IMPLEMENTATION OF A TRAPEZOIDAL RING ELEMENT IN NASTRAN FOR ELASTIC-PLASTIC ANALYSIS

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SUMMARY

The explicit expressions for an elastic-plastic trapezoidal ring element are presented and implemented in NASTRAN computer program. The material is assumed to obey the von Mises' yield criterion, isotropic hardening rule and the Prandtl-Reuss flow relations. For the purpose of demonstration, two elastic-plastic problems are solved and compared with previous results. The first is a plane-strain tube under uniform internal pressure and the second, a finite-length tube loaded over part of its inner surface. A very good agreement has been found in both test problems.

INTRODUCTION

In recent years finite element method has been widely used for solving complex nonlinear problems and many large-scale general purpose computer programs have been developed (ref. 1). The MARC and ANSYS systems have found wide applications, yet they are quite expensive. The piecewise linear analysis option of the NASTRAN program provides an algorithm for solving nonlinear problems in material plasticity (ref. 2). The load is applied in increments such that the stiffness properties can be assumed to be constant over each increment. However, the usefulness of this option is quite limited because only a few elements have been implemented. These include rod, tube, bar elements for one-dimensional problems and plate elements for two-dimensional plane stress problems. This paper describes the implementation of a trapezoidal ring element in NASTRAN for solving elastic-plastic problems of rotational symmetry.

The theoretical basis of our implementation follows the approach first developed by Swedlow (ref. 3). A unique relationship between the octahedral stress and the plastic octahedral strain is assumed to exist. The material is assumed to obey the Mises' yield criterion, isotropic hardening rule and the Prandtl-Reuss flow relations. The explicit expressions for axisymmetric plasticity are derived. The element stiffness matrix and the stress data recovery routines for the trapezoidal ring element are developed. Seven new subroutines are implemented and added to the NASTRAN code.

For the purpose of demonstration, two elastic-plastic test problems are solved. The first is an infinitely long tube under uniform internal pressure. The NASTRAN results are in excellent agreement with an exact solution based on a finite-difference approach (ref. 4). The second problem is a thick-walled cylinder of finite length loaded over part of its inner surface. The NASTRAN results are compared with those obtained by a two-dimensional code with the use of quadrilateral ring elements (ref. 5). A good agreement between the two results has also been achieved.

CONSTITUTIVE RELATIONS

Following the development by Swedlow (ref. 1), the constitutive relations to be used in our formulation for solving elastic-plastic problems of rotational symmetry will be presented here. In the development, a unique relationship between the octahedral stress and the plastic octahedral strain is assumed to exist and the use of ideally plastic materials is excluded. The total strain components (ε_r, ε_θ, ε_z and γ_{rz}) are composed of the elastic, recoverable deformations and the plastic portions (ε_r^p, ε_θ^p, ε_z^p, and γ_{rz}^p). The rates of plastic flow, ($\dot{\varepsilon}_r^p$, etc.), are independent of a time scale and are simply used for convenience instead of the incremental values. The definitions of the octahedral stress τ_o and the octahedral plastic strain rate $\dot{\varepsilon}_o^p$ in the case of rotational symmetry are:

$$\tau_o = (1/3)[(\sigma_r-\sigma_\theta)^2 + (\sigma_\theta-\sigma_z)^2 + (\sigma_z-\sigma_r)^2 + 6\tau_{rz}{}^2]^{1/2} \tag{1}$$

$$\dot{\varepsilon}_o^p = (1/3)[(\dot{\varepsilon}_r^p-\dot{\varepsilon}_\theta^p)^2 + (\dot{\varepsilon}_\theta^p-\dot{\varepsilon}_z^p)^2 + (\dot{\varepsilon}_z^p-\dot{\varepsilon}_r^p)^2 + (3/2)(\dot{\gamma}_{rz}^p)^2]^{1/2} \tag{2}$$

where (σ_r, σ_θ, σ_z, τ_{rz}) are the nonvanishing stress components.

A unique relationship between τ_o and ε_o^p is assumed and there exists a function, $M_T(\tau_o)$, such that

$$2M_T(\tau_o) = \dot{\tau}_o/\dot{\varepsilon}_o^p \tag{3}$$

The plastic modulus, $M_T(\tau_o)$, can be related to the slope, E_T, of the effective stress-strain ($\bar{\sigma}$-$\bar{\varepsilon}$) curve by

$$\frac{1}{3M_T(\tau_o)} = \frac{1}{E_T} - \frac{1}{E} \tag{4}$$

where E is the elastic modulus and

$$E_t = \dot{\bar{\sigma}}/\dot{\bar{\varepsilon}}$$

$$\bar{\sigma} = (3/\sqrt{2})\tau_o$$

$$\bar{\varepsilon} = \sqrt{2}\,\varepsilon_o + \bar{\sigma}/E\ . \tag{5}$$

The material is assumed to obey the Mises yield criterion and the Prandtl-Reuss flow rule. The matrix relationship for the plastic flow in the case of rotational symmetry is

$$\begin{Bmatrix} \dot{\varepsilon}_r \\ \dot{\varepsilon}_\theta \\ \dot{\varepsilon}_z \\ \dot{\gamma}_{rz} \end{Bmatrix} = [C] \begin{Bmatrix} \dot{\sigma}_r \\ \dot{\sigma}_\theta \\ \dot{\sigma}_z \\ \dot{\tau}_{rz} \end{Bmatrix} \tag{6}$$

$$[C] = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 \\ -\nu & 1 & -\nu & 0 \\ -\nu & -\nu & 1 & 0 \\ 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} + \frac{1}{6\tau_o^{\,2} M_T(\tau_o)}\begin{bmatrix} S_r^{\,2} & S_r S_\theta & S_r S_z & 2S_r\tau_{rz} \\ & S_\theta^{\,2} & S_\theta S_z & 2S_\theta\tau_{rz} \\ & & S_z^{\,2} & 2S_z\tau_{rz} \\ \text{SYM} & & & 4\tau_{rz}^{\,2} \end{bmatrix} \tag{7}$$

where ν is the Poisson's ratio,

$$S_r = (2\sigma_r - \sigma_\theta - \sigma_z)/3 \ ,$$

$$S_\theta = (2\sigma_\theta - \sigma_z - \sigma_r)/3 \ ,$$

$$S_z = (2\sigma_z - \sigma_r - \sigma_\theta)/3 \ . \tag{8}$$

For strain-hardening materials, M_T (or E_T) $\neq 0$, we can obtain the inverse of [C] numerically and this procedure is chosen in developing NASTRAN program. For ideally-plastic materials, $M_T = E_T = 0$, matrix [C] does not exist and the NASTRAN program fails. However, its inverse $[C]^{-1}$ still exists and the closed form has been derived in Ref. 6. In the axisymmetric case, the explicit form is (ref. 7)

$$[D] = \frac{2G}{1-2\nu}\begin{bmatrix} 1-\nu & & & \text{SYM} \\ \nu & 1-\nu & & \\ \nu & \nu & 1-\nu & \\ 0 & 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} - \frac{2G}{A}\begin{bmatrix} S_r^{\,2} & & & \text{SYM} \\ S_r S_\theta & S_\theta^{\,2} & & \\ S_r S_z & S_\theta S_z & S_z^{\,2} & \\ S_r\tau_{rz} & S\,\tau_{tz} & S_z\tau_{rz} & \tau_{rz}^{\,2} \end{bmatrix} \tag{9}$$

where

and

$$A = 3\tau_o^{\,2}(1 + M_T/G)$$

$$G = \frac{1}{2}\,E/(1+\nu) \quad . \tag{10}$$

If we want to remove the limitation that the use of ideally-plastic materials is excluded, we have to use Eq. (9) instead of Eq. (7).

TRAPEZOIDAL RING ELEMENT

The incremental displacement field employed for the trapezoidal ring element are

$$\Delta u(r,z) = \beta_1 + \beta_2 r + \beta_3 z + \beta_4 rz \; ,$$

$$\Delta w(r,z) = \beta_5 + \beta_6 r + \beta_7 z + \beta_8 rz \; . \tag{11}$$

The transformation from grid point coordinates to generalized coordinates is

$$\{\beta\} = [\Gamma_{\beta q}]\{\Delta q\} \tag{12}$$

where

$$\{\Delta q\}^T = [\Delta u_1,\ \Delta w_1,\ \Delta u_2,\ \Delta w_2,\ \Delta u_3,\ \Delta w_3,\ \Delta u_4,\ \Delta w_4] \; ,$$

$$\{\beta\}^T = [\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \beta_5,\ \beta_6,\ \beta_7,\ \beta_8] \; . \tag{13}$$

and the elements of the inverse of the transformation matrix $[\Gamma_{\beta q}]^{-1}$ are the coefficients of the β's in Equations (11), evaluated at the corners of the element.

The stiffness matrix is formed in the same manner as that for the anisotropic elastic element. The final form referred to grid point coordinates is

$$[K] = [\Gamma_{\beta q}]^T[\bar{K}][\Gamma_{\beta q}] \; ,$$

where

$$[\bar{K}] = 2\pi\int r[B]^T[D][B]dzdr \; . \tag{14}$$

[D], the matrix of material coefficients, is defined by Equation (9). The [B] matrix is the same as the elastic case, but now it expresses the incremental strains in terms of generalized coordinates

$$\{\Delta\varepsilon\} = [B]\{\beta\} \; . \tag{15}$$

NASTRAN IMPLEMENTATION

The NASTRAN implementation for an elastic-plastic trapezoidal ring element follows the steps outlined in section 6.8, "Adding a structural element," of reference 8. Changes were required in the functional modules PLA1, PLA3, and PLA4, which included the writing of several new subroutines. These new routines could easily be modeled after the existing code for the linear portion of the program. There are two major differences in the nonlinear subroutines. First is the new code for the calculation of the material stiffness matrix and second is that thermal stresses and element force calculations are eliminated in the nonlinear code.

The changes in PLA1 allows this module to identify the new element as a member of the piecewise linear element set and properly initialize the non-linear Element Summary and Element Connection Property Tables. Three element stress recovery subroutines were added to PLA3: PSAPRG, a driver for stress data recovery; PSTRR1 and PSTRR2, phase I and II stress recovery routines. Element stiffness calculations in PLA4 require four new subroutines: PKAPRG, a driver for nonlinear trapezoidal ring elements in PLA4, PKTRR1 and PKTRR2, stress recovery routines which generate stresses for the computation of the nonlinear material matrix; and PKTRAP, the stiffness matrix generation routine for nonlinear trapezoidal ring elements.

The computer system available for this work is IBM 360 Model 44. In order to add the new code into Link 13, it became necessary to add a new branch to the overlay trees to contain the new elements.

NUMERICAL EXAMPLES

For the purpose of demonstration, two elastic-plastic problems of rotational symmetry were solved and compared with other results (refs. 4 and 5). The first is an infinitely long tube under uniform internal pressure and the second, a thick-walled cylinder of finite length loaded over part of its inner surface. The material properties for both problems are the same. The elastic constants are $E = 30 \times 10^6$ psi, $\nu = 0.3$ and the effective stress-strain curve is represented by three line segments connecting the four points in the $(\bar{\varepsilon}-\bar{\sigma})$ plane, $(\bar{\varepsilon},\bar{\sigma}) = (0.0,0.0)$. (0.005, 150,000 psi), (0.055, 225,000 psi), (0.1, 225,000 psi).

EXAMPLE 1. Consider an infinitely long tube subjected to uniform internal pressure p. The plane strain condition is assumed. The tube of outside radius 2" and inside radius 1" has been divided into 25 trapezoidal ring elements. The numerical results based on the NASTRAN program have been obtained. For this problem, a new finite-difference approach (ref. 4) can be used to generate exact solution and to assess the accuracy of the NASTRAN code. Some of the results for the displacements and stresses are presented graphically in Figures 1 and 2. Twenty-five load increments are used in NASTRAN as shown in Figure 1. The radial displacements at the inside as well as outside surface are shown as functions of internal pressure. Figure 2 shows the distributions of radial, tangential and axial stress components in a pressurized tube when half of the tube is plastic. The pressure required to achieve this state is 0.7378 σ_0 based on NASTRAN code and 0.7356 σ_0 based on the finite-difference solution (ref. 4). Both codes indicate that the maximum tangential stress occurs at the elastic-plastic interface. As demonstrated in Figures 1 and 2, the NASTRAN results are in excellent agreement with those based on the finite-difference approach (ref. 4).

EXAMPLE 2. Consider a two-dimensional elastic-plastic thick-walled cylinder problem as shown in Figure 3. The tube with inner radius (1"), outer radius (2") and length (4") is loaded uniformly over a middle portion (2") of the inner surface. The mesh generation and the loading for the half of the undeformed structure is shown in the figure. This problem was solved in (ref. 5) based on a scale loading approach. The first load factor is the upper limit of the elastic solution and ten additional increments were needed until one of the outside elements becomes yielded. The same load factors were used to obtain the NASTRAN solution. Both programs indicate that the sequence in which the elements become plastic is 1,5,9,2,13,6,10,3,7,14,11,17,4. Some steps will cause more than one element to become plastic and those elements with effective stress $\bar{\sigma} \geq 0.99\ \bar{\sigma}_0$ have been considered as yielded. The numerical results for the radial displacement at the inside, u_a (point 1) and outside, u_b (point 5) as functions of internal pressure are shown in Figure 4. The stress components at the centroid of one inside element (No. 1) are shown in Figure 5. The effect of loading history on the displacements and stresses can be seen from Figures 4 and 5. A comparison of the results between NASTRAN program and reference 5 indicates that very good agreement has been achieved.

CONCLUSION

An elastic-plastic trapezoidal ring element has been implemented in NASTRAN computer program. Its application to elastic-plastic problems of rotational symmetry has been demonstrated by solving two thick-walled tube problems. The NASTRAN results for both problems are in excellent agreement with the other results.

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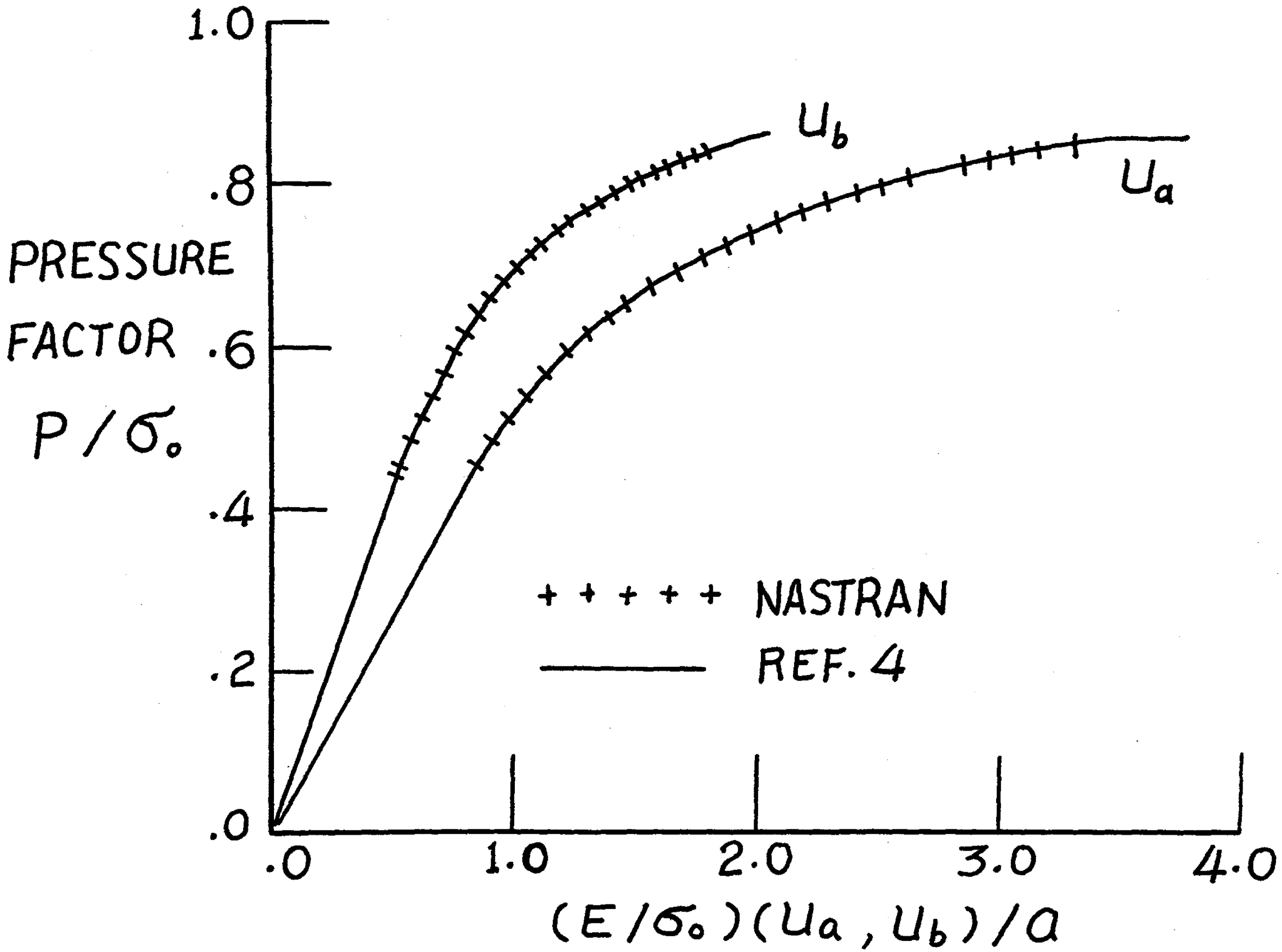
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.6
.4
.2
.0
PRESSURE
FACTOR
P / σ₀
Ub
Ua
+ + + + + NASTRAN
REF. 4
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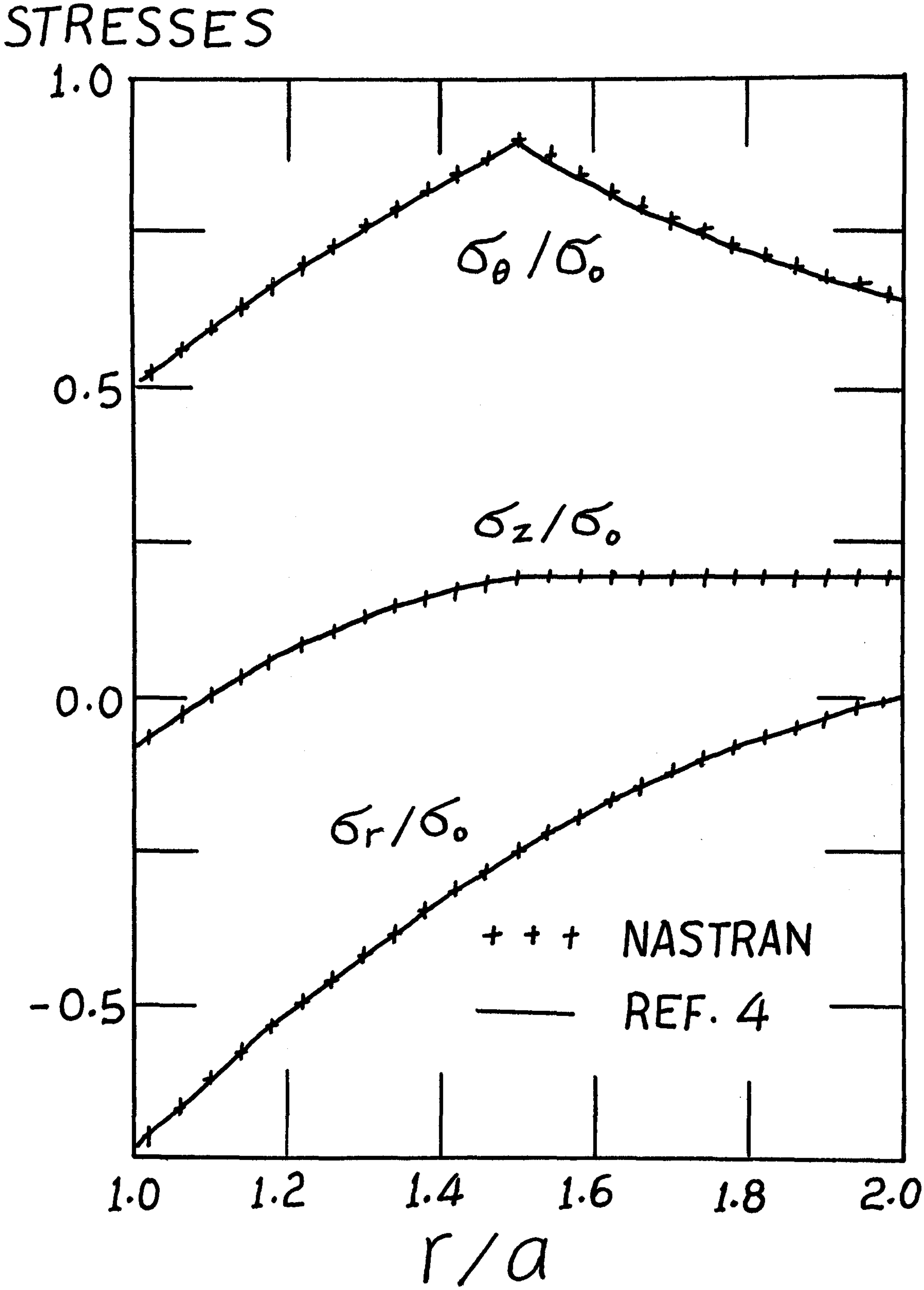
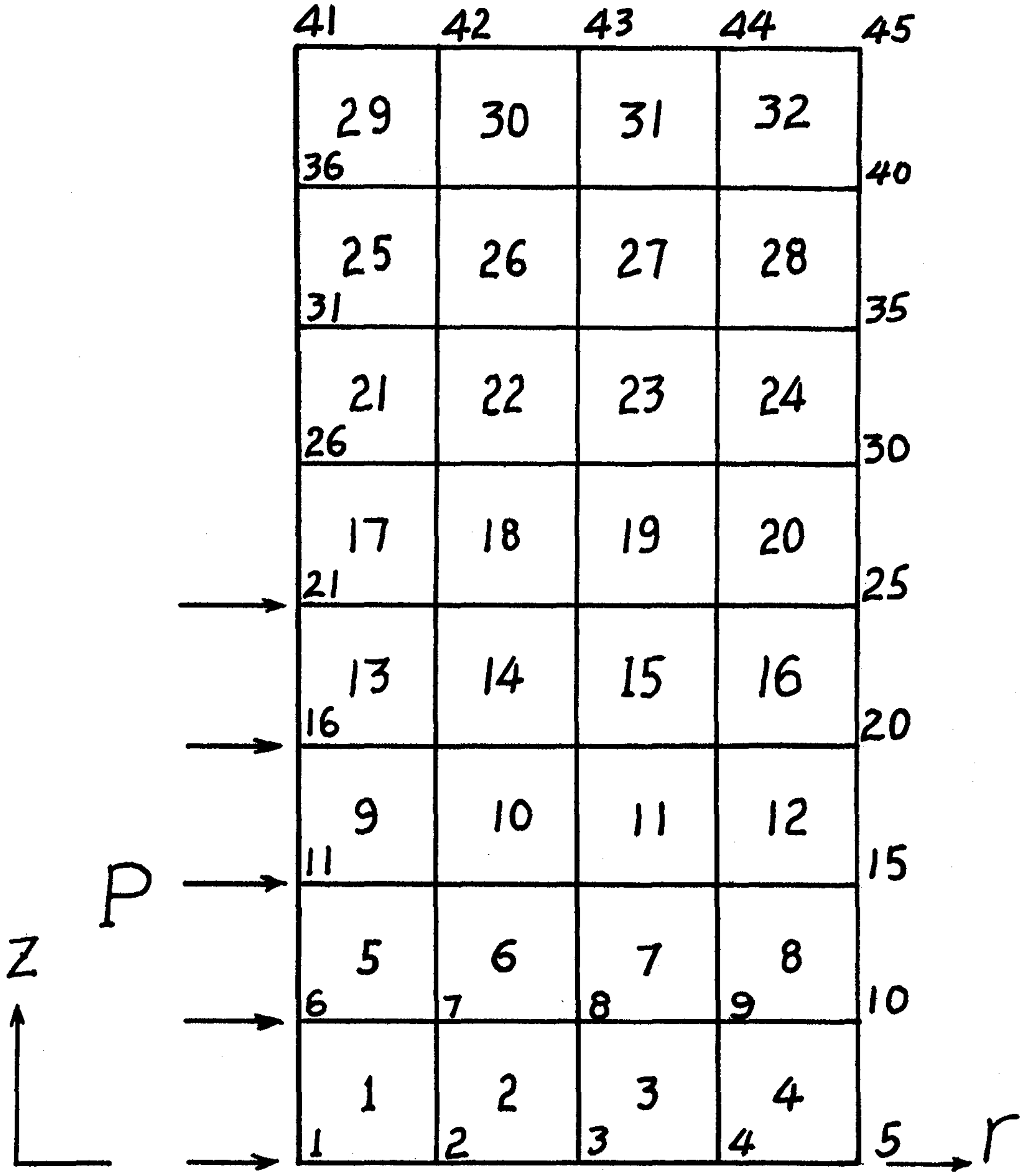
1.0
2.0
3.0
4.0
(E/σ₀)(Ua, Ub)/a

FIGURE -1



STRESSES
1.0
0.5
0.0
-0.5
σ_θ/σ_o
σ_z/σ_o
σ_r/σ_o
+ + + NASTRAN
REF. 4
1.0
1.2
1.4
1.6
1.8
2.0
r/a

FIGURE - 2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36 40
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31 35
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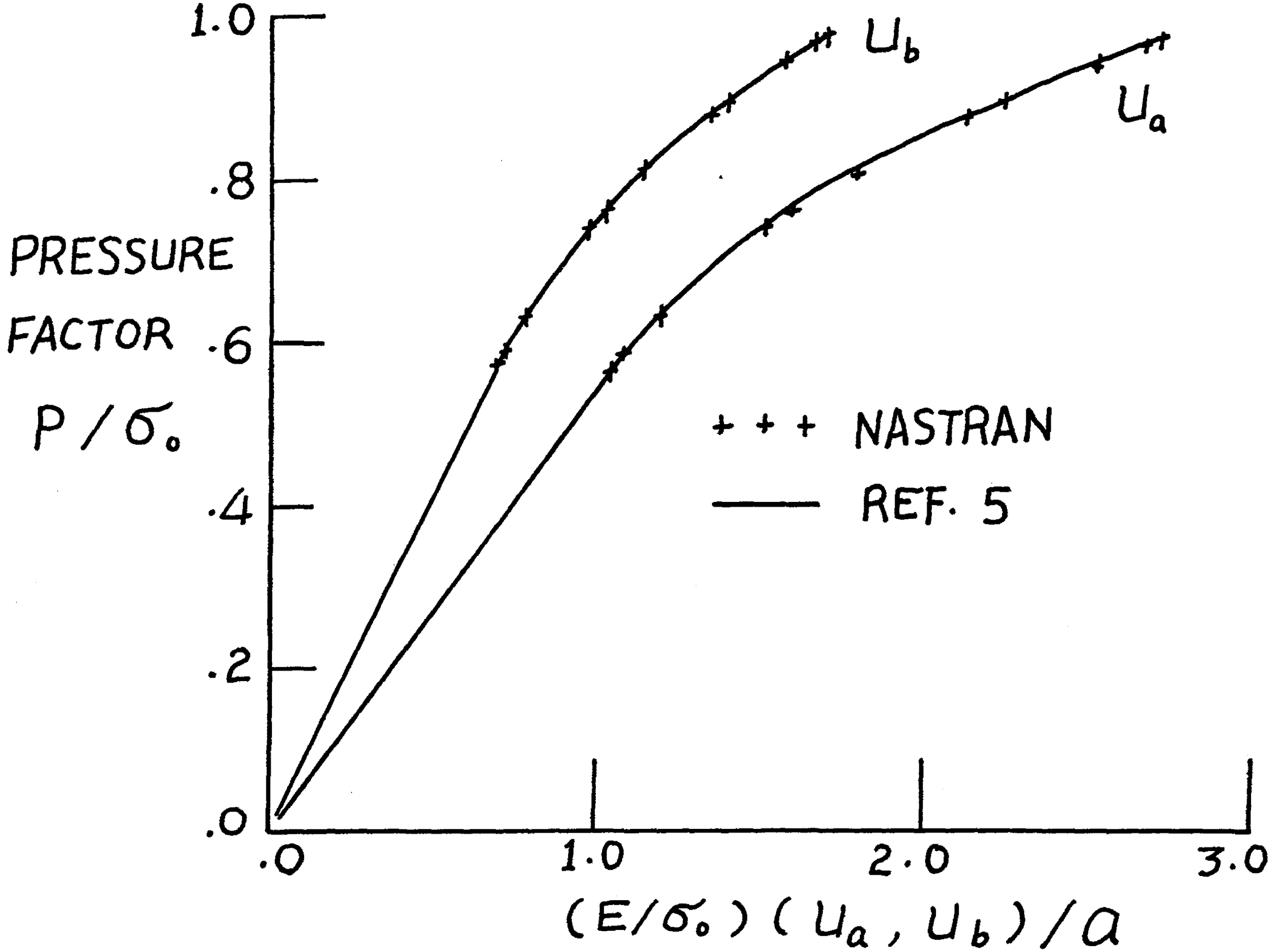
26 30
17 18 19 20
21 25
13 14 15 16
16 20
9 10 11 12
11 15
P
Z
5 6 7 8
6 7 8 9 10
1 2 3 4
1 2 3 4 5
r

FIGURE - 3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FACTOR
P / σ₀
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.8
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.0
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1.0
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3.0
(E/σ₀)(Ua, Ub)/a
Ub
Ua
+ + + NASTRAN
REF. 5

FIGURE - 4

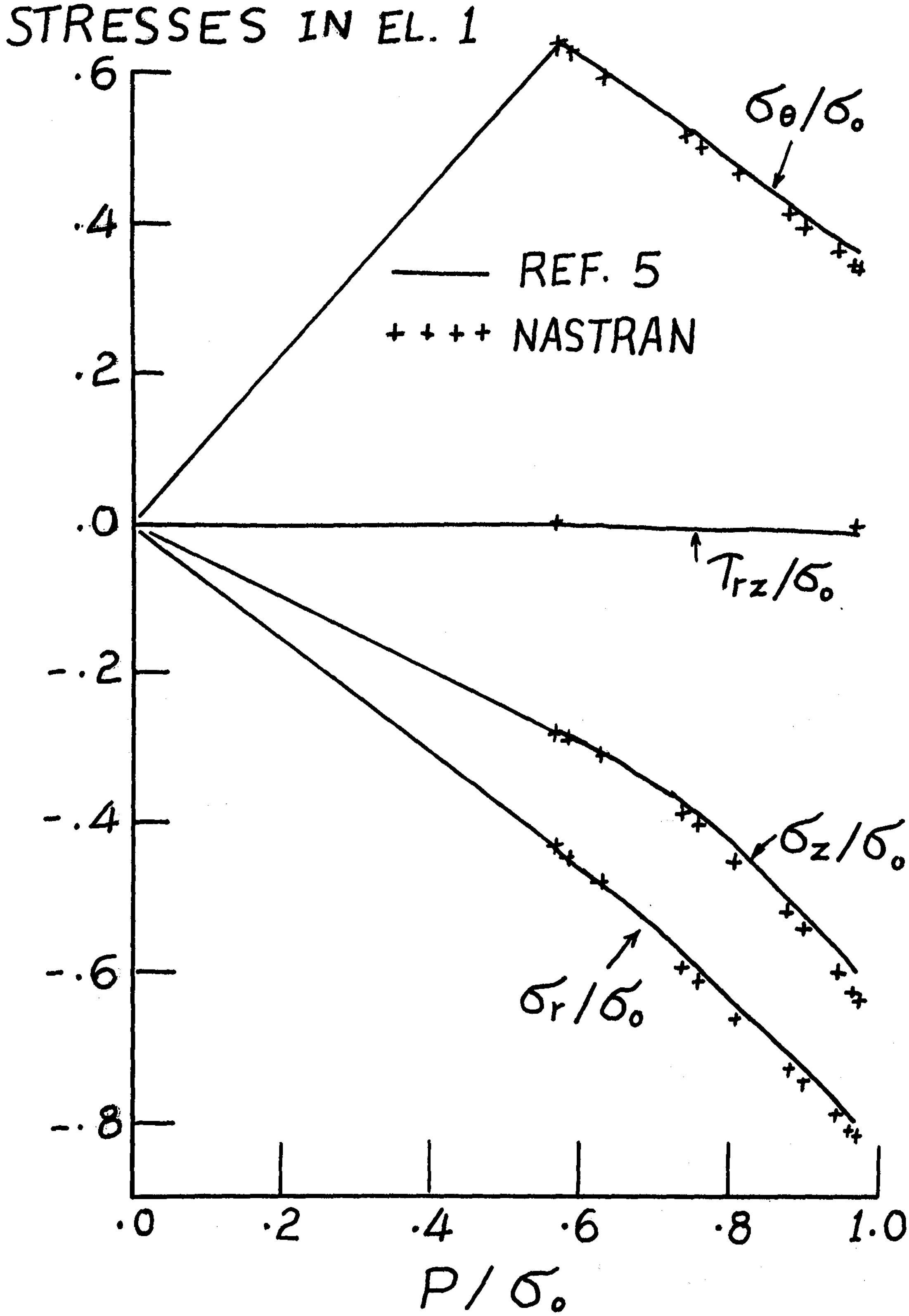
STRESSES IN EL. 1
·6
·4
·2
·0
-·2
-·4
-·6
-·8
σ_θ/σ_o
REF. 5
+ + + + NASTRAN
τ_{rz}/σ_o
σ_z/σ_o
σ_r/σ_o
·0
·2
·4
·6
·8
1.0
P/σ_o

FIGURE - 5