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MEMORY EFFECTS IN TURBULENCE

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16. Abstract

Memory effects in turbulent flow are studied. Based on experimental investigations of the wake flow of a hemisphere and cylinder, it is shown that such memory effects can be substantial and have a significant influence on momentum transport. Memory effects are described in terms of suitable memory functions.

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List of Symbols

$$\bar{D}_{ji} = \frac{1}{2}[\partial \bar{U}_j/\partial x_i + \partial \bar{U}_i/\partial x_j]$$

E_1	one-dimensional spectral energy distribution
G	memory function
k_1	wave number in x_1-direction
L_1, L_t	length scale
L_o	characteristic length
$P_t = \frac{1}{3}\varrho\overline{u_j u_i}$	turbulent pressure
p	turbulent pressure fluctuation
$[Q_{2k}]_L$	Lagrangian autocorrelation
$q' = (q^2)^{1/2} = (\overline{u_i u_i})^{1/2}$	
R_L	Lagrangian autocorrelation coefficient
t	time
U	characteristic velocity
[illeg.]	x_i-component of mean velocity
[illeg.]	x_i-component of turbulent velocity fluctuation
$u_1' = (u_1^2)^{1/2}$	intensity of u_1
u^*	shear stress velocity
x	coordinates
y	distance traveled in x_1-direction
δ	boundary layer thickness
δ_{ji}	Kronecker delta
ε_m	kinematic eddy viscosity
Λ_1	memory length scale
Λ_{1f}	axial integral length scale
Λ_{1L}	Lagrangian integral length scale
ν	kinematic viscosity
Ω	intermittence factor
ϱ	density
I_{1L}	Lagrangian integral time scale

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It is shown that the effects in a <u>turbulent flow</u> are <u>not</u>   /403*
<u>strictly local</u> and that <u>memory effects</u> play an essential role.
Thus, momentum-transfer processes are not determined by the local
gradient of the mean velocity alone.

The Reynolds stresses can be thought of as consisting of two
parts. The first part is the stress that would occur if it were
determined solely by the local state of the flow, expressed in the
local mean velocity gradient. The second part is an additional
contribution from a <u>memory</u> of the influence of a change in this
local gradient in the memory region of interest.

A transfer of the gradient type will often occur in fully de-
veloped flows, because either the additional memory effects are
relatively small, or the flow is self-preserving.

These <u>additional memory effects</u> can be directly detected in
the development regions of turbulent flows, where the turbulence
has no opportunity to adapt to new local flow conditions due to
the rapid fluctuation of the main flow direction.

The memory effect can be described with a suitable <u>memory
function</u>. As an approximation, a simple exponential function is
assumed for this. In this case the Reynolds stress is subject to
a relaxation equation whose solution yields the axial distribution
of the stress.

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* Numbers in the margin indicate pagination in the foreign text.

Based on experimental studies of the wake flow behind a hemisphere in a turbulent boundary layer as well as the wake flow of a cylinder, it is shown that such additional memory effects can be substantial. Both cases involve a flow in the development region which fluctuates rapidly in the main flow direction. If the shear stress under consideration is expressed by the velocity gradient and an eddy viscosity, this eddy viscosity is equal to the value of the fully developed flow after correction for the additional memory effect.

1. General

Flows which exhibit memory effects have been known for some time, mainly for non-Newtonian fluids. In this case the fluid should at least have viscoelastic properties which, as in rheology, are determined by its molecular structure.

But memory effects are also present in the turbulent flow of Newtonian fluids. The phenomenological theories of Prandtl and Taylor on the momentum transfer associated with turbulent or Reynolds shear stress essentially involve a memory effect determined by the Lagrangian autocorrelation of fluid particles in turbulent motion. Subsequent studies have shown that various phenomena in turbulence can be explained in principle by the assumption of a viscoelastic turbulence behavior, as in the formation of secondary flows in turbulent flow through a pipe of noncircular cross-section. It is apparent, then, that turbulence may be regarded as a hypothetical non-Newtonian fluid. A basic departure from rheology must be made, however.

First, the principle of local action is basic to rheology. According to this principle, the stresses in a particle are determined or influenced by motions of the medium outside a certain very small material environment of the particle. "Particle" here refers to a small quantity of matter with a specific identity and a

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volume which is very small in relation to the macroscopic length /404
scale of the motions. The state of stress at a particular point
in time is determined entirely by the history of the motion in
the immediate environment of the particle up to that time.

For "rheologically simple" fluids, the deformation history is
described completely by the local deformation gradient (velocity
gradient). Simple fluids are isotropic, and they "remember" the
past only via the deformation tensor up to the point of time under
consideration.

Second, a constitutive equation, i.e., a relation between
deformation and stress, must satisfy various conditions. The
equation must be invariant with respect to a transformation of
the coordinate system. It should also be invariant with respect
to time-dependent rigid motions such as translation and rotation;
in other words, it should be objective.

A century ago, Boussinesq regarded turbulent flow as a
hypothetical fluid, specifically a Newtonian fluid. He assumed
a linear relation between the stress tensor and the deformation.
tensor by using an effective "eddy viscosity" as a scalar quantity.
With a small extension, Boussinesq's relation can be written

$$ -\overline{u_j u_i} = -\frac{1}{\varrho}\,\bar{P}_t \cdot \delta_{ji} + \varepsilon_m\left(\frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial \bar{U}_j}{\partial x_i}\right). \tag{1} $$

where $\bar{P}_t = \frac{1}{3}\,\varrho\overline{u_j u_i}$ is the mechanical pressure, and ε_m is the
kinematic eddy viscosity.

Because the turbulent velocity fluctuations u_i are differ-
ence velocities and thus objective, as is the deformation tensor,
relation (1) is objective.

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If turbulence is regarded as a rheologically simple fluid with non-Newtonian properties, relation (1) can be further supplemented as follows:

$$- \overline{u_j u_i} = F_{ji}(x_k, t, \overline{U}_k, \overline{P}_t/\varrho, \partial \overline{U}_k/\partial x_m).$$

(2)

It can be shown that if the objectivity conditions are met, the following relation can be derived [1]:

$$- \overline{u_j u_i} = \varepsilon_0 \delta_{ji} + 2\varepsilon_m \overline{D}_{ji} + 4\varepsilon_c \overline{D}_{jk} \overline{D}_{ki}$$

(3)

where $\quad\quad\quad\quad \overline{D}_{ji} = 1/2 \, (\partial \overline{U}_j/\partial x_i + \partial \overline{U}_i/\partial x_j).$

The quantities ε_0, ε_m and ε_c are still functions of the principal invariants of $\overline{D}_{ji}$ and of the possible scalar invariants such as turbulent pressure. This nonlinear relation is analogous to the rheological equation of state for viscous fluids of Stokes and of Reiner-Rivlin. This relation satisfies the condition of local action but contains no "memory."

For a local action in turbulent flows, one must also consider very small particles which do not exceed the Kolmogroff micro length scale in order of magnitude. Now it happens that such small particles lose their identity (momentum) too quickly to make a significant contribution to turbulent transport. To be effective in this regard, the lumps of fluid must be rather large -- of the same order of magnitude as the energy-containing eddies, i.e., the order of magnitude of the integral length scale. The action is essentially no longer local, and so higher derivatives of the mean velocity must also be considered. The radius of action is taken to be proportional to the eddy size under consideration. In this case the size of a unit area for momentum

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transport ($\overline{u_j u_1}$) can no longer be taken as small, but must be of the order of magnitude of these lumps of fluid.

Memory effects in turbulence, expressed in a memory or relaxation distance, are determined in part by the size of the eddies. A classic example of the dependence of the relaxation or memory distance on eddy size is found in the studies of Clauser [2]. Fig. 1 shows the result of these studies. A rod (d = 12.5 mm) is placed in a turbulent boundary layer of thickness δ = 235 mm in a cross-stream direction, thereby producing a flow which is characterized by the local velocity defect $\Delta \overline{U}_1$. The decay of this defect occurs much more rapidly near the wall (x_2/δ = 0.16; smaller eddies) than at a point more distant from the wall (x_2/δ = 0.59; larger eddies).
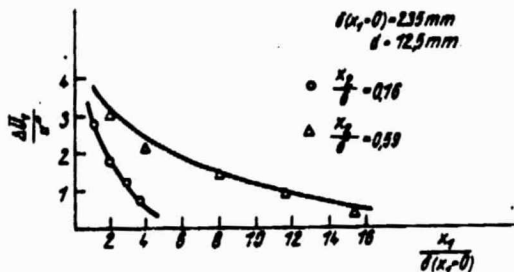

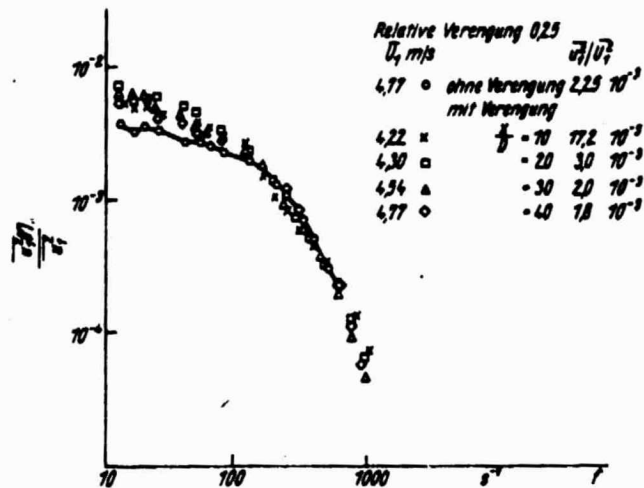Fig. 1. Decay of a mean-velocity disturbance in a turbulent boundary layer.

Fig. 2. Spectral energy distribution of turbulence along a pipe axis.

Key: a - Relative constriction 0.25; b- Without constriction; c - With constriction.

In his wave theory of turbulence, Landahl [3] showed that the only suitable length scale for the spatial behavior of an eddy is the size of the eddy itself. A wave component loses its identity at a distance of approximately six of its own wavelengths. Measurements in a boundary layer showed that at x_2/δ = 0.5, the Lagrangian integral length scale is approximately 4 δ. Thus, the disturbance will decay completely only after about 24 δ. The dependence of the decay of velocity disturbances on eddy size was nicely confirmed by the experimental results of Lissenburg [4]. The turbulent flow of air in a straight pipe of circular cross-section (D = 20 mm) was disturbed by a local constriction of the pipe. At various distances downstream of the constriction, measurements were made of mean velocity distribution and the intensity of turbulent velocity along the pipe axis and close to the wall. The Reynolds number relative to diameter and mean velocity was approximately 5000.

The results of this experiment are shown in Fig. 2. After /405 x = 20 D downstream of the constriction, the spectral energy distribution is already equal to the undisturbed flow for frequencies higher than 100 s^{-1}. For lower frequencies, the spectral distribution is not yet equal to the undisturbed flow even after 40 D. If, as an approximation, the propagation velocity of the disturbances is assumed to equal the mean velocity, Landahl's estimate for 20 D (= 0.4 m) leads to a frequency of 70 s^{-1}, which is of the same order of magnitude as 100 s^{-1}.

It can be concluded from Landahl's work that the more the transfer of size is governed by the fine structure of the turbulence, the more local is the effect. This validates the assumption of a transfer of gradient type which is made in many new turbulence theories in order to determine the influence of velocity correlations of third and higher order.

Because the larger eddies are effective in momentum transfer, with an action that is not limited to small regions, it may be asked whether the requirements of objectivity, and especially the condition of invariance under rotation, should still be satisfied, because centrifugal and Coriolis effects can no longer be neglected. Therefore, Lumley [1] has suggested that a constitutive equation for turbulence as a hypothetical fluid no longer be subjected to this last condition.

For the momentum transfer u_1, we may write the following general expression for the <u>time-averaged transport through a unit of area at right angles to the x_2-direction</u>:

$$\overline{u_2 u_1} = \frac{1}{T} \int_0^T dt_0 \, u_2(t_0) \, U_1(t_0). \tag{4}$$

The value $\overline{U}_1(t_o)$ is determined by the prior history of the particle for all $t \leq t_o$. This prior history is determined in part by the distribution of mean velocity in the entire region under consideration and by the scale of the interaction between the particle and its environment during their respective motions. Averaged over many particles, this is associated with the Lagrangian autocorrelation. In principle, this aftereffect can be expressed in terms of a suitable memory or relaxation function. But the determination of such a suitable function generally presents a difficulty which has not yet been overcome.

For a particle which has been in motion for a time t and has traveled a distance y_k, a series expansion of $U_1(t_o;t)$ yields

$$U_1(t_0; t) = (\overline{U}_1)_0 - y_k(t_0; t)\left(\frac{\partial \overline{U}_1}{\partial x_k}\right)_0 + \frac{1}{2} y_k y_l(t_0; t)\left(\frac{\partial^2 \overline{U}_1}{\partial x_k \partial x_l}\right)_0 - \frac{1}{6} y_k y_l y_m(t_0; t)\left(\frac{\partial^3 \overline{U}_1}{\partial x_k \partial x_l \partial x_m}\right)_0 + \cdots. \tag{5}$$

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If the first derivative of the mean velocity is sufficient, relation (4), with

$$y_k(t_0; t) = \int_0^t dt' \, u_k(t_0 - t'),$$

yields

$$-\overline{u_1 u_1} = \frac{\partial \bar{U}_1}{\partial x_k} \int_0^t dt' \, \overline{u_2(t_0) \, u_k(t_0 - t')} = \frac{\partial \bar{U}_1}{\partial x_k} \int_0^t dt' \, [Q_{2k}]_L \, (t') = \varepsilon_{2k}(t) \frac{\partial \bar{U}_1}{\partial x_k} \, ; \qquad (6)$$

$[Q_{2k}]_L = u_2(t_o) \, u_k(t_o - t)$ is the Lagrangian autocorrelation, and ε_{2k} is an eddy viscosity which can have various values for the various directions x_k.

For $t \to \infty$, we obtain the relation of Boussinesq, but with various possible values of the eddy viscosity ε_{2k}. The relation is not objective. Since the relation contains only the first velocity derivative, it appears as if the action is local in character. However, it must be noted that the appearance of the Lagrangian autocorrelation points to an influence of prior history. This influence has been nicely illustrated by Philip [5]. He imagines the velocity of a particle at each point in time as consisting of two parts. The first part corresponds to that which the particle still "remembers" of its velocity u(0) at time t = 0, while the second part consists of the momentum contributed by the exchange with its momentary environment:

$$u(t) = u(0) \, R_L(t) + u^*(t).$$

$R_L(t)$ is the Lagrangian autocorrelation coefficient and describes the influence of the past.

For a stationary homogenous turbulence, it should be true that

$u^2(t) = \overline{u^2}(0) =$ const. Philip calls the first part $u_1(t) = u(0) R_L(t)$ the _drift velocity_. For very large times t, the memory of the original velocity beccmes vanishingly small.

It is assumed in the simple relation (6) that the velocity gradient in the region through which a particle passes during the memory period is sufficiently constant. In most turbulent flows of interest, however, this is not the case. Corrsin [6] recently made a critical study of the _conditions_ which must be satisfied _so that the assumption of a turbulent diffusion of the gradient type may be made_. It is shown that of the various conditions, the most important is that _the length scale of the transport mechanism should be small relative to the distance over which the curvature of_ $\partial\overline{U}_1/\partial x_2$, i.e. $\partial^3 U_1/\partial x_2^3$, _undergoes a substantial change_. Corrsin considered a one-dimensional diffusion process in the x_2-direction and a spatially symmetrical transport mechanism, so that the term with the second derivative is absent in a series expansion such as eqn. (5). Experience has shown that the error in assuming a diffusion of the gradient type is not large. This is due to the fact that the diffusion is essentially one-dimensional in most cases.

In the cases of three-dimensional transport considered here, the terms with the second derivative must also be considered in series expansion (6) as a second approximation, without assuming a diffusion of the gradient type. If this is not done, a non-negligible error can be made locally in the distribution of the mean velocity. As early as 1942, Prandtl [7] pointed this out in a modification of his mixing-length theory for the environment of the maximum velocity of a free jet, in which the first velocity derivative vanishes. In this modification he proposed a statistical mean for the first and second derivatives.

Based on his measurements of turbulent diffusion of the

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kinetic energy of the three turbulent velocity components in the
wake flow of a cylinder, Townsend [8] concluded that there are
places in the wake where diffusion takes place against the gradi-
ent -- specifically, near the energy-distribution maximum of the
turbulence in a cross-section of the wake flow.

It is also known that near a mean-velocity maximum in the
case of asymmetrical flows, the turbulent shear stress and the
mean-velocity gradient may have opposite signs, which can be
interpreted as a "negative energy production" of the turbulence
[9-13]. The explanations advanced by Beguier [12], Hinze [14],
Eskinazi [15] and other authors are all based on a non-negligible
influence of large eddies with a memory effect that requires at
least that the second derivative of the mean velocity be taken
into account.

Apparently the turbulent shear stress, or more generally,
the Reynolds stresses, cannot be regarded as a local property of
the flow. This led Philips [16] to suggest that one should regard
the gradient of the Reynolds stress, rather than the Reynolds
stress itself, as a local property.

In the cases previously mentioned, one must take into
account the second velocity derivative, i.e., the change in the
velocity gradient perpendicular to the main flow. But there are
also cases in which the change of this gradient in the main
flow occurs relatively quickly with a time or length scale which
is no longer large in comparison with the scale of the large
energy-containing eddies. In this respect the turbulence is no
longer "adapted" to or "in equilibrium" with the local conditions
of the flow.

In his study of boundary layers which undergo rapid spatial
changes in the main-flow direction, Deissler [17] showed that the
Reynolds stress remains approximately constant along one streamline

and may be equated with the initial value. In these necessarily short boundary layers, the memory distance extends over the entire length of the boundary layer considered.

Other examples in which memory effects of the type mentioned have been detected are the wake flow behind a hemisphere in a turbulent boundary layer and the wake flow immediately behind a cylinder. These two cases will be discussed in greater detail below.

The memory effects can be allowed for if the series expansion of $\overline{U}_1$ according to eqn. (5) is continued in relation (4), and first and higher derivatives of $\partial\overline{U}_1/\partial x_k$ are also considered. /407 Relation (4) then becomes

$$-\overline{u_2 u_1} = \int_0^t dt' \left[\overline{u_2(t_0)\, u_k(t_0 - t')} - \frac{1}{2}\overline{u_2(t_0)\, u_k(t_0 - t')\, y_l(t_0; t')} \frac{\partial}{\partial x_l} + \frac{1}{6}\overline{u_2(t_0)\, u_k(t_0 - t')\, y_l y_m(t_0; t')} \frac{\partial^2}{\partial x_l \partial x_m} + \cdots \right] \frac{\partial\overline{U}_1}{\partial x_k}. \qquad (7)$$

In most cases it is sufficient to take into account only the terms up to the second derivative in a series expansion as a second approximation. In a manner analogous to relation (6), we find from eqn. (7)

$$-\overline{u_2 u_1} = \varepsilon_{2k}(t) \frac{\partial\overline{U}_1}{\partial x_k} - \frac{1}{2} \int_0^t dt'\, \overline{u_2(t_0)\, u_k(t_0 - t')\, y_l(t_0; t')} \frac{\partial^2\overline{U}_1}{\partial x_l \partial x_k}$$

$$= \varepsilon_{2k}(t) \frac{\partial\overline{U}_1}{\partial x_k} - \frac{1}{2} \varepsilon_{2k}^0(t)\, L_l(t) \frac{\partial^2\overline{U}_1}{\partial x_l \partial x_k}.$$

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The last term in an approximation based on the assumption that the integral of a triple correlation can be written as the product of a suitable length $L_1(t)$ and an eddy viscosity. A further approximation is that a scalar eddy viscosity is sufficient for ε_{2k} and ε_{2k}^0, because it is also assumed that $t \to \infty$:

$$- \overline{u_2 u_1} = \left[\epsilon_m \frac{\partial \overline{U}_1}{\partial x_2} - \frac{1}{2} \epsilon_m^0 L_1 \frac{\partial^2 \overline{U}_1}{\partial x_i \partial x_2} \right]. \tag{8}$$

This equation can be generalized to a relation for $-\overline{u_j u_1}$, one which is symmetrical in j and i. Because $\partial U_1 / \partial x_1 = 0$ for an incompressible medium, we write the following expression so that the relation for $\overline{u_j u_1}$ contains no contradiction:

$$- \overline{u_j u_i} = - \frac{\overline{P}_t}{\varrho} \delta_{ji} + 2 \left[\epsilon_m \overline{D}_{ji} - \frac{1}{2} \epsilon_m^0 L_i \frac{\partial}{\partial x_i} \overline{D}_{ji} \right]. \tag{9}$$

The second velocity derivative in eqn. (8) and eqn. (9) may signify a nonlocal action. This depends on what the length scale L_1 is determined by. At right angles to the mean-flow direction, inclusion of this term signifies a nonlocal action. In the main-flow direction, however, L_1 can become relatively large due to the high convection velocity, and the action may be local in character.

The influence of a spatial change in $\overline{D}_{ji}$ on momentum transfer can also be taken into account with a memory function. Instead of eqn. (9) we then consider

$$- \overline{u_j u_i} = - \frac{\overline{P}_t}{\varrho} \delta_{ji} + 2 \int_0^\infty dt' \, \epsilon_m(t') \, \overline{D}_{ji}(t - t') \, G(t'). \tag{10}$$

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For large times t, the normalized memory function G(t) decays to zero. It is characteristic of the turbulent flow under consideration. Eqn. (10) points essentially to a local action. As mentioned earlier, however, relatively large lumps of fluid play a role in momentum transfer. If $\overline{D}_{j1}$ varies noticeably in the range of the order of magnitude of these lumps, the action is not local. However, a description based on eqn. (10) may be possible if the transfer takes place as if the action were local.

Relation (10) is similar to that for a viscoelastic fluid. Crow [18] has suggested such a relation for describing the properties. of the fine structure of a turbulent flow.

2. Theory

In this section we shall examine relations (9) and (10) more closely. Of particular interest for the cases investigated experimentally is the <u>shear stress per unit mass</u> $(-\overline{u_2 u_1})$, because these cases involve nearly one-dimensional flows with a pronounced main-flow direction. The mean flow is two-dimensional with the velocity components $\overline{U}_1$ in the main-flow direction x_1, and $\overline{U}_2$ in the direction x_2 at right angles to the main-flow direction, where $\overline{U}_1 \gg \overline{U}_2$. The mean flow is independent of the x_3-direction.

As a second approximation, it is assumed that the memory effect of $\partial \overline{U}_1 / \partial x_2$, i.e., the influence of a relatively rapid change of $\partial \overline{U}_1 / \partial x_2$, is sufficient in the main-flow direction. According to eqn. (9),

$$-\overline{u_2 u_1} = \left[\varepsilon_m \frac{\partial \overline{U}_1}{\partial x_2} - \Lambda_1 c_m^{\circ} \frac{\partial^2 \overline{U}_1}{\partial x_1 \partial x_2} \right] \tag{11}$$

with $\Lambda_1 = 1/2\ L_1$.

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$$- \overline{u_2 u_1}(t) = \int_0^\infty dt'\, \varepsilon_m(t') \frac{\partial \overline{U}_1}{\partial x_2}(t - t')\, G(t')_{\bullet}. \tag{12}$$

The last expression is more general than eqn. (11). The significance of these relations is that the shear stress may be envisioned as consisting of two parts, hence:

$$- \overline{u_2 u_1} = (- \overline{u_2 u_1})_g + (- \overline{u_2 u_1})_e .$$

The first part $(-\overline{u_2 u_1})_g$ is the shear stress which would occur if it were determined entirely by the "local" state of the flow. Here "local" does not necessarily refer to a small region. It may be said that this part is "in equilibrium" with the local state. It may, for example, be equated with the shear stress according to the Boussinesq relation $\varepsilon_m\, \partial \overline{U}_1/\partial x_2$ if the memory remains confined to a region in which $\partial \overline{U}_1/\partial x_2$ is practically constant.

The second part $(-\overline{u_2 u_1})_e$ is an additional contribution which a lump of fluid "remembers" during the transfer process from the influence of a variable $\partial \overline{U}_1/\partial x_2$ in the memory region considered.

Usually the relative intensity of the turbulence is not large, and it may be assumed that Taylor's hypothesis of a frozen turbulence is valid: $x_1 \simeq \overline{U}_1 t$. Eqn. (12) can then be written in a Eulerian description:

$$- \overline{u_2 u_1}(x_1) = \int_{-\infty}^{x_1} dx_1'\, \varepsilon_m(x') \frac{\partial \overline{U}_1}{\partial x_2}(x_1')\, G(x_1 - x_1') . \tag{13}$$

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This expression contains the contributions $(-\overline{u_2 u_1})_g$ and $(-\overline{u_2 u_1})_e$. If $\varepsilon_m \partial \overline{U}_1 / \partial x_2 =$ const. in the region in which $G(x_1 - x_1') > 0$, we obtain with $\int_{\infty}^{\cdot} dx_1' G(x_1 - x_1') = 1$ the Boussinesq relation, and $(-\overline{u_2 u_1})_e = 0$.

As mentioned, eqn. (13) implies essentially a local action. In the present work this relation is successfully applied to several turbulent flows. In these cases it is apparently correct to assume that the momentum transfer is local in character but has a memory.

An obvious step is to associate the memory function with a Lagrangian autocorrelation. Although in principle the Lagrangian autocorrelation is not equal to a simple exponential function, such an assumption is sometimes made with satisfactory results in order to simplify calculations. We shall make the same assumption here for the memory function; thus, in normalized form, we obtain

$$G(x_1 - x_1') = \frac{1}{\Lambda_1} \exp\left(-\frac{x_1 - x_1'}{\Lambda_1}\right); \tag{14}$$

Λ_1 is then an <u>effective memory length</u>. With this expression for G, eqn. (13) becomes

$$- \overline{u_2 u_1}(x_1) = \frac{1}{\Lambda_1} \int_{-\infty}^{x_1} dx_1' \, \varepsilon_m(x_1') \frac{\partial \overline{U}_1}{\partial x_2}(x_1') \exp\left(-\frac{x_1 - x_1'}{\Lambda_1}\right). \tag{15}$$

Owing to the exponential function, expression (15) can now be regarded as the solution of an inhomogenous differential equation of first order, which is compatible with the relaxation process [19] considered above. This differential equation is

$$\Lambda_1 \frac{\partial}{\partial x_1}(- \overline{u_2 u_1}) + (- \overline{u_2 u_1}) = \varepsilon_m \frac{\partial \overline{U}_1}{\partial x_2}. \tag{16}$$

$$\bar{U}_1 \frac{\partial}{\partial x_1}(-\overline{u_2 u_1}) = \overline{u_2^2}\frac{\partial \bar{U}_1}{\partial x_2} + \frac{\partial}{\partial x_2}\overline{u_1\left(u_2^2 + \frac{p}{\varrho}\right)} - \frac{\overline{p}}{\varrho}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) + \nu\left[2\overline{\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_2}} + \frac{\partial^2}{\partial x_2^2}(-\overline{u_2 u_1})\right].$$

The second term on the right side is usually negligibly small.
Disregarding viscosity effects, the equation reduces to

$$\bar{U}_1 \frac{\partial}{\partial x_1}(-\overline{u_2 u_1}) \simeq \overline{u_2^2}\frac{\partial \bar{U}_1}{\partial x_2} - \frac{\overline{p}}{\varrho}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right).$$

r the pressure term, Rotta [21] suggested the following
combination with the shear stress:

$$-\frac{\overline{p}}{\varrho}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = -A\frac{q'}{L}(-\overline{u_2 u_1}),$$

where $q' = (\overline{q^2})^{1/2} = (\overline{u_i u_i})^{1/2}$; L is a length scale of the size of
an integral length scale; and A is a numerical constant. Hence,

$$\frac{\partial}{\partial x_1}(-\overline{u_2 u_1}) \simeq \frac{\overline{u_2^2}}{\bar{U}_1}\frac{\partial \bar{U}_1}{\partial x_2} - \frac{A}{L}\frac{q'}{\bar{U}_1}(-\overline{u_2 u_1}), \tag{21}$$

L is, as mentioned, of the size of the integral length scale. For
this we select Λ_{1f} as an axial integral length scale and set L =
const Λ_{1f}. We also set $\Lambda_1 = \bar{U}_1 I_{1L} = (\bar{U}_1/u_1')\Lambda_{1L} = \text{const } \bar{U}_1/u_1'\,\Lambda_{1L}$ with $u_1' =$
$(\overline{u_1^2})^{1/2}$; I_{1L} is the Lagrangian integral time scale, Λ_{1L} is the
Lagrangian integral length scale. If now $\Lambda/L \; q'/\bar{U}_1 = \text{const} \cdot$
$\cdot q'/u_1' \cdot 1/\Lambda_1$, we obtain eqn. (16) from eqn. (21) if const $q'/u_1' = 1$.
Further let $\overline{u_2^2} \; \Lambda_1/\bar{U}_1 = \overline{u_2^2} \; I_{1L} = \varepsilon_m I_{1L}/I_{2L}$, and for the Lagrangian
integral time scale, $I_{1L} \sim I_{1L}$, which is the case only in an iso-
tropic state of turbulence and is true only in rough approximation
for a nonisotropic turbulence. In any case, the relaxation

equation (16) is not incompatible with the dynamic equation for shear stress.

It is our intention to show <u>memory effects in flows which fluctuate rapidly in the mean flow direction</u>. It may be asked whether a memory effect can be present in flows which do not fluctuate rapidly in this way. According to relations (11) and (16), the memory effects are negligible if

$$\Lambda_1 \delta^2 \overline{U}_1 / \partial x_1 \partial x_2 \ll \partial \overline{U}_1 / \partial x_2 \tag{22a}$$

or if

$$\Lambda_1 \partial (- \overline{u_2 u_1}) / \partial x_1 \ll - \overline{u_2 u_1} . \tag{22b}$$

Now there is a class of flows for which these conditions are not satisfied, and yet there is no demonstrable memory effect. These are flows which possess the property of <u>self-preservation</u>.

Self-Preserving Flows

<u>These are flows with a pronounced main flow direction, whose structure maintains its similarity during development or decay in this direction</u>. Many free turbulent flows such as free jets and wake flows show such a similarity in the fully developed state. It is convenient to make a distinction between complete and incomplete similarity. In the case of <u>complete similarity</u>, a length scale and a velocity scale are sufficient to make the thus-reduced velocity distributions coincide completely. In the case of <u>incomplete similarity</u>, more than one velocity scale and/or more than one length scale is required.

To determine the possibility of the existence of a self-

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preserving flow, let us set

$$\bar{U}_1 = U\left(\frac{x_1}{L_0}\right)^p f(\eta), \qquad \bar{U}_2 = U\left(\frac{x_1}{L_0}\right)^r g(\eta), \qquad -\overline{u_1 u_2} = U^2\left(\frac{x_1}{L_0}\right)^s h(\eta),$$

$$\eta = \frac{x_2/L_0}{(x_1/L_0)^n}, \qquad \epsilon_m = U L_0\left(\frac{x_1}{L_0}\right)^n k(\eta) \quad \text{mit} \quad n = q - p + s. \tag{23}$$

Here L_0 is a characteristic length, and U a characteristic velocity.

By substitution in the Reynolds equations, it can be shown that complete similarity exists only if

$$q = 1; \; p = r = s/2.$$

With the relations in (23), it follows from the dynamic equations that a plane free jet and a round free jet may have complete similarity, where n = 1/2 for the plane case, and n = 0 for the round case. On the other hand, the plane and the round wake flows can have only incomplete similarity, but in both cases n = 0, and so ϵ_m is independent of x_1.

To study memory effects in self-preserving flows, we must also determine the possibility of similarity for the relaxation equation (16). For this we set

$$\Lambda_1 = L_0\left(\frac{x_1}{L_0}\right)^t l(\eta). \tag{24}$$

Inserting relations (25) and (24) into eqn. (16), we find that t = 1 is a necessary condition for similarity. Thus, the memory length scale must vary linearly with x_1. For a fully developed plane wake flow, s = -1, q = 1/2 and p = -1/2, and so n = 0.

Hence, the memory effect present can no longer be directly detected, even if $\Lambda_1 \propto x_1$ and conditions (22a) and (22b) were not satisfied. This follows from·eqn. (10), where for this similar flow the effect of the first term can be allowed for with an apparently larger ε_m.

3. Experimental Investigations

Two different flows with pronounced memory effects are considered: the restoration of a strongly disturbed boundary layer and the development region of the wake flow behind a cylinder.

Restoration of a Disturbed Boundary Layer

The boundary layer in question is that along a straight glass plate set up in the plane of symmetry of a wind tunnel. The test section of the wind tunnel is 4.5 m long and has a cross-section of 0.8 x 0.7 m^2. In the experiments the external velocity of the boundary layer was kept constant: U_o = 10.5 m/s. A hemisphere 40 mm in diameter was attached to the glass plate with the midpoint of its flat undersurface 3.65 m from the leading edge of the plate. The hemisphere was thus entirely submerged in the boundary layer. The three-dimensional wake flow of the hemisphere produces a considerable disturbance of the boundary-layer flow, as shown in Fig. 3.

The velocity measurements were performed with a constant-temperature hot-wire anemometer. The heating wire consisted of platinum-plated tungsten wire 5 μm in diameter. The length of the sensing element was 2 mm in most cases, but occasionally 1 mm. The distance between the wire supports was 10 mm. Measurements were made of: the three components of the mean velocity and the turbulent velocity, the turbulent shear stresses, spatial velocity correlations and one-dimensional spectral distributions of the

energy of the axial turbulent component. Wall shear stresses were measured with a Preston tube wherever possible.

Results of these investigations have already been published [22]. Here we shall again present some of the most important results together with results of further investigations. Equations (11) and (16) form the starting point for interpreting the results. It was assumed as a second approximation that only the change of $\partial \bar{U}_1/\partial x_2$ in the main flow direction [that is $\Lambda_1 \partial^2 \bar{U}_1/\partial x_1 \partial x_2$ in eqn. (11)] can be considered. This corresponds to the concept that fluid particles which contribute to momentum transfer travel only relatively short transverse distances in the x_2- and x_3-direction. However, closer investigation has shown that the neglected terms in eqn. (8) are actually not negligibly small. Since the individual contributions have positive and negative values, though, the sum total proves to be small.

An important assumption is now made regarding eqn. (11): that the eddy viscosities ε_m and ε_m^0 are equal and have a constant value in the important x_1-direction.
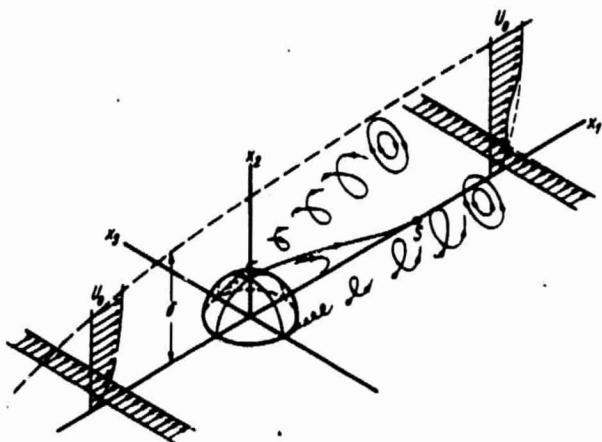
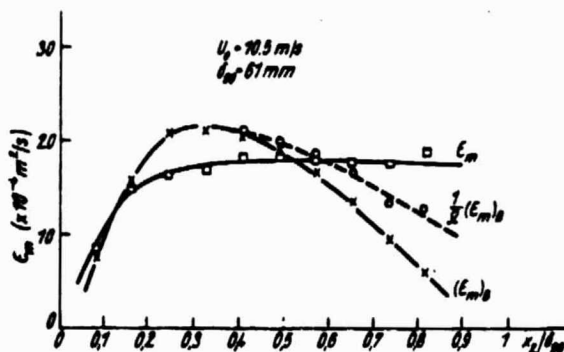

Fig. 3. Wake flow of a hemisphere in a boundary layer.

Fig. 4. Distributions of $(\varepsilon_m)_\beta$, $1/\Omega(\varepsilon_m)_\beta$ and ε_m acc. to eqn. (11), $x_1 = 0.50$ m, $x_2 = 0.02$ m.

Fig. 4 shows ϵ_m as a function of x_2/δ_{99} for $x_1 = 0.5$ m and $x_3 = 0.02$ m, first calculated from the Boussinesq relation (first approximation)

$$(\epsilon_m)_\beta = -\overline{u_2 u_1}/(\partial \overline{U_1}/\partial x_2),$$

Second, ϵ_m is shown corrected for the influence of intermittent turbulence in the outer region of the boundary layer $1/\Omega(\epsilon_m)_\beta$, with Ω as the intermittence factor; third, the value is calculated from eqn. (11) and is again corrected for intermittence effects. Lacking our own measurements of Ω, we assumed that the values measured by Klebanoff [23] in an undisturbed boundary layer were approximately valid for the disturbed boundary layer as well. It was found that the distribution of ϵ_m calculated from eqn. (11), unlike $(\epsilon_m)_\beta$, is similar to the distribution in an undisturbed boundary layer. This is shown in Fig. 5, which also gives the distributions of the dimensionless quantity $\epsilon_m/u^*\delta_{99}$ for $x_1 = 0.25$ m, $x_3 = 0.02$ m; $x_1 = 0.50$ m, $x_3 = 0.02$ m of the disturbed boundary layer and for the same x_1 distances, but $x_3 = -0.15$ m of the undisturbed boundary layer; u^* is the shear stress velocity. All values have been corrected with the intermittence factor Ω, i.e., $(-\overline{u_2 u_1})/\Omega$ was used in eqn. (11). The length Λ_1 was calculated from the equation

$$\Lambda_1 = 0.4 \frac{\overline{U_1}}{u_1^*} \Lambda_{1f} \tag{25}$$

The value 0.4 of the constant is a mean value of measured $\beta = \Lambda_{1L}/\Lambda_{1f}$ and agrees with the theoretical values found by Saffman [24]. In addition, Λ_{1f} was set equal to 0.4 δ_{99} in Fig. 4 and 5.
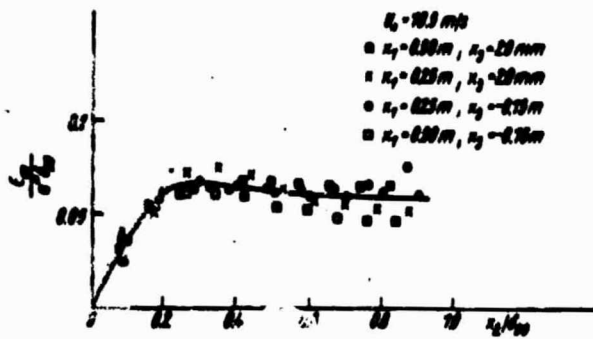
Fig. 5. Distributions of the dimensionless quantity $\epsilon_m/u^* \cdot \delta_{99}$; •——• calculated from $(-\overline{u_2 u_1})$ measurements of Klebanoff [23].
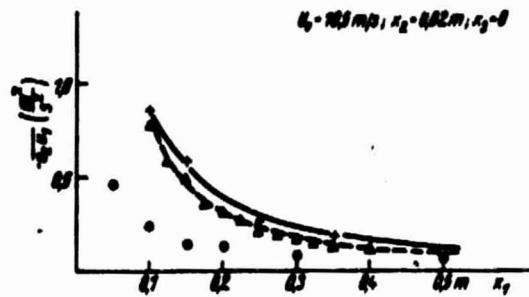
Fig. 6. Turbulent shear stress, measured (+) and calculated by Boussinesq relation (o) and by eqn. (20) (Δ). (Boussinesq: $\epsilon_m = 0.065\, u^* \delta_{99}$; eqn. (20): $\epsilon_m = \langle\epsilon_m\rangle = 14.2\cdot10^{-4}$ m^2/s).

According to our own measurements of the one-dimensional spectral energy distribution of u_1, i.e. of $E_1(k_1)$ (where k_1 is the wavenumber in the x_1-direction), a value of 0.6 appears to be a better approximation than 0.4. The locally measured value of the disturbed boundary layer was taken as u^*. Because the quotient $\partial^2\overline{U}_1/\partial x_1 \partial x_2$ was not constant along a distance equal to Λ_1, a weighted mean was assumed. It was also found that Λ_1 was not constant along a distance equal to Λ_1, as was predicted by eqn. (25). Thus, eqn. (16) [or eqn. (20)] is more suitable if one wishes to consider a variable Λ_1. From measurements of $\overline{U}_1$, u_1' and Λ_{1f} follows the relation

$$\Lambda_1 = 0{,}27 z_1^{0.7} \quad (\Lambda_1 \text{ und } z_1 \text{ in m}).\tag{26}$$

Because the result of calculations with various relations for Λ_1 has proved to be largely insensitive to changes in this relation, the approximate relation shown above is suitable.

The solution of eqn. (16) with Λ_1 according to eqn. (26) is given by eqn. (20) with $a = 0.27$ and $n = 0.7$. Also in eqn. (20), $x_{10} = 0$ and $-u_2 u_1(0) = 0.107$ m^2/s^2. The calculation was performed

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along a line x = 0.02 m, x = 0 (plane of symmetry). Actually
the calculation should be carried out along a streamline. But
because the difference between a streamline and a line x_2 = const
is quite small, the calculation was performed with x_2 = const for
simplicity. According to Fig. 5, $\epsilon_m/u^*\delta_{,,}$ is independent of x_1.
In eqn. (20) the variable value of $u^*\delta_{,,}$ should be used for ϵ_m.
It was found, however, that for 0.1 m < x_1 < 0.6 m, $u^*\delta_{,,}$ and
thus ϵ_m changes only very little, on the order of $\epsilon_m \propto x_1^{0.15}$.
Again, for simplicity, a constant mean value ϵ_m = 14.2 $\propto$ 10^{-4} m^2/s
was used for the calculation by eqn. (20). Fig. 6 shows the re-
sult of the calculation of $(-\overline{u_2u_1})$ together with the measured
values. The turbulent shear stress was also calculated by the
Boussinesq formula, with the local value $(\epsilon_m)_\beta$ = 0.065 $u^*\delta_{,,}$.

Although the shear stress $(-\overline{u_2u_1})$ calculated by eqn. (20) is
still somewhat smaller than the measured shear stress, eqn. (20)
shows considerable improvement over calculation by the Boussinesq
relation. The shear stress calculated with the local velocity
gradient $\partial\overline{U}_1/\partial x_2$ is much too small, particularly in the first re-
gion of the wake flow of the hemisphere. For large x_1, at which
the disturbed boundary layer is nearly restored, the two calcu-
lated shear stresses approach the measured value.

It may be asked whether memory effects in the sense understood
here might not be present in the undisturbed boundary layer. Re-
lation (26) does not satisfy condition (24) with t = 1 for simi-
larity. Hence, memory effects must be negligibly small. This
means that conditions (22a) and (22b) must be satisfied. With
Λ_1 calculated according to eqn. (25), we find for the undisturbed
boundary layer

$$\left| \frac{\Lambda_1}{-\overline{u_2u_1}} \frac{\delta}{\delta x_1}(-\overline{u_2u_1}) \right| \approx O(0.01),$$

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so that condition (22b) may be satisfied with certainty.

Wake Flow of a Cylinder

This investigation was conducted in a wind tunnel similar to that described above. The measurements were made with cylinders of various diameters. Here we shall consider only the results obtained for a cylinder of D = 0.04 m. With the velocity of the undisturbed wind tunnel flow U_o = 10.5 m/s, the Reynolds number $Re_D = U_o/D$ = 26,000.

The "classical" investigations of Townsend [8] in a cylinder wake (with Re_D = 1360) already showed that self-preservation of the mean flow velocity occurs no sooner than x_1/D = 70 to 100, because even greater downstream distances are necessary for turbulence. The investigations performed here pertain to the region 2 < x_1/D < 90; the region 20 < x_1/D < 50 was of particular interest in terms of the intended memory effects. For x_1/D < 20, effects of more or less discrete Kármán eddies are noticeable, depending on the Reynolds number.

The hot-wire anemometer described above was used to measure the mean velocity components $\overline{U}_1$ and $\overline{U}_2$, the turbulence intensities u_1' and u_2', the shear stress ($-\overline{u_2 u_1}$) and spectral distributions of u_1. From the measured values of $\overline{U}_1$, u_1' and the spectral distributions, $\Lambda_1(x_1,x_2)$ was calculated.

For x_2 = const, Λ_1 = const x_1, and thus Λ_1 is a linear function of x_1. The eddy viscosity ϵ_m was computed for the cross-section x_1 = 22.5 D as a function of $x_2/(x_2)_{99}$. Here $(x_2)_{99}$ is the lateral distance for which the velocity difference with respect to the free stream is one percent of the maximum local velocity difference. The calculation was performed only for the inner region, in which the turbulence still shows practically no

intermittence. Fig. 7 shows $\varepsilon_m/U_o D$ according to Boussinesq, according to eqn. (11) with $\varepsilon_m = \varepsilon_m^o$ and according to eqn. (16). The equilibrium value for larger x_1, at which the flow is fully developed is equal to 0.018. A memory effect is clearly evident. The values calculated by eqn. (11) and eqn. (16) are almost equal to this equilibrium value. This value is somewhat higher than 0.0164, the value that follows from the measurements of Townsend [20] with $Re_D = 1360$.
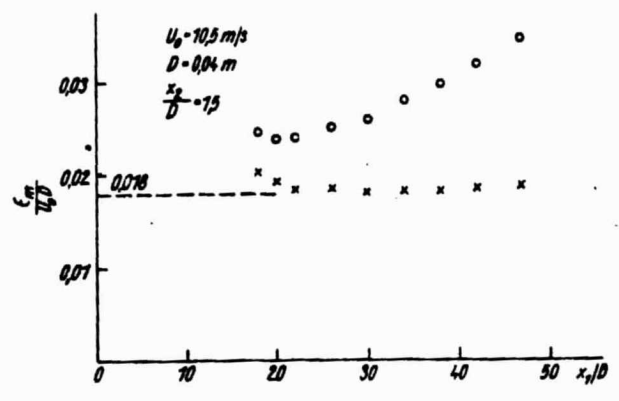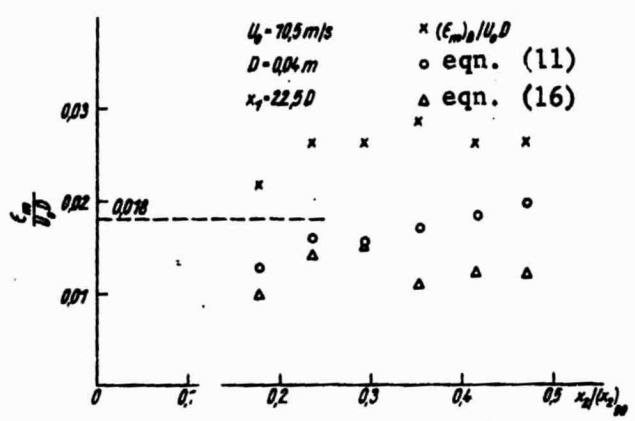
Fig. 7. Distribution of $\varepsilon_m/U_o D$ according to Boussinesq (x, eqn. (11) (o) and eqn. (16) (Δ).

Fig. 8. Distribution of $\varepsilon_m/U_o D$ in a cylinder wake along the line $x_2/D = 1.5$; calculated according to Boussinesq (o) and eqn. (11) (x).

The variation of $\varepsilon_m/U_o D$ in an axial direction was calculated according to eqn. (11) along a line $x_2/D = 1.5$. This line passes through the point in cross-section $x_1 = 18\,D$ at which the shear stress attains a maximum. In eqn. (11) Λ_1 was calculated from eqn. (25), where $\Lambda_{1f} = 0.5$ to $0.6\,(x_2)_{99}$ according to the spectral measurements for $x_2 = 1.5\,D$. This leads to $\Lambda_1 \backsim 0.3$ to $0.36\,x_1$, and thus the linear relation previously mentioned. Along the line $x_2 = 1.5\,D$, the intermittence factor $\Omega = 1$. Fig. 8 shows $\varepsilon_m/U_o D$ calculated according to Boussinesq and according to eqn. (11). After $x_1/D = 20$ the value calculated from this equation is
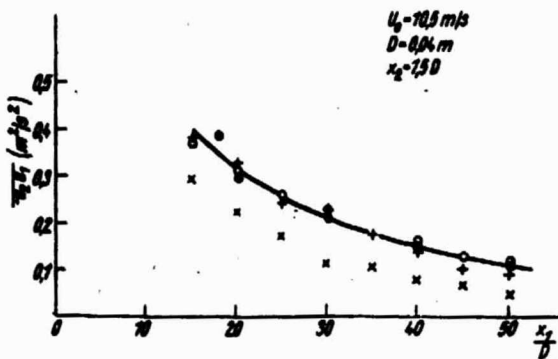
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Fig. 9. Axial shear stress distribution in a cylinder wake, measured (•) and calculated according to Boussinesq (x), eqn. (19) (o) and eqn. (11) (+).

practically constant and equal to the equilibrium value of 0.018. The memory effect according to eqn. (19) was investigated along the same line $x_2 = 1.5$ D. Here, according to Fig. 8, ε_m = const along this line. At point $x_{10} = 0$, it was assumed that $-\overline{u_2 u_1}(0) = 0$. The result of the calculation with $\varepsilon_m = 0.018\ U_0 D = 0.0076$ m²/s and $a = 0.3$ is shown in Fig. 9, together with $(-\overline{u_2 u_1})$ calculated according to Boussinesq with the same value of ε_m. This figure also shows $(-\overline{u_2 u_1})$ calculated according to eqn. (11) and the measured variation of $(-\overline{u_2 u_1})$.

The linear relation $\Lambda_1 \simeq 0.3\ x_1$ can only be an approximation. In reality Λ_1 must increase somewhat parabolically with x_1 before it can vary linearly with x_1 in the fully-developed equilibrium region in which similarity obtains. According to eqn. (25), $\Lambda_1 \propto \Lambda_{1f} U_0/u_1'$, because $\overline{U}_1 \simeq U_0$ for very large x_1/D. In the similarity region, $\Lambda_{1f} \propto (x_1 D)^{1/2}$ and $u_1'/U_0 \propto (D/x_1)^{1/2}$; thus, $\Lambda_1 \propto x_1$. From the measurements of Townsend [8] it can be calculated that $\Lambda_1 \simeq 0.26\ x_1$.

With a linear relation for Λ_1, eqn. (16) can satisfy the conditions for similarity. The following values for the exponents in relation (23) have been obtained for self-preserving wake flow: $p = -1/2$, $q = 1/2$ and $s = -1$; thus, $n = 0$, and ε_m is independent of x_1. As stated earlier, a memory effect is no longer directly detectable under these circumstances. To determine the possibility of its existence, one must weigh the quantities of the terms in

eqn. (16) against each other. A rough estimate based on the measurements of Townsend yields

$$\left| \frac{A_1}{-\overline{u_2 u_1}} \frac{\partial}{\partial x_1}(-\overline{u_2 u_1}) \right| \simeq 0{,}2 \text{ bis } 0{,}3 \, ,$$

It is apparent that this term is not small. It is possible, then, that the memory effect in question is present in the entire wake flow, but is not directly detectable due to the similarity of the flow. In this case a theoretical solution with the Boussinesq relation for shear stress and an adjusted value of ε_m without a memory effect may be satisfactory.

4. Concluding Remarks

The above theoretical considerations, supported by experimental findings, have clearly shown that the additional memory effects in turbulence can play an essential role, especially if we are dealing with flows which fluctuate relatively rapidly in the flow direction, so that the turbulence has no opportunity to adapt completely to the local state of the mean flow. Then the Reynolds stresses, for example, can be thought of as consisting of two parts. The first part is practically "local" in character and is determined by the local state of the flow, while the second part is an additional contribution from the flow's prior history, i.e., an additional memory effect due to a relatively rapid change in the flow state. These additional memory effects can still be present in fully developed turbulent flows which are practically in equilibrium. If the flow is self-preserving, however, these memory effects are not directly detectable. The above investigations suggest that memory effects are always present in self-preserving wake flows, because these effects are negligibly small in a fully developed turbulent wall flow (boundary layer).

When we consider relations which are local in character, such as the Boussinesq relation, as a first approximation, the above expansions are nothing more than a second approximation. This point must be reemphasized.

As stated earlier, memory effects are directly detectable in the development region of a turbulent flow, and not only in a wake flow, but also in a free jet. An investigation of memory effects in the development region of a grid flow is currently in progress. It is known that the turbulence of the fully developed grid flow is axially symmetrical with $\overline{u_1^2} > \overline{u_2^2} = \overline{u_3^2}$. If these quantities are interpreted as normal stresses, the normal stress in the axial direction $\overline{u_1^2}$ is greater than the normal stresses in the transverse direction. Thus, the shear stress must be maximum in areas which form an angle $\pi/4$ with the axial direction. This is incompatible with the absence of a mean velocity gradient, if we assume a simple relation between shear stress and deformation rate. The mean flow and turbulence are not yet homogenous in the development region, and memory effects can produce the aforementioned anomaly, which may exist for an unlimited distance downstream provided the memory effects are sustained.

One difficulty in making a quantitative determination of memory effects is our lack of knowledge with respect to relaxation time, memory function and Lagrangian autocorrelation. Relation (25) is nothing more than an approximation. Moreover, the determination of the axial derivative of $\partial \overline{U}_1/\partial x_2$ in eqn. (11) and of $\partial/\partial x_1(-\overline{u_2 u_1})$ in eqn. (16) can be performed only with a low degree of accuracy. The large scatter of the points, as in Fig. 5 and 7, is due largely to this fact. In this respect a quantitative determination of the memory effect by means of eqn. (17), which requires integration instead of differentiation, offers significant advantages.

Our considerations here have been limited to turbulent momentum transfer. Naturally, such memory effects also play a role in the transport of other transferrable quantities such as heat and mass.

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