NASA-TM-81064 19800016563

A Reproduced Copy

OF

Reproduced for NASA

by the

NASA Scientific and Technical Information Facility

FFNo 672 Aug 65

80-FM-20

JSC-16499

(NASA-TM-81064) INVERSION AND APPROXIMATION OF LAPLACE TRANSFORMS (NASA) 37 p HC A03/MF A01 CSCL 12A

N80-25056

Unclas

G3/64 21651

Inversion and Approximation of Laplace Transforms

Mission Planning and Analysis Division

April 1980

National Aeronautics and Space Administration

Lyndon B. Johnson Space Center
Houston, Texas

80-FM-20

JSC-16499

SHUTTLE PROGRAM

INVERSION AND APPROXIMATION OF LAPLACE TRANSFORMS

By William M. Lear, TRW

JSC Task Monitor: P. Pixley, Mathematical Physics Branch

Approved: Emil R. Schiesser, Chief
Mathematical Physics Branch

Approved: Ronald L. Berry, Chief
Mission Planning and Analysis Division

Mission Planning and Analysis Division

National Aeronautics and Space Administration

Lyndon B. Johnson Space Center

Houston, Texas

April 1980

CONTENTS

FIGURES

1.0 INTRODUCTION

Included in this report is a novel method of inverting Laplace transforms by using a new set of orthonormal functions. As a byproduct of the inversion, it is seen how to approximate very complicated Laplace transforms by a transform with a series of simple poles along the left-half plane real axis. The inversion and approximation process is simple enough to be put on a programable hand calculator.

2.0 INVERSION AND APPROXIMATION

Let $f(s)$ be a Laplace transform and $F(t)$ its exact inverse. ${}_NF(t)$ will be the approximate inverse, given by

$$ {}_NF(t) = A_1L_1(st) + A_2L_2(st) + \cdots + A_NL_N(st) \tag{1} $$

where the $L_n(st)$ are the new orthonormal functions (described below and in the appendix). The A_n values are the Fourier coefficients and are given by

$$ A_n(s) = \int_0^{\infty} F(t)L_n(st)dt \tag{2} $$

s is a free parameter chosen to produce the best approximation, as shown below.

The integral square approximation error is given by

$$ E(s) = \int_0^{\infty} \{{}_NF(t) - F(t)\}^2 dt = \int_0^{\infty} F(t)^2 dt - \sum_{n=1}^{N} A_n^2(s) \geq 0 . \tag{3} $$

To minimize the integral square error, s is chosen such that

$$ C = \sum_{n=1}^{N} A_n^2(s) \quad \text{is maximum} \tag{4} $$

The new orthonormal functions are shown below.

$$L_n = {}_na_1e^{-st} + {}_na_2e^{-2st} + {}_na_3e^{-3st} + \ldots + {}_na_ne^{-nst} \qquad (5)$$

The values of ${}_na_i$ are chosen such that

$$\int_0^\infty L_nL_m dt = 0 \quad \text{for } n \neq m$$
$$= 1 \quad \text{for } n = m \qquad (6)$$

The first 10 orthonormal functions are listed below.

$$L_1 = \sqrt{2s}e^{-st} \qquad s > 0$$

$$L_2 = \sqrt{4s}(-2e^{-st} + 3e^{-2st})$$

$$L_3 = \sqrt{6s}(3e^{-st} - 12e^{-2st} + 10e^{-3st})$$

$$L_4 = \sqrt{8s}(-4e^{-st} + 30e^{-2st} - 60e^{-3st} + 35e^{-4st})$$

$$L_5 = \sqrt{10s}(5e^{-st} - 60e^{-2st} + 210e^{-3st} - 280e^{-4st} + 126e^{-5st})$$

$$L_6 = \sqrt{12s}(-6e^{-st} + 105e^{-2st} - 560e^{-3st} + 1260e^{-4st}$$
$$- 1260e^{-5st} + 462e^{-6st})$$

$$L_7 = \sqrt{14s}(7e^{-st} - 168e^{-2st} + 1260e^{-3st} - 4200e^{-4st} + 6930e^{-5st}$$
$$- 5544e^{-6st} + 1716e^{-7st})$$

$$L_8 = \sqrt{16s}(-8e^{-st} + 252e^{-2st} - 2520e^{-3st} + 11550e^{-4st} - 27720e^{-5st}$$
$$+ 36036e^{-6st} - 24024e^{-7st} + 6435e^{-8st})$$

$$L_9 = \sqrt{18s}(9e^{-st} - 360e^{-2st} + 4620e^{-3st} - 27720e^{-4st} + 90090e^{-5st}$$
$$- 168168e^{-6st} + 180180e^{-7st} - 102960e^{-8st} + 24310e^{-9st})$$

$$L_{10} = \sqrt{20s}(-10e^{-st} + 495e^{-2st} - 7920e^{-3st} + 60060e^{-4st} - 252252e^{-5st} + 630630e^{-6st} - 960960e^{-7st} + 875160e^{-8st} - 437580e^{-9st} + 92378e^{-10st})$$

Figure 1 shows plots of the first four L_n.

The values of the $_na_i$ coefficients are given by

$$_na_i = (-1)^{n+i}\sqrt{2sn}\frac{(n+i-1)!}{i!(i-1)!(n-i)!} \tag{7}$$

or

$$_na_i = (-1)^{n+i}\sqrt{2sn}\frac{n}{i!(i-1)!}\prod_{j=1}^{i-1}(n^2 - j^2) \tag{8}$$

where $\quad _na_1 = (-1)^{n+1}\sqrt{2ns}\; n \qquad (9)$

The recursion relationship for the $_na_i$ is given by

$$_na_1 = (-1)^{n+1}\sqrt{2ns}\; n \qquad n = 1, 2, \cdots, N \tag{10}$$

$$_na_i = -\frac{n^2 - (i-1)^2}{i(i-1)}\, {_na_{i-1}} \qquad i = 2, 3, \cdots, n \quad n = 2, 3, \cdots, N \tag{11}$$

The recursion relationship for the L_n is given by

$$U_n = 2\frac{2n-1}{\sqrt{n(n-1)}} \qquad n = 2,3, \cdots, N \tag{12}$$

$$V_n = \frac{n(n-1)}{2n-1} \qquad n = 2, 3, \cdots, N \tag{13}$$

$$L_1 = \sqrt{2s}\, e^{-st} \qquad (14)$$

$$L_2 = \sqrt{s}\, e^{-st}(6e^{-st} - 4) \qquad (15)$$

$$L_n = U_n\left((e^{-st} - V_n + V_{n-1})L_{n-1} - L_{n-2}/U_{n-1}\right) \quad n = 3, 4, \cdots, N \qquad (16)$$

This is the relationship that should be used to compute the L_n in a computer program. It is simple, fast, and accurate.

Equation 2 gave the Fourier coefficients in terms of F(t). In terms of the Laplace transform, f(s), they are given by

$$A_n = \sum_{i=1}^{n} {}_n a_i f(is) \qquad (17)$$

Note that as n increases, so does the magnitude of the ${}_n a_i$, which has an oscillating sign. This can cause serious roundoff error problems in computing the A_n. It is speculated that the maximum value of $n = N$ be limited to approximately the number of significant decimal digits of accuracy used by a particular computer. One way to evaluate this problem for a particular computer is to set[a]

$$f(s) = \frac{1}{s + 1}$$

Let $s = 1$ and compute the A_n. Theoretically

$$A_1 = 1/\sqrt{2}$$

$$A_n = 0 \qquad \text{for } n > 1$$

[a]Also see theorem 15 in the appendix.

and

$$C = \sum_{n=1}^{N} A_n^2 = 0.5$$

Due to roundoff error, the theoretical values will not be achieved for N large.

Perhaps a better way of computing the A_n (which may be slightly less affected by roundoff error) is to use the algorithm shown below, which also computes C.

```
    C = 0
    DO c  n = 1, N
    A_n = f(ns)
    IF (n.EQ.1) GOTO b
    δ = 1
    DO a  i = 1, n - 1
    A_n = [i(2n - i) / ((n + 1 - i)(n - i))] A_n - δf((n - i)s)
a   δ = -δ
b   A_n = √(2ns) nA_n
    C = C + A_n^2
c   PRINT C
```

Note that s should be chosen such that C is maximum.

All the L_n approach zero as t approaches infinity. Therefore, the approximations work well only when $F(t) \to 0$ as t approaches infinity. This will be the case for stable system weighting functions - an important application. An example of what to do when $F(t)$ does not decay to zero is shown below. Let

$$g(s) = \frac{1 - e^{-2s}}{s^2}$$

Apply the final value theorem.

$$G(\infty) = \lim_{s \to 0} sg(s) = 2$$

So instead of inverting $g(s)$, invert

$$f(s) = g(s) - \frac{2}{s}$$

Now $F(t) \to 0$ as t approaches infinity and $G(t) = F(t) + 2$. Thus

$$_N G(t) = 2 + {_N F(t)}$$

3.0 EXAMPLES

As the first example, let

$$f(s) = \frac{s + 1}{(s + 1)^2 + \pi^2} \tag{18}$$

The exact inverse is

$$F(t) = e^{-t} \cos(\pi t) \tag{19}$$

Figures 2 through 9 show the values of

$$C = \sum_{n=1}^{N} A_n^2 \tag{20}$$

versus s for values of N from 1 to 14. The maximum value that C can obtain (neglecting roundoff errors) is 0.27300 since

$$\int_0^\infty F(t)^2 dt = 0.27300 \tag{21}$$

It is seen that each value of N has its own optimum value of s, and the choice of s can greatly influence the accuracy of the fit.

Figure 10 shows plots of $F(t)$, ${}_3F(t)$, and ${}_6F(t)$. For $N = 3$ the optimum value of s was $s = 2.2$. In this case

$$A_1 = 0.33378\ 95910$$

$$A_2 = 0.28719\ 57089$$

$$A_3 = -0.21481\ 58487$$

$${}_3F(t) = -3.345e^{-2.2t} + 11.921e^{-4.4t} - 7.805e^{-6.6t} \tag{22}$$

The approximate Laplace transform is thus seen to be

$${}_3f(s) = -\frac{3.345}{s + 2.2} + \frac{11.921}{s + 4.4} - \frac{7.805}{s + 6.6} \tag{23}$$

For $N = 6$ the optimum value of $s = 0.9$ and

$A_1 = 0.18930\ 92215$ $\qquad$ $A_2 = 0.36507\ 5176$[illegible]

$A_3 = 0.22324\ 71254$ $\qquad$ $A_4 = -0.14742\ 23756$

$A_5 = -0.11047\ 63746$ $\qquad$ $A_6 = 0.11492\ 41046$

$$
{}_6F(t) = -1.916e^{-0.9t} + 43.527e^{-1.8t} - 252.178e^{-2.7t} + 554.831e^{-3.6t} - 517.636e^{-4.5t} + 174.488e^{-5.4t} \qquad (24)
$$

From figure 8 it is seen that $N = 10$ and $s = 0.65$ will give an excellent fit. For this case

$A_1 = 0.14940\ 23073$ $\qquad$ $A_2 = 0.31134\ 50651$

$A_3 = 0.31771\ 10153$ $\qquad$ $A_4 = 0.03684\ 84058$

$A_5 = -0.18661\ 71768$ $\qquad$ $A_6 = -0.03590\ 26947$

$A_7 = 0.10304\ 31558$ $\qquad$ $A_8 = -0.02596\ 24761$

$A_9 = -0.03656\ 09018$ $\qquad$ $A_{10} = 0.04125\ 86606$

and

$$
\begin{aligned}
{}_{10}F(t) = {} & -0.832e^{-0.65t} + 59.853e^{-1.3t} - 1195.825e^{-1.95t} \\
& + 10\ 138.374e^{-2.6t} - 44\ 250.068e^{-3.25t} + 110\ 056.528e^{-3.9t} \\
& - 169\ 440.633e^{-4.55t} + 142\ 526.134e^{-5.2t} \\
& - 68\ 134.644e^{-5.85t} + 13\ 742.171e^{-6.5t}
\end{aligned} \qquad (25)
$$

For the next example

$$
f(s) = \ln\left(\frac{s + 2}{s + 1}\right) - \frac{1}{s + 2} \qquad (26)
$$

The exact inverse is

$$F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t} \tag{27}$$

Figure 11 shows plots of $C = \sum_{n=1}^{N} A_n^2$ versus s for N = 4, 8, and 12. It is clear that for s = 0.5, only four terms are needed to give an excellent fit. In this case

$$_4F(t) = -0.00261e^{-0.5t} + 0.17291e^{-t} + 0.66316e^{-1.5t} - 0.83353e^{-2t} \tag{28}$$

Figure 11 shows plots of F(t) and $_4F(t)$. There is no visible difference between F(t) and $_4F(t)$.

Let

$$g(s) = \ln\left(\frac{s + 2}{s + 1}\right) \tag{29}$$

Then from equations 26 and 28, g(s) is approximated by

$$_4g(s) = -\frac{0.00261}{s + 0.5} + \frac{0.17292}{s + 1} + \frac{0.66316}{s + 1.5} + \frac{0.16647}{s + 2} \tag{30}$$

For s > 0, $_4g(s)$ is an excellent approximation of g(s), as seen below.

s	$g(s)$	${}_4g(s)$
0	0.69315	0.69304
0.1	.64663	.64660
1	.40547	.40547
2	.28768	.28769
5	.15415	.15415
10	.08701	.08701

Note

$$ {}_4G(t) = -0.00261e^{-0.5t} + 0.17292e^{-t} + 0.66316e^{-1.5t} + 0.16647e^{-2t} \quad (31) $$

where

$$ G(t) = \frac{1}{t}(e^{-t} - e^{-2t}) \quad (32) $$

Note $G(0) = 1$ and ${}_4G(0) = 0.99994$.

For the final example

$$ f(s) = e^{-\sqrt{s}} \quad (33) $$

which has an exact inverse of

$$ F(t) = \frac{1}{2\sqrt{\pi t^3}} \exp\left(-\frac{1}{4t}\right) \quad (34) $$

Figure 13 shows plots of $C = \sum_{n=1}^{N} A_n^2$ versus s for values of $N = 6$, 10, and 14.

For $N = 6$ the optimum value of $s = 0.8$, and

$$ {}_6F(t) = 1.4551e^{-0.8t} - 15.6761e^{-1.6t} + 83.8937e^{-2.4t} $$
$$ - 204.8879e^{-3.2t} + 232.5782e^{-4t} - 97.7713e^{-4.8t} \qquad (35) $$

As seen from figure 14, ${}_6F(t)$ is a very good approximation of $F(t)$, which is remarkable since $F(t)$ is a complicated function of time that is very dissimilar to a power series in $e^{-0.8t}$.

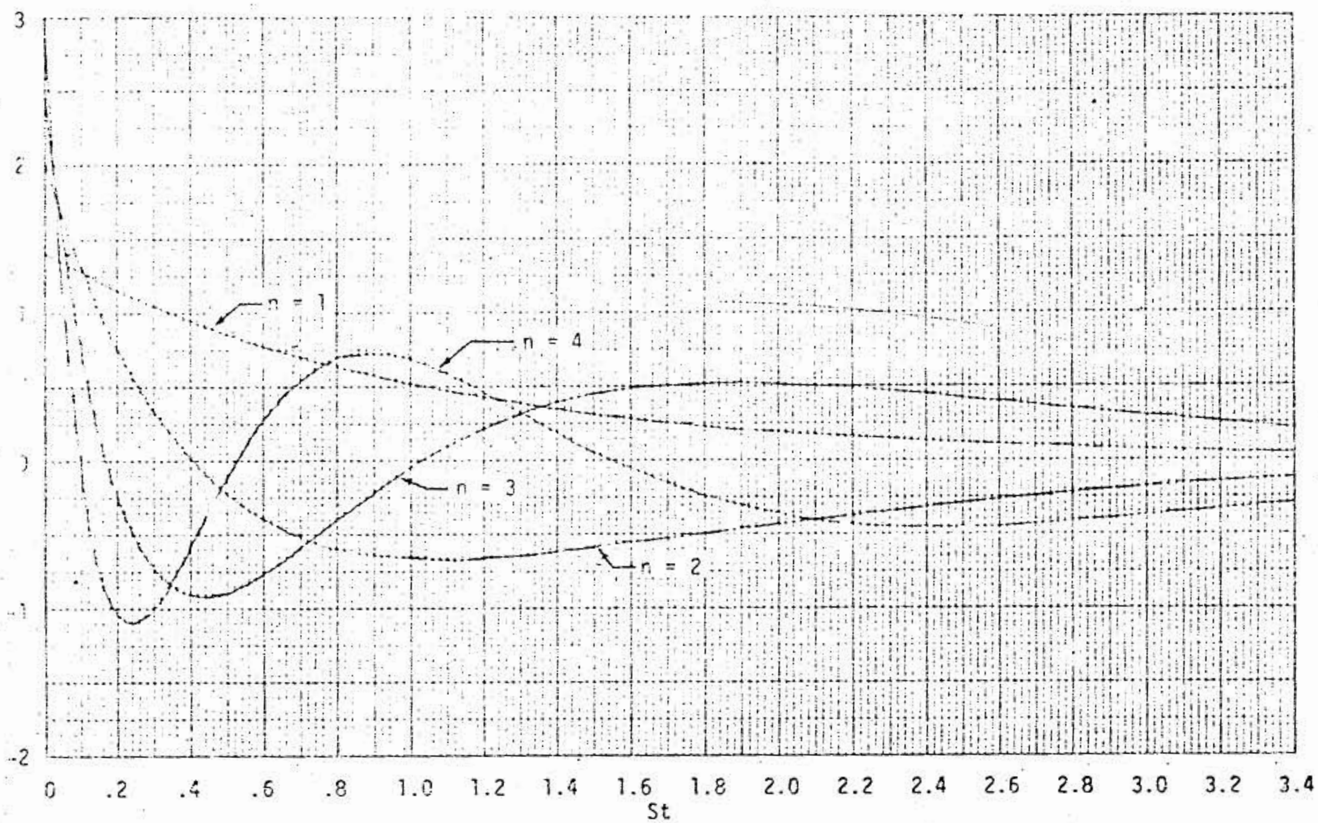

Figure 1.- Plot of first four orthonormal functions.

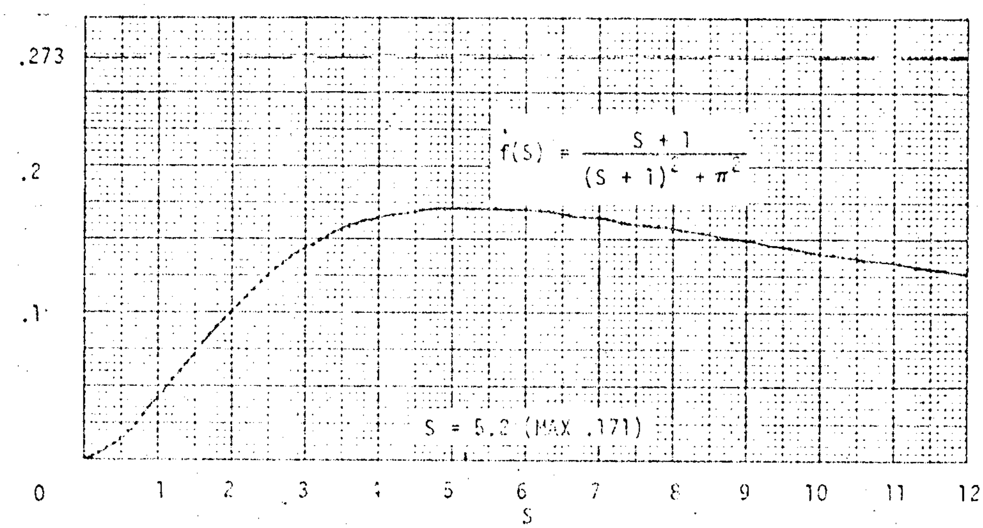

Figure 2.–$\sum_{n=1}^{1} A_n^2$ for $F(t) = e^{-t} \cos(\pi t)$

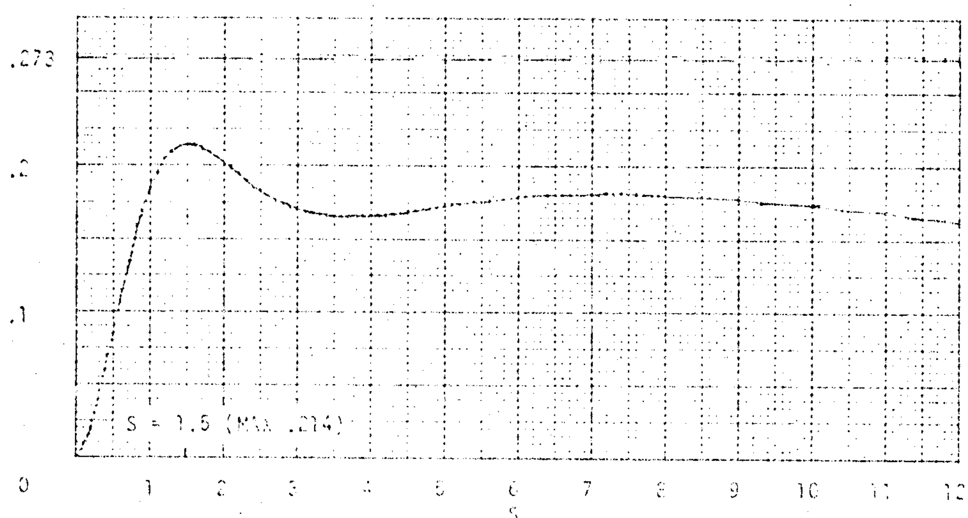

Figure [illegible].–$\sum_{n=1} A_n^2$ for F(t) [illegible]

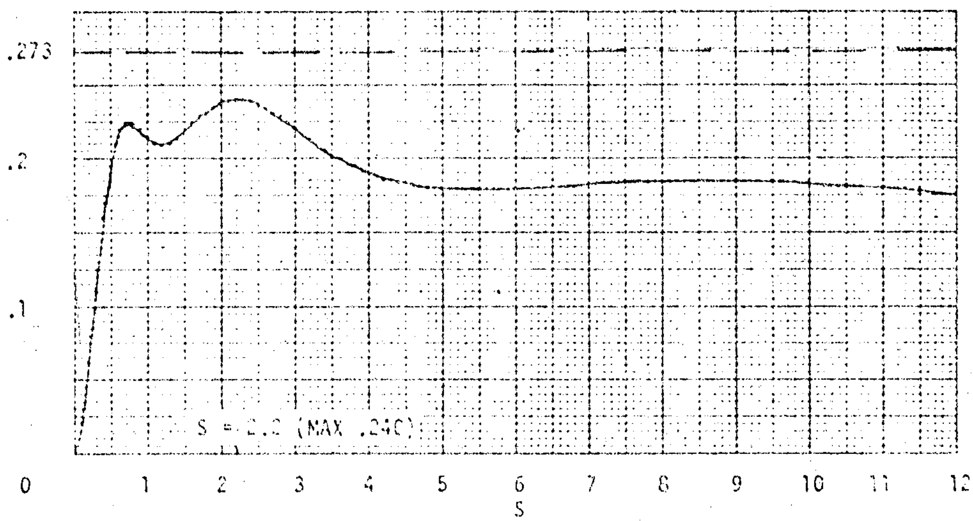

Figure 4.- $\sum_{n=1}^{3} A_n^2$ for $F(t) = e^{-t} \cos(\pi t)$.

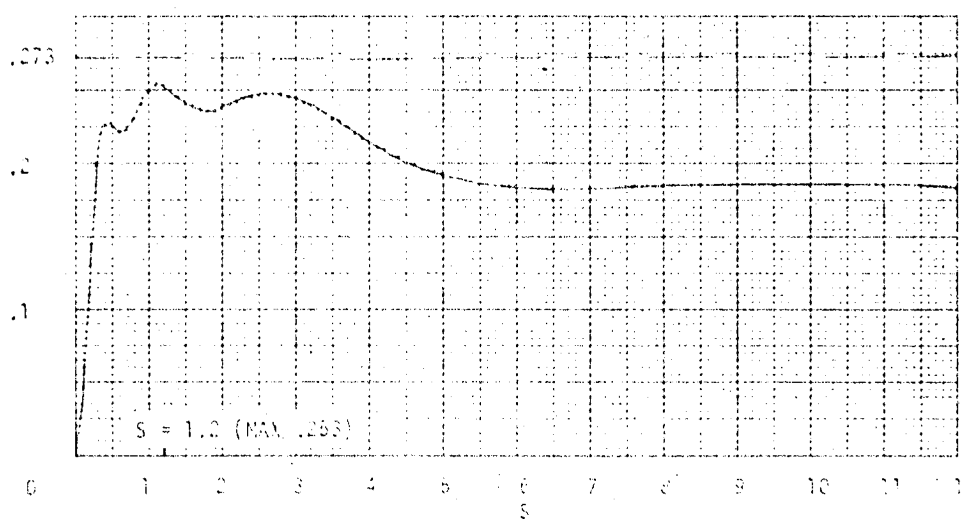

Figure 5.- $\sum_{n=1}^{4} A_n^2$ for $F(t) = e^{-t} \cos(\pi t)$.

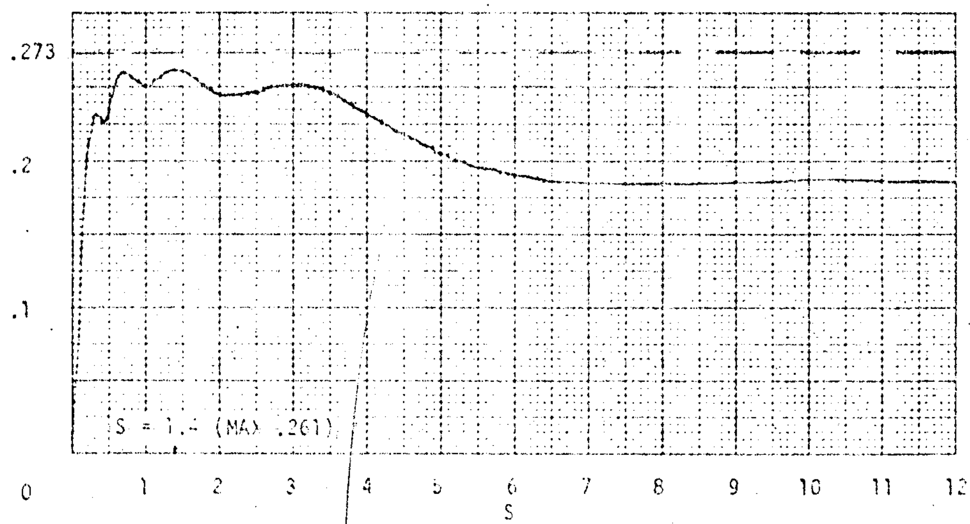

Figure 6.- $\sum_{n=1}^{5} A_n^2$ for $F(t) = e^{-t}\cos(\pi t)$.

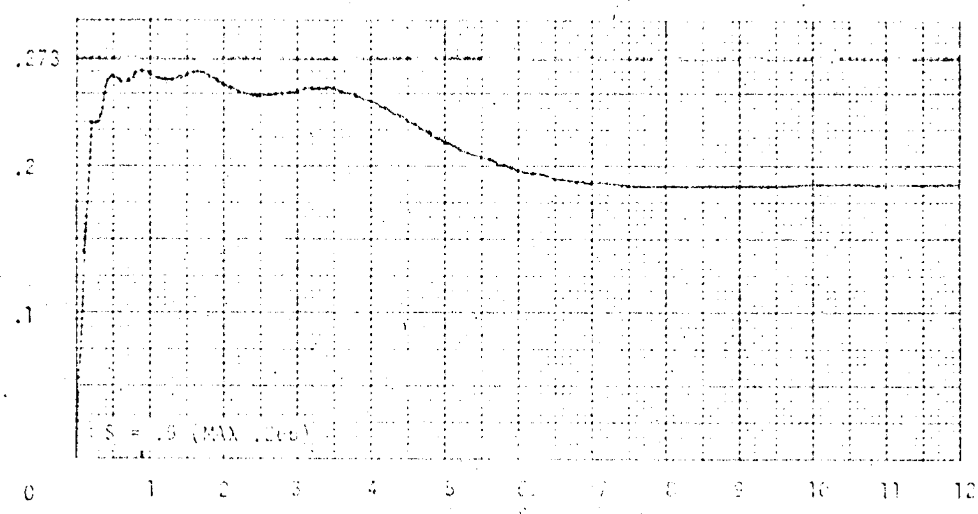

Figure [illegible].- $\sum$ [illegible] for [illegible] = e^{-t} [illegible].

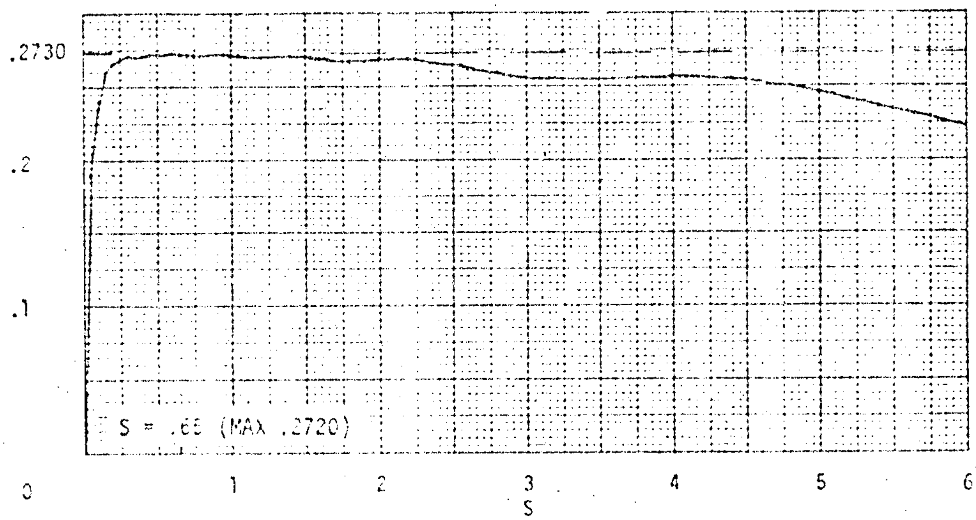

Figure 8.- $\sum_{n=1}^{10} A_n^2$ for $F(t) = e^{-t} \cos(\pi t)$.

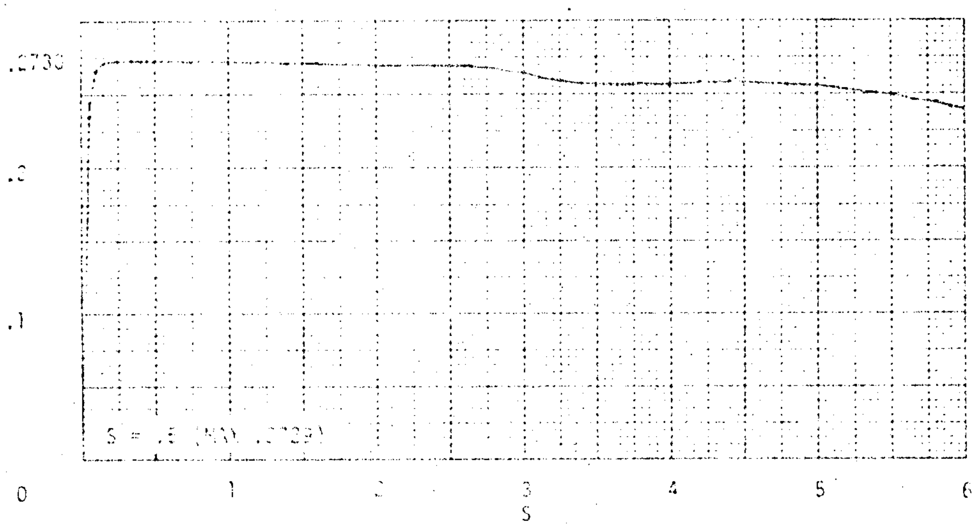

Figure [illegible].- $\sum A_n^2$ for $F(t) = e^{-t} \cos(\pi t)$.

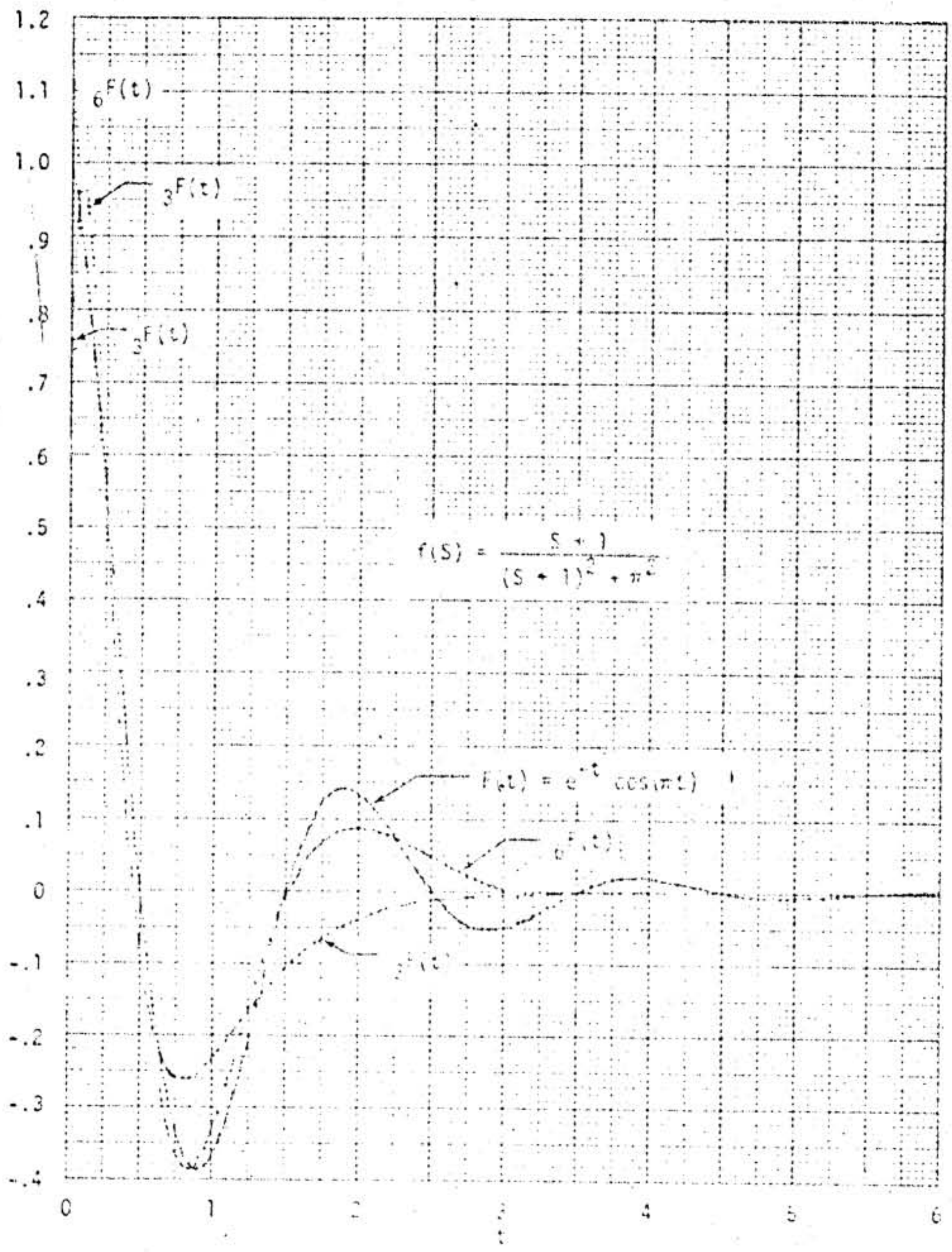

Figure 10.- Approximations of $F(t) = e^{-t}\cos(\pi t)$.

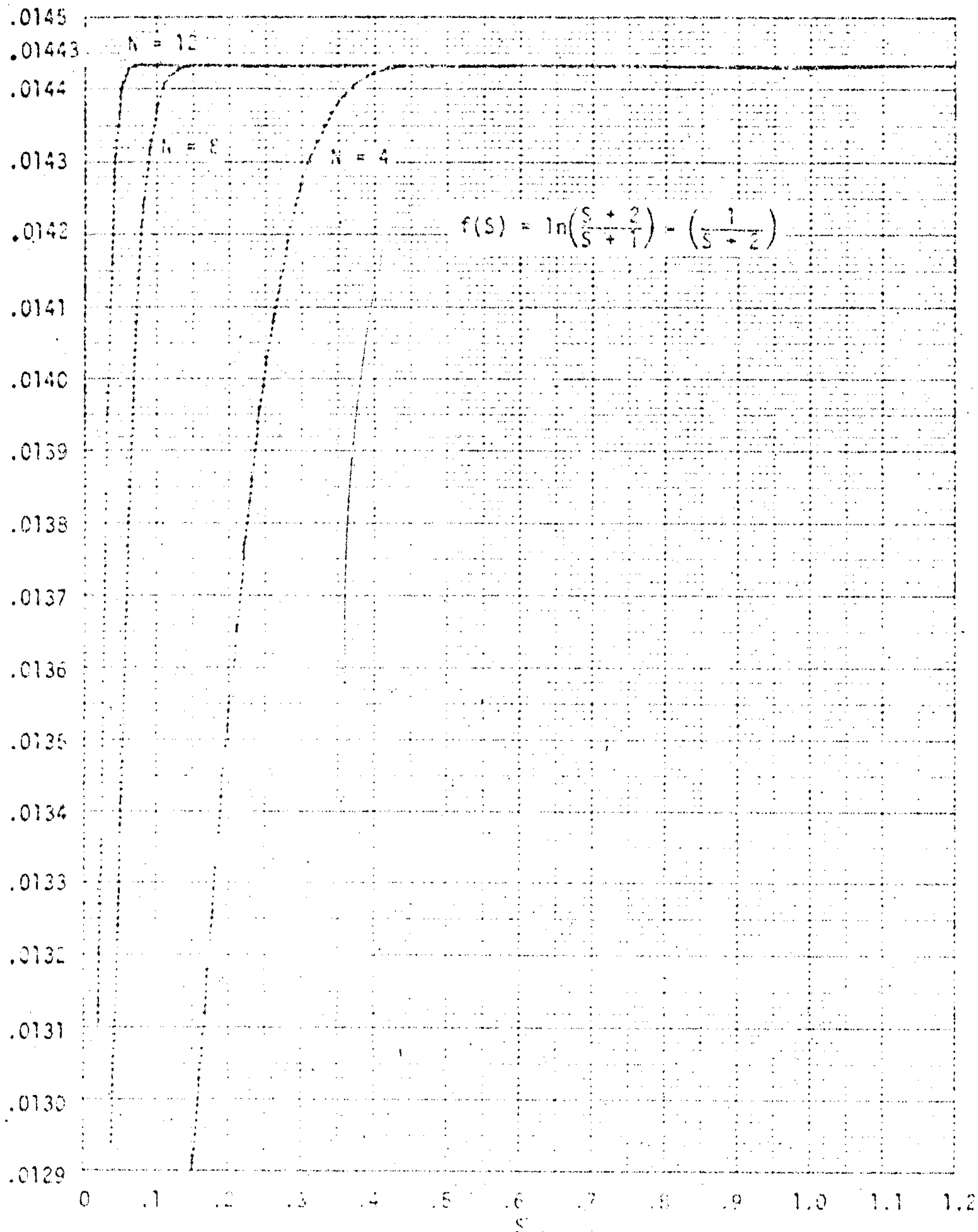

Figure 11.- $\sum A_n$ for $f(t) = \frac{1}{t}(e^{-t}$ [illegible]

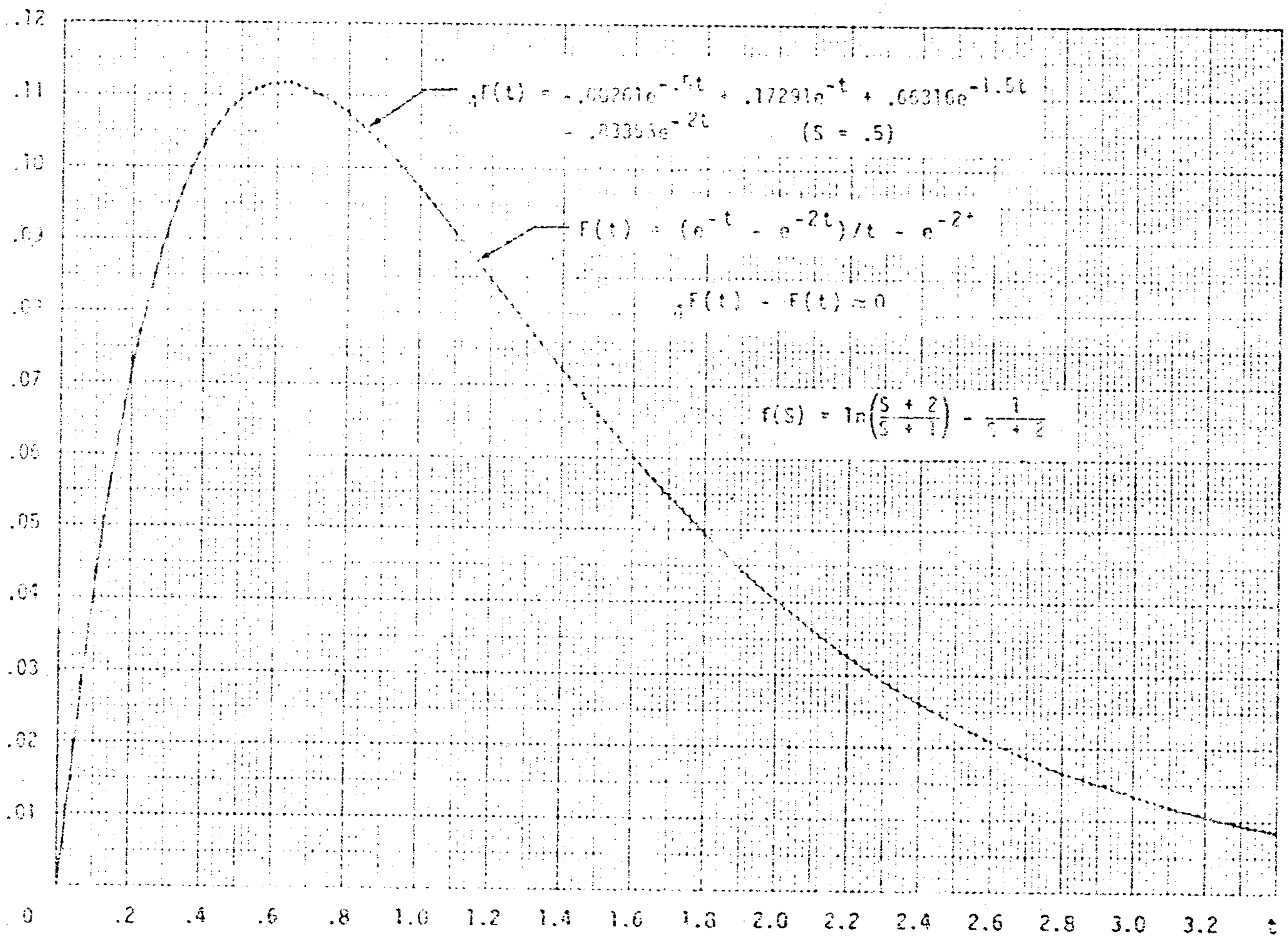

Figure 12.- Approximation of $F(t) = \frac{1}{t}(e^{-t} - e^{-2t}) - e^{-2t}$.

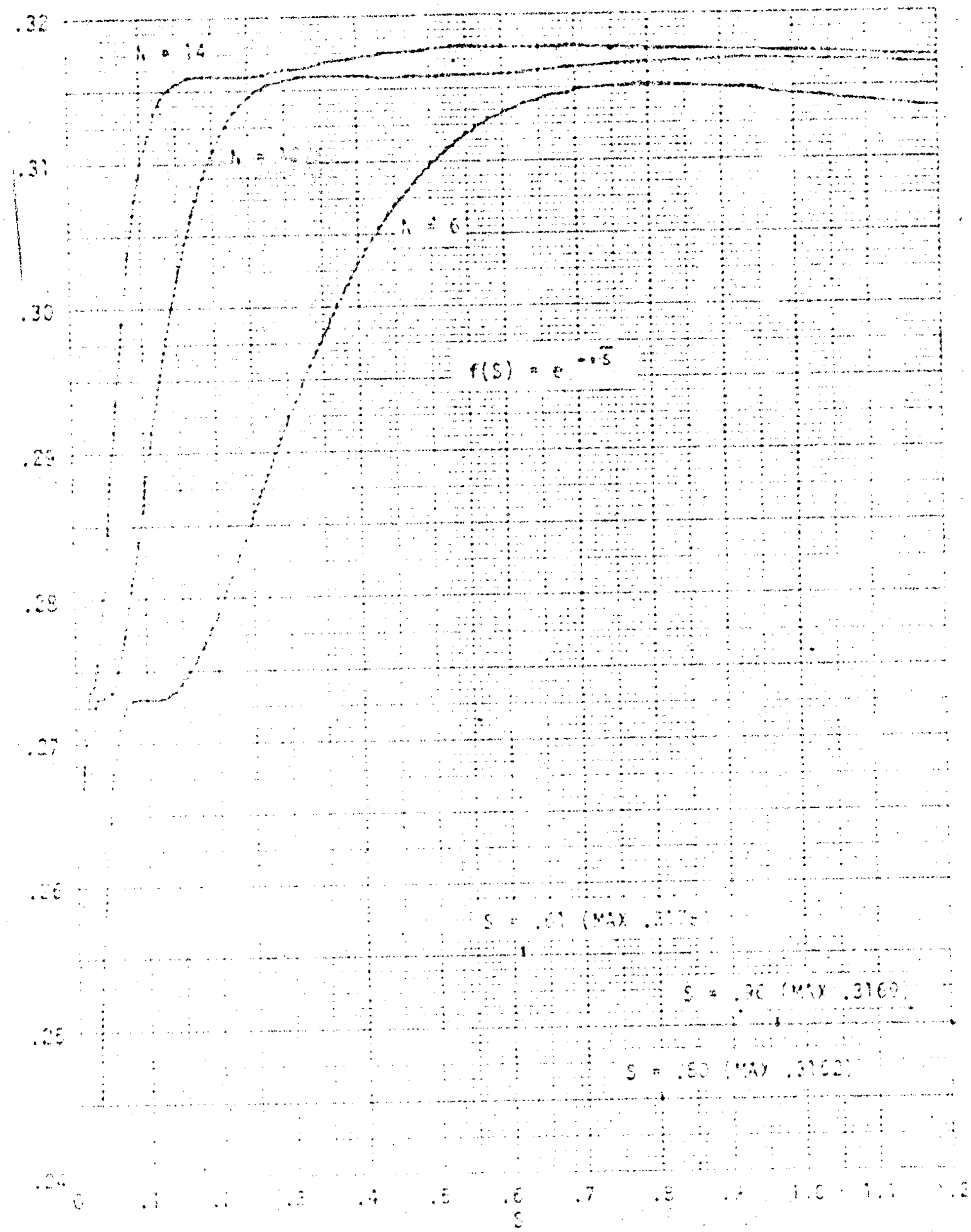

Figure [illegible] $\sum$ for [illegible]

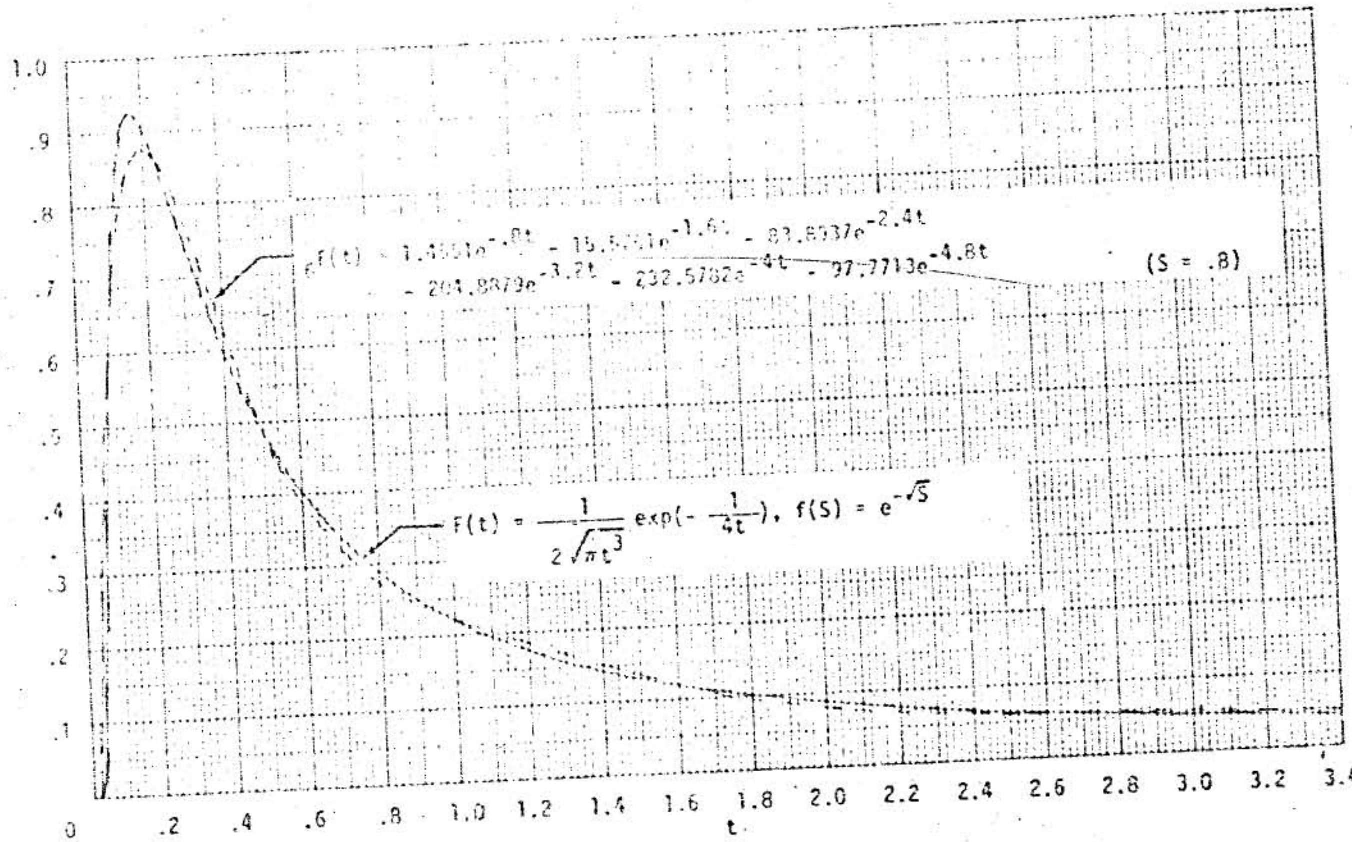

Figure 14.- Approximation of $F(t) = \frac{1}{2\sqrt{\pi t^3}} \exp(-\frac{1}{4t})$.

APPENDIX

THE L_n FUNCTIONS AND PROPERTIES

For brevity, the theorems and lemmas presented here will be shown without proof.

Definition 1:

The scalar product of $f(t)$ and $g(t)$ will be defined by

$$(f,g) = \int_0^\infty f(t)g(t)dt \tag{1}$$

Definition 2:

Define $L_n(st)$ by

$$L_n(st) = \sum_{i=1}^{n} {}_na_i e^{-ist} \tag{2}$$

where

$$ {}_na_i = (-1)^{n+i}\sqrt{2sn}\,\frac{(n+i-1)!}{i!(i-1)!(n-i)!} \tag{3}$$

Alternately

$$ {}_na_i = (-1)^{n+i}\sqrt{2sn}\,\frac{n}{i!(i-1)!}\prod_{j=1}^{i-1}(n^2-j^2) \tag{4}$$

where

$$ {}_na_1 = (-1)^{n+1}\sqrt{2sn}\;n \tag{5}$$

Lemma 1:

For $n > 1$

$$\sum_{i=1}^{n} \frac{n^{a_i}}{x + i} = \sqrt{2sn} \frac{(x - 1)(x - 2)\cdots(x - (n - 1))}{(x + 1)(x + 2)\cdots(x + (n - 1))(x + n)} \tag{6}$$

Corollary A:

$$\int_0^\infty L_n(st)dt = (-1)^{n+1} \frac{2}{\sqrt{2ns}} \tag{7}$$

or

$$\sum_{i=1}^{n} \frac{n^{a_i}}{i} = (-1)^{n+1}\sqrt{2sn}\ \frac{1}{n} \tag{8}$$

Corollary B:

$$L_n(0) = \sqrt{2ns} \tag{9}$$

or

$$\sum_{i=1}^{n} n^{a_i} = \sqrt{2ns} \tag{10}$$

Theorem 1:

The system of functions $L_n(st)$ are orthonormal. That is

$$(L_n, L_m) = 0 \quad \text{for } n \neq m$$
$$= 1 \quad \text{for } n = m \tag{11}$$

Definition 4:

The generating function $g(z,t)$ is defined as

$$g(z,t) = 1 - \frac{1}{\sqrt{1 + \frac{4z}{(1-z)^2} e^{-st}}} = g(1/z,t) \tag{12}$$

Theorem 2:

Expansion of $g(z,t)$ into Maclaurin's series gives

$$g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2ns}} z^n L_n(st) \qquad z^2 \leq 1 \tag{13}$$

$$g(z,t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2ns}} \frac{1}{z^n} L_n(st) \qquad z^2 \geq 1 \tag{14}$$

Theorem 3:

The difference equation satisfied by $L_n(st)$ is

$$L_n = 2\,\frac{2n-1}{\sqrt{n(n-1)}}\left\{\left[e^{-st} - \frac{n(n-1)}{2n-1} + \frac{(n-1)(n-2)}{2n-3}\right]L_{n-1} - \frac{1}{2}\,\frac{\sqrt{(n-1)(n-2)}}{2n-3}L_{n-2}\right\} \tag{15}$$

Theorem 4:

The differential equation satisfied by L_n is

$$(e^{st} - 1)\ddot{L}_n + se^{st}\dot{L}_n + s^2n^2L_n = 0 \tag{16}$$

Also of interest is

$$L_n = (-1)^{n+1}\,\frac{\sqrt{2ns}}{(n-1)!}\quad\frac{d^{n-1}}{d(e^{-st})^{n-1}}\left[e^{-nst}(1 - e^{-st})^{n-1}\right] \tag{17}$$

Theorem 5:

$$e^{-nst} = \frac{2}{s}\,n!(n-1)!\sum_{i=1}^{n}\frac{\sqrt{i}}{(n+i)!(n-i)!}L_i(st) \tag{18}$$

Definition 5:

Let

$$\int_0^\infty F(t)^2dt$$

be finite.

Let ${}_NF(t)$ be an approximation of $F(t)$. The integral square error is defined by

$$E = \int_0^\infty ({}_N F(t) - F(t))^2 dt \tag{19}$$

Theorem 6:

The best approximation of $F(t)$ in the integral square error sense (E minimized) is given by

$${}_N F(t) = \sum_{n=1}^{N} A_n L_n(st) \tag{20}$$

where

$$A_n(s) = \int_0^\infty F(t) L_n(st) dt \tag{21}$$

The integral square error is now given by

$$E = \int_0^\infty F(t)^2 dt - \sum_{n=1}^{N} A_n^2 \geq 0 \tag{22}$$

E is minimized by choosing s such that $\sum_{n=1}^{N} A_n^2$ is maximum.

Theorem 7, completeness theorem:

If

$$\int_0^\infty F(t)^2 dt$$

is finite, and the Laplace transform of $F(t)$, $f(s)$ exists, then

$$E \to 0 \quad \text{as} \quad N \to \infty$$

Theorem 8:

Let the Laplace transform of $F(t)$ be

$$f(s) = \int_0^\infty F(t)e^{-st}dt \tag{23}$$

Then

$$A_n(s) = \sum_{i=1}^{n} {}_n a_i f(is) \tag{24}$$

Theorem 9:

${}_N F(t)$ can be written as

$$_N F(t) = \sum_{n=1}^{N} {}_N B_n e^{-nst} \tag{25}$$

where

$$_N B_n = 2s \left[{}_N b_{n1} f(s) + {}_N b_{n2} f(2s) + {}_N b_{n3} f(3s) + \cdots + {}_N b_{nN} f(Ns) \right] \tag{26}$$

and where

$$ {}_N b_{ij} = {}_N b_{ji} = \frac{1}{2s} \sum_{k=1}^{N} {}_k a_i \; {}_k a_j \qquad ({}_k a_m = 0 \quad \text{for} \quad m > k) \tag{27} $$

or

$$ {}_N b_{ij} = {}_N b_{ji} = \frac{(-1)^{i+j}}{2(i+j)} \frac{1}{i!(i-1)!} \frac{1}{j!(j-1)!} \frac{(N+i)!}{(N-i)!} \frac{(N+j)!}{(N-j)!} \tag{28} $$

Lemma 2:

$$ \sum_{i=1}^{N} \frac{{}_N b_{ij}}{x+i} = \frac{(-1)^{N-j}}{2} \frac{(N+j)!}{(N-j)!j!(j-1)!} $$

$$ \frac{(x-1)(x-2)\cdots(x-N)}{(x+1)(x+2)\cdots(x+N)} \frac{1}{x-j} \tag{29} $$

Theorem 10:

$$ f(is) = \frac{1}{s} \sum_{n=1}^{N} \frac{{}_N B_n}{i+n} \tag{30} $$

Theorem 11:

$$ \sum_{n=1}^{N} A_n^2 = \sum_{n=1}^{N} {}_N B_n f(ns) \tag{31} $$

where ${}_N B_n$ was given by equation 26.

Theorem 12:

$$_{N}F(0) = \sum_{n=1}^{N} \sqrt{2ns}\, A_n \tag{32}$$

Theorem 13:

$$\int_0^{\infty} G(t)L_m(st)L_n(st)dt = \sum_{j=1}^{m} {}_m a_j \; {}_n g_j \tag{33}$$

where

$$_{n}g_j = \sum_{i=1}^{n} {}_n a_i g\left((i + j)s\right) \tag{34}$$

where $g(s)$ is the Laplace transform of $G(t)$.

Theorem 14:

The best approximation to the jth derivative of $F(t)$ is

$$_{N}F^{(j)}(t) = \sum_{n=1}^{N} {}_j A_n L_n(st) \tag{35}$$

where

$$ {}_jA_n = \sum_{i=1}^{n} {}_na_i(is)^j f(is) - F(+0) \sum_{i=1}^{n} {}_na_i(is)^{j-1} \tag{36} $$

$$ - \left.\frac{dF}{dt}\right|_{t=+0} \sum_{i=1}^{n} {}_na_i(is)^{j-2} - \left.\frac{d^2F}{dt^2}\right|_{t=+0} \sum_{i=1}^{n} {}_na_i(is)^{j-3} $$

$$ - \cdots - \left.\frac{d^{j-1}F}{dt^{j-1}}\right|_{t=+0} \sum_{i=1}^{n} {}_na_i $$

Note

$$ {}_NF^{(j)}(t) \neq \frac{d^j{}_NF(t)}{dt^j} \tag{37} $$

For example, if $j = 1$, the first derivative, then

$$ {}_1A_n = \sum_{i=1}^{n} {}_na_i(is)f(is) - \sqrt{2sn}\, F(+0) \tag{38} $$

Note equation 10, corollary B,

$$ \sum_{i=1}^{n} {}_na_i = \sqrt{2ns} $$

was used to obtain equation 38. The value of $F(+0)$ can be obtained from the initial value theorem.

$$ F(+0) = \lim_{s \to \infty} sf(s) \tag{39} $$

If $j = 2$, the second derivative, then

$$_2A_n = \sum_{i=1}^{n} n^{a_i}(is)^2 f(is) - \sqrt{2sn}\, n^2 sF(+0) - \sqrt{2sn} \left.\frac{dF}{dt}\right|_{t=+0} \qquad (40)$$

If $j = 3$

$$_3A_n = \sum_{i=1}^{n} n^{a_i}(is)^3 f(is) - \sqrt{2sn}\, \frac{n^2}{2}(n^2 + 1)s^2F(+0)$$

$$- \sqrt{2sn}\, n^2 s \left.\frac{dF}{dt}\right|_{t=+0} - \sqrt{2sn} \left.\frac{d^2F}{dt^2}\right|_{t=+0} \qquad (41)$$

For $j = 4$

$$_4A_n = \sum_{i=1}^{n} n^{a_i}(is)^4 f(is) - \sqrt{2sn}\, \frac{n^2}{6}(n^4 + 4n^2 + 1)s^3F(+0)$$

$$- \frac{1}{2}\sqrt{2sn}\, n^2(n^2 + 1)s^2 \left.\frac{dF}{dT}\right|_{t=+0} - \sqrt{2sn}\, n^2 s \left.\frac{d^2F}{dt^2}\right|_{t=+0}$$

$$- \sqrt{2sn} \left.\frac{d^3F}{dt^3}\right|_{t=+0} \qquad (42)$$

Theorem 15:

If

$$f(s) = \frac{A}{s + a} \qquad (43)$$

then

$$A_n = (-1)^{n+1} A \sqrt{2ns} \frac{(s - a)(2s - a)\cdots\left((n - 1)s - a\right)}{(s + a)(2s + a)\cdots(ns + a)} \tag{44}$$

$$A_1 = A \sqrt{2s} \frac{1}{s + a} \tag{45}$$

Note the results for $A = 1$ and $a = 0$, $F(t)$ a unit step function. In this case

$$A_n = (-1)^{n+1} \frac{2}{\sqrt{2ns}} \tag{46}$$

Hence

$$_NF(t) = 2 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{\sqrt{2ns}} L_n(st) \tag{47}$$

From corollary B, $L_n(0) = \sqrt{2ns}$, hence

$$\begin{aligned} {}_NF(0) &= 0 \qquad N \text{ even} \\ &= 2 \qquad N \text{ odd} \end{aligned} \tag{48}$$

The equations shown in theorem 15 are useful for testing the accuracy of computer computations.

END

DATE

FILMED

JUL 24 1980