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NUMERICAL INSTABILITY AND STABILIZATION OF THE KINEMATIC DIFFERENTIAL EQUATION USING QUATERNIONS

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16. Abstract

Discussion of difficulty involved in describing the position of a rigid body or gyroscope with given rotating vector, including examination of the possibility of using quaternions for this purpose.

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NUMERICAL INSTABILITY AND STABILIZATION OF THE KINEMATIC DIFFERENTIAL EQUATION USING QUATERNIONS

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1. Introduction

/T 118*

Because of the singularity of the Eulerian angles (frame barrier) other coordinates often must be used to describe the position of a rigid body or gyroscope with a given rotating vector $\omega = \omega(t)$ respectively. One possibility is using rotating quaternions. These satisfy the linear differential equation (DGL)

$$\dot{q} = \frac{1}{2}\Omega q \qquad \text{oder} \qquad \begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -w_1 & -w_2 & -w_3 \\ w_1 & 0 & w_3 & -w_2 \\ w_2 & -w_3 & 0 & w_1 \\ w_3 & w_2 & -w_1 & 0 \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{1}$$

The w_i are here the normalized coordinates of the rotating vector w in the rigid system. The present text is concerned with this kinematic differential equation using quaternions (KQD).

The four coordinates of q must fulfill the requirement

$$N := \|q\|^2 - 1 = q_0^2 + q_1^2 + q_2^2 + q_3^2 - 1 = 0 \tag{2}$$

From KQD follows $\dot{N} = 0$, thus N = constant is a first integral of KQD[1]. It is physically stabilized on the boundary.

2. Numerical Instability of the Normalized Integral

An unsymmetrical force-free gyroscope was employed as an example. Its motion is represented exactly by the Eulerian angles ψ, ϑ, φ [1]. This concerns a periodic nutation in ϑ of the

* Numbers in the margin indicate pagination in the foreign text.

[1] The designation "integral" is also applied in the following for N.

figure axis. The Renge-Kutta procedure of the fourth order with constant step H was employed as integration procedure.

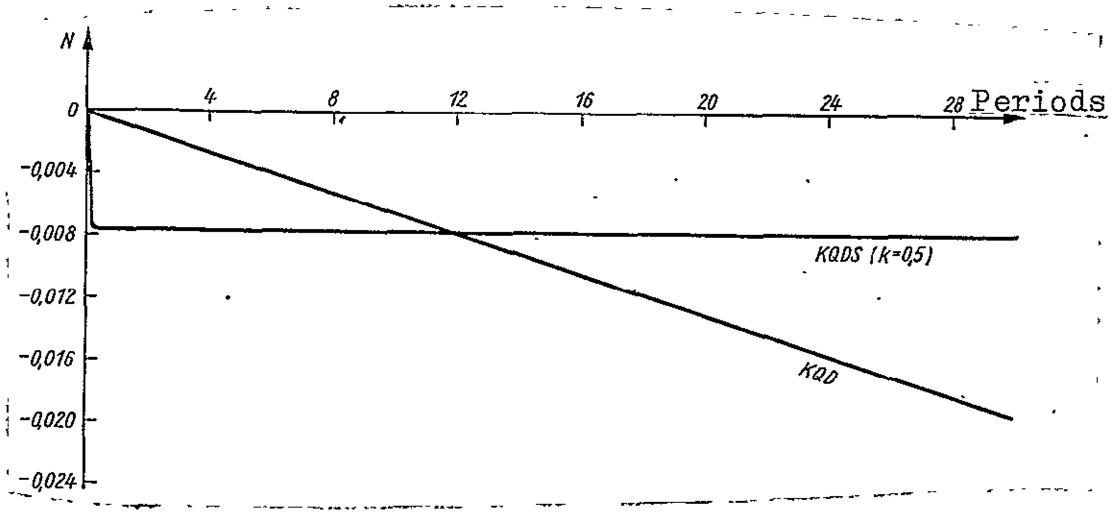

Figure 1: Normalized Error N(t)

The normalized integral deviated in a nearly linear manner from the desired value. The course (designated by KQD) is given as a function of time, where the first 30 periods of ϑ were calculated with 30 steps for each period. Because of the physical stability of the normalized integral it is only a matter of a numerical instability here. The procedural error must always have an effect in one direction. As proof the quaternion q_{j+1} from the disrupted Taylor series may be calculated after step (j+1) under the assumption (V1) that the rotating vector may be assumed to be constant during an integration step with the result

$$\begin{aligned} &\|q_{j+1}\|^2 = (1-\beta_j)\|q_j\|^2, \text{ with } \beta_j = \frac{(H|w_j|)^6}{4608}\left(1-\frac{(H|w_j|)^2}{32}\right) \\ &w_j = w(t_j), \quad t_j = jH, \quad j = 0, 1, 2, \ldots \end{aligned} \qquad (3).$$

where for all possible steps $\beta_j > 0$. Corresponding values apply for procedures of the third and fifth order. The quaternion norm thereby decreases with each integration step and asymtotically approaches zero.

Remarks: /T 119

a) A rather good approximation of precondition V1 was achieved in the example employed. The value of rotating vector could

even further be assumed as always constant (V2).
Then

$$\beta_i = \text{const.} = \beta, \quad \|q_i\|^2 = (1-\beta)i, \tag{4}$$

and this function describes the actual course of the normalized integral with a high degree of accuracy.

b) If as an especially simple example a permanent rotation is calculated with $w_1 = w_2 = 0, w_3 = 1$, equations (3) and (4) describe exactly the behavior of the normalized integral.

c) The proof resulting from (3) requires in addition to V1 only that the matrix Ω of KQD is symmetrical and involutoric[2], so that other differential equations with such coefficient matrices will exhibit identical characteristics.

When the Eulerian angle ϑ is calculated from the quaternion according to the usual formula

$$\cos\vartheta = q_0^2 - q_1^2 - q_2^2 + q_3^2 \tag{5},$$

then the normalized error effects a very large error in this angle, for

$$\cos\vartheta \approx \|q\|^2 \cos\vartheta^*, \quad \text{thus} \quad \vartheta \approx \vartheta^* - \frac{N}{\tan\vartheta^*} \tag{6}$$

when ϑ^* is the exact angle. This causes a numerical singularity in the range of the frame barrier.

Independent of these results, suitable steps must be taken to stabilize this integral numerically because of the numerical instability of the normalized integral.

3. Stabilisation according to Baumgarte

Baumgarte [2] has proposed a gradient stabilisation procedure for systems of usual differential equations of the first order. In the case of the requirement for stabilisation designated here "on average" the differential equation is altered with the aid of known

2) $\Omega^2 = -|w|^2 E$, E = unit matrix

first integrals in such a manner that the stabilized differential equation leads to a differential equation for the error in integrals with the result that these are reduced asymtotically on the average when they have deviated from desired value zero through procedural error. Applied to kinematic differential equation using quaternions, the stabilized differential equation (KQDS) becomes

$$\dot{q} = \tfrac{1}{2}(\Omega - 4kNE)\,q \qquad (7),$$

where k is a positive, otherwise arbitrarily selected stabilisation factor. The normalized integral would have to behave according to

$$\dot{N} = -4kN(1+N) \qquad (8).$$

Actually, however, the typical course shown in Fig. 1 with k = 0.5 as an example occurs (designated by KQDS). During the first integration step a jump in the norm occurs. Thereafter the integral stabiliyes on an erroneous value, the error is therefore not reduced asymptotically.

Examination of a Renge-Kunge step demonstrates that the series (N_j) of the normalized error fulfills the corresponding law of formation

$$N_{j+1} = N_j - (1+N_j)\,\beta_j\,, \qquad \beta_j = \beta_j(H, k, |w_j|, N_j) \qquad (9).$$

The expression for β_j can no longer be stated explicitly. It is a polynominal of the 80th order in N_j. The coefficients are in each case polynominals in k with parameters H and $|w_j|$. Here V1 is applied again.

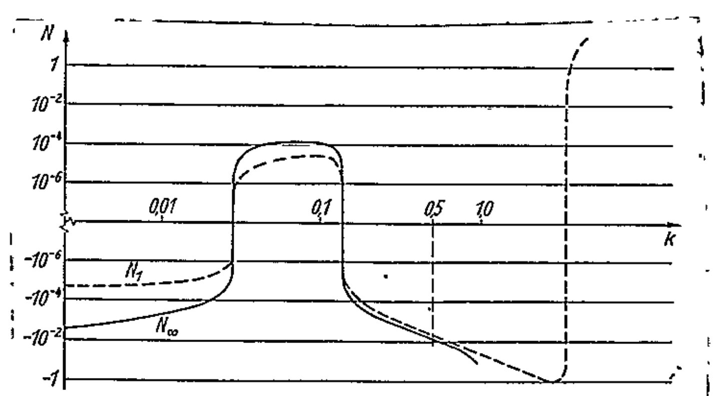

Figure 2: Jump in Norm

Since the absolute member of $\boxed{\beta_1}$ does not disappear, the value of N_1 already differs clearly from zero (Fig. 2), although $N_0 = 0$. Stabilisation would probably have no influence, as long as the solution meets the requirement. Because erroneous aids, subsequently used in the averaging, are, however, intentionally applied to the integration procedure, a correct solution via stabilisation is made incorrect.

The numerical examination of the convergental behavior of the sequence (N_j) suceeds, with the application of the additional assumption V2 although this entails a great deal of work. It converges toward a value $\boxed{N_\infty}$ or demonstrates divergent behavior according to the value of the stabilisation factor k[3]. The result is presented in Fig. 2 (logarithmic scaling, step size and rotating vector as in Fig. 1). There are only two optimal stabilisation factors, in particular the zero positions of $\boxed{N_\infty}$. In order to maintain $\boxed{|N_\infty| < 10^{-6}}$ it is necessary to select the stabilisation factor with a minimum of three decimal places exactly equal to one of the two optimal stabilisation factors. /T 120

The criticism pertaining to the stabilisation procedure[4] described in [2] may be summarized as follows: a) the accuracy depends to a great extent on the stabilisation factor; b) in some cases stabilisation factors have to be determined very exactly. The search for these definitely entails a great deal of work and these are above all dependent on the problem. They undergo changes with the differential equation supplied at the beginning, commencing conditions, integration procedure and stepsize chosen; c) proof assumptions are questionable from the standpoint of numerics. The asymtotic differential equation applied for the proof of the error in first integrals (here:(8)) describes the behavior of the integrals only when the integration procedure functions with no error. This differential equation, however, is only actually fulfilled

3) $\boxed{N_\infty}$ is a zero place of function $\boxed{\beta}$ only dependent on N.

4) The regulation designated here as "individual" stabilisation leads to the same qualitative results.

in the framework of the integration error. Since the commencing condition also remains within this order, it can scarcely describe the numerical behavior of first integrals.

4. Stabilisation through Normalisation

It has been shown that the difficulties resulting from the numerical instability of the quaternion norm are avoided completely, When the quaternion is divided by its norm after each integration step. A proof of this procedure will be published separately.

REFERENCES

[1] Magnus, K., Gyroscopes, Theory and Applications, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[2] Baumgarte, J., "Asymtotic Stabilisation of Integrals in the case of Normal Differential Equations of the First Order," Zeitschrift für Angewandte Mathematik und Mechanik, 701-704 (1973).