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ASYMPTOTIC EIGENSTRUCTURES*

by

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ABSTRACT

The behavior of the closed loop eigenstructure of a linear system with output feedback is analyzed as a single parameter multiplying the feedback gain is varied. An algorithm is presented that computes the asymptotically infinite eigenstructure, and it is shown how a system with high gain feedback decouples into single input single output systems. Then a synthesis algorithm is presented which uses full state feedback to achieve a desired asymptotic eigenstructure.

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I. Introduction

The closed looop eigenstructure of a linear system has long been recognized as an important consideration in the design of control systems. Most attention in the control literature has been given to the behavior of closed loop eigenvalues as parameters of the control system are varied, and the root locus has emerged as a tool to study this behavior. The behavior of closed loop eigenvectors has also received a lot of attention because for multi-input multi-output (MIMO) systems the placement of eigenvectors can be used to shape transient responses, decouple inputs and outputs, and reject certain types of disturbances.

The design procedure we have in mind is to specify an asymptotic eigenstructure that can be achieved with high gain state feedback. Then the design can be simplified to choosing a single gain so that bandwidth constraints are satisfied. A way to do this using the linear quadratic regulator (LQR) is given in [1,2]. The design procedure suggested there is to specify an asymptotic eigenstructure, choose quadratic weights based on these specifications, and then vary a single parameter multiplying the control weights. In this paper we propose the following alternative procedure: specify an asymptotic eigenstructure, choose a state feedback matrix based on these specifications, and then vary a single parameter multiplying this matrix. The alternative procedure does not have the advantages of guarenteed stability, phase and gain margins of the LQR; but is important none-the-less. Both procedures suffer the serious drawback of requiring full state feedback. A precursor to both of these procedures is in [3,4] where state feedback is used to achieve a non-asymptotic eigenstructure.

This paper starts with a review of the finite and infinite zero structure of the open loop system, and treats the infinite zeros as well defined quantities. The behavior of the closed loop eigenstructure is reviewed, and then an algorithm is presented to compute the asymptotically infinite eigenstructure. Next we show how a MIMO system can be decomposed into separate single-input single-output (SISO) systems that have the same asymptotically infinite eigenstructures. Then we present the previously mentioned algorithm for achieving a desired asymptotic eigenstructure with full state feedback, and then we finish with several examples.

Previous analysis techniques for multivariable root loci have appeared in [5,6]. In the former an algorithm is presented to compute the asymptotically infinite patterns of the closed loop eigenvalues. The first of our algorithms is similar to this, but differs importantly in that we compute the asymptotic behavior of the eigenvectors and we introduce a subspace decomposition of R^m which can be used decompose the system into SISO parts. In the latter reference the root locus is interpreted as living on a Riemann surface. We have not used this approach because of computational reasons and because we are not yet convinced that an engineer trying to design a control system need concern himself with Riemann surfaces.

The linear systems we consider are restricted to having the same number of inputs and outputs, being controllable and observable, being nondegenerate, and having distinct finite zeros. Further restrictions are placed on the allowable asymptotically infinite behavior of the root locus.

In later work we hope to remove some or all of these restrictions and to study in more detail the synthesis of output feedback.

Notation

Matrices are denoted by capital letters, scalars and vectors by lower case letters. A^T and y^H are the transpose of A and the Hermitian transpose of y. Subspaces are denoted by script letters, with the exception of $\mathbb{R}^n$. "Im A" and "ker A" are the image and kernel of the linear map A. The dimension of $\mathcal{U}$ is dim $\mathcal{U}$, subspace inclusion is $\subset$, subspace intersection is $\cap$, and linear combination of subspaces is $\mathcal{U} + \mathcal{V}$.

II. Pole Zero Configuration of an Open Loop System

We are concerned with the following linear time invariant system:

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx \tag{2}$$

where

$$x \in \mathbb{R}^n$$

$$u \in \mathbb{R}^m$$

$$y \in \mathbb{R}^m \ .$$

The number of inputs and outputs are equal. We assume that the realization is minimal, which is equivalent to assuming that (A,B) is controllable and (A,C) is observable. The open loop system has n poles and n zeros associated with it, and in this section we review their dynamic interpretations.

The open loop poles are the n eigenvalues of A. The open loop poles are the complex frequencies that can appear in the output without appearing in the input.

There are n zeros associated with the open loop system, of which p are finite and n-p are infinite. The zeros are sometimes called "transmission zeros." The finite zeros are defined to be those finite values of s which reduce the rank of

$$\begin{bmatrix} A-sI & B \\ -C & 0 \end{bmatrix} \ .$$

We assume that the finite zeros are distinct and that the system is not degenerate in the sense that not all values of s in the complex plane reduce the rank of the matrix. The finite zeros are the finite solutions s_i^0 of the generalized eigenvalue problem [7],

$$\begin{bmatrix} A-s_i^0 I & B \\ -C & 0 \end{bmatrix} \begin{bmatrix} x_i^0 \\ \nu_i^0 \end{bmatrix} = 0 \qquad i = 1,\ldots, p. \tag{3}$$

Under our assumptions the number of finite zeros is $0 \leq p \leq n-m$. Associated with each finite zero are right zero directions x_i^0 and ν_i^0. It is also possible to define the generalized eigenvalue problem

$$[y_i^{0H} \;\; \eta_i^{0H}] \begin{bmatrix} A-s_i^0 I & B \\ -C & 0 \end{bmatrix} = 0 \qquad i=1,\ldots, p, \tag{4}$$

and associated with each finite zero are also the left zero directions y_i^{0H} and η_i^{0H}. The finite zeros are the complex frequencies "absorbed" by the system in the following sense. If at time $t=0$ the system is at state $x(0) = x_i^0$, then an input of the form $u(t) = \nu_i^0 e^{s_i t}$ for $t \geq 0$ will result in $y(t) = 0$ for $t \geq 0$ [8].

The n-p zeroes at infinity are well defined and can be given the following interpretation [9]. If the input is of the form

$$u(t) = \sum_{i=0}^{\ell} u_i \delta^{(i)}(t) \; ; \tag{5}$$

where $\delta^0(t)$ is an impulse, $\delta^1(t)$ is a doublet, and so on; and if the initial state of time $t = 0^-$ is

$$x(0^-) = -\sum_{i=0}^{\ell} A^i B u_i \; , \tag{6}$$

where u_0 is arbitrary and the remaining u_i vectors satisfy

$$\begin{bmatrix} CB & CAB & & CA^{\ell-1}B \\ & \ddots & \ddots & \\ & & \ddots & CAB \\ 0 & & & CB \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_\ell \end{bmatrix} = 0 \; ; \tag{7}$$

then the output $y(t) = 0$ for $t \geq 0$. The large matrix in (7) is called a Toeplitz matrix. The integer ℓ is characteristic of the open loop system and will be interpreted in the next section.

III. The Asymptotic Eigenstructure of the Closed Loop System

We use output feedback of the form

$$u = -\frac{1}{k} Ky , \tag{8}$$

where K is an mxm invertiable matrix and k is a real number in the range $0 \leq k \leq \infty$. The closed loop system matrix is

$$A_{c\ell} = A - \frac{1}{k} BKC , \tag{9}$$

and its eigenvalues, right, and left eigenvectors are defined in the usual way by

$$(A_{c\ell} - s_i I)x_i = 0 \qquad i=1,\ldots,n \tag{10}$$

$$y_i^H(A_{c\ell} - s_i I) = 0 \qquad i=1,\ldots,n \quad . \tag{11}$$

As k is varied from infinity down to zero the n closed loop eigenvalues trace out a root locus on the complex s plane and the right and left eigenvectors "spin" in $\mathbb{R}^n$. We are particularly interested in asymptotic behavior as $k \to 0$, which we now review.

As $k \to 0$ p of the n branches of the root locus approach finite zeros. The right and left eigenvectors associated with the eigenvalues on the finite branches approach the right and left zero directions. The finite zeros and zero directions can be computed using the generalized eigenvalue problems (3,4), and for notational convience we group the solutions into

$$S^0 = \text{diag } (s_1^0,\ldots, s_p^0)$$

$$X^0 = [x_1^0, \ldots, x_p^0]$$

$$Y^0 = [y_1^0, \ldots, y_p^0] .$$

The remaining n-p branches approach the infinite zeros. The n-p right eigenvectors associated with the eigenvalues on the infinite branches will asymptotically span the same subspace of R^n spanned by all possible $x(0^-)$ of (6), and a similar property can be derived for the left eigenvectors.

The information we have given so far about the finite and infinite eigenstructure can be computed knowing only the open loop system, and does not depend on the output feedback gain matrix K. Unfortunately this information is not enough for a good analysis of a control system, and this is especially true for an analysis of the asymptotically infinite eigenstructure. In the rest of this section we list properties of the asymptotically infinite eigenstructure, most of which depend on K, and in the next section we show how these propertiies can be computed.

The infinite branches of the root locus break into m patterns. The order of the i^{th} pattern is n_i, where "order" is defined to be the number of closed loop eigenvalues in the pattern. The following identity must be true:

$$\sum_{i=1}^{m} n_i = n-p \quad ,$$

and the highest order pattern is r. It turns out that $r = \ell+1$, where ℓ is given in (5,6,7). There are n_i asymptotes in the i^{th} pattern. The closed loop eigenvalues approach these asymptotes and have a magnitude approximately equal to

$$\left(\frac{|s_i^\infty|}{k}\right)^{\frac{1}{n_i}} \qquad .$$

The asymptotes are spaced $(360/n_i)°$ apart, make angles with the positive real axis of

$$\left(\frac{\arg\,(-s_i^\infty) + \ell 360}{n_i}\right)° \qquad \ell = 0,1,\ldots,\, n_i-1 \ ,$$

and have a center of gravity ψ_i. The n_i right and left eigenvectors associated with the i^{th} pattern asymptotically span the same subspaces of $\mathbb{R}^n$ as

$$B\nu_i^\infty,\ AB\nu_i^\infty,\ldots,\ A^{n_i-1}\,B\nu_i^\infty$$

$$C^T\eta_i^\infty,\ (AC)^T\eta_i^\infty,\ldots,\ (A^{n_i-1}C)^T\eta_i^\infty \ .$$

For notational convenience we group the asymptotic properties into

$$S^\infty = \mathrm{diag}(s_1^\infty,\ldots,\, s_m^\infty)$$

$$N^\infty = [\nu_1^\infty,\ldots,\, \nu_m^\infty]$$

$$M^\infty = [\eta_1^\infty,\ldots,\, \eta_m^\infty]$$

$$\psi = (\psi_1,\ldots,\, \psi_m)$$

$$\ell = (01,\ 11,\ 21,\ldots,\ [n_1-1]1,$$
$$02,\ 12,\ 22,\ldots,\ [n_2-1]2,\ldots[n_m-1]m).]$$

Each part ij of the multi-index γ describes the $A^i B \nu_j^\infty$ vector. The complex numbers s_i^∞ and ψ_i^∞ and the complex m-vectors ν_i^∞ and η_i^∞ must occur in complex conjugate pairs.

IV. An Algorithm to Compute the Asymptotically Infinite Eigenstructure

The problem of finding the asymptotically infinite eigenstructure can be concisely stated as: Given A,B,C, and K find S^{∞}, N^{∞}, M^{∞}, ψ, and γ. We first go through a series of definitions and then solve the problem. The algorithm can be programmed on a computer using the stable numerical algorithms of EISPACK [10] (specifically singular value decomposition (SVD) and the eigenvalue problem), but we make no claims that our algorithm is the "fastest" or "best". We apologize for the notational nightmare that follows, but we do not yet know of any way to avoid it.

Define the matrices

$$G_i = CA^{i-1}B \qquad i=1,\ldots, r$$

where r is the highest order pattern, and define the subspaces of $\mathbb{R}^n$

$$\mathcal{U}_0 = \mathbb{R}^m$$

$$\mathcal{U}_1 = \ker G_1$$

$$\vdots$$

$$\mathcal{U}_r = \ker G_1 \cap \ldots \cap \ker G_r$$

$$\mathcal{V}_0 = \mathbb{R}^m$$

$$\mathcal{V}_1 = \ker G_1^T$$

$$\vdots$$

$$\mathcal{V}_r = \ker G_1^T \cap \ldots \cap \ker G_r^T \quad .$$

The subspaces are nested so that

$$0 = \mathcal{U}_r \subseteq \cdots \subseteq \mathcal{U}_0 = \mathbb{R}^m$$

$$0 = \mathcal{V}_r \subseteq \cdots \subseteq \mathcal{V}_0 = \mathbb{R}^m ,$$

and their dimensions are

$$\ell_i = \dim \mathcal{U}_i = \dim \mathcal{V}_i \qquad i = 0,\ldots, r .$$

Define the indices

$$m_i = \ell_{i-1} - \ell_i \qquad i = 1,\ldots, r ,$$

and it follows that

$$\sum_{i=1}^{r} m_i = m .$$

We restrict our attention to systems for which m_i is the number of i^{th} order patterns and ℓ_i is the number of patterns greater than i^{th} order. Next we define the matrices

$$U_i , \quad V_i \qquad i=0,1,\ldots, r-1$$

where the columns of U_i form a basis for $\mathcal{U}_i$ and the columns of V_i form a basis for $\mathcal{V}_i$. The dimensions of both U_i and V_i are $m\times\ell_i$. We decompose S^∞, N^∞, and M^∞ into

$$S^\infty = \operatorname{diag} (S_1^\infty,\ldots, S_r^\infty)$$

$$N^\infty = [N_1^\infty,\ldots, N_r^\infty]$$

$$M^\infty = [M_1^\infty,\ldots, M_r^\infty] .$$

In general each S_i^∞ is a $m_i x m_i$ block diagonal matrix, but we restrict our attention to systems for which each S_i^∞ is diagonal. The diagonal elements are the s_j^{∞}'s associated with i^{th} order patterns. Each of the N_i^{∞} and M_i^{∞} matrices have dimensions $m \times m_i$ and have as columns the ν_j^{∞}'s and η_j^{∞}'s associated with i^{th} order patterns. As we will see later,

$$\left.\begin{aligned} U_i &= \text{Im } N_{i+1}^{\infty} + \ldots + \text{Im } N_r^{\infty} \\ V_i &= \text{Im } M_{i+1}^{\infty} + \ldots + \text{Im } M_r^{\infty} \end{aligned}\right\} \quad i=0, \ldots, r-1$$

and in general

$$\left.\begin{aligned} \text{Im } N_i^{\infty} \cap \text{Im } N_j^{\infty} \neq 0 \\ \text{Im } M_i \cap \text{Im } M_j^{\infty} \neq 0 \end{aligned}\right\} \quad i \neq j \quad .$$

In words the top relationships tell us that one of the possible sets of basis vectors of U_i is the ν_j^{∞}'s associated with patterns of greater than i^{th} order, and that in general the ν_j^{∞}'s are not orthogonal. A similar statement can be made about the η_j^{∞}'s. The last definitions are

$$T_i = (V_{i-1}^H K^{-1} U_{i-1})^{-1} (V_{i-1}^H G_i U_{i-1}) \qquad i=1,\ldots, r$$

and the Jordan form decompositions

$$T_i = \begin{bmatrix} W_{i1} & W_{i2} \end{bmatrix} \begin{bmatrix} \Lambda_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_{i3}^H \\ W_{i4}^H \end{bmatrix} \qquad i=1,\ldots, r \ .$$

The matrices W_{i1} and W_{i3} have as many columns as there are i^{th} order patterns, and W_{i2} and W_{i4} have as many columns as there are greater than i^{th} order patterns. The dimensions of T_i are $\ell_{i-1} \times \ell_{i-1}$, W_{i1} and W_{i3} are $\ell_{i-1} \times m_i$, and W_{i2} and W_{i4} are $\ell_{i-1} \times \ell_i$. Finally we conclude this long paragraph with the fact that if there are no ith order patterns then $m_i=0$, $\ell_{i-1} = \ell_i$, $\mathcal{U}_{i-1} = \mathcal{U}_i$, $\mathcal{V}_{i-1} = \mathcal{V}_i$, $T_i = 0$, and the S_i^{∞}, N_i^{∞}, M_i^{∞}, W_{i1}, and W_{i3} matrices vanish.

The algorithm to find S^{∞}, N^{∞}, M^{∞}, γ, and ψ is:

1) Use SVD to compute U_i, V_i, and m_i for $i=1,\ldots,r$.

2) Use m_i for $i=1,\ldots,r$ to compute γ.

3) Compute T_i for $i=1,\ldots, r$.

4) Compute the Jordan form decompositions of T_i for $i=1,\ldots,r$.

5) Compute S_i^{∞}, N_i^{∞}, and M_i^{∞} for $i=1,\ldots,r$ by

$$S_i^{\infty} = \Lambda_i$$

$$N_i^{\infty} = U_{i-1}W_{i1}$$

$$M_i^{\infty} = V_{i-1}W_{i3} \quad .$$

6) Compute ψ_i for $i=1,\ldots, m$ by

$$\psi_i = \frac{\beta_i}{\eta_i} \quad ,$$

where $$\beta_i = \frac{\eta_i^{\infty H} CA^{\eta_i} B\nu_i^{\infty}}{\eta_i^{\infty H} CA^{\eta_i - 1} B\nu_i^{\infty}} \quad .$$

The following explanation of the first step may be helpful. The columns of U_i form a basis for the kernel of

$$\begin{bmatrix} CB \\ \vdots \\ CA^{i}B \end{bmatrix} \qquad i=1,\ldots, r-1 \; ,$$

so to compute U_i we can use SVD to compute an orthonormal basis of this tall skinny matrix. Likewise V_i can be found by computing the kernel of

$$\begin{bmatrix} (CB)^T \\ \vdots \\ (CA^{i}B)^T \end{bmatrix} \qquad i=1,\ldots,r-1 \quad .$$

It is not necessary to fill U_i and V_i with orthonormal vectors, but this is a convenient by-product of SVD. There are many other ways to compute the kernel of a matrix, and SVD is almost certainly not the method to use when computing the U_i's and V_i's by hand. However, there may be difficulty determining the dim $\mathcal{U}_i$, and SVD is the most reliable way to do this. If there is difficulty determining the dim $\mathcal{U}_i$ then the root locus may have "strange" behavior such as asymptotically infinite patterns that shift orders at large radii.

We note that we have used the last column of a Toeplitz matrix to find the order of each of the asymptotically infinite root locus patterns. We have restricted our attention to systems for which this can be done. In general the entire Toeplitz matrix must be used.

The generic case is when Rank (CB) = m. This should be viewed as a mathematical property, see [11] for a precise definition of "generic." If the system to be analyzed is generic then it has n-m transmission zeros and m first order infinite patterns. The s_i^∞'s, ν_i^∞'s , and η_i^∞'s make up the eigenstructure of KCB. For design purposes the generic case is too restrictive to be of interest. This is easiest to see for SISO systems, for which root loci with second and higher order infinite patterns are commonplace.

V. Proof of Analysis Algorithm

In this section we prove the fifth step of the algorithm to find the asymptotically infinite eigenstructure. The proof of the sixth step we defer until the next section. The first four steps do not need to be proved.

The proof of the first step is by induction and uses the fact that all of the closed loop eigenvalues s_j and the associated ν_j vectors must satisfy

$$[kK^{-1} + \Phi(s_j)]\nu_j = 0 \qquad j = 1,\ldots, n \tag{12}$$

where

$$\Phi(s) = C(sI-A)^{-1}B$$

$$= \sum_{i=1}^{\infty} \frac{1}{s^i} G_i \ .$$

To show that (12) is true we note from (1,2,8,9,10, and 11) that

$$\begin{bmatrix} A-s_jI & B \\ -C & -kK^{-1} \end{bmatrix} \begin{bmatrix} x_j \\ \nu_j \end{bmatrix} = 0 \qquad j=1,\ldots, n$$

and therefore for $j=1,\ldots,n$ we have that

$$(A-s_jI)x_j + B\nu_j = 0$$

$$Cx_j = -kK^{-1}\nu_j = C(s_jI-A)^{-1}B\nu_j$$

$$[kK^{-1} + C(s_jI-A)^{-1}B]\nu_j = 0.$$

The first step of the induction proof is to show that the fifth step is valid for S_1^∞, N_1^∞, and M_1^∞. We assume without loss of generality that first order patterns exist. Equation (12) can be rewritten

$$[kK^{-1} + \frac{1}{s_j} G_1 + o(\frac{1}{s_j})]\nu_j = 0 \qquad j=1,\ldots, n$$

As $k \to 0$ this becomes

$$[s_j^\infty I - T_1]\nu_j^\infty = 0 \qquad j=1,\ldots,m$$

where

$$s_j^\infty = -ks_j$$

The nonzero eigenvalues of T_1 correspond to first order patterns because the closed loop eigenvalues s_j are solutions of

$$s_j = \frac{-s_j^\infty}{k} .$$

The single solution s_j traces out a first order pattern. The right and left eigenvectors of T_1 are the ν_j^∞'s and η_j^∞'s associated with first order patterns, and therefore the fifth step is true for i=1. The ν_j^∞'s associated with second and higher order patterns lie in the kernel of T_1, which is U_1. Heuristically speaking, there ν_j^∞'s are not "trapped" by the s_j^{-1} term.

The next step in the induction proof is to assume that S_{i-1}^∞, N_{i-1}^∞, and M_{i-1}^∞ are valid and then show that S_i^∞, N_i^∞, and M_i^∞ are valid. If N_{i-1}^∞ is valid then the ν_j^∞ vectors associated with $\geq i^{th}$ order patterns form a basis for U_{i-1}. Therefore for each of these ν_j^∞'s there exists a ω_j such that $\nu_j^\infty = U_{i-1}\omega_j$. Substituting into (12) we get

$$[kK^{-1} + \Phi(s_j)]U_{i-1}\omega_j = 0 \qquad j = 1,\ldots, \ell_{i-1} .$$

Multiply on the left to get

$$V_{i-1}^H[kK^{-1} + \Phi(s_j)]\, U_{i-1}\omega_j = 0 \qquad j = 1,\ldots, \ell_{i-1} ,$$

which reduces to

$$\left[kV_{i-1}^H K^{-1} U_{i-1} + \frac{1}{s_j^i} V_{i-1}^H G_i U_{i-1} + o\left(\frac{1}{s_j^i}\right)\right] \omega_j = 0$$

$$j=1,\ldots,\ell_{i-1} \quad .$$

As $k \to 0$ this becomes

$$(s_j^\infty I - T_i)\omega_j = 0 \qquad j=1,\ldots, \ell_{i-1}$$

where

$$s_j^\infty = -ks_j^i \quad .$$

The nonzero eigenvalues of T_i correspond to i^{th} order patterns because the closed loop eigenvalues s_j are the i solutions of

$$s_j^i = \frac{-s_j^\infty}{k} \quad .$$

The ω_j's can be used to compute $\nu_j^\infty = U_{i-1}\omega_j$ and therefore $N_{i1}^\infty = U_{i-1}W_{i1}$. Using a similar argument $M_i^\infty = V_{i-1}W_{i3}$. The ν_j^∞'s associated with greater than i^{th} order patterns are not "trapped" by the s_j^{-i} term of (12) and therefore lie in U_i. This completes the proof.

VI. <u>A High Gain Decomposition of the System into m SISO Systems</u>

There are m asymptotically infinite patterns of the root locus, and as we will now show we can decompose the MIMO system into m SISO systems, each of which has one of the infinite patterns. After doing this we will prove the sixth step of the analysis algorithm (finding the center of gravity of the infinite patterns).

Use the ν_i^∞ and η_i^∞ vectors to define the following SISO systems:

$$\left.\begin{aligned} \dot{x} &= Ax + b_i u \\ y &= c_i x \\ u &= -\frac{1}{k_i} y \end{aligned}\right\} \qquad i = 1, \ldots, m$$

where

$$b_i = B\nu_i^\infty$$

$$c_i = \eta_i^{\infty H} C$$

$$k_i = k\eta_i^{\infty H} K^{-1} \nu_i^\infty \quad .$$

The return difference equation of each of the SISO systems is $1 + g_i(s)/k_i$ where

$$g_i(s) = c_i(sI - A)^{-1} b_i$$

$$= \sum_{j=n_i}^{\infty} \frac{1}{s^j} \eta_i^{\infty H} CA^{j-1} B\nu_i^\infty \qquad i=1,\ldots, m.$$

Since the closed loop poles are the zeroes of the return difference equation we have that

$$1 + g_i(s)/k_i = 0 \qquad i=1,\dots, m \quad ,$$

which can be rewritten as

$$\frac{k_i}{g_i(s)} + 1 = 0 \qquad i=1,\dots, m \quad .$$

Carry out the long division and the result is

$$s^{n_i} - \beta_i s^{n_i-1} + \dots + \frac{1}{k} s_i^{\infty} + \dots = 0 \qquad i=1,\dots,m \tag{13}$$

where

$$\beta_i = \frac{\eta_i^{\infty} C A^{n_i} B \nu_i^{\infty}}{\eta_i^{\infty} C A^{n_i-1} B \nu_i^{\infty}}$$

$$s_i^{\infty} = \frac{\eta_i^{\infty} C A^{n_i-1} B \nu_i^{\infty}}{\eta_i^{\infty} K^{-1} \nu_i^{\infty}} \quad .$$

To verify that the asymptotically infinite pattern of each of the SISO systems is the same as one of the patterns of the larger system we need only note that as $k \to 0$ (13) can be approximated by

$$s^{n_i} = \frac{-1}{k} s_i^{\infty} \qquad i=1,\dots, m \quad .$$

To verify the sixth step of the analysis algorithm of section IV we approximate (13) by

$$s^{n_i} - \beta_i s^{n_i-1} + \frac{1}{k} s_i^{\infty} = 0 \qquad i=1,\dots, m \quad .$$

The sum of the n_i infinite eigenvalues in this pattern must equal β_i and therefore the center of gravity is

$$\psi_i = \frac{\beta_i}{n_i} \qquad i=1,\dots, m \quad .$$

For $n_i = 1$ the term "center of gravity" could be more appropriately replaced by "starting point."

Some of the SISO systems may have complex coefficients. When this happens a second system exists which has the complex conjugate coefficients. To work with only real coefficients a 2x2 system must be used. The closed loop eigenvalues of the combined asymptotic pattern are solutions of

$$(s^{n_i} + \frac{1}{k} s_i^{\infty})(s^{n_i} + \frac{1}{k} \bar{s}_i^{\infty}) = 0,$$

which is

$$s^{2n_i} + \frac{2}{k} \text{Re}(s_i) s^{n_i} + \frac{1}{k^2} |s_i^{\infty}|^2 = 0 \quad .$$

The center of gravity of the combined asymptotic pattern can be found to be $\text{Re}(\psi_i)$.

VII. An Algorithm to Synthesis an Asymptotic Eigenstructure with Full State Feedback

We consider a system with full state feedback:

$$\dot{x} = Ax + Bu$$

$$u = -\frac{1}{k} Fx$$

where

$$x \in \mathbb{R}^n$$

$$u \in \mathbb{R}^m \quad .$$

We assume that (A,B) is controllable. The problem is to choose a feedback matrix F such that as $k \to 0$ the closed loop system has achieved a described eigenstructure. Concisely stated the problem is: Given S^0, X^0, S^∞, N^∞, and γ find F.

The solution is

$$F = N^\infty S^\infty [0 \quad I][X^0,\ U^\gamma P]^{-1} \tag{14}$$

where

$$U^\gamma = [B\nu_1^\infty,\ AB\nu_1^\infty, \ldots, A^{n_1-1} B\nu_1^\infty$$
$$B\nu_2^\infty\ ,\ AB\nu_2^\infty, \ldots,\ A^{n_2-1} B\nu_2^\infty, \ldots,\ A^{n_m-1} B\nu_m^{\infty}]$$

P = Permutation matrix that rearranges the columns of U^γ in an arbitary way except that the last m columns of U^γ are

$$A^{n_1-1} B\nu_1^\infty,\ A^{n_2-1} B\nu_2^\infty, \ldots,\ A^{n_m-1} B\nu_m^\infty$$

In general the X^0, S^∞, N^∞, and U^γ contain complex numbers. To work just with real numbers then replace the complex conjugate columns of X^0, N^∞, and U^γ with their real and imaginary parts, and replace the 2x2 blocks

$$\begin{bmatrix} s_i^\infty & \\ & \bar{s}_i^\infty \end{bmatrix}$$

in S^∞ by

$$\begin{bmatrix} \mathrm{Re}\ s_i^\infty & \mathrm{Im}\ s_i^\infty \\ -\mathrm{Im}\ s_i^\infty & \mathrm{Re}\ s_i^\infty \end{bmatrix} .$$

One way to interpret this result is that state feedback is used to place the finite and infinice zeros and zero directions, and as the feedback gain is increased the closed loop eigenvalues and eigenvectors approach these zeros and zero directions.

The desired asymptotic eigenstructure is not completely arbitrary. We restrict our attention to cases where the finite zeros are distinct and the S^∞ matrix is diagonal. More fundamental are the restrictions that

(1) the complex s_i^0's, s_i^∞'s, x_i^0's, and ν_i^∞'s occur in complex conjugate pairs;

(2) for each zero direction x_i^0 there must exist a ν_i^0 such that $(A-s_i^0 I)x_i^0 + B\nu_i^0 = 0$;

(3) the number of finite zeros must be $0 \leq p \leq n-m$;

(4) the columns of X^0 and U^γ must be linearly independent; and

(5) the multi-index γ must specify m asymptotically infinite patterns whose orders total to n-p.

To prove that (14) is true we compute the asymptotic eigenstructure using the previous methods described in this paper. Once F is fixed we can treat the state feedback problem as an output feedback problem

$$\dot{x} = Ax + Bu$$

$$y = Fx$$

$$u = -\frac{1}{k} Iy \quad .$$

The asymptotically finite eigenstructure must satisfy

$$\begin{bmatrix} A-s_i^0 & B \\ -F & 0 \end{bmatrix} \begin{bmatrix} x_i^0 \\ \nu_i^0 \end{bmatrix} = 0 \qquad i = 1,\dots, p \quad .$$

This is true because we have assumed that a ν_i^0 exists such that $(A-s_i^0 I)x_i^0 + B\nu_i^0 = 0$, and because F is constructed in such a way that $Fx_i^0 = 0$. Next we show (for $i=1,\dots,m$) that the i^{th} asymptotically infinite pattern is $n_i{}^{th}$ order and has s_i^∞ and ν_i^∞ associated with it. By the way F is constructed $FU^\gamma P = [0 \quad N^\infty S^\infty]$ and in particular $FA^{n_i-1} B\nu_i^\infty = \nu_i^\infty s_i^\infty$. So if we start with

$$[kI + \Phi(s_i)]\nu_i = 0 \qquad i=1,\dots, n \tag{15}$$

where $\quad \Phi(s) = F(sI-A)^{-1}B$, and if we let $\nu_i = \nu_i^\infty$ for $i=1,\dots,m$, then as $k\to 0$ (15) can be rewritten

$$\left[kI + \frac{s_i^\infty}{s_i^{n_i}} I\right] \nu_i^\infty = 0 \qquad i=1,\dots, m \quad . \tag{16}$$

Equation (16) is true if $s_i^{n_i} = -s_i^{\infty}/k$, which is the equation for an i^{th} order pattern. This completes the proof of the synthesis algorithm.

We note that once we have chosen S^0, X^0, S^{∞}, N^{∞}, and γ there does not exist any extra freedom to place the center of gravities ψ_i. This is easiest to see for the SISO case because it is a well known classical root locus result that the open loop poles and finite zeros determine the center of gravity of the single asymptotically infinite pattern.

Equation (14) can be simplified in the generic case (Rank (CB) = m). Then we have $U^{\gamma} = \Sigma N^{\infty}$ and (14) becomes

$$F = N^{\infty}S^{\infty}(N^{\infty})^{-1}[0 \quad I][X^0B]^{-1} . \tag{17}$$

Equation (14) can be simplified even more in the SISO case when the state space realization is in controllable canonical form. For SISO systems there is only one asymptotically infinite pattern, it suffices to set $\nu_i^{\infty} = 1$, and there does not exist any freedom to place the zero directions x_i^0, $i=1,\ldots, p$. The procedure is to choose the finite zeros s_i^0 for $i=1,\ldots, p$ where $0 \le p \le n-1$; then form the polynominal

$$\prod_{i=1}^{p} (s-s_i^0) = s^p + \beta_{p-1}s^{p-1} +\ldots+ \beta_0 ;$$

and then the state feedback matrix is

$$F = [\beta_0, \beta_1,\ldots, \beta_{p-1}, 0,\ldots, 0] . \tag{18}$$

VIII. Examples

In the first example we analyze the asymptotically infinite eigenstructure of a linear system. Then we use full state feedback to achieve a specified asymptotic eigenstructure.

Example 1

We use the same system as example 2 of [5]. Given the following A,B,C, and K matrices we use the algorithm of section IV to find the asymptotically infinite eigenstructure.

$$A = \begin{bmatrix} 12 & -60 & 28 & -4 & 32 & 36 & -54 \\ 28 & -75 & 12 & 30 & 0 & 45 & -92 \\ -8 & -134 & 33 & 48 & 12 & 71 & 0 \\ -8 & -14 & -6 & 21 & -18 & 5 & 24 \\ -4 & 62 & -21 & -12 & -18 & -35 & 24 \\ 64 & -90 & 18 & 24 & 12 & 60 & -200 \\ 9 & -22 & 10 & -2 & 12 & 14 & -33 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 0 & 8 \\ 2 & 1 & 3 \\ 6 & 4 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & 1 \\ 2 & 1 & 5 \\ 2 & 0 & 3 \end{bmatrix} \qquad C^T = \begin{bmatrix} -3 & 2 & 3 \\ 11 & 7 & -1 \\ 1 & 4 & 1 \\ -9 & -11 & -1 \\ 5 & 10 & 2 \\ -6 & -3 & 1 \\ 9 & -7 & -9 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The open loop eigenvalues are -3, -2, -1, 0, 1, 2, 3. To find the U_i and V_i matrices we need

$$CB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad CAB = \begin{bmatrix} -2 & 3 & -5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$CA^2B = \begin{bmatrix} 4 & -5 & 17 \\ 0 & -1 & -7 \\ 0 & 0 & -8 \end{bmatrix} \qquad CA^3B = \begin{bmatrix} -8 & -3 & -23 \\ 0 & -11 & 35 \\ 0 & 0 & 24 \end{bmatrix} .$$

It can easily be verified that

$$U_1 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad U_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$m_1 = 1 \qquad m_2 = 1 \qquad m_3 = 1$$

At this point we know there will be a first, second, and third order infinite pattern. The multi-index γ is therefore

$$\gamma = (01,\ 02,\ 12,\ 03,\ 13,\ 23)$$

where

$$n_1 = 1,\ n_2 = 2,\ \text{and}\ n_3 = 3 \quad .$$

The T_i's and their Jordan form decompositions are

$$T_1 = KCB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = (V_1^H K^{-1} U_1)^{-1}\ V_1^H CABU_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$T_3 = (V_2^H K^{-1} U_2)^{-1} V_2^H CA^2 BU_2 = -8 \quad .$$

Therefore we have that

$$s_1^\infty = 1 \qquad \nu_1^\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \eta_1^\infty = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$s_2^\infty = 1 \qquad \nu_2^\infty = U_1W_{21} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \eta_2^\infty = V_1W_{23} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$s_3^\infty = -8 \qquad \nu_3^\infty = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad \eta_3^\infty = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad .$$

The center of gravities are

$$\psi_1 = -2$$

$$\psi_2 = - 1/2$$

$$\psi_3 = -1 \quad .$$

In summary, there is one first order pattern along the negative real axis with a radius of k^{-1}; there is one second order pattern with angles $\pm$ 90°, radius $k^{-1/2}$, and center of gravity -1/2; and there is one third order pattern with angles 0°, ± 120°, radius $2k^{-1/2}$, and center of gravity -1.

Example 2

Consider a system with

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -5 & -4 & 0.1 & 1 \\ 0.1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For three different cases we specify the asymptotically infinite eigenstructure and then compute the state feedback matrix F. Since this is a generic example we use equation (17) to compute F. The asymptotically finite eigenstructure is the same in each case.

$$s^0_{1,2} = -3\pm 2i \qquad x^0_{1,2} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -4 \end{bmatrix} \pm \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix} i \ .$$

The infinite eigenstructure differs for each case. The specifications and the resulting F matrices are shown in Table 1. The root loci are shown in Figure 1.

In the first case we have $s^\infty_1 = 1$ and $s^\infty_2 = 2$. The two first order patterns stay on the negative real axis. In the second case $s^\infty_{1,2} = 1 \pm \sqrt{3}\, i$ The two first order patterns make angles of ±120° with the positive real axis. In the third case $s^\infty_{1,2} = -\sqrt{3} \pm i$. Again there are two first order patterns, this time making angles of ±30°, and the system is unstable for high gain.

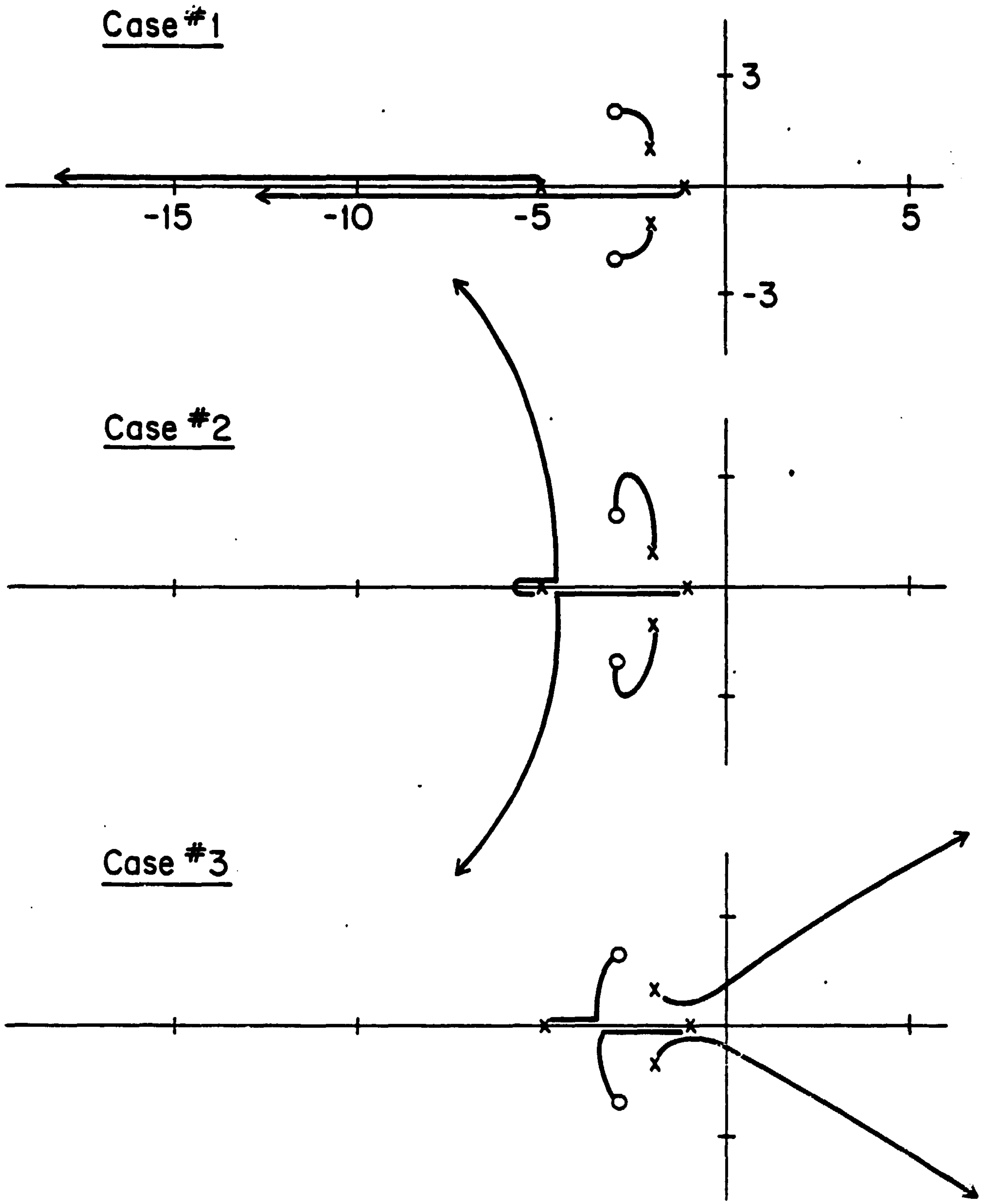
Table 1

Matrices Used in Example 2

Case	$\tilde{N}\tilde{S}(\tilde{N})^{-1}$	F
1	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 16 & 4 & 0 & 2 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 1 \\ -4 & 2 \end{bmatrix}$	$\begin{bmatrix} 8 & 2 & 0 & 1 \\ 16 & 4 & -4 & 2 \end{bmatrix}$
3	$\begin{bmatrix} 0 & 1 \\ -4 & -2\sqrt{3} \end{bmatrix}$	$\begin{bmatrix} 8 & 2 & 0 & 1 \\ -27.713 & -6.928 & -4 & -3.464 \end{bmatrix}$

Figure 1

Root Loci for Example 2



IX. Conclusions

For a restricted class of linear systems we have shown how to analyze the asymptotic eigenstructure as control gains get very large. More importantly, we have shown how to decompose MIMO systems into SISO systems that have the same asymptotically infinite eigenstructures, and we have shown how to use state feedback to achieve a desired asymptotic eigenstructure.

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