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SUDDEN STRETCHING OF A FOUR LAYERED-COMPOSITE PLATE

BY

G. C. SIH AND E. P. CHEN

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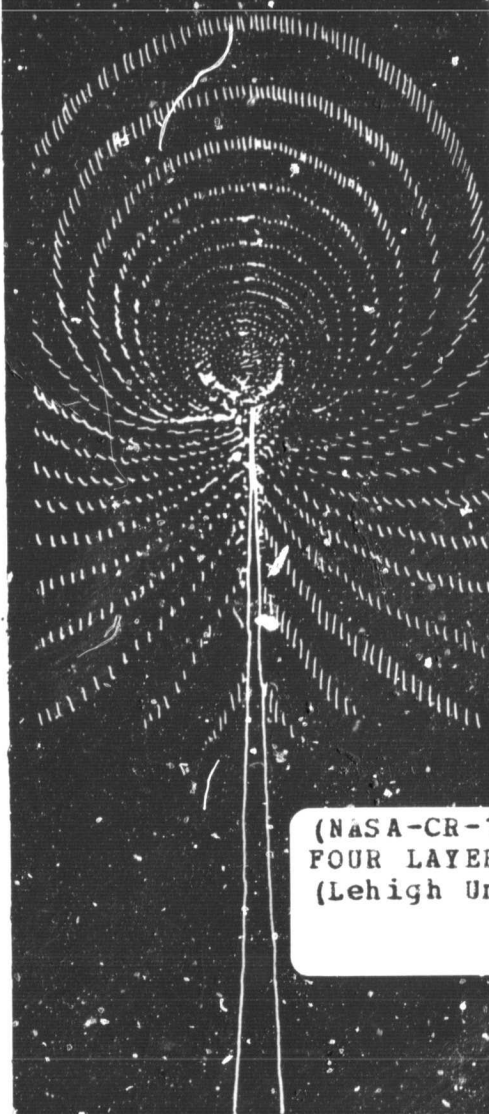
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FOREWORD

The results in this report on the sudden stretching of a four-layered composite plate were obtained during the course of research supported by the NASA-Lewis Research Center in Cleveland, Ohio, for the period February 13, 1979 through February 12, 1980 under Grant NSG 3179 with the Institute of Fracture and Solid Mechanics at Lehigh University. The Principal Investigator of the Project is Professor George C. Sih who wishes to acknowledge Dr. Christos C. Chamis, the NASA Project Manager, for the encouragement he provided during this Project.

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LIST OF SYMBOLS

a	- half crack length
A, B, \dots, D	- unknowns in integrals, functions of (s, p)
Br	- Bromwich contour in complex p -plane
c_{21}	- shear wave speed for material 1
F, G	- known functions of (s, p)
h	- laminate thickness
H	- potential function
H^*	- Laplace transform of H
$H(t)$	- Heaviside unit step function
J_0	- Bessel function of order zero
$k_1(t)$	- dynamic stress intensity factor
$k_1^*(p)$	- Laplace transform of $k_1(t)$
$L(\xi, n, p)$	- kernel in Fredholm integral equation
N_0	- constant stress resultant
N_x, N_y, \dots, N_{xy}	- stress resultants
p	- Laplace transform variable
R_x, R_y	- transverse shear forces
r, θ	- crack tip polar coordinates
s	- variable of integration
s_j	- parameter defined in equation (26) with $j = 1, 2$
t	- time
u_x, v_y, w_z	- displacement components in the (x, y, z) coordinate system
v_x, v_y, v_z	- displacement functions of x and y
x, y, z	- rectangular coordinates

$\alpha_0, \gamma_0, \delta_0, \rho_0, \omega_0$	- material parameters
β, γ, δ	- material constants
Δ_j	- defined in equation (28) with $j = 1, 2$
$\epsilon_x, \epsilon_y, \dots, \gamma_{yz}$	- strain components
κ	- correction factor $\pi/\sqrt{12}$ in plate theory
μ_j	- shear moduli with $j = 1, 2$
ν_j	- Poisson's ratio with $j = 1, 2$
ξ, η	- variables of integration
ρ_j	- mass density for material j
λ_j, μ_j	- Lamé coefficients with $j = 1, 2$
$(\Lambda_x)_j, (\Lambda_y)_j, \dots, (\Lambda_{xy})_j$	- strain resultants
$\sigma_x, \sigma_y, \dots, \tau_{yz}$	- stress components
ϕ^*	- potential function in Laplace transform plane
$\phi^*(\xi, \rho)$	- unknown in Fredholm integral equation

SUDDEN STRETCHING OF A FOUR-LAYERED COMPOSITE PLATE

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ABSTRACT

A research effort primarily concerned with the understanding of laminated composite plates with cracks subjected to time-dependent extensional loads is reported here. When loads are applied suddenly to a laminate, waves are reflected and refracted through the laminae and give rise to stresses and strains throughout the composite system. The process is three-dimensional in character and presents a formidable problem in the theory of elastodynamics, particularly in the presence of crack-like imperfections.

An approximate theory of laminated plates is developed by assuming that the extensional and thickness mode of vibration are coupled. The mixed boundary value crack problem of a four-layered composite plate is solved. Dynamic stress intensity factors for a crack subjected to suddenly applied stress are found to vary as a function of time and depend on the material properties of the laminate. Stress intensification in the region near the crack front can be reduced by having the shear modulus of the inner layers to be larger than that of the outer layers.

INTRODUCTION

The current interest in laminates for structural application is associated with the high strength-to-weight ratio which can be developed in laminates. These laminates are generally composed of layers which have been reinforced by embedding unidirectional fibers. The layers are adhered to each other such that the fiber direction varies from one layer to the next in a previously determined manner. The freedom of choice for fiber orientation in the layers of the composite system enables the development of laminates with special preferential directional properties for particular applications. Because of this characteristic of fibrous composites, the employment of these systems rather than equivalent homogeneous members will be clearly advantageous in many applications.

Because of the complicated internal structure of composite systems, stress analysis is much more difficult than for equivalent single-phase material. One fact which emerges very clearly from laminate studies is that the stress field in composite systems is truly three-dimensional in character. Thus, even the stress field in a symmetric laminate subjected to in-plane loading cannot be accurately modeled by standard two-dimensional methods of analysis. The previous work in this area further indicates that relatively little effort has been made to formulate laminate plate theories that can effectively solve for the redistribution of stresses and strains due to the presence of mechanical imperfections such as cracks.

One possible means of simplifying the three-dimensional equations of elasticity is to invoke the concept adopted in the formulation of plate theory. Approximate stress and strain dependence on the plate thickness coordinate are assumed such that the governing differential equations possess only two independent

space variables. In addition, special attention must be given to the state of affairs near the crack when formulating plate theories for analyzing crack problems. With this in mind, Hartranft and Sih [1] developed an approximate three-dimensional theory for a single material plate containing a through crack. The condition of plane strain was preserved ahead of the crack as suggested by Sih [2]. This theory was later extended to laminates by Badaliance, Sih and Chen [3] to solve the problem of a through crack in a laminar plate subjected to in-plane loading. The through crack configuration represents a preliminary effort to model the damage of composite plates. Additional complications arise when the load is time dependent. These considerations will be taken into account in the development of a new dynamic theory of laminated composite plates subjected to extensional loads.

This work is concerned with the formulation of a dynamic theory of laminated plates and reduces to that of Kane and Mindlin [4] for the single material plate. The idealized condition of stress and displacement continuity across the interface is replaced by assigning certain conditions of material nonhomogeneity in the thickness direction of the laminated plate as if it were a single layered nonhomogeneous plate. The nonhomogeneity is made equivalent to a symmetric laminate balanced with reference to its mid-plane. A through crack is assumed to exist in a four-layered laminate. Dynamic stress intensity factors are obtained for the case of a suddenly applied uniform in-plane loading and shown to vary as a function of time. Discussed are also the influence of material properties of the layers on the local stresses.

BASIC FORMULATION

The elastodynamic equations of generalized plane stress are adequate only if the frequency of vibration is lower than that of the first thickness mode and the wave length is large in comparison with the plate thickness. In other words, the coupling between extensional and thickness mode of vibration can be neglected. When laminated composite plates are stressed dynamically, loads are transmitted through the laminae by the reflection of thickness refraction of stress waves. The mode of vibration cannot be ignored, particularly in the vicinity of a crack-like imperfection where the stress state acquires a three-dimensional character.

A dynamic laminate plate theory will be developed to solve the problem of a four-layered composite plate with a through crack subjected to a suddenly applied uniform extensional load. The theory is a generalization to that given by Kane and Mindlin [4] for a single layer plate in which the extensional and thickness mode of vibration are assumed to be coupled. Accounted for is the lowest thickness-stretch mode such that the displacement is normal to the plate surface. Mindlin and Medick [5] have also considered a formulation in which the thickness-shear mode of vibration with displacement parallel to the plate surface is also included. The mid-plane of the plate is taken as the nodal plane of vibration.

Consider a four-layered composite plate of thickness h as shown in Figure 1 where each layer has the same thickness $h/4$. The two outer layers have material properties (μ_2, ν_2) or (λ_2, μ_2) while the two inner layers have material properties (μ_1, ν_1) or (λ_1, μ_1) . The Lamé coefficients are denoted by λ_j and μ_j ($j = 1, 2$). The layers are stacked such that symmetry prevails across the mid-plane of the laminate composite. The crack of width $2a$ cuts through the entire thickness of the laminate.

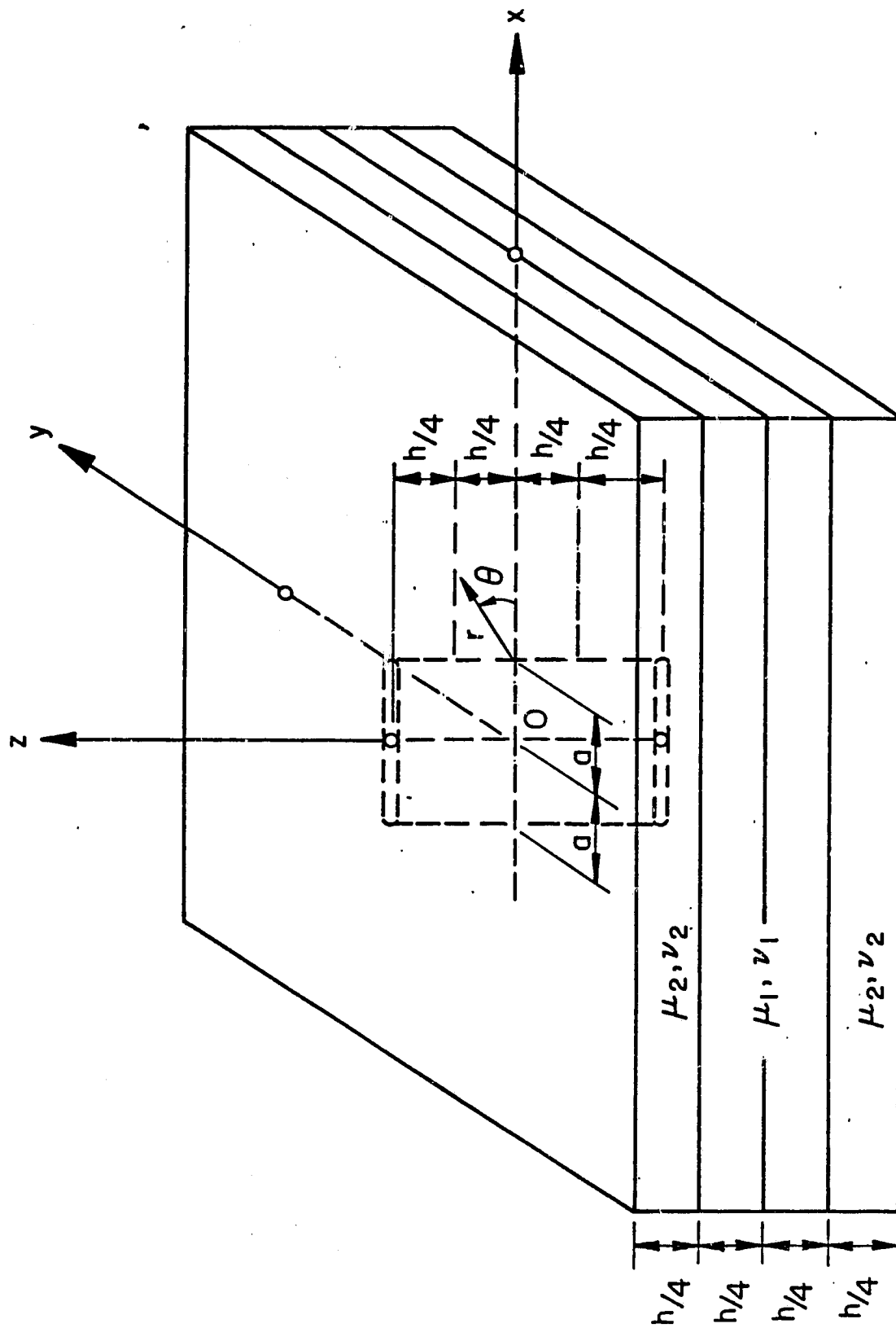


Figure 1 - A four-layered composite plate with a crack

The time-dependent displacement field is assumed to be

$$\begin{aligned}
 u_x &= v_x(x,y,t) \\
 v_y &= v_y(x,y,t) \\
 w_z &= \frac{2z}{h} v_z(x,y,t)
 \end{aligned}
 \tag{1}$$

It follows that the strain components can be written as

$$\begin{aligned}
 \epsilon_x(x,y,t) &= \frac{\partial v_x}{\partial x} \\
 \epsilon_y(x,y,t) &= \frac{\partial v_y}{\partial y} \\
 \epsilon_z(x,y,t) &= \frac{2}{h} v_z \\
 \gamma_{xy}(x,y,t) &= \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \\
 \gamma_{yz}(x,y,t) &= \frac{2z}{h} \frac{\partial v_z}{\partial y} \\
 \gamma_{zx}(x,y,t) &= \frac{2z}{h} \frac{\partial v_z}{\partial x}
 \end{aligned}
 \tag{2}$$

in which the transverse normal and shear strains are assumed to be linear in the thickness coordinate z . If each layer of the laminated composite plate is isotropic, then the following stress-strain relationships may be used:

$$\begin{aligned}
 \sigma_x &= (\lambda+2\mu)\epsilon_x + \lambda(\epsilon_y + \kappa\epsilon_z) \\
 \sigma_y &= (\lambda+2\mu)\epsilon_y + \lambda(\epsilon_x + \kappa\epsilon_z) \\
 \sigma_z &= (\lambda+2\mu)\kappa^2\epsilon_z + \lambda\kappa(\epsilon_x + \epsilon_y)
 \end{aligned}$$

$$\tau_{yz} = \mu \gamma_{yz}$$

$$\tau_{zx} = \mu \gamma_{zx}$$

$$\tau_{xy} = \mu \gamma_{xy}$$

(3)

The constant

$$\kappa = \pi/\sqrt{12}$$

(4)

accounts for the coupling between the extensional and thickness mode of vibration.

It is determined from the three-dimensional equations of elasticity. As in the

development of plate theories, the resultant strain quantities $(\Lambda_x)_j$, $(\Lambda_y)_j$,

..., $(\Lambda_{xy})_j$ ($j = 1, 2$) will be defined:

$$[(\Lambda_x)_1, (\Lambda_y)_1, (\Lambda_z)_1, (\Lambda_{xy})_1] = \frac{2}{h} \int_{-h/4}^{h/4} [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}] dz$$

$$[(\Lambda_x)_2, (\Lambda_y)_2, (\Lambda_z)_2, (\Lambda_{xy})_2] = \frac{2}{h} \left\{ \int_{h/4}^{h/2} [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}] dz \right.$$

$$\left. + \int_{-h/2}^{-h/4} [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}] dz \right\}$$

(5)

$$[(\Lambda_{xz})_1, (\Lambda_{yz})_1] = \frac{96}{h^3} \int_{-h/4}^{h/4} [\gamma_{xz}, \gamma_{yz}] z dz$$

$$[(\Lambda_{xz})_2, (\Lambda_{yz})_2] = \frac{96}{7h^3} \int_{-h/2}^{-h/4} [\gamma_{xz}, \gamma_{yz}] z dz + \frac{h/2}{h/4} \int_{h/4}^{h/2} [\gamma_{xz}, \gamma_{yz}] z dz$$

Substituting equations (2) into (5), it is found that

$$(\Lambda_x)_1 = (\Lambda_x)_2 = \frac{\partial v_x}{\partial x}$$

$$\begin{aligned}
(\Lambda_y)_1 &= (\Lambda_y)_2 = \frac{\partial v_x}{\partial y} \\
(\Lambda_z)_1 &= (\Lambda_z)_2 = \frac{2}{h} v_z \\
(\Lambda_{xy})_1 &= (\Lambda_{xy})_2 = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \\
(\Lambda_{xz})_1 &= (\Lambda_{xz})_2 = \frac{2}{h} \frac{\partial v_z}{\partial x} \\
(\Lambda_{yz})_1 &= (\Lambda_{yz})_2 = \frac{2}{h} \frac{\partial v_z}{\partial y}
\end{aligned} \tag{6}$$

The laminate plate theory can be most conveniently formulated in terms of the stress resultants

$$(N_x, N_y, N_z, N_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_z, \tau_{xy}) dz \tag{7}$$

and the transverse shears

$$(R_x, R_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) z dz \tag{8}$$

The stress-strain relations in equations (3) when enforced yield

$$\begin{aligned}
N_x(x, y, t) &= \frac{h}{2} [(\beta+2\gamma) \frac{\partial v_x}{\partial x} + \beta \frac{\partial v_y}{\partial y}] + \beta \kappa v_z \\
N_y(x, y, t) &= \frac{h}{2} [(\beta+2\gamma) \frac{\partial v_y}{\partial y} + \beta \frac{\partial v_x}{\partial x}] + \beta \kappa v_z \\
N_z(x, y, t) &= (\beta+2\gamma) \kappa^2 v_z + \frac{1}{2} \beta \kappa h \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) \\
N_{xy}(x, y, t) &= \frac{1}{2} \gamma h \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)
\end{aligned} \tag{9}$$

and

$$R_x(x,y,t) = \frac{1}{48} \delta h^2 \frac{\partial v_z}{\partial x}$$

$$R_y(x,y,t) = \frac{1}{48} \delta h^2 \frac{\partial v_z}{\partial y}$$
(10)

In equations (9) and (10), β , γ and δ stand for

$$\beta = \lambda_1 + \lambda_2, \quad \gamma = \mu_1 + \mu_2, \quad \delta = \mu_1 + 7\mu_2$$
(11)

Denoting ρ_1 and ρ_2 as the mass density of the inner and outer layers of the laminate, the three equations of motion governing N_x , N_y , ..., R_y become

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \frac{1}{2} h(\rho_1 + \rho_2) \frac{\partial^2 v_x}{\partial t^2}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \frac{1}{2} h(\rho_1 + \rho_2) \frac{\partial^2 v_y}{\partial t^2}$$

$$\frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} - N_z = \frac{1}{48} h^2(\rho_1 + 7\rho_2) \frac{\partial^2 v_z}{\partial t^2}$$
(12)

The result of substituting equations (9) and (10) into (12) renders

$$\gamma \nabla^2 v_x + (\beta + \gamma) \frac{\partial}{\partial x} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) + \frac{2\beta\kappa}{h} \frac{\partial v_z}{\partial x} = (\rho_1 + \rho_2) \frac{\partial^2 v_x}{\partial t^2}$$

$$\gamma \nabla^2 v_y + (\beta + \gamma) \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) + \frac{2\beta\kappa}{h} \frac{\partial v_z}{\partial y} = (\rho_1 + \rho_2) \frac{\partial^2 v_y}{\partial t^2}$$

$$\delta \nabla^2 v_z - \frac{48}{h^2} (\beta + 2\gamma) \kappa^2 v_z - \frac{24\beta\kappa}{h} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = (\rho_1 + 7\rho_2) \frac{\partial^2 v_z}{\partial t^2}$$
(13)

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator in two dimensions.

METHOD OF SOLUTION

Equations (13) will be solved by introducing two potential functions $\phi(x,y,t)$ and $H(x,y,t)$ as

$$v_x(x,y,t) = \frac{\partial \phi}{\partial x} + \frac{\partial H}{\partial y} \quad (14)$$

$$v_y(x,y,t) = \frac{\partial \phi}{\partial y} - \frac{\partial H}{\partial x}$$

Making the appropriate algebraic manipulations, the governing equations for the potential functions can be derived by enforcing equations (13):

$$\begin{aligned} \gamma \nabla^2 H &= (\rho_1 + \rho_2) \frac{\partial^2 H}{\partial t^2} - \delta(\beta + 2\gamma) \nabla^4 \phi + \frac{192 \kappa^2 \gamma (\beta + 2\gamma)}{h^2} \nabla^2 \phi \\ &= \{(\rho_1 + \rho_2) [(\rho_1 + 7\rho_2) \frac{\partial^4 \phi}{\partial t^4} + \frac{48}{h^2} (\beta + 2\gamma) \frac{\partial^2 \phi}{\partial t^2}] \\ &\quad - [\delta(\rho_1 + \rho_2) + (\beta + 2\gamma)(\rho_1 + 7\rho_2) \frac{\partial^2}{\partial t^2} (\nabla^2 \phi)]\} \end{aligned} \quad (15)$$

Once $\phi(x,y,t)$ and $H(x,y,t)$ are known, v_x and v_y can be obtained from equations (14) and

$$v_z(x,y,t) = \frac{h}{2\beta\kappa} [(\rho_1 + \rho_2) \frac{\partial^2 \phi}{\partial t^2} - (\beta + 2\kappa) \nabla^2 \phi] \quad (16)$$

Suppose that a uniform stress resultant N_0 is applied suddenly to the crack surfaces and the resulting deformation is symmetrical with respect to the x-axis, then the following conditions are to be specified:

$$N_y(x,0,t) = -N_0 H(t), \quad x < a \quad (17)$$

$$v_y(x,0,t) = 0, \quad x \geq a$$

where $H(t)$ is the Heaviside unit step function. The condition of symmetry further requires that

$$N_{xy}(x,0,t) = R_y(x,0,t) = 0, \quad \text{for all } x \quad (18)$$

Use will now be made of the Laplace transform. Let $\phi^*(x,y,p)$, $H^*(x,y,p)$, etc., denote the Laplace transforms of the functions $\phi(x,y,t)$, $H(x,y,t)$, etc. Equation (15) when expressed in the Laplace transform domain become

$$\begin{aligned} (\nabla^2 - \omega_1^2) \phi_1^*(x,y,p) &= 0 \\ (\nabla^2 - \omega_2^2) \phi_2^*(x,y,p) &= 0 \\ (\nabla^2 - \omega_3^2) H^*(x,y,p) &= 0 \end{aligned} \quad (19)$$

where the potential $\phi(x,y,t)$ has been separated into two parts:

$$\phi(x,y,t) = \phi_1(x,y,t) + \phi_2(x,y,t) \quad (20)$$

in terms of time t or

$$\phi^*(x,y,p) = \phi_1^*(x,y,p) + \phi_2^*(x,y,p) \quad (21)$$

in terms of the Laplace transform variable p . The parameters ω_j ($j = 1, 2, 3$) in equations (19) are defined as

$$\omega_{1,2}^2 = \frac{6\kappa^2}{h\delta_0} [(\alpha_0 + \delta_0) \left(\frac{p}{\omega_0}\right)^2 + \rho_0 \pm \{[(\alpha_0 + \delta_0) \left(\frac{p}{\omega_0}\right)^2 + \rho_0]^2 - 4\alpha_0\delta_0 \left(\frac{p}{\omega_0}\right)^2 [(\frac{p}{\omega_0})^2 + \rho_0]\}^{1/2}] \quad (22)$$

$$\omega_3^2 = (\rho_1 + \rho_2) p^2 \gamma^{-1}$$

in which the newly defined quantities are

$$\alpha_0 = \frac{\beta + 2\gamma}{(\rho_1 + \rho_2)\gamma_0^2}, \quad \delta_0 = \frac{\delta}{(\rho_1 + 7\rho_2)\gamma_0^2} \quad (23)$$

$$\rho_0 = \frac{4(\rho_1 + \rho_2)}{\rho_1 + 7\rho_2}, \quad \omega_0^2 = \frac{12\kappa^2(\beta + 2\gamma)}{h^2(\rho_1 + \rho_2)}$$

and γ_0 takes the form

$$\gamma_0^2 = \frac{4\gamma(\beta + \gamma)}{(\rho_1 + \rho_2)(\beta + 2\gamma)} \quad (24)$$

Equations (19) then give

$$\phi_1^*(x, y, p) = \frac{2}{\pi} \int_0^\infty A(s, p) \cos(sx) \exp(-s_1 y) ds$$

$$\phi_2^*(x, y, p) = \frac{2}{\pi} \int_0^\infty B(s, p) \cos(sx) \exp(-s_2 y) ds \quad (25)$$

$$H^*(x, y, p) = \frac{2}{\pi} \int_0^\infty C(s, p) \sin(sx) \exp(-s_3 y) ds$$

with s_j being given by

$$s_j = \sqrt{s^2 + \omega_j^2}, \quad j = 1, 2, 3 \quad (26)$$

The dynamic problem has now been reduced to finding the three unknown functions $A(s,p)$, $B(s,p)$ and $C(s,p)$.

DUAL COUPLED INTEGRAL EQUATIONS

Before the boundary and symmetry conditions can be enforced, it is necessary to obtain $v_x^*(s,y,p)$, $v_y^*(x,y,p)$, etc., in terms of the unknowns in equations (25). With the help of equations (14) and (16), it can be shown that

$$\begin{aligned}
 v_x^*(x,y,p) &= -\frac{2}{\pi} \int_0^{\infty} [sA(s,p) \exp(-s_1 y) + sB(s,p) \exp(-s_2 y) \\
 &\quad + s_3 C(s,p) \exp(-s_3 y)] \sin(sx) ds \\
 v_y^*(x,y,p) &= -\frac{2}{\pi} \int_0^{\infty} [s_1 A(s,p) \exp(-s_1 y) + s_2 B(s,p) \exp(-s_2 y) \\
 &\quad + sC(s,p) \exp(-s_3 y)] \cos(sx) ds
 \end{aligned} \tag{27}$$

$$v_z^*(x,y,p) = \frac{2}{\pi} \int_0^{\infty} [\Delta_1 A(s,p) \exp(-s_1 y) + \Delta_2 B(s,p) \exp(-s_2 y)] \cos(sx) ds$$

The quantities Δ_j ($j = 1,2$) are given by

$$\Delta_j = \frac{h(\beta+2\gamma)}{2\beta\kappa} \left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega_j^2 \right], \quad j = 1,2 \tag{28}$$

Similarly, the Laplace transform of $N_x^*(x,y,p)$, $N_y^*(x,y,p)$ become

$$\begin{aligned}
 N_x^*(x,y,p) &= \frac{2}{\pi} \gamma h \int_0^{\infty} \left\{ \left[\frac{(\rho_1+\rho_2)p^2}{2\gamma} - s_1^2 \right] A(s,p) \exp(-s_1 y) \right. \\
 &\quad \left. + \left[\frac{(\rho_1+\rho_2)p^2}{2\gamma} - s_2^2 \right] B(s,p) \exp(-s_2 y) - s s_3 C(s,p) \exp(-s_3 y) \right\} \cos(sx) ds
 \end{aligned}$$

$$N_y^*(x,y,p) = \frac{2}{\pi} \gamma h \int_0^{\infty} \left\{ \left[s^2 + \frac{(\rho_1 + \rho_2)p^2}{2\gamma} \right] [A(s,p) \exp(-s_1 y) + B(s,p) \exp(-s_2 y)] \right. \\ \left. + s s_3 C(s,p) \exp(-s_3 y) \right\} \cos(sx) ds \quad (29)$$

$$N_{xy}^*(x,y,p) = \frac{2}{\pi} \gamma h \int_0^{\infty} \left\{ s s_1 A(s,p) \exp(-s_1 y) + s s_2 B(s,p) \exp(-s_2 y) \right. \\ \left. + \frac{1}{2} (s^2 + s_3^2) C(s,p) \exp(-s_3 y) \right\} \sin(sx) ds$$

while $R_x^*(x,y,p)$ and $R_y^*(x,y,p)$ take the forms

$$R_x^*(x,y,p) = - \frac{\delta h^2}{24\pi} \int_0^{\infty} s [\Delta_1 A(s,p) \exp(-s_1 y) + \Delta_2 B(s,p) \exp(-s_2 y)] \sin(sx) ds \quad (30)$$

$$R_y^*(x,y,p) = - \frac{\delta h^2}{24\pi} \int_0^{\infty} [s_1 \Delta_1 A(s,p) \exp(-s_1 y) + s_2 \Delta_2 B(s,p) \exp(-s_2 y)] \cos(sx) ds$$

The symmetry conditions in equations (18) when applied show that $A(s,p)$, $B(s,p)$ and $C(s,p)$ can be expressed in terms of a single unknown $D(s,p)$:

$$A(s,p) = \frac{s^2 + s_3^2}{s_1} D(s,p) \\ B(s,p) = -s_1 \left[\frac{(\rho_1 + \rho_2)p^2}{\beta + 2\gamma} - \omega_1^2 \right] / \left\{ s_2 \left[\frac{(\rho_1 + \rho_2)p^2}{\beta + 2\gamma} - \omega_2^2 \right] \right\} A(s,p) \quad (31) \\ C(s,p) = - \frac{2s s_1 (\omega_1^2 - \omega_2^2)}{(s^2 + s_3^2) \left[\frac{(\rho_1 + \rho_2)p^2}{\beta + 2\gamma} - \omega_2^2 \right]} A(s,p)$$

Application of the mixed boundary conditions in equations (17) leads to a system of dual integral equations

$$\int_0^{\infty} D(s,p) \cos(sx) ds = 0, \quad x > a \quad (32)$$

$$\int_0^{\infty} sF(s,p) D(s,p) \cos(sx) ds = -\frac{\pi N_0}{2\gamma hp}, \quad x < a$$

The function $F(s,p)$ is known:

$$F(s,p) = \frac{s^2+s_3^2}{ss_1} \left[\left[s^2 + \frac{(\rho_1+\rho_2)p^2}{2\gamma} \right] \left\{ 1 - \frac{s_1 \left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega_1^2 \right]}{s_2 \left[\frac{(\rho_1+\rho_2)p^2}{\beta+2\gamma} - \omega_2^2 \right]} \right\} - \frac{2s^2s_1s_3}{s^2+s_3^2} \left[\frac{\omega_1^2-\omega_2^2}{(\rho_1+\rho_2)p^2/(\beta+2\gamma)-\omega_2^2} \right] \right] \quad (33)$$

The standard procedure by Copson [6] may be applied to solve equations (32) and the result is

$$D(s,p) = -\frac{\pi N_0 a^2}{2\gamma hp} \left\{ \frac{[(\rho_1+\rho_2)p^2/(\beta+2\gamma)]-\omega_2^2}{(\rho_1+\rho_2)p^2} \frac{1}{\gamma} \left(1 - \frac{\gamma}{\beta+2\gamma} \right) (\omega_1^2-\omega_2^2) \right\} \times \int_0^1 \sqrt{\xi} \phi^*(\xi,p) J_0(sa\xi) d\xi \quad (34)$$

in which $\phi^*(\xi,p)$ can be computed from a Fredholm integral equation of the second kind:

$$\phi^*(\xi,p) + \int_0^1 \phi^*(\eta,p) L(\xi,\eta,p) d\eta = \sqrt{\xi} \quad (35)$$

The kernel $L(\xi,\eta,p)$ is

$$L(\xi,\eta,p) = \sqrt{\xi\eta} \int_0^{\infty} s \left[G\left(\frac{s}{a}, p\right) - 1 \right] J_0(s\xi) J_0(s\eta) ds \quad (36)$$

while the function $G(s,p)$ is related to $F(s,p)$ in equation (33):

$$G(s,p) = \frac{\{[(\rho_1 + \rho_2)p^2 / (\beta + 2\gamma)] - \omega_2^2\}^\gamma}{p^2 [1 - \gamma / (\beta + 2\gamma)] (\omega_1^2 - \omega_2^2)} F(s,p) \quad (37)$$

DYNAMIC STRESS INTENSITY FACTORS

Of interest is the intensification of the dynamic stresses ahead of the crack. Hence, the integrals in equations (29) and (30) must be evaluated for large values of s which corresponds to distances near the crack edge $x = \pm a$ and $y=0$. In terms of the polar coordinates r and θ in Figure 1, the Laplace transform of the stress resultants for small r are found:

$$\begin{aligned} N_x^*(r, \theta, p) &= \frac{k_1^*(p)}{\sqrt{2r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) + O(r^0) \\ N_y^*(r, \theta, p) &= \frac{k_1^*(p)}{\sqrt{2r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) + O(r^0) \\ N_{xy}^*(r, \theta, p) &= \frac{k_1^*(p)}{\sqrt{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^0) \end{aligned} \quad (38)$$

$$R_x^*(r, \theta, p) = R_y^*(r, \theta, p) = O(r^0)$$

in which $k_1^*(p)$ is the Laplace transform of $k_1(t)$:

$$k^*(p) = \frac{\phi^*(1,p)}{p} N_0 \sqrt{a} \quad (39)$$

The Laplace inversion theorem may now be applied to give

$$\begin{aligned}
N_x(r, \theta, t) &= \frac{k_1(t)}{\sqrt{2r}} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) + O(r^0) \\
N_y(r, \theta, t) &= \frac{k_1(t)}{\sqrt{2r}} \cos \frac{\theta}{2} (1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) + O(r^0) \\
N_{xy}(r, \theta, t) &= \frac{k_1(t)}{\sqrt{2r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + O(r^0) \\
R_x(r, \theta, t) &= R_y(r, \theta, t) = O(r^0)
\end{aligned}
\tag{40}$$

Equations (40) reveal that dynamic loading does not affect the functional relationship of r and θ . The stress intensity factor, however, is a function of time:

$$k_1(t) = \frac{N_0 \sqrt{a}}{2\pi i} \int_{Br} \frac{\phi^*(1, p)}{p} \exp(pt) dp \tag{41}$$

where Br denotes the Bromwich path of integration. Once $\phi^*(\xi, p)$ is calculated from equation (35) and evaluated at $\xi=1$, equation (41) may be solved numerically.

Figure 2 gives a plot of $\phi^*(1, p)$ as a function of c_{21}/pa where $c_{21} = (\mu_1/\rho_1)^{1/2}$ is the shear wave velocity referred to the material in the inner layers. For $\rho_1 = \rho_2$, $\nu_1 = \nu_2 = 0.3$ and $\rho_1 = \rho_2$, $\phi^*(1, p)$ is seen to increase monotonically with c_{21}/pa . Three different ratios of $\mu_2/\mu_1 = 0.2, 1.0$ and 5.0 are considered. Making use of the results in Figure 2, $k_1(t)$ in equation (41) may be computed. Refer to Figure 3 for a display of $k_1(t)/N_0 \sqrt{a}$ versus $c_{21}t/a$. The resultant stress intensity factors are observed to vary as a function of time. Their amplitude rise quickly reaching a peak and then declines. The solution for a homogeneous plate corresponds to $\mu_2/\mu_1 = 1.0$ as the Poisson's ratio and mass density

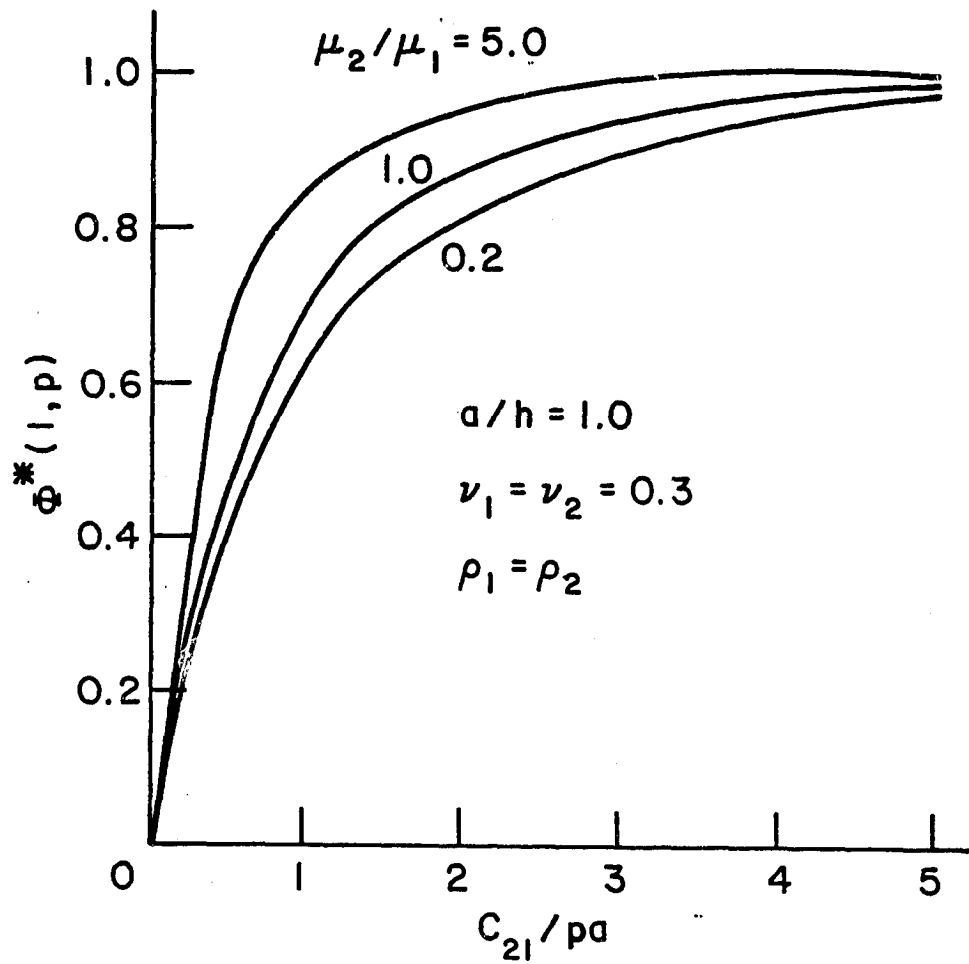


Figure 2 - Numerical values of $\phi^*(1,p)$ as a function of c_{21}/pa for $a/h = 1.0$

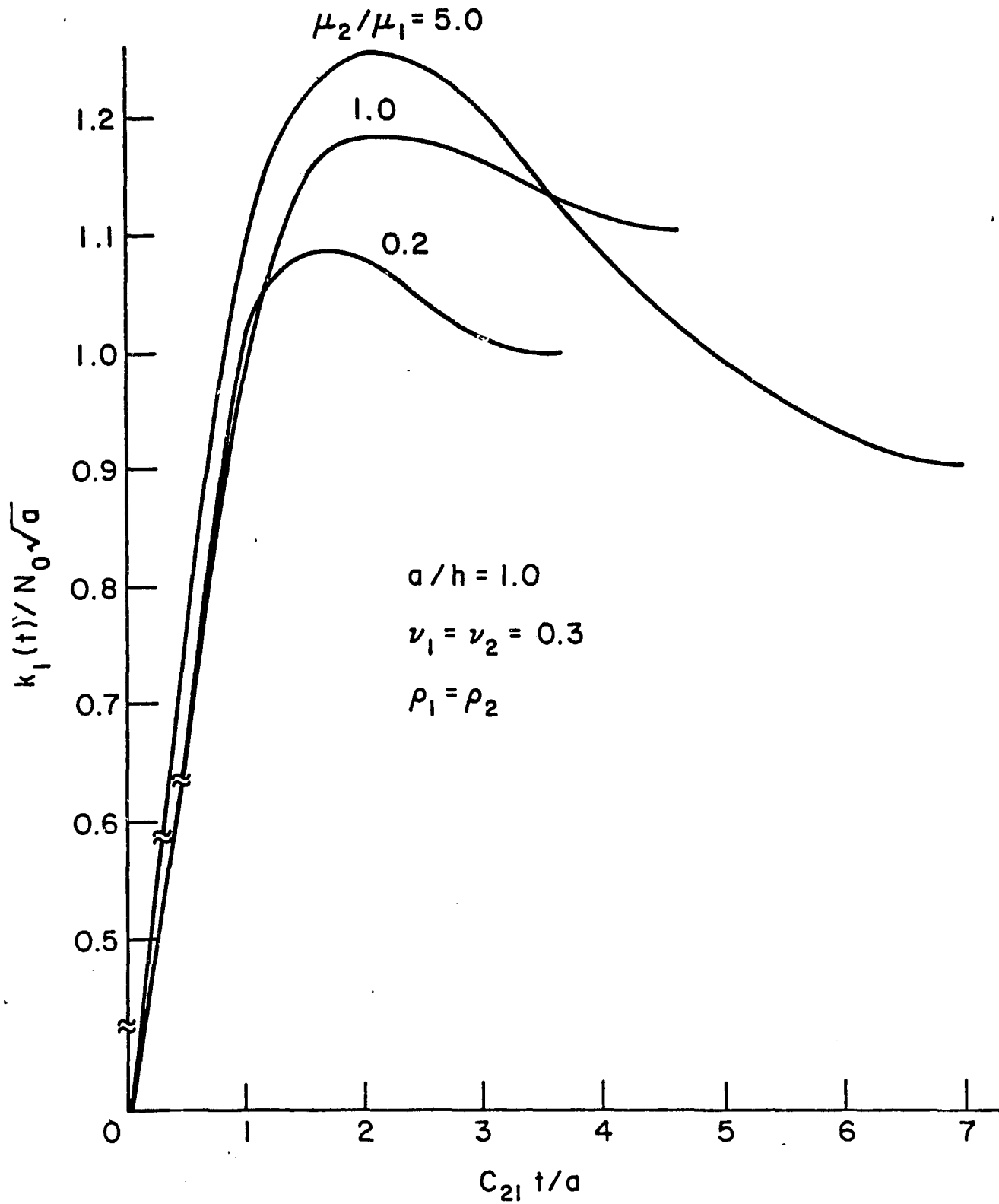


Figure 3 - Normalized resultant stress intensity factor versus $c_{21} t/a$ for $a/h = 1.0$

for the inner and outer layers are assumed to be equal. The peak value of $k_1(t)$ is greater than that of the homogeneous plate solution for $\mu_2 > \mu_1$ while the opposite is found for $\mu_2 < \mu_1$. Hence, the intensity of the crack border stress field can be reduced by having the shear modulus of the outer layers to be smaller than that of the inner layers.

CONCLUDING REMARKS

A dynamic laminate plate theory has been developed for solving crack boundary value problems. The complexity of the problem owing to material nonhomogeneity and dynamic stress analysis necessitates certain simplifying assumptions so that effective analytical solutions can be obtained. It is shown that the dynamic stresses near a mechanical imperfection such as a crack are intensified depending on the stacking sequence of the laminae. In general, this intensity tends to increase quickly for small time reaching a peak and then decreases to the static solution for sufficiently long time. When the modulus of the outer layers are smaller than that of the inner layers, the crack border stress intensity reaches a maximum quicker than the homogeneous solution but with a smaller magnitude. The opposite holds for the case when the outer layers are stiffer than the inner layers. Information of this type is useful for evaluating the resistance of laminate plates to impact loadings.

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COMPUTER PROGRAM: DYNAMIC LAMINATE PLATE THEORY WITH A CRACK

```

1      PROGRAM FLAP(INPUT,OUTPUT)
      REAL NON(4),F(4,4,2),G(4,4),D(4),PT(4)
      REAL B(4),C(4)
      REAL LP(19),DTA(19)
5      EQUIVALENCE (NON,R)
      COMMON K1,K2,K3,K4
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      LP(1)=0.0
      DTA(1)=0.0
10     READ 2,K1,K2,K3,K4
      2  FORMAT(I2)
      * K1 = ORDER OF SYSTEM OF EQUATIONS
      * K2 = NO. OF DISTINCT KERNELS
      * K3 = NO. OF DATA POINTS
15     * K4 = NO. OF DATA SETS TO BE EVALUATED
      * SET UP DATA POINTS
      AK=K3
      DO 5 N=1,K3
      AN=N
20     5  PT(N)=AN/AK
      * SET UP INTEGRATION MATRIX
      M=K3-2
      N=K3-1
      A=K3
25     A=1./(3.*A)
      DO 10 K=2,M,2
10     D(K)=2.*A
      DO 15 K=1,N,2
15     D(K)=4.*A
30     D(K3)=A
      * CALCULATE NONHOMOGENEOUS TERMS
      RHS=1.0
      DO 22 I=1,K2
      PRINT 9
35     9  FORMAT(IH1)
      DO 999 II=1,K4
      DO 35 N=1,K3
35     NON(N)=RHS*SQRT(PT(N))
      CALL CONST(I)
40     * CALCULATE KERNEL MATRICES
      DO 20 N=1,K3
      DO 20 M=1,K3
      F(M,N,I)=FU(I,PT(M),PT(N))
45     20 CONTINUE
      CALL CHANGE(F,G,D,1)
      CALL LINEQ(G,B,C,K3)
      DO 40 L=1,K3
      PRINT 6,PT(L),NON(L)
50     6  FORMAT(5X,F8.4,F15.6)
      40 CONTINUE
      LP(II+1)=NON(K3)
      DTA(II+1)=P
      999 CONTINUE
      CALL LAPINV(DTA,LP)
55     22 CONTINUE
      FND

```

```

1      FUNCTION SIMP(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      MXYZ=2**15
      DEL=0.25*(B-A)
5      IF(DEL)40,45,50
      45 SIMP=0.0
      RETURN
      50 CONTINUE
      SA=Z(I,A)+Z(I,B)
10     SB=Z(I,A+2.*DEL)
      SC=Z(I,A+DEL)+Z(I,A+3.*DEL)
      S1=(DEL/3.)*(SA+2.*SB+4.*SC)
      IF(S1.EQ.0.0) GO TO 45
      K=8
15     35 SB=SB+SC
      DEL=0.5*DEL
      SC=Z(I,A+DEL)
      J=K-1
      DO 5 N=3,J,2
20     5 SC=SC+Z(I,A+AN*DEL)
      S2=(DEL/3.)*(SA+2.*SB+4.*SC)
      DIF=ABS((S2-S1)/S1)
      ER=0.01
25     IF(DIF-ER)30,25,25
      30 SIMP=S2
      RETURN
      25 K=2*K
      S1=S2
30     IF(K-MXYZ)35,35,40
      40 PRINT 42,I,A,B
      42 FORMAT(5X,' INT. DOES NOT CONVERGE ',I3,2F9.4)
      PRINT 60,X,Y
      60 FORMAT(2F10.5)
35     DO 70 J=1,10
      DIP=J
      DIP=DIP/10.
      W=Z(I,DIP)
      PRINT 60,W
40     70 CONTINUE
      CALL EXIT
      END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS

4 SIMP

VARIABLES	SN	TYPE	RELOCATION			
0 A		REAL	F.P.	260	AN	REAL
0 B		REAL	F.P.	4	BMU	REAL
250 DEL		REAL		262	DIF	REAL
264 DIP		REAL		263	ER	REAL

```

1          SUBROUTINE CHANGE(F,G,D,I)
          COMMON K1,K2,K3,K4
          REAL F(4,4,2),G(4,4),D(4)
          DO 10 N=1,K3
5           DO 10 M=1,K3
             G(M,N) =F(M,N,I)*D(N)
          10 CONTINUE
             DO 20 N=1,K3
          20 G(N,N)=G(N,N)+1.0
          10 RETURN
          END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 CHANGE

VARIABLES	SN	TYPE	RELOCATION			
0 D		REAL	ARRAY	F.P.	0	F REAL
0 G		REAL	ARRAY	F.P.	0	I INTEGER
0 K1		INTEGER	/ /		1	K2 INTEGER
2 K3		INTEGER	/ /		3	K4 INTEGER
53 M		INTEGER			52	N INTEGER

STATEMENT LABELS
0 10

0 20

LOOPS	LABEL	INDEX	FROM-TO	LENGTH	PROPERTIES
17	10	* N	4 7	17B	NOT INNER
30	10	M	5 7	3B	INSTACK
43	20	N	8 9	4B	INSTACK

COMMON BLOCKS LENGTH
/ / 4

STATISTICS
PROGRAM LENGTH 65B 53
SCM BLANK COMMON LENGTH 4B 4
47000B SCM USED


```

1          SURROUTINE LINEQ(A,B,T,N)
          REAL A(N,N),R(N),T(N)
          DO 5 I=2,N
5          A(I,1)=A(I,1)/A(1,1)
          DO 10 K=2,N
          M=K-1
          DO 15 I=1,N
15         T(I)=A(I,K)
          DO 20 J=1,M
10         A(J,K)=T(J)
          J1=J+1
          DO 20 I=J1,N
          T(I)=T(I)-A(I,J)*A(J,K)
20        CONTINUE
          A(K,K)=T(K)
          IF(K.EQ.N) GO TO 10
          M=K+1
          DO 25 I=M,N
25         A(I,K)=T(I)/A(K,K)
20        CONTINUE
          * BACK SUBSTITUTE
          DO 31 I=1,N
          T(I)=B(I)
          M=I+1
25         IF(M.GT.N) GO TO 31
          DO 30 J=M,N
          B(J)=B(J)-A(J,I)*T(I)
30        CONTINUE
31        CONTINUE
          DO 35 I=1,N
30         K=N+1-I
          R(K)=T(K)/A(K,K)
          K1=K-1
          IF(K1.EQ.0) GO TO 35
          DO 36 J1=1,K1
          J=K-J1
          T(J)=T(J)-A(J,K)*R(K)
35         36 CONTINUE
          35 CONTINUE
          RETURN
40         END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 LINEQ

VARIABLES	SN	TYPE	RELOCATION		
0 A		REAL	ARRAY	F.P.	0 B REAL
172 I		INTEGER			175 J INTEGER
176 J1		INTEGER			173 K INTEGER
177 K1		INTEGER			174 M INTEGER
0 N		INTEGER		F.P.	0 T REAL

```

1      FUNCTION FU(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,RMU,X,Y
      X=A
      Y=B
5      IF(A*B)5,10,5
      10  FU=0.0
      RETURN
      5   SUM=SIMP(I,0.0,5.0)
      ER=0.01
10     DEL =5.0
      20  UP=DEL+5.0
      ADDL=SIMP(I,DEL,UP)
      DEL =UP
      TEST=ABS(ADDL/SUM)
15     SUM=SUM+ADDL
      IF(TEST-ER)15,20,20
      15  FU=SQRT(X*Y)*SUM
      RETURN
      END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4 FU

VARIABLES	SN	TYPE	RELOCATION			
0 A		REAL	F.P.	62	ADDL	REAL
0 B		REAL	F.P.	4	BMU	REAL
60 DEL		REAL		57	ER	REAL
55 FU		REAL		0	H	REAL
0 I		INTEGER	F.P.	1	P	REAL
2 PK1		REAL	AUX	3	PK2	REAL
56 SUM		REAL		63	TEST	REAL
61 UP		REAL		5	X	REAL
6 Y		REAL	AUX			

EXTERNALS	TYPE	ARGS		
SIMP	REAL	3		SQRT REAL

INLINE FUNCTIONS	TYPE	ARGS	
ABS	REAL	1	INTRIN

STATEMENT LABELS			
14 5		0 10	INACTIVE
22 20			

COMMON BLOCKS	LENGTH
AUX	7

STATISTICS		
PROGRAM LENGTH	648	52
SCM LABELED COMMON LENGTH	78	7
47000B SCM USED		

```

1      FUNCTION BESJO(A)
      IF (A-3.)5,5,10
5      B=A*A/9.
      W=1.-2.2499997*B
5      Z=B*B
      W=W+1.2656208*Z
      Z=Z*B
      W=W-.3163866*Z
      Z=Z*B
10     W=W+.0444479*Z
      Z=Z*B
      W=W-.0039444*Z
      Z=Z*B
      BESJO=W+.00021*Z
15     RETURN
      10  B=3./A
      W=.79788456-.00000077*B
      V=A-.78539816-.04166397*B
      Z=B*B
20     W=W-.0055274*Z
      V=V-.00003954*Z
      Z=Z*B
      W=W-.00009512*Z
      V=V+.00262573*Z
25     Z=Z*B
      W=W+.00137237*Z
      V=V-.00054125*Z
      Z=Z*B
      W=W-.00072805*Z
30     V=V-.00029333*Z
      Z=Z*B
      W=W+.00014476*Z
      V=V+.00013558*Z
      BESJO=W/SQRT(A)*COS(V)
35     RETURN
      END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4 BESJO

VARIABLES	SN	TYPE	RELOCATION			
0 A		REAL	F.P.	114	B	REAL
113 BESJO		REAL		117	V	REAL
115 W		REAL		116	Z	REAL

EXTERNALS	TYPE	ARGS			
COS	REAL	1 LIBRARY		SQRT	REAL

STATEMENT LABELS				
0 5	INACTIVE		26	10

```

1          SUBROUTINE CONST(I)
          COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
          PK1=0.3
          PK2=0.3
5          BMU=50.0
          H=1.0
          READ 2,P
2          FORMAT(F10.5)
          HH=1./H
10         PRINT 1,BMU,PK1,PK2,HH,P
1          FORMAT(/////5X,* MU2/MU1 =*F6.2,* NU1 =*F4.2,* NU2 =
          1A/H =*F4.2,* C21/PΔ =*F4.2//)
          RETURN
          END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 CONST

VARIABLES	SN	TYPE	RELOCATION			
4 BMU		REAL	AUX	0	H	REAL
55 HH		REAL		0	I	INTEGER
1 P		REAL	AUX	2	PK1	REAL
3 PK2		REAL	AUX	5	X	REAL
6 Y		REAL	AUX			

FILE NAMES	MODE				
INPUT	FMT	OUTPUT	FMT		

STATEMENT LABELS					
37 1	FMT	25	2	FMT	

COMMON BLOCKS	LENGTH
AUX	7

STATISTICS		
PROGRAM LENGTH	568	46
SCM LABELED COMMON LENGTH	78	7
47000B SCM USED		

```

1      FUNCTION Z(I,S)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      COMPLEX DA,DL1,DL2,SA,SB,SC,SD
      COMPLEX GA,GB,CA,CR,CC,F,G
5      PI=3.1415926
      PP=P*P
      PG=2./PP/(1.+BMU)
      AA=2.*(1.-PK1)/(1.-2.*PK1)
      AB=2.*(1.-PK2)/(1.-2.*PK2)
10     PA=2./PP/(AA+BMU*AB)
      PO=2.*H/H/PI/PI/(AA+BMU*AB)/PP
      RA=(1.+BMU)/(AA+BMU*AB)
      BB=1.-BA
      BC=(1.+7.*BMU)/4./(1.+BMU)
15     BD=PI*PI/2./H/H
      ALP=1./4./RA/BB
      DLP=BC/4./RB
      DD=((ALP+DLP)*PO+1.)**2-4.*ALP*DLP*PO*(PO+1.)
      G=CMPLX(DD,0.0)
20     DA=CSQRT(G)
      DL1=BD/DLP*((ALP+DLP)*PO+1.+DA)
      DL2=BD/DLP*((ALP+DLP)*PO+1.-DA)
      SC=S*S+DL1
      SD=S*S+DL2
25     GA=CSQRT(SC)
      GB=CSQRT(SD)
      GC=SQRT(S*S+PG)
      SA=(PA-DL2)/(DL1-DL2)
      SB=(PA-DL1)/(PA-DL2)
30     CA=SA/PG/BB
      CB=2.*(S*S+PG/2.)**2/GA*(1.-GA/GB*SB)
      CC=2.*S*S*GC/SA
      F=CA*(CB-CC)
      Q=REAL(F)
35     QA=AIMAG(F)
      IF(QA-0.0)5,10,5
10     Z=(Q-S)*BESJO(S*X)*BESJO(S*Y)
      RETURN
5     PRINT 9,P,S,F
40    9 FORMAT(4F10.5)
      CALL EXIT
      END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
4 Z

VARIABLES	SN	TYPE	RELOCATION		
276 AA		REAL	277	AB	REAL
306 ALP		REAL	302	BA	REAL
303 BB		REAL	304	BC	REAL
305 BD		REAL	4	BMU	REAL

```

1          SUBROUTINE LAPINV(GLAM,PHI)
C          THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
C          OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
C          INVERSION INTEGRAL
5          REAL MUL
          DIMENSION A(50),GLAM(50),PHI(50),C(4,50)
          DIMENSION BK(101),TT(101)
          COMMON/2/TT,TF,DT,MN,BK,TT
          READ 1,NN,MN,MM
10         1 FORMAT(3I2)
          READ 2,TI,TF,DT
          2 FORMAT(3F10.5)
          PRINT 99
          99 FORMAT(1H1)
15         CALL SPLICE(GLAM,PHI,MM,C)
          PRINT 101
          101 FORMAT(/////5X,*   GLAM           PHI   *)
          PRINT 102,(GLAM(I),PHI(I),I=1,MM)
          102 FORMAT(5X,F10.5,5X,F10.5)
20         M11=MM-1
          PRINT 99
          DO 10 I=1,NN
          READ 3,BET,DEL
          3 FORMAT(2F10.5)
          PRINT 98,BET,DEL
25         98 FORMAT(/////5X,*BETA =*F5.3,* DELTA =*F5.3)
          DO 11 L=1,MM
          AL=L
          S=1./(AL+BET)/DEL
          CALL SPLINE(GLAM,PHI,MM,C,S,G)
          F=G*S
          IF(AL-2.)81,82,83
          81 A(1)=(1.+BET)*DEL*F
          GO TO 11
          82 A(2)=((2.+BET)*DEL*F-A(1))*(3.+BET)
          GO TO 11
          83 CONTINUE
          TOP=1.
          L1=L-1
          AL1=L1
          DO 12 J=1,L1
          AJ=J
          TOP=AJ*TOP
          12 CONTINUE
          L2=2*L-1
          BOT=1.
          DO 13 J=L,L2
          AJ=J
          BOT=(AJ+BET)*BOT
          13 CONTINUE
          MUL=BOT/TOP
          SUM=0.0
          DO 14 N=1,L1
          AN=N
          IF(AN-2.)85,86,87
          85 TOD=1.
          GO TO 88

```

```

86 TOD=AL1
GO TO 88
60 87 CONTINUE
TOD=1.
ICH=L1-(N-2)
DO 15 J=ICH,L1
AJ=J
65 TOD=AJ*TOD
15 CONTINUE
88 CONTINUE
BOD=1.
JA=L1+N
70 DO 16 J=L,JA
AJ=J
BOD=BOD*(AJ+BET)
16 CONTINUE
CO=TOD/BOD
75 SUM=SUM+CO*A(N)
14 CONTINUE
A(L)=MUL*(DEL*F-SUM)
11 CONTINUE
CALL JACSER(DEL,A,BET)
80 10 CONTINUE
999 CONTINUE
RETURN
END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 LAPINV

VARIABLES	SN	TYPE	RELOCATION			
377 A		REAL	ARRAY		364	REAL
354 AL		REAL			362	REAL
371 AN		REAL			351	REAL
4 BK		REAL	ARRAY	2	374	REAL
366 BOT		REAL			461	REAL
376 CO		REAL			352	REAL
2 DT		REAL		2	357	REAL
356 G		REAL			0	REAL
347 I		INTEGER			373	INTEGER
363 J		INTEGER			375	INTEGER
353 L		INTEGER			361	INTEGER
365 L2		INTEGER			346	INTEGER
3 MN		INTEGER		2	344	REAL
350 M11		INTEGER			370	INTEGER
345 NN		INTEGER			0	REAL
355 S		REAL			367	REAL
1 TF		REAL		2	0	REAL
372 TOD		REAL			360	REAL
151 TT		REAL	ARRAY	2		REAL

```

1      SUBROUTINE JACSER(D,C,R)
      DIMENSION C(50),SF(50),P(50)
      DIMENSION BK(101),TT(101)
      COMMON/2/TI,TF,DT,MN,BK,TT
5      TT(1)=0.0
      BK(1)=0.0
      LM=1
      T=TI
12     T=T+DT
10     X=2.*EXP(-D*T)-1.
      CALL JACOBI(MN,X,R,P)
      SF(1)=C(1)*P(1)
      DO 10 L=2,MN
      L1=L-1
15     AL=L
      SF(L)=SF(L1)+C(L)*D(L)
10     CONTINUE
      LM=LM+1
      BK(LM)=SF(5)
      TT(LM)=T
20     IF(T.LE.TF) GO TO 12
      PRINT 97
97     FORMAT(/////5X,* T K T K
25     1 T K *)
      DO 31 MY=1,25
      MA=MY+1
      MB=MA+25
      MC=MB+25
      MD=MC+25
30     PRINT 96,TT(MA),BK(MA),TT(MB),BK(MB),TT(MC),BK(MC),T1
96     FORMAT(5X,F5.2,3X,F7.5,3X,F5.2,3X,F7.5,3X,F5.2,3X,F7.
1F7.5)
31     CONTINUE
      RETURN
35     END

```

SYMBOLIC REFERENCE MAP (P=1)

ENTPY. POINTS
3 JACSER

VARIABLES	SN	TYPE	RELOCATION				
151 AL		REAL		0	B	REAL	
4 BK		REAL	ARRAY	2	C	REAL	
0 D		REAL			DT	REAL	
147 L		INTEGER		144	LM	INTEGER	
150 L1		INTEGER		153	MA	INTEGER	
154 MB		INTEGER		155	MC	INTEGER	
156 MD		INTEGER		3	MN	INTEGER	
152 MY		INTEGER		241	P	REAL	
157 SF		REAL	ARRAY	145	T	REAL	
1 TF		REAL		0	TI	REAL	
151 TT		REAL	ARRAY	2	146	X	REAL


```

1      SUBROUTINE JACOBI(N,X,B,PB)
C      THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER
C      K-1 WITH ARG X AND PARAMETER B GT -1
      DIMENSION PB(N)
5      AN=N
      IF(AN-2.)1,2,3
1      PB(1)=1.
      RETURN
2      PB(1)=1.
10     PB(2)=X-B*(1.-X)/2.
      RETURN
3      BSO=B*B
      BONE=B+1.
      PB(1)=1.
15     PB(2)=X-B*(1.-X)/2.
      DO 4 K=3,N
      AK=K
      AK1=AK-1.
      AK2=AK-2.
20     K1=K-1
      K2=K-2
      C01=((2.*AK1)+B)*X
      C01=((2.*AK2)+B)*C01
      C01=((2.*AK2)+BONE)*(C01-BSQ)
25     C02=2.*AK2*(AK2+B)*((2.*AK1)+B)
      C0=2.*AK1*(AK1+B)*((2.*AK2)+B)
4      PB(K)=(C01*PB(K1)-C02*PB(K2))/C0
      RETURN
      END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 JACOBI

VARIABLES	SN	TYPE	RELOCATION			
106 AK		REAL		107	AK1	REAL
110 AK2		REAL		102	AN	REAL
0 B		REAL	F.P.	104	BONE	REAL
103 BSO		REAL		115	C0	REAL
113 C01		REAL		114	C02	REAL
105 K		INTEGER		111	K1	INTEGER
112 K2		INTEGER		0	N	INTEGER
0 PB		REAL	ARRAY F.P.	0	X	REAL

STATEMENT LABELS

0	1	INACTIVE	24	2
0	4			

LOOPS	LABEL	INDEX	FROM-TO	LENGTH	PROPERTIES
47	4	K	16 27	258	OPT

```

1      SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
      DIMENSION X(50),Y(50),C(4,50)
      IF(XINT-X(1))1,10,11
10     YINT=Y(1)
5      RETURN
11     CONTINUE
      IF(X(M)-XINT)1,12,13
12     YINT=Y(M)
      RETURN
10     13 CONTINUE
      K=M/2
      N=M
2      CONTINUE
      IF(X(K)-XINT)3,14,5
14     YINT=Y(K)
      RETURN
3      CONTINUE
      IF(XINT-X(K+1))4,15,7
15     YINT=Y(K+1)
20    RETURN
4      CONTINUE
      YINT=(X(K+1)-XINT)*(C(1,K)*(X(K+1)-XINT)**2+C(3,K))
      YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
      RETURN
25    5 CONTINUE
      IF(X(K-1)-XINT)6,16,17
6      K=K-1
      GO TO 4
16     YINT=Y(K-1)
30    RETURN
17     N=K
      K=K/2
      GO TO 2
7      LL=K
35    K=(N+K)/2
8      CONTINUE
      IF(X(K)-XINT)3,14,18
18     CONTINUE
      IF(X(K-1)-XINT)6,16,19
40    19 N=K
      K=(LL+K)/2
      GO TO 8
1      PRINT 101
45    101 FORMAT(* OUT OF RANGE FOR INTERPOLATION *)
      STOP
      END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 SPLINE

```

1      SUBROUTINE SPLICE(X,Y,M,C)
      DIMENSION X(50),Y(50),D(50),P(50),E(50),C(4,50)
      DIMENSION A(50,3),R(50),Z(50)
      MM=M-1
5      DO 2 K=1,MM
      D(K)=X(K+1)-X(K)
      P(K)=D(K)/6.
      2 E(K)=(Y(K+1)-Y(K))/D(K)
      DO 3 K=2,MM
10     3 B(K)=E(K)-E(K-1)
      A(1,2)=-1.-D(1)/D(2)
      A(1,3)=D(1)/D(2)
      A(2,3)=P(2)-P(1)*A(1,3)
      A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
15     A(2,3)=A(2,3)/A(2,2)
      B(2)=B(2)/A(2,2)
      DO 4 K=3,MM
      A(K,2)=2.*(P(K-1)+D(K))-P(K-1)*A(K-1,3)
      B(K)=B(K)-P(K-1)*B(K-1)
20     A(K,3)=P(K)/A(K,2)
      4 B(K)=B(K)/A(K,2)
      Q=D(M-2)/D(M-1)
      A(M,1)=1.+Q*A(M-2,3)
      A(M,2)=-Q-A(M,1)*A(M-1,3)
25     R(M)=B(M-2)-A(M,1)*B(M-1)
      Z(M)=B(M)/A(M,2)
      MN=M-2
      DO 6 I=1,MN
      K=M-I
30     6 Z(K)=B(K)-A(K,3)*Z(K+1)
      Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
      DO 7 K=1,MM
      Q=1./(6.*D(K))
      C(1,K)=Z(K)*Q
35     C(2,K)=Z(K+1)*Q
      C(3,K)=Y(K)/D(K)-Z(K)*P(K)
      7 C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
      RETURN
      END

```

SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
3 SPLICE

VARIABLES	SN	TYPE	RELOCATION			
373 A		REAL	ARRAY		621 R	REAL
0 C		REAL	ARRAY	F.P.	145 D	REAL
311 E		REAL	ARRAY		144 I	INTEGER
141 K		INTEGER			0 M	INTEGER
140 MM		INTEGER			143 MN	INTEGER
227 P		REAL	ARRAY		142 Q	REAL
0 X		REAL	ARRAY	F.P.	0 Y	REAL