta \\ \hline \cos\zeta\sin\beta & \sin\zeta & \cos\zeta\cos\beta \end{array}\right] \tag{A15}$$

The quantities β, ζ, and θ are to be expressed in terms of the variables u, v, w, ϕ, and the components of the initial curvature of the elastic axis. To this end, let $\bar{R}$ and $\bar{R}_1$ be the position vectors of the points P and P' in Fig. 3, and $\Delta\bar{R}$ be the displacement of P. Then,

$$\bar{R}_1 = \bar{R} + \Delta\bar{R} = \bar{R} + u\bar{e}_{X_{B3}} + v\bar{e}_{Y_{B3}} + w\bar{e}_{Z_{B3}} \tag{A16}$$

Differentiating the expression for $\Delta\bar{R}$ in Eq. (A16) with respect to s leads to

$$\frac{\partial(\Delta\bar{R})}{\partial s} = \alpha_x\bar{e}_{X_{B3}} + \alpha_y\bar{e}_{Y_{B3}} + \alpha_z\bar{e}_{Z_{B3}} \tag{A17}$$

where

$$\begin{aligned} \alpha_x &= u' - k_{z_{B3}}v + k_{y_{B3}}w \\ \alpha_y &= v' + k_{z_{B3}}u - k_{x_{B3}}w \\ \alpha_z &= w' - k_{y_{B3}}u + k_{x_{B3}}v \end{aligned} \tag{A18}$$

Differentiating Eq. (A16) with respect to s and substituting Eq. (A17) into the resulting expression, the expression for $\partial\bar{R}_1/\partial s$ is

$$\frac{\partial\bar{R}_1}{\partial s} = \frac{\partial\bar{R}}{\partial s} + \alpha_x\bar{e}_{X_{B3}} + \alpha_y\bar{e}_{Y_{B3}} + \alpha_z\bar{e}_{Z_{B3}} \tag{A19}$$

Now, from calculus

$$\frac{\partial\bar{R}}{\partial s} = \bar{e}_{Z_{B3}} \tag{A20}$$

Substitution of Eq. (A20) into Eq. (A19) gives

$$\frac{\partial \bar{R}_1}{\partial s} = \alpha_x \bar{e}_{X_{B3}} + \alpha_y \bar{e}_{Y_{B3}} + (1 + \alpha_z)\bar{e}_{Z_{B3}} \tag{A21}$$

The relation between the extensional component of the Green's strain tensor on the elastic axis and $\partial \bar{R}_1/\partial s$ is given by

$$\varepsilon = \frac{1}{2}\left(\frac{\partial \bar{R}_1}{\partial s} \cdot \frac{\partial \bar{R}_1}{\partial s} - 1\right) \tag{A22}$$

Substituting Eq. (A21) into Eq. (A22), the expression for ε reduces to

$$\varepsilon = \alpha_z + \frac{1}{2}\left(\alpha_z^2 + \alpha_x^2 + \alpha_y^2\right) \tag{A23}$$

The relation between ds and ds_1 can be written as

$$ds_1 = (1 + 2\varepsilon)^{1/2} ds \tag{A24}$$

Differentiating Eq. (A16) with respect to s_1, and substituting Eq. (A24) into the resultant expression, one can write

$$\frac{\partial \bar{R}_1}{\partial s_1} = \bar{e}_{Z_{B6}} = (1 + 2\varepsilon)^{-1/2}\left[(1 + \alpha_z)\bar{e}_{Z_{B3}} + \alpha_x \bar{e}_{X_{B3}} + \alpha_y \bar{e}_{Y_{B3}}\right] \tag{A25}$$

Invoking the assumption that the elongations and shears (and hence strains) are negligible compared to unity leads to

$$\bar{e}_{Z_{B6}} = \alpha_x \bar{e}_{X_{B3}} + \alpha_y \bar{e}_{Y_{B3}} + (1 + \alpha_z)\bar{e}_{Z_{B3}} \tag{A26}$$

From Eqs. (A1), (A15), and (A26), one can write

$$\begin{aligned} \ell_3 &= \cos\zeta \sin\beta = \alpha_x \\ m_3 &= \sin\zeta = \alpha_y \\ n_3 &= \cos\zeta \cos\beta = 1 + \alpha_z \end{aligned} \tag{A27}$$

The orthogonality condition between ℓ_3, m_3, and n_3

$$\ell_3^2 + m_3^2 + n_3^2 = 1 \tag{A28}$$

is satisfied before invoking the small strain assumption. This condition must also be satisfied under the small strain assumption. However, Eq. (A26) which assumes small strains leads to an interesting result. From Eqs. (A23) and (A27), one can write for small strains

$$n_3 = 1 + \alpha_z = 1 + \varepsilon - \frac{1}{2}\left(\alpha_x^2 + \alpha_y^2 + \alpha_z^2\right) \simeq 1 - \frac{1}{2}\left(\alpha_x^2 + \alpha_y^2 + \alpha_z^2\right) \tag{A29}$$

An alternative expression for n_3 in terms of α_x and α_y follows from Eqs. (A27) and (A28) and is

$$n_3 = \left[1 - \left(\alpha_x^2 + \alpha_y^2\right)\right]^{1/2} \simeq 1 - \frac{1}{2}\left(\alpha_x^2 + \alpha_y^2\right) \tag{A30}$$

Thus, Eqs. (A29) and (A30) show that there are two slightly different expressions for n_3. These two expressions must be equal. Therefore, one should impose an additional assumption that the quantity α_z^2 is negligible compared to α_x^2 and α_y^2 and/or unity when the small strain assumption is invoked. This assumption is made in the present development. Accordingly,

$$n_3 = 1 - \frac{1}{2}\left(\alpha_x^2 + \alpha_y^2\right) \tag{A31}$$

From Eqs. (A27) and (A31), the expressions for the trigonometric functions involving β and ζ are

$$\sin\beta = \frac{\alpha_x}{1 - \alpha_y^2/2} \simeq \alpha_x \qquad \cos\beta \simeq 1 - \frac{\alpha_x^2}{2}$$

$$\sin\zeta = \alpha_y \qquad \cos\zeta \simeq 1 - \frac{\alpha_y^2}{2} \tag{A32}$$

The retention of the terms $\alpha_x^2/2$ and $\alpha_y^2/2$ in Eqs. (A31) and (A32) is consistent with the fact that some of the rotations must be regarded as substantially exceeding the strain components for a slender rotating beam. This implies that the right hand side of Eq. (A23) represents a small difference of large terms. This is discussed in Ref. 13, page 203. The implications of discarding these terms while deriving the nonlinear aeroelastic equations of a helicopter rotor blade were

discussed in Ref. 11. The third Eulerian angle θ is associated with torsion of the blade and, hence, is given by

$$\theta = \phi \tag{A33}$$

The expressions for the transformation matrix $[T]$ and for the components of curvature are given in terms of the direction cosines in Eqs. (A10), (A11), and (A14), and those for direction cosines in terms of the Eulerian angles are given in Eqs. (A1) and (A15). The Eulerian angles are expressed in terms of the quantities α_x, α_y, α_z, and ϕ in Eqs. (A32) and (A33), and the expressions for α_x, α_y, and α_z are given in terms of u, v, w, $k_{x_{B3}}$, $k_{y_{B3}}$, and $k_{z_{B3}}$ in Eq. (A18). Combining Eqs. (A1), (A15), (A32), and (A33), the second-degree expression for the transformation matrix is

$$[T] = \begin{bmatrix} 1 - \frac{\phi^2}{2} - \frac{\alpha_x^2}{2} & \phi & -\alpha_x - \phi\alpha_y \\ -\phi - \alpha_x\alpha_y & 1 - \frac{\phi^2}{2} - \frac{\alpha_y^2}{2} & \phi\alpha_x - \alpha_y \\ \alpha_x & \alpha_y & 1 - \frac{\alpha_x^2}{2} - \frac{\alpha_y^2}{2} \end{bmatrix} \tag{A34}$$

Combining Eqs. (A5), (A10), (A11), (A14), and (A34), the second-degree expressions for the components of curvature are

$$k_{x_{B6}} = k_{x_{B3}} - \alpha_x k_{z_{B3}} - \alpha_y' - k_{x_{B3}} \frac{\alpha_x^2}{2} - k_{x_{B3}} \frac{\phi^2}{2} + \phi\alpha_x' - \phi\alpha_y k_{z_{B3}} + \phi k_{y_{B3}} \tag{A35}$$

$$k_{y_{B6}} = k_{y_{B3}} - k_{z_{B3}}\alpha_y + \alpha_x' - k_{y_{B3}} \frac{\left(\alpha_y^2 + \phi^2\right)}{2} + \phi\alpha_y' + k_{z_{B3}}\phi\alpha_x - \phi k_{x_{B3}} - k_{x_{B3}}\alpha_x\alpha_y \tag{A36}$$

$$k_{z_{B6}} = k_{x_{B3}}\alpha_x + k_{y_{B3}}\alpha_y + k_{z_{B3}} - k_{z_{B3}} \frac{\left(\alpha_x^2 + \alpha_y^2\right)}{2} + \phi' + \alpha_y\alpha_x' \tag{A37}$$

For the case of zero section pitch angle, Eq. (9) leads to

$$k_{y_{B3}} = -\theta_o'$$

$$k_{x_{B3}} = k_{z_{B3}} = 0 \tag{A38}$$

and Eqs. (A34) and (A35)-(A37) simplify to

$$[T] = \begin{bmatrix} 1 - \frac{\phi^2}{2} - \frac{(u' - \theta_o' w)^2}{2} & \phi & -(u' - \theta_o' w) - \phi v' \\ -\phi - (u' - \theta_o' w)v' & 1 - \frac{\phi^2}{2} - \frac{v'^2}{2} & \phi(u' - \theta_o' w) - v' \\ (u' - \theta_o' w) & v' & 1 - \frac{1}{2}\left[(u' - \theta_o' w)^2 + v'^2\right] \end{bmatrix} \tag{A39}$$

$$k_{x_{B6}} = -v'' + \phi(u' - \theta_o' w)' - \phi\theta_o'$$

$$k_{y_{B6}} = -\theta_o' + (u' - \theta_o' w)' + \frac{\theta_o'}{2}(v'^2 + \phi^2) + \phi v''$$

$$k_{z_{B6}} = -\theta_o' v' + \phi' + v'(u' - \theta_o' w)' \tag{A40}$$

APPENDIX B

DERIVATION OF STRAIN-DISPLACEMENT RELATIONS

This Appendix will develop the second-degree nonlinear expressions for the strains. To this end, let $\bar{r}_o$ and $\bar{r}_1$ be the position vectors before and after deformation of an arbitrary mass point in the cross section of the blade. These vectors can be written as

$$\bar{r}_o = \bar{R} + x_3\bar{e}_{X_{B3}} + y_3\bar{e}_{Y_{B3}}$$

$$\bar{r}_1 = \bar{R}_1 + x_3\bar{e}_{X_{B6}} + y_3\bar{e}_{Y_{B6}} \tag{B1}$$

where the effect of warping of the blade is neglected.

The differential of the vector, $\bar{r}_o$, is given by

$$d\bar{r}_o = \left[\frac{d\bar{R}}{ds} + x_3\frac{d\bar{e}_{X_{B3}}}{ds} + y_3\frac{d\bar{e}_{Y_{B3}}}{ds}\right]ds + \bar{e}_{X_{B3}}dx_3 + \bar{e}_{Y_{B3}}dy_3 \tag{B3}$$

The derivatives of the unit vectors can be written as

$$\frac{d\bar{e}_{X_{B3}}}{ds} = \bar{\omega}_{X_{B3}Y_{B3}Z_{B3}} \times \bar{e}_{X_{B3}} \tag{B4}$$

$$\frac{d\bar{e}_{Y_{B3}}}{ds} = \bar{\omega}_{X_{B3}Y_{B3}Z_{B3}} \times \bar{e}_{Y_{B3}} \tag{B5}$$

Substituting Eq. (8) into Eqs. (B4) and (B5), and the resulting equations into Eq. (B3), yield

$$d\bar{r}_o = (dx_3 - y_3k_{z_{B3}}ds)\bar{e}_{X_{B3}} + (dy_3 + x_3k_{z_{B3}}ds)\bar{e}_{Y_{B3}} + (1 - x_3k_{y_{B3}} + y_3k_{x_{B3}})ds\bar{e}_{Z_{B3}} \tag{B6}$$

The same procedure leads to the following expression for $d\bar{r}_1$

$$d\bar{r}_1 = \left(dx_3 - y_3 k_{z_{B6}} ds_1\right)\bar{e}_{X_{B6}} + \left(dy_3 + x_3 k_{z_{B6}} ds_1\right)\bar{e}_{Y_{B6}} + \left(1 - x_3 k_{y_{B6}} + y_3 k_{x_{B6}}\right) ds_1 \bar{e}_{Z_{B6}} \tag{B7}$$

The Lagrangian strain tensor $[\varepsilon_{ij}]$ is defined as follows:

$$d\bar{r}_1 \cdot d\bar{r}_1 - d\bar{r}_o \cdot d\bar{r}_o = 2\lfloor dx_3 \; dy_3 \; dz_3 \rfloor [\varepsilon_{ij}] \begin{Bmatrix} dx_3 \\ dy_3 \\ dz_3 \end{Bmatrix} \tag{B8}$$

Combining Eqs. (A24), (A34), (B6), (B7) and (B8) and collecting terms, the expressions for the three strain components of interest become

$$\varepsilon_{z_3 z_3} = \varepsilon - x_3\left(k_{y_{B6}} - k_{y_{B3}}\right) + y_3\left(k_{x_{B6}} - k_{x_{B3}}\right) + \frac{x_3^2}{2}\left(k_{y_{B6}}^2 - k_{y_{B3}}^2\right) + \frac{y_3^2}{2}\left(k_{x_{B6}}^2 - k_{x_{B3}}^2\right) + \frac{\left(x_3^2 + y_3^2\right)}{2}\left(k_{z_{B6}}^2 - k_{z_{B3}}^2\right) - x_3 y_3\left(k_{y_{B6}} k_{x_{B6}} - k_{x_{B3}} k_{y_{B3}}\right) \tag{B9}$$

$$\varepsilon_{z_3 x_3} = -\frac{y_3}{2}\left(k_{z_{B6}} - k_{z_{B3}}\right) \tag{B10}$$

$$\varepsilon_{z_3 y_3} = \frac{x_3}{2}\left(k_{z_{B6}} - k_{z_{B3}}\right) \tag{B11}$$

Substituting Eqs. (A23), (A35), (A36), and (A37) into Eqs. (B9), (B10), and (B11), the second-degree expressions for the strain components reduce to

$$\varepsilon_{z_3 z_3} = \alpha_z + \frac{1}{2}\left(\alpha_x^2 + \alpha_y^2\right) + \frac{x_3^2}{2}\Big[k_{x_{B3}}^2\phi^2 - 2k_{y_{B3}}k_{z_{B3}}\alpha_y + 2k_{y_{B3}}\alpha_x' - k_{y_{B3}}^2\phi^2 + 2k_{y_{B3}}\phi\alpha_y' + 2k_{y_{B3}}k_{z_{B3}}\phi\alpha_x - 2k_{x_{B3}}k_{y_{B3}}\phi + 2k_{x_{B3}}k_{z_{B3}}\alpha_y\phi - 2k_{x_{B3}}\phi\alpha_x'\Big] + \frac{1}{2}y_3^2\Big[k_{y_{B3}}^2\phi^2 - 2k_{x_{B3}}k_{z_{B3}}\alpha_x - 2k_{x_{B3}}\alpha_y' - k_{x_{B3}}^2\phi^2$$

$$+ 2k_{x_{B3}}k_{y_{B3}}\phi - 2\phi\alpha_x k_{y_{B3}}k_{z_{B3}} - 2k_{x_{B3}}k_{z_{B3}}\phi\alpha_y + 2k_{x_{B3}}\phi\alpha_x' - 2k_{y_{B3}}\phi\alpha_y'\Big]$$

$$- x_3\Big[- k_{z_{B3}}\alpha_y + \alpha_x' - \frac{k_{yB3}}{2}\left(\alpha_y^2 + \phi^2\right) + \phi\alpha_y' + k_{z_{B3}}\phi\alpha_x - \phi k_{x_{B3}}$$

$$- k_{x_{B3}}\alpha_x\alpha_y\Big] + y_3\Big[- k_{z_{B3}}\alpha_x - \alpha_y' - \frac{\alpha_x^2}{2}\,k_{x_{B3}} - \frac{\phi^2}{2}k_{x_{B3}} + \phi\alpha_x'$$

$$- \phi\alpha_y k_{z_{B3}} + \phi k_{y_{B3}}\Big] - x_3y_3\Big[- k_{x_{B3}}k_{z_{B3}}\alpha_y + k_{x_{B3}}\alpha_x' - k_{x_{B3}}k_{y_{B3}}\frac{\phi^2}{2}$$

$$+ k_{x_{B3}}\phi\alpha_y' + k_{x_{B3}}k_{z_{B3}}\phi\alpha_x - k_{x_{B3}}^2\phi - \alpha_x k_{y_{B3}}k_{z_{B3}} + k_{x_{B3}}k_{z_{B3}}\alpha_x\phi$$

$$- k_{y_{B3}}\alpha_y' + k_{x_{B3}}\phi\alpha_y' - k_{x_{B3}}k_{y_{B3}}\frac{\phi^2}{2} + k_{y_{B3}}\phi\alpha_x' - k_{y_{B3}}k_{z_{B3}}\phi\alpha_y + k_{y_{B3}}^2\phi$$

$$+ \phi k_{y_{B3}}\alpha_x' - k_{x_{B3}}k_{y_{B3}}\phi^2\Big] + \frac{\left(x_3^2 + y_3^2\right)}{2}\Big[\phi'^2 + 2k_{x_{B3}}k_{z_{B3}}\alpha_x$$

$$+ 2k_{x_{B3}}\alpha_x\phi' + 2k_{y_{B3}}k_{z_{B3}}\alpha_y + 2k_{y_{B3}}\alpha_y\phi' + 2k_{y_{B3}}\phi'\Big] \qquad \text{(B12)}$$

$$\varepsilon_{z_3x_3} = -\frac{y_3}{2}\left[k_{x_{B3}}\alpha_x + k_{y_{B3}}\alpha_y - k_{z_{B3}}\frac{\left(\alpha_x^2 + \alpha_y^2\right)}{2} + \phi' + \alpha_y\alpha_x'\right]$$

(B13)

$$\varepsilon_{z_3y_3} = \frac{x_3}{2}\left[k_{x_{B3}}\alpha_x + k_{y_{B3}}\alpha_y - k_{z_{B3}}\frac{\left(\alpha_x^2 + \alpha_y^2\right)}{2} + \phi' + \alpha_y\alpha_x'\right]$$

(B14)

It should be pointed out that in arriving at the expressions given in Eqs. (B12) to (B14) several terms have been discarded based either on

considerations related to the small strain assumption or on considerations related to the approximations which can be made because of the assumed slenderness of the blade.

For the case of zero section pitch angle, the expressions for the initial curvature components are given by Eq. (A38). Substituting Eqs. (A18) and (A38) into Eqs. (B12)-(B14), the expressions for the strains are

$$\varepsilon_{z_3z_3} = w' + \theta_o'u + \frac{1}{2}\left(u'^2 + \theta_o'^2w^2 - 2\theta_o'u'w + v'^2\right)$$

$$+ \frac{x_3^2}{2}\left[-2\theta_o'(u'' - \theta_o''w - \theta_o'w') - \theta_o'^2\phi^2 - 2\theta_o'\phi v''\right]$$

$$+ \frac{y_3^2}{2}\left(\theta_o'^2\phi^2 + 2\theta_o'\phi v''\right) - x_3\left[u'' - \theta_o''w - \theta_o'w' + \frac{\theta_o'}{2}\left(v'^2 + \phi^2\right) + \phi v''\right]$$

$$+ y_3\left[-v'' + \phi(u'' - \theta_o''w - \theta_o'w') - \phi\theta_o'\right] - x_3y_3\left[\theta_o'v''\right.$$

$$\left. - 2\theta_o'\phi(u'' - \theta_o''w - \theta_o'w') + \theta_o'^2\phi\right] + \frac{\left(x_3^2 + y_3^2\right)}{2}\left(\phi' - 2\theta_o'\phi'v'\right) \tag{B15}$$

$$\varepsilon_{z_3x_3} = -\frac{y_3}{2}\left[\phi' + v'(u'' - w_o'\theta' - \theta_o''w) - \theta_o'v'\right] \tag{B16}$$

$$\varepsilon_{z_3y_3} = \frac{x_3}{2}\left[\phi' + v'(u'' - w'\theta_o' - \theta_o''w) - \theta_o'v'\right] \tag{B17}$$

APPENDIX C

DERIVATION OF THE VELOCITY COMPONENTS OF A BLADE ELEMENT

The resultant velocities of a point on the elastic axis of the blade in the deformed and the undeformed coordinate systems are related according to

$$\bar{V}_{X_{B6}Y_{B6}Z_{B6}} = [T]\bar{V}_{X_{B3}Y_{B3}Z_{B3}} \tag{C1}$$

where, from Fig. 5,

$$\bar{V}_{X_{B6}Y_{B6}Z_{B6}} = -U_P\bar{e}_{X_{B6}} - U_T\bar{e}_{Y_{B6}} + U_R\bar{e}_{Z_{B6}} \tag{C2}$$

and $[T]$ is given in Appendix A. The total relative velocity (aerodynamic + dynamic) of a point on the elastic axis is given by

$$\bar{V}_{X_{B3}Y_{B3}Z_{B3}} = \left[\bar{V}_a - \frac{d\bar{r}_1}{dt}\right]_{X_{B3}Y_{B3}Z_{B3}} \tag{C3}$$

Neglecting wind shear and gusts in the wind, the aerodynamic velocity, $\bar{V}_a$, consists of two parts: (1) the free-stream velocity V_∞; and (2) the induced velocity v_i. As shown in Fig. 1, both V_∞ and $-v_i$ are parallel to the X_I-axis. Thus,

$$(\bar{V}_a)_{X_IY_IZ_I} = (V_\infty - v_i)\bar{e}_{X_I} \tag{C4}$$

Defining two nondimensional parameters

$$\mu = V_\infty/\Omega R \tag{C5}$$

and

$$\mu_i = v_i/\Omega R \tag{C6}$$

Eq. (C4) reduces to

$$(\bar{V}_a)_{X_IY_IZ_I} = \Omega R(\mu - \mu_i)\bar{e}_{X_I} \tag{C7}$$

Substituting for $\bar{e}_{X_I}$ in the above equation from Eqs. (2), (5), and (6), Eq. (C7) reduces to

$$(\bar{V}_a)_{X_{B3}Y_{B3}Z_{B3}} = (\mu - \mu_i)\Omega R \left[(\cos\theta_o \cos\psi \cos\gamma - \sin\gamma \sin\psi)\bar{e}_{X_{B3}}\right.$$

$$\left. - (\cos\theta_o \cos\psi \sin\gamma + \sin\psi \cos\gamma)\bar{e}_{Y_{B3}} - \sin\theta_o \cos\psi\bar{e}_{Z_{B3}}\right] \qquad (C8)$$

For zero section pitch angle, Eq. (C8) simplifies to

$$(\bar{V}_a)_{X_{B3}Y_{B3}Z_{B3}} = (\mu - \mu_i)\Omega R\left[\cos\theta_o \cos\psi\bar{e}_{X_{B3}} - \sin\psi\ \bar{e}_{Y_{B3}}\right.$$

$$\left. - \sin\theta_o \cos\psi\bar{e}_{Z_{B3}}\right] \qquad (C9)$$

For zero section pitch angle, the position vector of a point on the quarter-chord point from Eq. (25) is

$$\bar{r}_1{}_{x_3=y_3=0} = (x_o \cos\theta_o + z_o \sin\theta_o + u)\bar{e}_{X_{B3}} + v\bar{e}_{Y_{B3}}$$

$$+ (-x_o \sin\theta_o + z_o \cos\theta_o + w)\bar{e}_{Z_{B3}} \qquad (C10)$$

The angular velocity of the B3-system from Eq. (29) is

$$\bar{\omega} = \Omega \sin\theta_o\bar{e}_{Y_{B3}} + \Omega \cos\theta_o\bar{e}_{Z_{B3}} \qquad (C11)$$

The dynamic velocity of a point on the quarter-chord point, from Eqs. (C10) and (C11), is

$$\frac{d\bar{r}_1}{dt} = (\dot{u} - v\Omega \cos\theta_o)\bar{e}_{X_{B3}} + \left[\dot{v} + (x_o \sin\theta_o - z_o \cos\theta_o - w)\Omega \sin\theta_o\right.$$

$$\left. + (x_o \cos\theta_o + z_o \sin\theta_o + u)\Omega \cos\theta_o\right]\bar{e}_{Y_{B3}} + (\dot{w} + v\Omega \sin\theta_o)\bar{e}_{Z_{B3}}$$

(C12)

Substituting Eqs. (C9) and (C12) into Eq. (C3) yields

$$\bar{V}_{X_{B3}Y_{B3}Z_{B3}} = \left[(\mu - \mu_i)\,\Omega R \cos\theta_o \cos\psi - (\dot{u} - v\Omega\cos\theta_o)\right]\bar{e}_{X_{B3}}$$

$$+ \left[-(\mu - \mu_i)\Omega R \sin\psi - \dot{v} - (x_o\cos\theta_o + z_o\sin\theta_o\right.$$

$$\left. + u)\Omega\cos\theta_o - (x_o\sin\theta_o - z_o\cos\theta_o - w)\Omega\sin\theta_o\right]\bar{e}_{Y_{B3}}$$

$$- \left[(\mu - \mu_i)\sin\theta_o\cos\psi + (\dot{w} + v\Omega)\sin\theta_o\right]\bar{e}_{Z_{B3}} \qquad (C13)$$

Substituting Eqs. (A39) and (C13) into Eq. (C1) and using Eq. (C2), the second-degree expressions for U_P and U_T in terms of u, v, w, and ϕ and their time derivatives, are

$$U_P = -(\mu - \mu_i)\Omega R\cos\theta_o\cos\psi\left[1 - \frac{\phi^2}{2} - \frac{(u' - \theta_o' w)^2}{2}\right]$$

$$+ (\dot{u} - v\Omega\cos\theta_o) + \phi(\mu - \mu_i)\,\Omega R\sin\psi + \dot{v}\phi + x_o\Omega\phi - w\phi\Omega\sin\theta_o$$

$$+ \phi u\Omega\cos\theta_o - \left[(u' - \theta_o' w) + \phi v'\right](\mu - \mu_i)\Omega R\sin\theta_o\cos\psi$$

$$- (u' - \theta_o' w)(\dot{w} + v\Omega\sin\theta_o)$$

$$U_T = \left[\phi + v'(u' - \theta_o' w)\right](\mu - \mu_i)\Omega R\cos\theta_o\cos\psi - \phi(\dot{u} - v\Omega\cos\theta_o)$$

$$+ \left(1 - \frac{\phi^2}{2} - \frac{v'^2}{2}\right)(\mu - \mu_i)\Omega R\sin\psi + \dot{v} + \left(1 - \frac{\phi^2}{2} - \frac{v'^2}{2}\right)x_o\Omega$$

$$- w\Omega\sin\theta_o + u\Omega\cos\theta_o + (\mu - \mu_i)\Omega R\sin\theta_o\cos\psi\left[-v'\right.$$

$$\left. + \phi(u' - \theta_o' w)\right] - v'(\dot{w} + v\Omega)\sin\theta_o \qquad (C14)$$

The quantity $\dot{\epsilon}$ appearing in Eqs. (60) and (62) is the angular velocity of the blade section about the local negative Z_{B6}-axis and, consistent with the present notation, can be written as $-\dot{\epsilon}_{Z_{B6}}$. The

expression for $\dot{\varepsilon}_{z_{B6}}$ can be regarded as composed of two parts: the first part arising from the angular velocity of the shaft in space; the second part arising from the angular velocity associated with the elastic deformations. The first part is due to Ω and can be obtained for zero section pitch angle from the relation

$$\begin{Bmatrix} \dot{\varepsilon}_{x_{B6}} \\ \dot{\varepsilon}_{y_{B6}} \\ \dot{\varepsilon}_{z_{B6}} \end{Bmatrix} = [T] \begin{Bmatrix} \Omega \sin \theta_o \\ 0 \\ \Omega \cos \theta_o \end{Bmatrix} \tag{C15}$$

Substituting Eq. (A39) into Eq. (C15), the first part of $\dot{\varepsilon}_{z_{B6}}$ is

$$\left(\dot{\varepsilon}_{z_{B6}}\right)_{\Omega} = (u' - \theta_o' w)\Omega \sin \theta_o + \left[1 - \frac{(u' - \theta_o' w)^2}{2} - \frac{v'^2}{2}\right]\Omega \cos \theta_o \tag{C16}$$

The second part is due to elastic deformation and is obtained by replacing ϕ' by $\dot{\phi}$, $(u' - \theta_o' w)'$ by $\frac{\partial}{\partial t}(u' - \theta_o' w)$, and θ_o' by $\dot{\theta}_o$ (in the first term only) in the expression for $k_{z_{B6}}$ given in Eq. (A40). Since $\dot{\theta}_o$ is zero, the second part simplifies to

$$\left(\dot{\varepsilon}_{z_{B6}}\right)_{\text{deformation}} = \dot{\phi} + v'(\dot{u}' - \theta_o' \dot{w}) \tag{C17}$$

Combining Eqs. (C16) and (C17), the total sectional pitching velocity of the airfoil is

$$\dot{\varepsilon}\left(= -\dot{\varepsilon}_{z_{B6}}\right) = -\left\{(u' - \theta_o' w)\Omega \sin \theta_o + \left[1 - \frac{(u' - \theta_o' w)^2}{2} - \frac{v'^2}{2}\right]\Omega \cos \theta_o + \dot{\phi} + v'(\dot{u}' - \theta_o' \dot{w})\right\} \tag{C19}$$

Thus far, the expressions for U_P, U_T and $\dot{\varepsilon}$ have been developed in terms of the quantities u, v, w, ϕ, and their derivatives, the free-stream velocity V_∞, the rotational speed Ω, the geometric properties of the blade, and the induced velocity v_i. Now, an explicit expression for the induced velocity will be developed.

The induced velocity v_i is determined from momentum theory in which the induced velocity at the rotor is one half its value in the wake. With this assumption, the thrust of the VAWT is

$$T = 2\rho A_p (V_\infty - v_i) v_i \tag{C20}$$

where A_p is the projected area of the rotor in the vertical plane. Another expression for the thrust will be developed in terms of the elemental forces acting on a blade section.

Using Eqs. (2), (5), (47), and (60), the elemental force acting on a blade section in the X_I directions is

$$F_{x_I} = \left(F_{x_{B6}} - F_{y_{B6}}\phi\right) \cos\psi \cos\theta_o - \left(F_{x_{B6}}\phi + F_{y_{B6}}\right) \sin\psi$$

$$+ \left[F_{x_{B6}}(u' - \theta_o' w) + F_{y_{B6}} v'\right] \sin\theta_o \cos\psi \tag{C21}$$

The expression for the average thrust can be written as

$$T = \frac{b}{2\pi} \int_{-H/2}^{H/2} \int_0^{2\pi} F_{x_I} dz_o d\psi \tag{C22}$$

Equating Eqs. (C20) and (C22), one obtains an integral equation for v_i which can be solved by an iterative procedure for a given V_∞.

REFERENCES

1. South, P.; and Rangi, R.: The Performance and Economics of the Vertical-Axis Wind Turbine Developed at the National Research Council, Ottawa, Canada. Presented at the 1973 Annual Meeting of the Pacific Northwest Region of the American Society of Agricultural Engineers, Calgary, Alberta, October 10-12, 1973, Paper No. PNW-73-303. (See also Agricultural Engineer, Feb. 1974, pp. 14-16.)

2. Blackwell, B. F.; et al.: Engineering Development Status of the Darrieus Wind Turbine. J. Energy, vol. 1, no. 1, Jan.-Feb. 1977, pp. 50-64.

3. Darrieus, G. J. M.: Turbine Having Its Rotating Shaft Transverse to the Flow of Current. U.S. Patent 1,835018, Dec. 1931.

4. Blackwell, B. F.; and Sheldahl, R. E.: Selected Wind Tunnel Test Results for the Darrieus Wind Turbine. J. Energy, vol. 1, Nov.-Dec. 1977, pp. 382-386.

5. Templin, R. J.; and South, P.: Canadian Wind Energy Program. Proceedings of Vertical-Axis Wind Turbine Technology Workshop, SAND-76-5586/CONF-7605511, 1976, pp. I57-I88.

6. Ham, N. D.: Aeroelastic Analysis of the Troposkien-Type Wind Turbine. Proceedings of the Vertical-Axis Wind Turbine Technology Workshop, SAND76-5568/CONF-7605511, 1976, pp. II185-II204.

7. Ashley, H.: Use of Asymptotic Methods in Vibration Analysis. Proceedings of the Wind Turbine Structural Dynamics Workshop, NASA CP-2034/CONF-771148, 1977, pp. 39-52.

8. Ottens, H. H.; and Zwaan, R. J.: Investigations on the Aeroelastic Stability of Large Wind Rotor. International Symposium on Wind Energy Systems, 2nd, Vol. 1, H. S. Stephens, ed., British Hydromechanics Research Association (Cranfield, England), 1978, pp. C3-31 to C3-48.

9. Vollan, A. J.: The Aeroelastic Behavior of Large Darrieus-Type Wind Energy Converters Derived from the Behaviour of a 5M Rotor. International Symposium on Wind Energy Systems, 2nd, Vol. 1, H. S. Stephens, ed., British Hydromechanics Research Association (Cranfield, England), 1978, pp. C5-67 to C5-88.

10. Kvaternik, R. G.; and Kaza, K. R. V.: Nonlinear Curvature Expressions for Combined Flapwise Bending, Chordwise Bending, Torsion, and Extension of Twisted Rotor Blades. NASA TM X-73997, 1976.

11. Kaza, K. R. V.; and Kvaternik, R. G.: Nonlinear Aeroelastic Equations for Combined Flapwise Bending, Chordwise Bending, Torsion and Extension of Twisted Nonuniform Rotor Blades in Forward Flight. NASA TM-74059, 1977.

12. Kaza, K. R. V.: Aeroelastic Stability of Wind Turbine Blades. Wind Turbine Structural Dynamics, NASA CP-2034/CONF-771148, 1977.

13. Novozhilov, V. V.: Foundations of the Nonlinear Theory of Elasticity. Graylock Press, Rochester, N.Y., 1953.

14. Bisplinghoff, R. L.; and Ashley, H.: Principles of Aeroelasticity. John Wiley and Sons, Inc., 1962.

15. Greenberg, J. M.: Airfoil in Sinusoidal Motion in a Pulsating Stream. NACA TN-1326, 1947.

16. Hunter, W. F.: The Integrating-Matrix Method for Determining the Natural Vibration Characteristics of Propeller Blades. NASA TN D-6064, 1970.

17. White, W. F., Jr.; and Malatino, R. E.: A Numerical Method for Determining the Natural Vibration Characteristics of Rotating Nonuniform Cantilever Blades. NASA TM X-72751, 1975.

18. Kvaternik, R. G.; White, W. F., Jr.; and Kaza, K. R. V.: Nonlinear Flap-Lag-Axial Equations of a Rotating Beam with Arbitrary Precone Angle. (AIAA Paper 78-491) AIAA/ASME Structures, Structural Dynamics and Materials, 17th Conference, Apr. 1978, pp. 214-227.

19. Cooley, J. W.; Lewis, P. A. W.; and Welch, P. D.: Application of the Fast Fourier Transform to Computation of Fourier Integrals, Fourier Series, and Convolution Integrals. IEEE Trans. Audio Electroacoustics, vol. AU-15, no. 2, June 1967, pp. 79-84.

20. Hohenemser, K. H.; and Yin, S.-K.: Some Applications of the Method of Multi-blade coordinates. J. Am. Helicopter Soc., vol. 17, no. 3, July 1972, pp. 3-12.

21. Hammond, C. E.: An Application of Floquet Theory to Prediction of Mechanical Instability. J. Am. Helicopter Soc., vol. 19, no. 4, Oct. 1974, pp. 14-23.

22. Kaza, K. R. V.; and Hammond, C. E.: An Investigation of Flap-Lag Stability of Wind Turbine Rotors in the Presence of Velocity Gradients and Helicopter Rotors in Forward Flight. AIAA/ASME/SAE Structures, Structural Dynamics and Materials Conference, 17th, American Institute of Aeronautics and Astronautics, 1976, pp. 421-431.

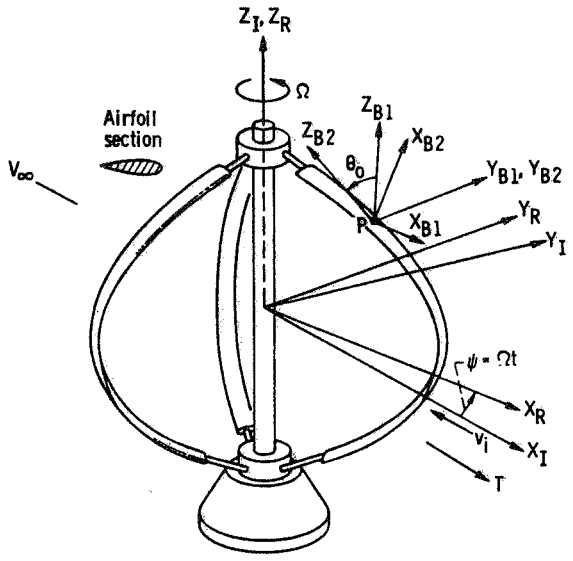

Figure 1. - Vertical-axis wind turbine and coordinate systems.

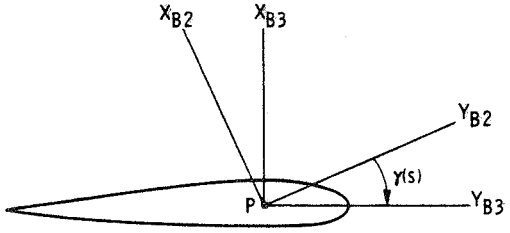

Figure 2. - Coordinate systems of blade cross section.

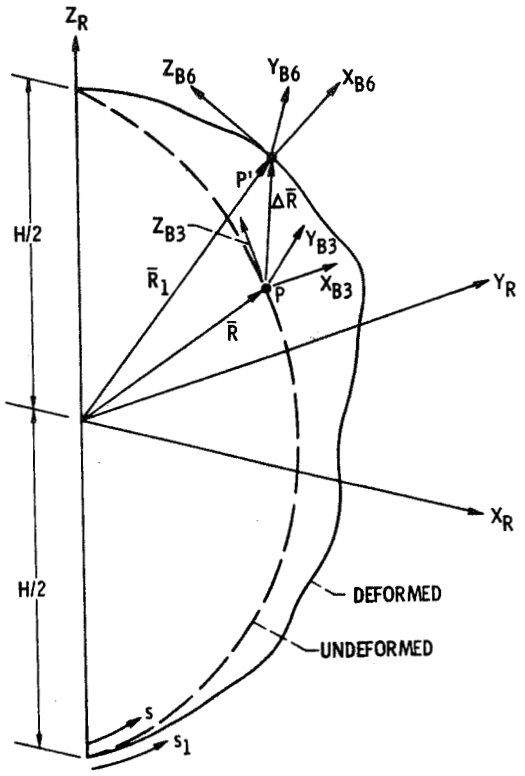

Figure 3. - Elastic axis of blade before and after deformation and coordinate systems.

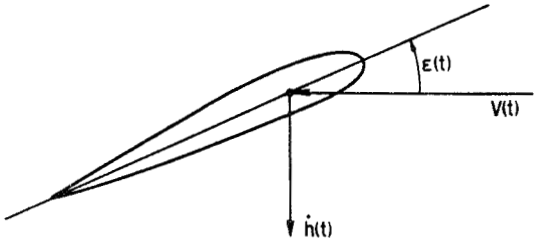

Figure 4. - Cross section of blade in general unsteady motion.

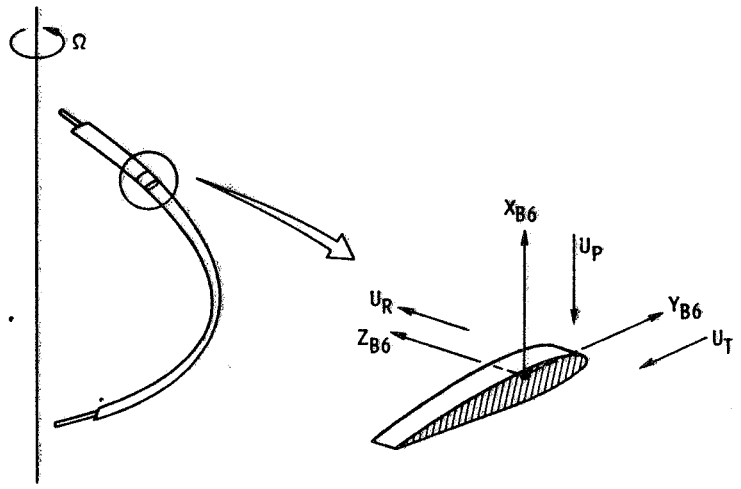

Figure 5. - Relative velocity components at blade cross section.

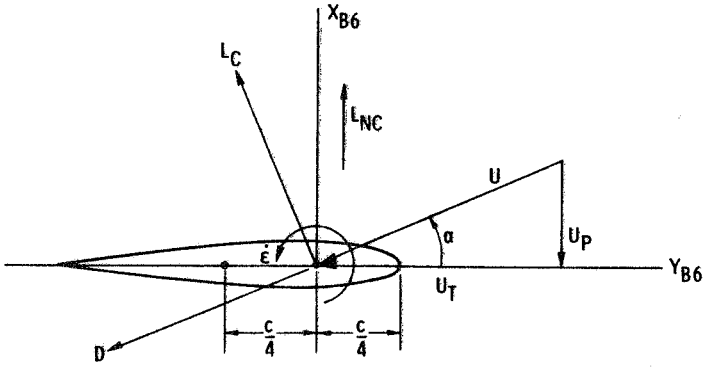

Figure 6. - Blade section inflow geometry and aerodynamic force components.

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TM-79295		

4. Title and Subtitle	5. Report Date
AEROELASTIC EQUATIONS OF MOTION OF A DARRIEUS VERTICAL-AXIS WIND-TURBINE BLADE	December 1979
	6. Performing Organization Code
7. Author(s)	8. Performing Organization Report No.
Krishna Rao V. Kaza and Raymond G. Kvaternik	E-238
	10. Work Unit No.
9. Performing Organization Name and Address	
National Aeronautics and Space Administration Lewis Research Center Cleveland, Ohio 44135	11. Contract or Grant No. NSG-3139
	13. Type of Report and Period Covered
12. Sponsoring Agency Name and Address	Technical Memorandum
U.S. Department of Energy Distributed Solar Technology Division Washington, D.C. 20545	14. Sponsoring Agency ~~Code~~ Report No. DOE/NASA/1028-79/25

15. Supplementary Notes Krishna Rao V. Kaza, associated with University of Toledo, presently assigned to Lewis Research Center; Raymond G. Kvaternik, Langley Research Center, Hampton, Va. Prepared under Interagency Agreement E(48-26) 1028. Project Manager, Christos C. Chamis, Materials and Structures Division, NASA Lewis Research Center.

16. Abstract

The second-degree nonlinear aeroelastic equations of motion for a slender, flexible, nonuniform, Darrieus vertical-axis wind-turbine blade which is undergoing combined flatwise bending, edgewise bending, torsion, and extension are developed using Hamilton's principle. The blade aerodynamic loading is obtained from strip theory based on a quasi-steady approximation of two-dimensional incompressible unsteady airfoil theory. The derivation of the equations has its basis in the geometric nonlinear theory of elasticity and the resulting equations are consistent with the small deformation approximation in which the elongations and shears (and hence strains) are negligible compared to unity. These equations are suitable for studying vibrations, both static and dynamic aeroelastic instabilities, and dynamic response. Several possible methods of solution of the equations, which have periodic coefficients, are discussed.

17. Key Words (Suggested by Author(s))	18. Distribution Statement
Darrieus wind turbine; Rotor blade; Aeroelasticity; Curved beam; Nonlinear beam curvatures; Structural dynamics	Unclassified - unlimited STAR Category 39 DOE Category UC-60

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified		

* For sale by the National Technical Information Service, Springfield, Virginia 22161

5 MAY 80 A

John Reich
352-335-106

UNITED STATES DEPARTMENT OF ENERGY
P.O. BOX 62
OAK RIDGE, TENNESSEE 37830

OFFICIAL BUSINESS
PENALTY FOR PRIVATE USE $300

POSTAGE AND FEES PAID
UNITED STATES
DEPARTMENT OF ENERGY

FS- 1

MCDONNELL DOUGLAS AIRCRAFT COMPANY
ATTN ENGINEERING LIB
DEPARTMENT 022
BOX 516
ST. LOUIS, MO 63166

MCDONNELL DOUGLAS
RESEARCH

1100 24 MAR RECD