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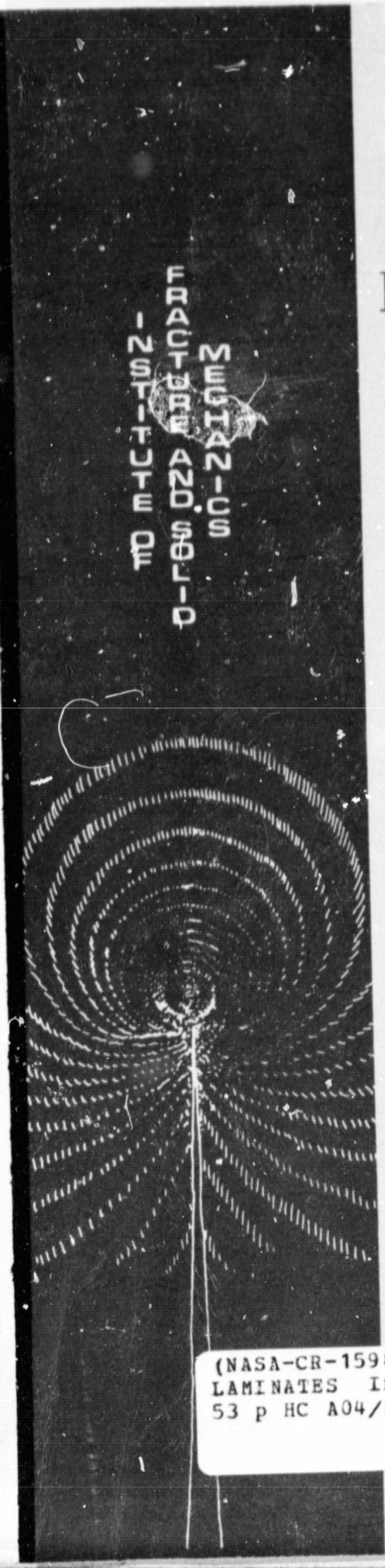

LEHIGH UNIVERSITY

SUDDEN BENDING OF CRACKED LAMINATES

BY

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16. Abstract

A dynamic approximate laminated plate theory is developed with emphases placed on obtaining effective solution for the crack configuration where the $1/\sqrt{r}$ stress singularity and the condition of plane strain are preserved. The radial distance r is measured from the crack edge. The results obtained show that the crack moment intensity tends to decrease as the crack length to laminate plate thickness is increased. Hense, a laminated plate has the desirable feature of stabilizing a through crack as it increases its length at constant load. Also, the level of the average load intensity transmitted to a through crack can be reduced by making the inner layers to be stiffer than the outer layers. The present theory, although approximate, is useful for anlayzing laminate failure to crack propagation under dynamic load conditions.

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FOREWORD

The research results in this report on the sudden bending of a laminated plate containing a through crack represent a portion of the work performed for the NASA - Lewis Research Center in Cleveland, Ohio for the period February 13, 1979 through February 12, 1980 under Grant NSG 3179 with the Institute of Fracture and Solid Mechanics at Lehigh University. The Principal Investigator of the project is Professor George C. Sih. The co-author, Dr. E. P. Chen, was a faculty member at Lehigh University and is now employed by the Sandia Laboratory in New Mexico. The encouragement and helpful comments made by Dr. Christos C. Chamis, the NASA Project Manager, are gratefully acknowledged.

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a	- half of the crack length
$B^{(1)}, B^{(2)}, B^{(3)}$	- unknowns in integrals, function of (s,p)
Br	- Bromwich contour in the complex p-plane
C	- unknown in dual integral equation, function of (s,p)
c_{21}	- shear wave speed for medium 1
D_0, D_1, D_2	- flexural rigidities of the layers in the laminate
$f^*(p)$	- Laplace transform of f(t)
G	- known function of (s,p) in the dual integral equation
h	- laminate thickness
H^*	- function related to displacement
$H(t)$	- Heaviside unit step function
J_0	- Bessel function of order zero
$\overline{K}$	- kinetic energy
$K_1(t)$	- moment intensity factor
$L(\xi,\eta,p)$	- kernel in Fredholm integral equation
M_0	- magnitude of applied moment
M_x, M_y, H_{xy}	- moments per unit length defined in the xy coordinate system
M_n, H_{ns}, Q_n	- moments and shear per unit length defined in the normal and tangential directions
p	- Laplace transform variable
q_j	- lateral loadings on laminate with j = 1,2
q	- defined as $q_1 - q_2$
Q_x, Q_y	- shear force per unit length
r, θ	- crack edge polar coordinates
R,S	- parameters in the laminate plate theory

t	-	time
$\overline{T},\overline{V}$	-	energy quantities at time t
$\overline{T}_0,\overline{V}_0$	-	energy quantities at time t=0
u_x,v_y,w_z	-	displacement components in the (x,y,z) coordinate system
w,w_j	-	displacement functions with j = 1,2
x,y,z	-	rectangular coordinates
α_j,β_j	-	parameters related to α and β with j = 1,2
γ_j	-	exponents for transform of solution with j = 1,2,3
$\Gamma_x,\Gamma_y,---,\Gamma_{yz}$	-	equivalent strains
$\delta_0,\overline{\delta}_0$	-	parameters in plate theory, function of p
$\epsilon_x,\epsilon_y,---,\gamma_{yz}$	-	strain components
κ	-	shear correction parameter in plate theory
μ_0	-	equivalent shear modulus
μ_j	-	shear modulus with j = 1,2
ν_0	-	equivalent Poisson's ratio
ν_j	-	Poisson's ratio with j = 1,2
ξ,η	-	variables of integration
$\rho_0,\overline{\rho}$	-	function of ρ_1 and ρ_2
ρ_j	-	mass density for medium j
$\rho(z)$	-	mass density as a function of z
$\sigma_x,\sigma_y,---,\tau_{yz}$	-	stress components
ϕ^*	-	displacement function in Laplace transform plane
ψ_x,ψ_y	-	displacement functions in the xy coordinate system
ψ_n,ψ_s	-	displacement functions in the ns coordinate system
$\Psi^*(\xi,p)$	-	unknown in Fredholm integral equation

$\Psi^*(1,p)$ - value of $\Psi^*(\xi,p)$ evaluated at $\xi=1$

ω - frequency parameter

∇^2 - Laplacian operator

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ABSTRACT

A number of laminated plate theories have been developed in recent times to analyze the static and dynamic response of composite laminates with or without the presence of stress concentrators such as holes, cracks, etc. Many of the theories tend to quickly become intractable when considering the determination of the state of affairs near the singular crack edges that are present in the laminate, particularly if the loading is time dependent. Additional uncertainties arise due to the lack of information on the mechanical properties of the interface through which load transfer takes place between the adjacent layers. This paper focuses attention on the intensification of stresses near a through crack in the laminate that suddenly undergoes bending. A dynamic plate theory is developed to include many of the essential features of the problem such as material nonhomogeneity in the thickness direction, realistic crack edge stress singularity and distribution while the parameter dependence of various significant quantities is also assessed. Of particular interest is the variation of the dynamic stress intensity factor with time. Numerical results for different

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geometric and material constants are displayed graphically to show how they can affect the transfer of load to the vicinity of a through crack in the laminate that undergoes sudden bending.

INTRODUCTION

The damage of laminated composite materials is, to say the least, very complex since it involves various modes of failure such as fiber breaking, matrix cracking, interface delamination, etc. Analytical modeling would be beyond approach if all these failure modes were to be accounted for. The spirit of fracture mechanics is to assume that a critical single flaw or damage zone exists and can lead to instability in terms of load applied to the laminate. Damage accumulated in the composite other than the dominant flaw may often be simulated by changing some of the mechanical properties of the composite which are usually the stiffness of the constituents. Although not all laminates can be identified with a single characteristic damage state, the single-flaw fracture mechanics approach will be taken in this analysis in order that a sensitivity study on the physical parameters affecting laminate fracture can be made possible. One of the main objectives of this investigation is to come forth with a feasible dynamic theory of the laminate plate for analyzing composite failure due to crack propagation.

As a consequence of increased use of laminate composites in aircraft and other high speed vehicles, the analysis of the fracture behavior of layered composites has attracted the attention of a considerable number of investigators [1,2]. A variety of diverse approaches has been proposed to analyze laminate failure and a collection of papers on this subject can be found in [3]. The role with which the interfaces play in transferring the load from one layer to the next in the laminate was emphasized. Because of the difference in the material properties of the adjacent layers, the stresses across the interface experience steep gradients. Only recently, a comprehensive study was made on how the conditions in the interface can influence composite failure [4]. Even though the interface may be relatively thin when compared with other dimensions of the composite, the resulting stresses can be sensitive to the material properties of the interface depending on the loading conditions. There exists no theory at the present which can relate the strength of a composite structure to the conditions in the interface. This aspect of the problem is emphasized in this report.

The aforementioned difficulties become even more overwhelming when the loading is time dependent. There is no need to emphasize the virtue for constructing approximate dynamic theories for laminate composites, particularly for handling crack problems. In the case of bending loads, it is essential that the three physical boundary conditions of bending moment, twisting moment and transverse shear stress be satisfied individually on the crack edge. Such a theory has been developed by Mindlin [5] for a single layered plate made of isotropic and homogeneous material and applied to solve a number of crack problems [6]. An equally effective theory is described herein for the dynamic bending of laminate plates. Each layer of the laminate assumes different elastic properties and is attached to the next layer with continuous strains across the interface. The problem of a through crack in a balanced symmetric laminate is solved for a moment applied suddenly on the crack surface. Not only are the qualitative features of the three-dimensional stress distribution preserved in the vicinity of the crack front, but, perhaps more significantly, the dynamic stress intensity factor, which is a quantitative measure of the load transmitted to the crack, is determined in terms of the significant material and geometric parameters such that an effective study on laminate fracture can be made.

DYNAMIC THEORY OF LAMINATED PLATE

Without loss in generality, a four layered composite plate will be considered as shown in Figure 1. The two middle layers are made of a material with shear modulus μ_1, Poisson's ratio ν_1 and mass density ρ_1 while the two outer layers have the properties μ_2, ν_2 and ρ_2. A set of rectangular Cartesian coordinates x, y and z are attached to the mid-plane of the laminate such that the layer properties are symmetric with respect to the xy-plane with z being the thickness coordinate. The total height of the laminate is h with each layer having the same thickness h/4. The outer edges of the laminate are sufficiently far away from the crack so that their influences can be neglected.

Basic assumptions and relations. The layers of the laminate in the thickness possess different material properties μ_j, ν_j and ρ_j (j = 1,2) such that (μ_1, ν_1,ρ_1) prevails in the range $0<|z|<h/4$ and (μ_2,ν_2,ρ_2) applies to $h/4<|z|<h/2$. The surfaces of the laminate are free from tangential tractions

$$\tau_{xz} = \tau_{yz} = 0 \text{ for } z = \pm h/2 \qquad (1)$$

but may be subjected to normal pressures q_1 and q_2 as follows:

$$\sigma_z = \begin{cases} -q_1(x,y,t) \text{ for } z = h/2 \\ -q_2(x,y,t) \text{ for } z = -h/2 \end{cases} \qquad (2)$$

In the sequel, the notation

$$q(x,y,t) = q_2(x,y,t) - q_1(x,y,t) \qquad (3)$$

will be used. In plate theory, it is more convenient to work with the moments M_x, M_y, H_{xy} and shearing forces Q_x, Q_y defined in the usual manner as

$$(M_x,M_y,H_{xy}) = \int_{-h/2}^{h/2} (\sigma_x,\sigma_y,\tau_{xy})zdz$$

$$(Q_x,Q_y) = \int_{-h/2}^{h/2} (\tau_{xz},\tau_{yz})dz \qquad (4)$$

From the stress and strain relations and equations (4), the expressions

$$M_x = D_1[(\Gamma_x)_1 + \nu_1(\Gamma_y)_1] + D_2[(\Gamma_x)_2 + \nu_2(\Gamma_y)_2]$$

$$M_y = D_1[(\Gamma_y)_1 + \nu_1(\Gamma_x)_1] + D_2[(\Gamma_y)_2 + \nu_2(\Gamma_x)_2] \qquad (5)$$

$$H_{xy} = \left(\frac{1-\nu_1}{2}\right) D_1(\Gamma_{xy})_1 + \left(\frac{1-\nu_2}{2}\right) D_2(\Gamma_{xy})_2$$

and

$$Q_x = \frac{\kappa^2}{2} h[\mu_1(\Gamma_{xz})_1 + \mu_2(\Gamma_{xz})_2]$$

$$Q_y = \frac{\kappa^2}{2} h[\mu_1(\Gamma_{yz})_1 + \mu_2(\Gamma_{yz})_2] \qquad (6)$$

are developed provided that the quantities $(\Gamma_x)_j$, $(\Gamma_y)_j$,---, $(\Gamma_{yz})_j$ $(j = 1,2)$ stand for

$$[(\Gamma_x)_1, (\Gamma_y)_1, (\Gamma_{xy})_1] = \frac{96}{h^3} \int_{-h/4}^{h/4} (\varepsilon_x, \varepsilon_y, \gamma_{xy}) z dz$$

$$[(\Gamma_x)_2, (\Gamma_y)_2, (\Gamma_{xy})_2] = \frac{96}{7h^3} [\int_{-h/2}^{-h/4} (\varepsilon_x, \varepsilon_y, \gamma_{xy}) z dz$$

$$+ \int_{h/4}^{h/2} (\varepsilon_x, \varepsilon_y, \gamma_{xy}) z dz] \tag{7}$$

$$[(\Gamma_{xz})_1, (\Gamma_{yz})_1] = \frac{2}{h} \int_{-h/4}^{h/4} (\gamma_{xz}, \gamma_{yz}) dz$$

$$[(\Gamma_{xz})_2, (\Gamma_{yz})_2] = \frac{2}{h} [\int_{-h/2}^{-h/4} (\gamma_{xz}, \gamma_{yz}) dz + \int_{h/4}^{h/2} (\gamma_{xz}, \gamma_{yz}) dz]$$

In equations (5), D_1 and D_2 are the flexural rigidities of the layers given by

$$D_1 = \frac{\mu_1 h^3}{48(1-\nu_1)}, \quad D_2 = \frac{7\mu_2 h^3}{48(1-\nu_2)} \tag{8}$$

The constant κ in equation (6) accounts for the thickness-shear motion of the plate and takes the value of $\pi/\sqrt{12}$ as given in [5].

Now, let the displacements be continuous through the interfaces by letting

$$u_x = z\psi_x(x,y,t), \quad v_y = z\psi_y(x,y,t), \quad w_z = w(x,y,t) \tag{9}$$

Making use of the strain-displacement relations together with equations (7) and (9), it is found that

$$(\Gamma_x)_1 = (\Gamma_x)_2 = \frac{\partial \psi_x}{\partial x}, \quad (\Gamma_y)_1 = (\Gamma_y)_2 = \frac{\partial \psi_y}{\partial y}$$

$$(\Gamma_{xy})_1 = (\Gamma_{xy})_2 = \frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y} \tag{10}$$

$$(\Gamma_{xz})_1 = (\Gamma_{xz})_2 = \psi_x + \frac{\partial w}{\partial x}, \quad (\Gamma_{yz})_1 = (\Gamma_{yz})_2 = \psi_y + \frac{\partial w}{\partial y}$$

Hence, the moments M_x, M_y and H_{xy} can be expressed in terms of the displacement functions ψ_x, ψ_y and w:

$$M_x = D_0\left[\frac{\partial \psi_x}{\partial x} + \nu_0 \frac{\partial \psi_y}{\partial y}\right]$$

$$M_y = D_0\left[\frac{\partial \psi_y}{\partial y} + \nu_0 \frac{\partial \psi_x}{\partial x}\right] \tag{11}$$

$$H_{xy} = \frac{D_0}{2} (1-\nu_0)\left(\frac{\partial \psi_y}{\partial x} + \frac{\partial \psi_x}{\partial y}\right)$$

The same applies to Q_x and Q_y which become

$$Q_x = \frac{\pi^2}{12} h\mu_0\left(\psi_x + \frac{\partial w}{\partial x}\right)$$

$$Q_y = \frac{\pi^2}{12} h\mu_0\left(\psi_y + \frac{\partial w}{\partial y}\right) \tag{12}$$

Note that D_0, ν_0 and μ_0 are defined as

$$D_0 = D_1 + D_2, \quad \nu_0 = \frac{D_1\nu_1 + D_2\nu_2}{D_0}, \quad \mu_0 = \frac{\mu_1 + \mu_2}{2} \tag{13}$$

Equations (11) and (12) are, in fact, similar to those derived in [5] for the case of a single layer homogeneous plate except that the constants D, ν and μ are now replaced by D_0, ν_0 and μ_0.

Governing differential equations. Consider the elastodynamic equations of motion given by

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho(z) \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho(z) \frac{\partial^2 v_y}{\partial t^2} \tag{14}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho(z) \frac{\partial^2 w_z}{\partial t^2}$$

in which the mass density may vary in the thickness direction of the laminate. Multiplying the first two equations by z, expressing the stresses in terms of moments and integrating the results with respect to z from -h/2 to h/2 lead to

$$\frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - Q_x = \frac{\rho_0}{12} h^3 \frac{\partial^2 \psi_x}{\partial t^2}$$

$$\frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = \frac{\rho_0}{12} h^3 \frac{\partial^2 \psi_y}{\partial t^2} \tag{15}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = \bar{\rho} h \frac{\partial^2 w}{\partial t^2}$$

in which

$$\rho_0 = \frac{1}{8}(\rho_1 + 7\rho_2), \quad \bar{\rho} = \frac{1}{2}(\rho_1 + \rho_2) \tag{16}$$

The result of inserting equations (11) and (12) into equations (15) is a system of second order partial differential equations

$$\frac{(1-\nu_0)}{2} D_0 \nabla^2 \psi_x + \frac{(1+\nu_0)}{2} D_0 \frac{\partial}{\partial x} \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right) - \frac{\pi^2 h}{12} \mu_0 \left(\psi_x + \frac{\partial w}{\partial x}\right) = \frac{\rho_0}{12} h^3 \frac{\partial^2 \psi_x}{\partial t^2}$$

$$\frac{(1-\nu_0)}{2} D_0 \nabla^2 \psi_y + \frac{(1+\nu_0)}{2} D_0 \frac{\partial}{\partial y} \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right) - \frac{\pi^2 h}{12} \mu_0 \left(\psi_y + \frac{\partial w}{\partial y}\right) = \frac{\rho_0}{12} h^3 \frac{\partial^2 \psi_y}{\partial t^2} \qquad (17)$$

$$\frac{\pi^2 h}{12} \mu_0 \left(\nabla^2 w + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y}\right) + q = \bar{\rho} h \frac{\partial^2 w}{\partial t^2}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator in two dimensions. Equations (17) may be combined to give a single equation

$$\left(D_0 \nabla^2 - \frac{\rho_0 h^3}{12} \frac{\partial^2}{\partial t^2}\right)\left(\nabla^2 - \frac{12\bar{\rho}}{\pi^2 \mu_0} \frac{\partial^2}{\partial t^2}\right) w + \bar{\rho} h \frac{\partial^2 w}{\partial t^2}$$

$$= \left[1 - \frac{12 D_0}{\pi^2 h \mu_0} \nabla^2 + \frac{h^2 \rho_0}{\pi^2 \mu_0} \frac{\partial^2}{\partial t^2}\right] q \qquad (18)$$

solving for the transverse displacement $w(x,y)$ of the laminated plate.

Boundary conditions. In order to derive the boundary conditions that must be specified on the crack, consider the energy stored in the laminate

$$\bar{W} = \int_{-h/2}^{h/2} W dz = \frac{1}{2} \left(M_x \Gamma_x + M_y \Gamma_y + H_{xy} \Gamma_{xy} + Q_x \Gamma_{xz} + Q_y \Gamma_{yz}\right) \qquad (19)$$

in which Γ_x, Γ_y,---, Γ_{yz} are related to ψ_x, ψ_y and w as indicated in equations (10). Equations (5) and (6) may thus be applied to render

$$4\overline{W} = D_0(1+\nu_0)(\Gamma_x+\Gamma_y)^2 + \frac{\pi^2}{6}\mu_0 h(\Gamma_{yz}^2+\Gamma_{xz}^2) + D_0(1-\nu_0)[(\Gamma_x-\Gamma_y)^2 + \Gamma_{xy}^2] \qquad (20)$$

Since the physical constants $D_0(1+\nu_0)$ and $D_0(1-\nu_0)$ are positive, $\overline{W}$ is a positive definite quantity. Hence, $\overline{W}$ vanishes if and only if the equivalent strains Γ_x, Γ_y, etc., vanish individually. Equation (20) also implies that

$$M_x = \frac{\partial\overline{W}}{\partial\Gamma_x},\ M_y = \frac{\partial\overline{W}}{\partial\Gamma_y},\ H_{xy} = \frac{\partial\overline{W}}{\partial\Gamma_{xy}} \qquad (21)$$

and

$$Q_x = \frac{\partial\overline{W}}{\partial\Gamma_{xz}},\ Q_y = \frac{\partial\overline{W}}{\partial\Gamma_{yz}} \qquad (22)$$

The kinetic energy in the laminate is

$$\overline{T} = \int_{-h/2}^{h/2} T\,dz = \frac{1}{2}\int_{-h/2}^{h/2} \rho(z)[(\frac{\partial u_x}{\partial t})^2 + (\frac{\partial v_y}{\partial t})^2 + (\frac{\partial w_z}{\partial t})^2]dz$$

which, when expressed in terms of ψ_x, ψ_y and w, takes the form

$$\overline{T} = \frac{\rho_0 h^3}{24}[(\frac{\partial\psi_x}{\partial t})^2 + (\frac{\partial\psi_y}{\partial t})^2] + \frac{\overline{\rho}h}{2}(\frac{\partial w}{\partial t})^2 \qquad (23)$$

It is now possible to write down the expression for the total energy of the laminate at time t:

$$\overline{T} + \overline{V} = \int_{t_0}^{t} dt \iint \{\frac{\rho_0 h^3}{24}[(\frac{\partial\psi_x}{\partial t})^2 + (\frac{\partial\psi_y}{\partial t})^2] + \frac{\overline{\rho}h}{2}(\frac{\partial w}{\partial t})^2\}dxdy$$

$$+ \int_{t_0}^{t} dt \iint \frac{\partial\overline{W}}{\partial t}dxdy + \overline{T}_0 + \overline{V}_0 \qquad (24)$$

where $\overline{V}$ is the total potential energy. Note that $\overline{T}_0$ and $\overline{V}_0$ are the values of $\overline{T}$ and $\overline{V}$ corresponding to time t_0. Equation (24) may be integrated by parts and the results may be arranged to read as*

$$\overline{T} + \overline{V} = \int_{t_o}^{t} dt \int_{s} \left(\frac{\partial\psi_n}{\partial t} M_n + \frac{\partial\psi_s}{\partial t} H_{ns} + \frac{\partial w}{\partial t} Q_n\right)ds$$

$$+ \int_{t_o}^{t} dt \iint q \frac{\partial w}{\partial t} dxdy + \overline{T}_0 + \overline{V}_0 \tag{25}$$

The above result may be interpreted as the total energy in the laminate at time t and consists of the initial energy at t_0 plus the work done by the external forces along the edges and over the surfaces of the laminate during the time interval $t-t_0$. The initial and boundary conditions for the laminate can now be easily extracted from equation (25). They can be summarized as follows:

(1) <u>On the laminate or crack edges</u>: Any combination containing one member of each of the three pairs $(\frac{\partial\psi_n}{\partial t}, M_n)$, $(\frac{\partial\psi_s}{\partial t}, H_{ns})$ and $(\frac{\partial w}{\partial t}, Q_n)$ may be specified on the crack or laminate edge.

(2) <u>Throughout the laminate</u>: The initial values of ψ_x, ψ_y and w and their time derivatives need be known.

(3) <u>Tractions and Displacements</u>: The external load q or the displacement w on the laminate may be specified.

This completes the development of the dynamic laminate plate theory which will be used to solve a crack problem.

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*Refer to page 45 for the derivation of equation (25).

A CRACKED LAMINATE PLATE

As an example, consider the laminate in Figure 1 to be initially at rest and bent suddenly by a moment with a constant magnitude of M_0 maintained on the crack surfaces. The conditions can be stated as

$$Q_y(x,o,t) = H_{xy}(x,o,t) = 0 \text{ for } 0 \leq |x| < \infty \tag{26}$$

and

$$M_y(x,o,t) = -M_0 H(t) \text{ for } |x| < a \text{ and } \psi_y(x,o,t) = 0 \text{ for } |x| > a \tag{27}$$

which is of the mixed type. The displacement functions are subjected to the conditions that

$$\lim_{x^2+y^2 \to \infty} [\psi_x(x,y,t),\ \psi_y(x,y,t),\ w(x,y,t)] = 0$$

No other external forces or constraints are present.

Laplace transform. The governing equations (17) will be solved by introducing the Laplace transform pair

$$\begin{aligned} f^*(p) &= \int_0^\infty f(t)e^{-pt}dt \\ f(t) &= \frac{1}{2\pi i}\int_{Br} f^*(p)e^{pt}dp \end{aligned} \tag{28}$$

where the second integral is over the Bromwich path. Applying the first of equations (28) to (17) yields

$$\frac{(1-\nu_0)}{2} D_0\nabla^2\psi_x^* + \frac{(1+\nu_0)}{2} D_0 \frac{\partial}{\partial x} (\frac{\partial\psi_x^*}{\partial x} + \frac{\partial\psi_y^*}{\partial y}) - \frac{\pi^2 h}{12} \mu_0(\psi_x^* + \frac{\partial w^*}{\partial x}) = \frac{\rho_0 h^3}{12} p^2\psi_x^*$$

$$\frac{(1-\nu_0)}{2} D_0\nabla^2\psi_y^* + \frac{(1+\nu_0)}{2} D_0 \frac{\partial}{\partial y} (\frac{\partial\psi_x^*}{\partial x} + \frac{\partial\psi_y^*}{\partial y}) - \frac{\pi^2 h}{12} \mu_0(\psi_y^* + \frac{\partial w^*}{\partial y}) = \frac{\rho_0 h^3}{12} p^2\psi_y^* \quad (29)$$

$$\frac{\pi^2 h}{12} \mu_0(\nabla^2 w^* + \frac{\partial\psi_x^*}{\partial x} + \frac{\partial\psi_y^*}{\partial y}) = \bar{\rho} h p^2 w^*$$

The analysis may be simplified by letting

$$\psi_x^* = \frac{\partial\phi^*}{\partial x} + \frac{\partial H^*}{\partial y}, \quad \psi_y^* = \frac{\partial\phi^*}{\partial y} - \frac{\partial H^*}{\partial x} \quad (30)$$

such that equations (29) simplify to

$$\frac{\partial}{\partial x} \{\nabla^2\phi^* - (R\delta_0^4 + S^{-1})\phi^* - S^{-1}w^*\} + \frac{1-\nu_0}{2} \frac{\partial}{\partial y} (\nabla^2-\omega^2)H^* = 0$$

$$\frac{\partial}{\partial y} \{\nabla^2\phi^* - (R\delta_0^4 + S^{-1})\phi^* - S^{-1}w^*\} - \frac{1-\nu_0}{2} \frac{\partial}{\partial x} (\nabla^2-\omega^2)H^* = 0 \quad (31)$$

$$\nabla^2(\phi^*+w^*) - S\bar{\delta}_0^4 w^* = 0$$

The new quantities introduced in equations (31) are defined as

$$R = \frac{h^2}{12}, \quad S = \frac{12D_0}{\pi^2 h\mu_0}, \quad \delta_0^4 = \frac{\rho_0 h p^2}{D_0}, \quad \bar{\delta}_0^4 = \frac{\bar{\rho} h p^2}{D_0} \quad (32)$$

and

$$\omega^2 = \frac{2(R\delta_0^4+S^{-1})(D_1+D_2)}{(1-\nu_1)D_1+(1-\nu_2)D_2} \quad (33)$$

Furthermore, if

$$\phi^* = (\beta-1)w^* \tag{34}$$

is introduced into equations (31), it can be shown that

$$\nabla^2 w^* - \alpha^2 w^* = 0 \tag{35}$$

while α and β are given by

$$\alpha^2 = R\delta_0^4 + S^{-1} + \frac{S^{-1}}{\beta-1}, \quad \beta = \frac{S\bar{\delta}_0^4}{\alpha^2} \tag{36}$$

Consequently, the functions ψ_x^* and ψ_y^* in equations (30) become

$$\begin{aligned} \psi_x^* &= (\beta_1-1)\frac{\partial w_1^*}{\partial x} + (\beta_2-1)\frac{\partial w_2^*}{\partial x} + \frac{\partial H^*}{\partial y} \\ \psi_y^* &= (\beta_1-1)\frac{\partial w_1^*}{\partial y} + (\beta_2-1)\frac{\partial w_2^*}{\partial y} - \frac{\partial H^*}{\partial x} \end{aligned} \tag{37}$$

and w^* may be written as

$$w^* = w_1^* + w_2^* \tag{38}$$

In equations (37), β_1 and β_2 are given as

$$\beta_{1,2} = (R\delta_0^4 + S^{-1})^{-1} \alpha_{2,1}^2 \tag{39}$$

in which

$$\alpha_{1,2}^2 = \frac{1}{2}\{(R\delta_0^4 + S\bar{\delta}_0^4) \pm [(R\delta_0^4 - S\bar{\delta}_0^4)^2 - 4\bar{\delta}_0^4]^{1/2}\} \tag{40}$$

It is now apparent that once H^*, w_1^* and w_2^* are found from

$$(\nabla^2-\omega^2)H^* = 0, \; (\nabla^2-\alpha_1^2)w_1^* = 0, \; (\nabla^2-\alpha_2^2)w_2^* = 0 \tag{41}$$

the problem is basically solved in the Laplace transform plane.

Integral equation. Taking advantage of the symmetry condition with respect to the y-axis, it is not difficult to show that the following integrals

$$w_1^*(x,y,p) = \frac{2}{\pi}\int_0^\infty B^{(1)}(s,p)\cos(sx)e^{-\gamma_1 y}ds$$

$$w_2^*(x,y,p) = \frac{2}{\pi}\int_0^\infty B^{(2)}(s,p)\cos(sx)e^{-\gamma_2 y}ds \tag{42}$$

$$H^*(x,y,p) = \frac{2}{\pi}\int_0^\infty B^{(3)}(s,p)\sin(sx)e^{-\gamma_3 y}ds$$

satisfy equations (41) provided that

$$\gamma_{1,2} = (s^2+\alpha_{1,2}^2)^{1/2}, \; \gamma_3 = (s^2+\omega^2)^{1/2} \tag{43}$$

The unknowns $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ must be determined from the boundary conditions in equations (26) and (27) whose Laplace transform are

$$Q_y^*(x,o,p) = H_{xy}^*(x,o,p) = 0 \text{ for } 0<x<\infty \tag{44}$$

and

$$M_y^*(x,o,p) = -\frac{M_0}{p} \text{ for } 0<x<a \text{ and } \psi_y^*(x,o,p) = 0 \text{ for } x>a \tag{45}$$

The appropriate quantities in equations (44) and (45) may be obtained by first putting equations (42) into (37) and (38). This gives

$$\psi_x^* = -\frac{2}{\pi}\int_0^\infty \{s[(\beta_1-1)B^{(1)}(s,p)e^{-\gamma_1 y} + (\beta_2-1)B^{(2)}(s,p)e^{-\gamma_2 y}]$$

$$+ \gamma_3 B^{(3)}(s,p)e^{-\gamma_3 y}\} \sin(sx)ds$$

$$\psi_y^* = -\frac{2}{\pi}\int_0^\infty \{(\beta_1-1)\gamma_1 B^{(1)}(s,p)e^{-\gamma_1 y} + (\beta_2-1)\gamma_2 B^{(2)}(s,p)e^{-\gamma_2 y}$$

$$+ sB^{(3)}(s,p)e^{-\gamma_3 y}\} \cos(sx)ds \qquad (46)$$

and

$$w^* = \frac{2}{\pi}\int_0^\infty \{B^{(1)}(s,p)e^{-\gamma_1 y} + B^{(2)}(s,p)e^{-\gamma_2 y}\} \cos(sx)ds \qquad (47)$$

The Laplace transform of equations (11) and (12) will clearly involve ψ_x^*, ψ_y^* and w^*. Equations (46) and (47) and equations (45) can be satisfied if the function $C(s,p)$ obeys the dual integral equations

$$\int_0^\infty C(s,p)\cos(sx)ds = 0 \qquad x \geq a$$

$$\int_0^\infty sG(s,p)C(s,p)\cos(sx)ds = \frac{\pi M_0}{D_0(1-\nu_0^2)p} \qquad x<a \qquad (48)$$

with $G(s,p)$ being a known function

$$\frac{(1-\nu_0^2)}{2}G(s,p) = \{(1-\beta_1)(\gamma_1^2-\nu_0 s^2)^2/(s\gamma_1) - (1-\beta_2)(\gamma_2^2-\nu_0 s^2)^2/(s\gamma_2)$$

$$- 2s\gamma_3(1-\nu_0)(\alpha_1^2-\alpha_2^2)/\omega^2\}/(\alpha_1^2-\alpha_2^2) \qquad (49)$$

The conditions in equations (44) may be used to relate the functions $B^{(1)}$, $B^{(2)}$ and $B^{(3)}$ to $C(s,p)$:

$$B^{(1)}(s,p) = \frac{(1-\nu_0)s^2+\alpha_1^2}{\gamma_1(\alpha_1^2-\alpha_2^2)} C(s,p)$$

$$B^{(2)}(s,p) = -\frac{(1-\nu_0)s^2+\alpha_2^2}{\gamma_2(\alpha_1^2-\alpha_2^2)} C(s,p) \tag{50}$$

$$B^{(3)}(s,p) = \frac{s(1-\nu_0)(\beta_2-\beta_1)}{\alpha_1^2-\alpha_2^2} C(s,p)$$

Without going into details, the solution for equations (48) is of the form [6]

$$C(s,p) = \frac{\pi M_0 a^2}{D_0(1-\nu_0^2)p} \int_0^1 \sqrt{\xi}\, \Psi(\xi,p) J_0(sa\xi)d\xi \tag{51}$$

where J_0 is zero order Bessel function of the first kind and the function $\Psi^*(\xi,p)$ can be found from a Fredholm integral equation of the second kind:

$$\Psi^*(\xi,p) + \int_0^1 L(\xi,\eta,p)\Psi^*(\eta p)d\eta = \sqrt{\xi} \tag{52}$$

The kernel $L(\xi,\eta,p)$ is symmetric in ξ and η and takes the form

$$L(\xi,\eta,p) = \sqrt{\xi\eta} \int_0^\infty s[G(\tfrac{s}{a}, p) - 1]J_0(s\xi)J_0(s\eta)ds \tag{53}$$

Equation (52) can be evaluated numerically for $\Psi^*(\xi,p)$ in the Laplace transform domain and then inverted into the time domain by using the second of equations (28).

Dynamic moment intensity factor. The time dependence of the solution may be recovered by two different procedures. The first is to apply the Laplace inversion formula to the quantities of interest and obtain the complete solution as a function of time. Such an approach is not only cumbersome and can often result in a considerable amount of difficulties in numerical calculations. In fracture mechanics, since it is only necessary to focus attention on the state of affairs near the crack front, Sih et al [7] have suggested to obtain the asymptotic stress solution in the Laplace transform domain such that the time inversion is applied only to the first term of the stress expansion near the crack tip. This approach has greatly simplified the analysis and will be used here.

The local solution may be found by expanding the integral in equation (51) for C(s,p) for large values of the argument s. Once the moments M_x^*, M_y^* and H_{xy}^* are expressed in terms of C(s,p), the resulting integrals may be evaluated to give the asymptotic expansions:

$$
\begin{aligned}
M_x^*(r,\theta,p) &= \frac{K_1^*(p)}{\sqrt{2r}} \cos\frac{\theta}{2}\{1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\} + O(r^0) \\
M_y^*(r,\theta,p) &= \frac{K_1^*(p)}{\sqrt{2r}} \cos\frac{\theta}{2}\{1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\} + O(r^0) \qquad (54) \\
H_{xy}^*(r,\theta,p) &= \frac{K_1^*(p)}{\sqrt{2r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + O(r^0)
\end{aligned}
$$

where Q_x^* and Q_y^* are nonsingular and remain finite as $r\to 0$, i.e.,

$$Q_x^* = Q_y^* = O(r^0) \qquad (55)$$

The polar coordinates r and θ are measured from the crack front as shown in Figure 1. The parameter

$$K_1^*(p) = M_0\sqrt{a}\,\frac{\psi^*(1,p)}{p} \tag{56}$$

is the Laplace transform of the dynamic moment intensity factor and $\psi^*(1,p)$ denotes the values of the function $\psi^*(\xi,p)$ near the crack border $\xi=1$.

Applying the Laplace inversion theorem to equations (54) yields the solution as a function of time:

$$\begin{aligned}
M_x(r,\theta,t) &= \frac{K_1(t)}{\sqrt{2r}} \cos\frac{\theta}{2}\left\{1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right\} + O(r^0) \\
M_y(r,\theta,t) &= \frac{K_1(t)}{\sqrt{2r}} \cos\frac{\theta}{2}\left\{1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right\} + O(r^0) \\
H_{xy}(r,\theta,t) &= \frac{K_1(t)}{\sqrt{2r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + O(r^0)
\end{aligned} \tag{57}$$

The dynamic moment intensity factor $K_1(t)$ may be computed from

$$K_1(t) = \frac{M_0\sqrt{a}}{2\pi i}\int_{Br} \frac{\psi^*(1,p)}{p}\, e^{pt}dp \tag{58}$$

once $\psi^*(1,p)$ is known.

Numerical results. Since the procedure for solving the Fredholm integral equation is already well known, it is not necessary to cover the details. The numerical values of $\psi^*(1,p)$ in equation (58) are given in Figures 2 to 4 for the three different values of μ_2/μ_1 = 0.1, 1.0 and 10.0. The Poisson's ratio and mass density for the layers are assumed to be the same as their variations in the thickness direction do affect the results appreciably. The function $\psi^*(1,p)$ is seen to increase monotonically with c_{21}/pa where $c_{21} = (\mu_1/\rho_1)^{1/2}$ is the shear wave speed of the material in the outer layers.

As an indication of the load intensity transmitted to the crack edge region as a function of time, the moment intensity factor $K_1(t)$ will be computed from equation (58) by using the results in Figures 2 to 4. Figures 5 to 7 display the variations of the normalized quantity $K_1(t)/M_0\sqrt{a}$ with the dimensionless time parameter $c_{21}t/a$ for μ_2/μ_1 = 0.1, 1.0 and 10.0 while the crack length to laminate thickness ratio 2a/h takes on the values of 1, 2 and 4. Generally speaking, $K_1(t)$ tends to increase with time reaching a peak and then acquires an oscillatory character. The peak value of $K_1(t)$ appears to be inversely proportional to the ratio of 2a/h, i.e., $K_1(t)$ maximum at 2a/h = 1 is larger than that at 2a/h = 4. The moment intensity tends to decrease as the crack length is increased. Also, $K_1(t)$ maximum occurs earlier when the shear moduli in the outer layers of the laminate is larger than those in the inner layers. Refer to the curves in Figure 7 for $\mu_2/\mu_1 > 1$ and those in Figure 5 for $\mu_2/\mu_1 < 1$. The influence of μ_2/μ_1 can be best illustrated by fixing the ratio of 2a/h and use μ_2/μ_1 as a varying parameter. Figure 8 shows a plot of $K_1(t)/M_0\sqrt{a}$ versus $c_{21}t/a$ as μ_2/μ_1 takes the values 0.1, 1.0 and 10.0. It is clear that the crack edge moment intensity can be reduced by letting $\mu_2 < \mu_1$, i.e., making the shear moduli of the inner layers to be larger than the moduli of the outer layers.

CONCLUDING REMARKS

A dynamic laminated plate theory is developed with emphases placed on obtaining effective solution for the crack configuration where the $1/\sqrt{r}$ stress singularity and the condition of plane strain are preserved. The radial distance r is measured from the crack edge. Although each layer in the laminate is assumed to be isotropic, it is a simple extension to include anisotropy simulating the directional properties of fiber reinforcement. This additional complexity was not thought to be necessary in this preliminary analysis.

Several revealing conclusions can be made from the numerical results of the example on the sudden bending of a cracked laminate when compared with a single layer homogeneous plate.

(1) The crack moment intensity tends to decrease as the crack length to laminate plate thickness is increased. Hence, a laminated plate has the desirable feature of stabilizing a through crack as it increases its length at constant load.

(2) The level of the average load intensity transmitted to a through crack can be reduced by making the inner layers to be stiffer than the outer layers.

The foregoing comments are strictly based on the concept of moment intensity factor as used in the theory of fracture mechanics. In the normal course of design, other considerations must also be accounted for. However, the point has been made that the present theory, although approximate, is useful for analyzing laminate failure due to crack propagation.

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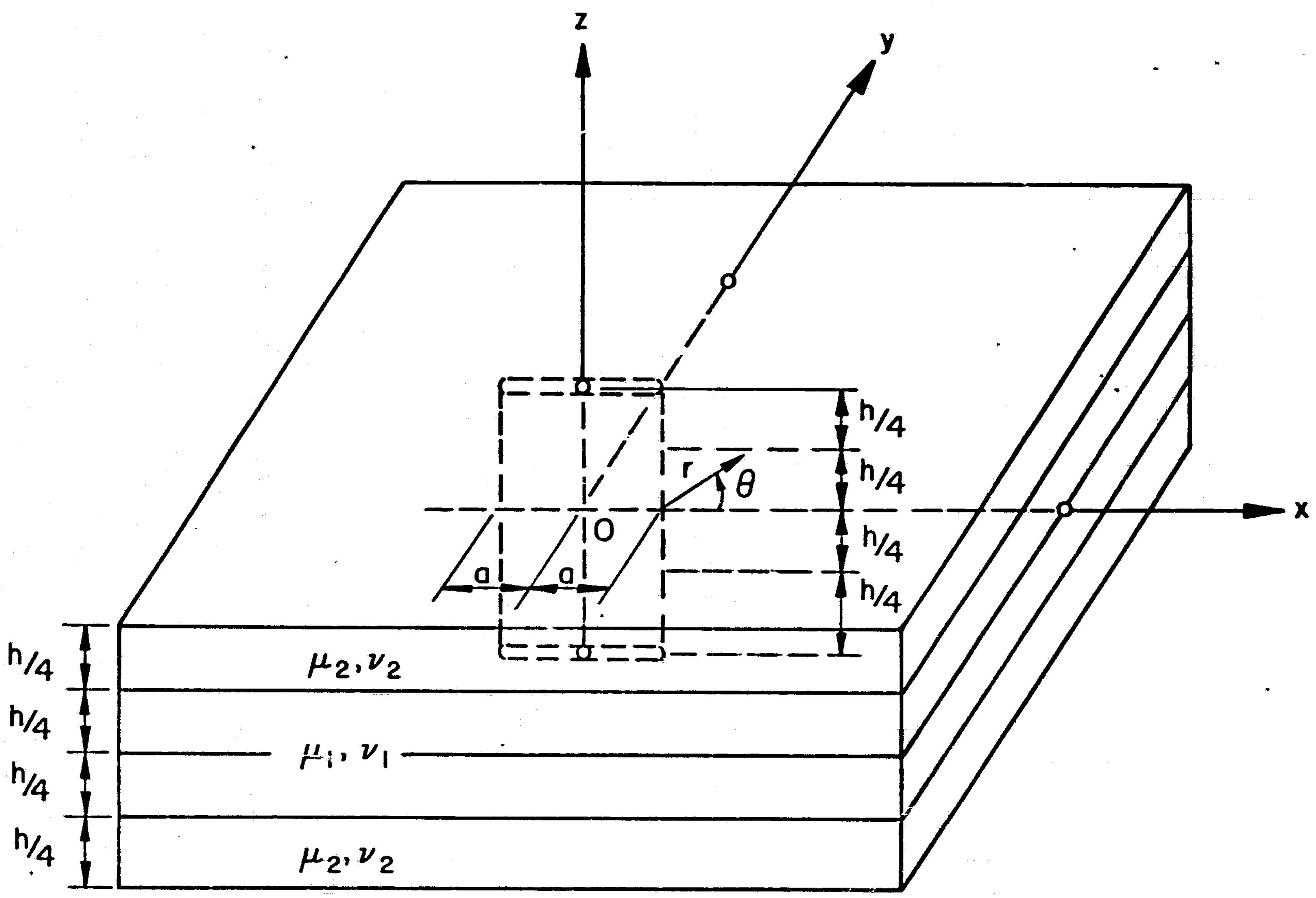

Figure 1 - A symmetrically layered plate with a through crack

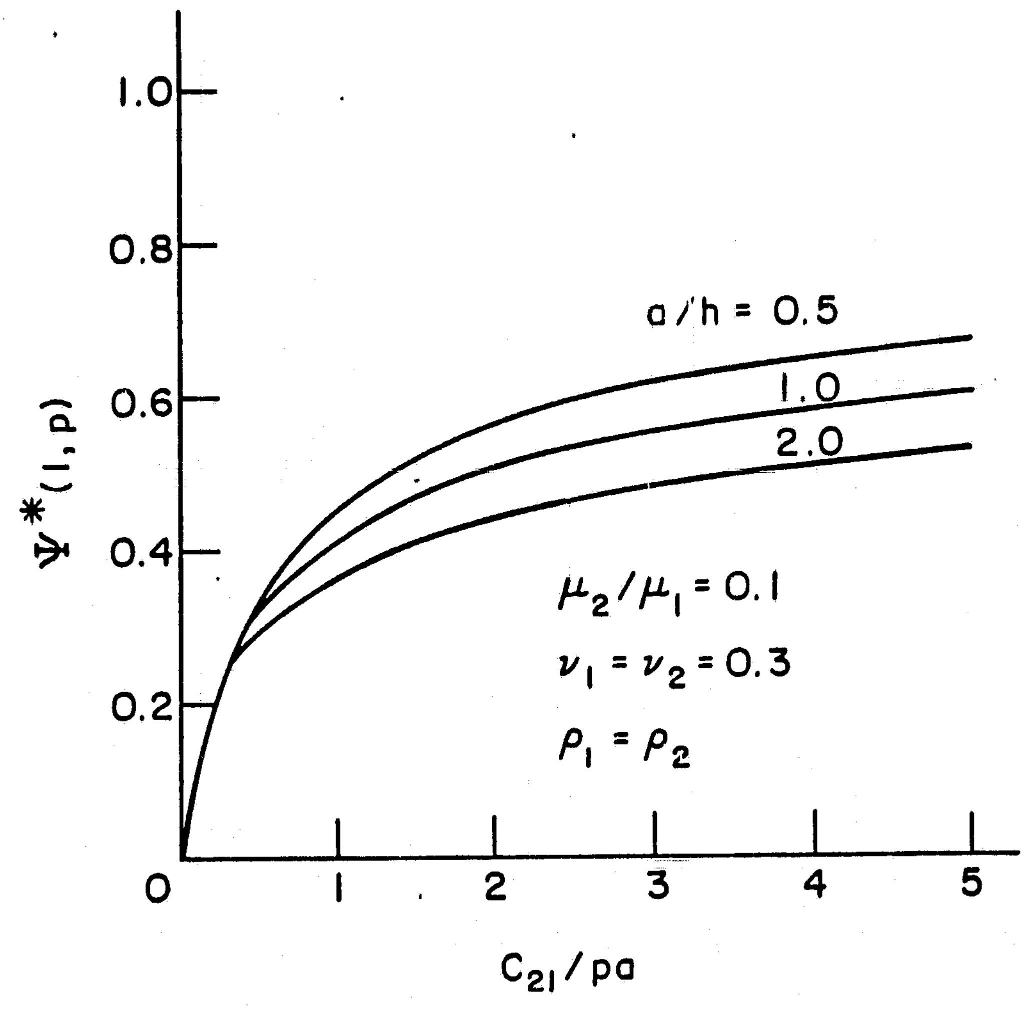

Figure 2 - Numerical values of $\Psi^*(1,p)$ as a function of c_{21}/pa for $\mu_2/\mu_1 = 0.1$

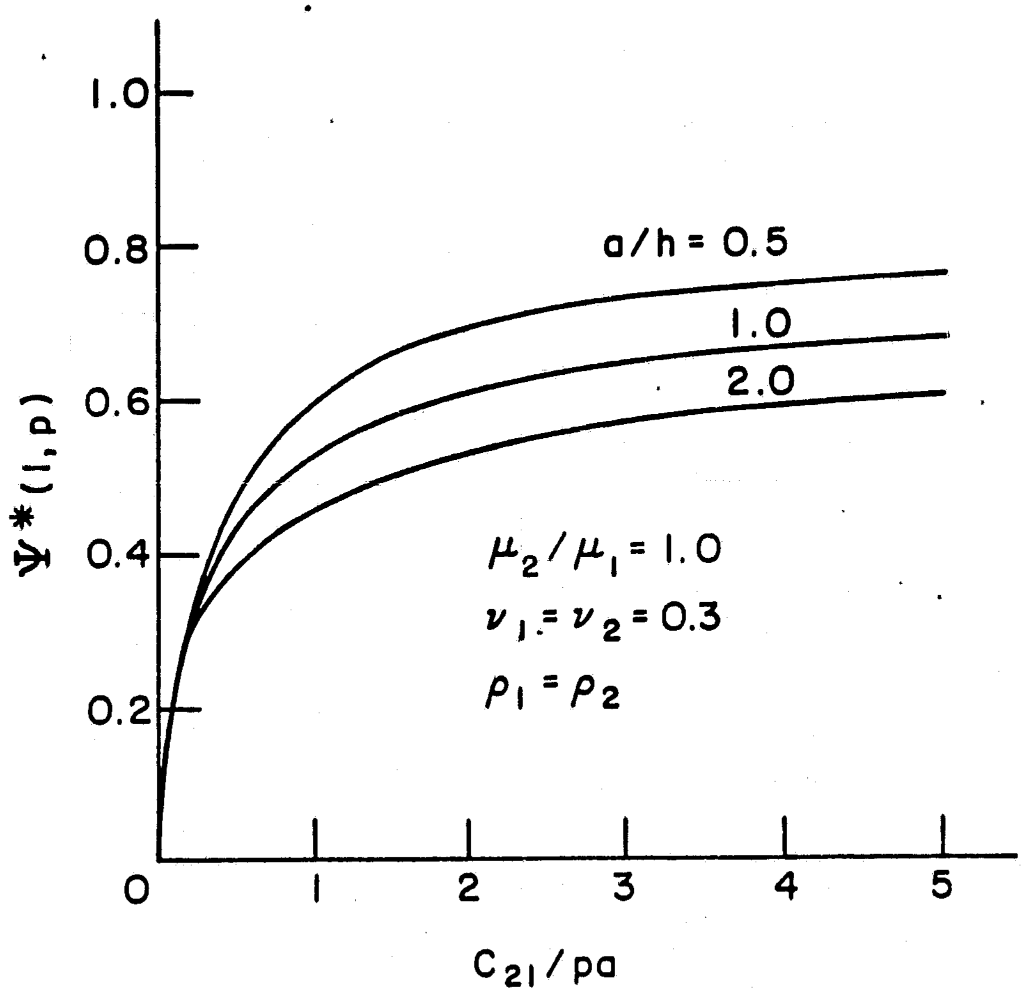

Figure 3 - Numerical values of $\Psi^*(1,p)$ as a function of c_{21}/pa for $\mu_2/\mu_1 = 1.0$

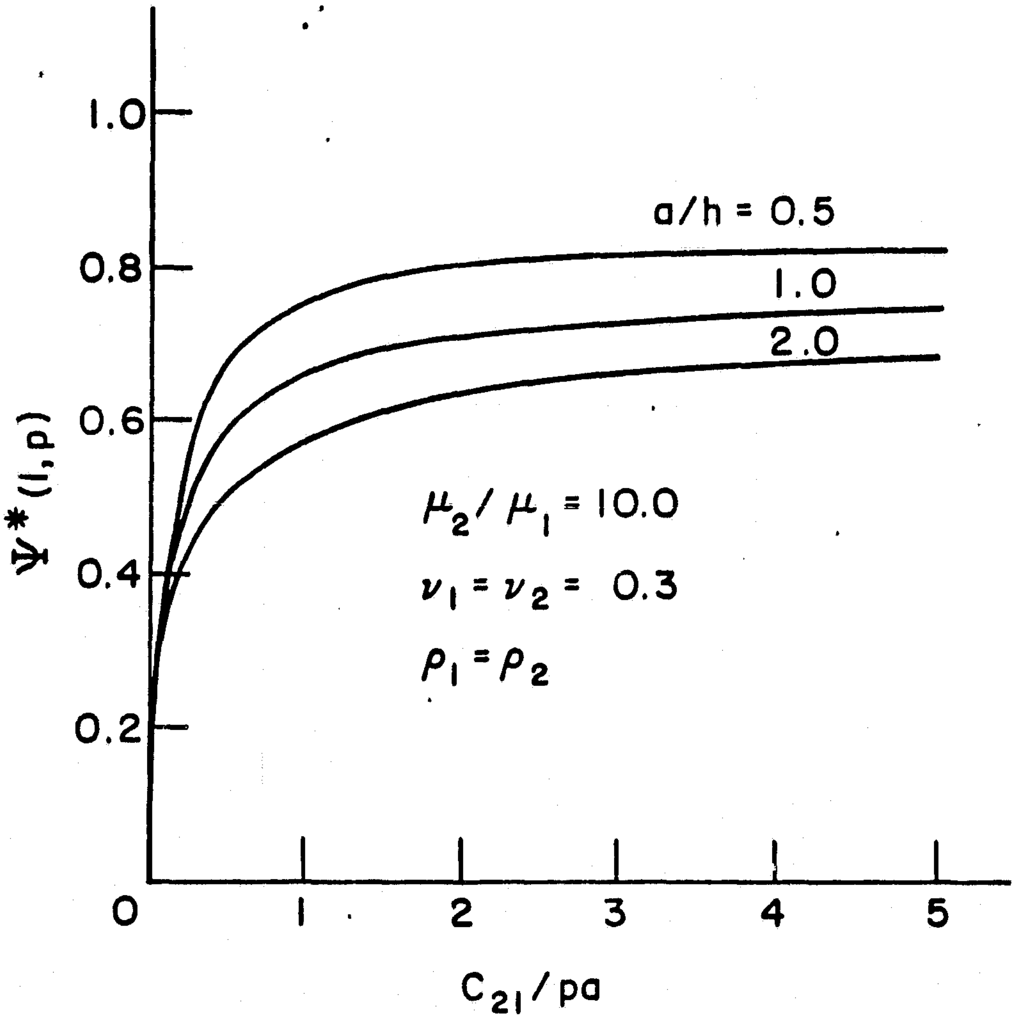

Figure 4 - Numerical values of $\Psi^*(1,p)$ as a function of c_{21}/pa for $\mu_2/\mu_1 = 10.0$

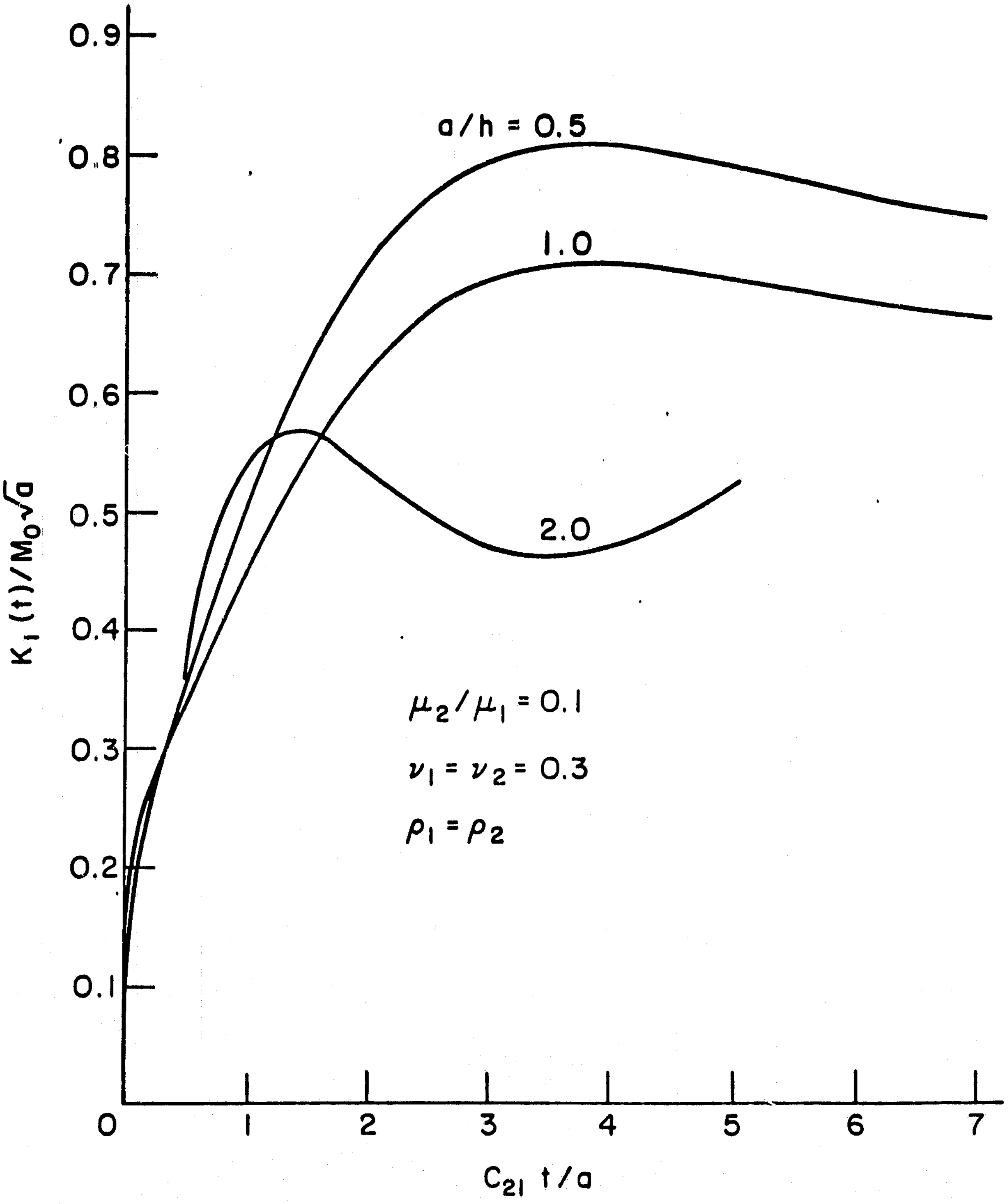

Figure 5 - Normalized moment intensity factor as a function of $c_{21}t/a$ for $\mu_2/\mu_1 = 0.1$

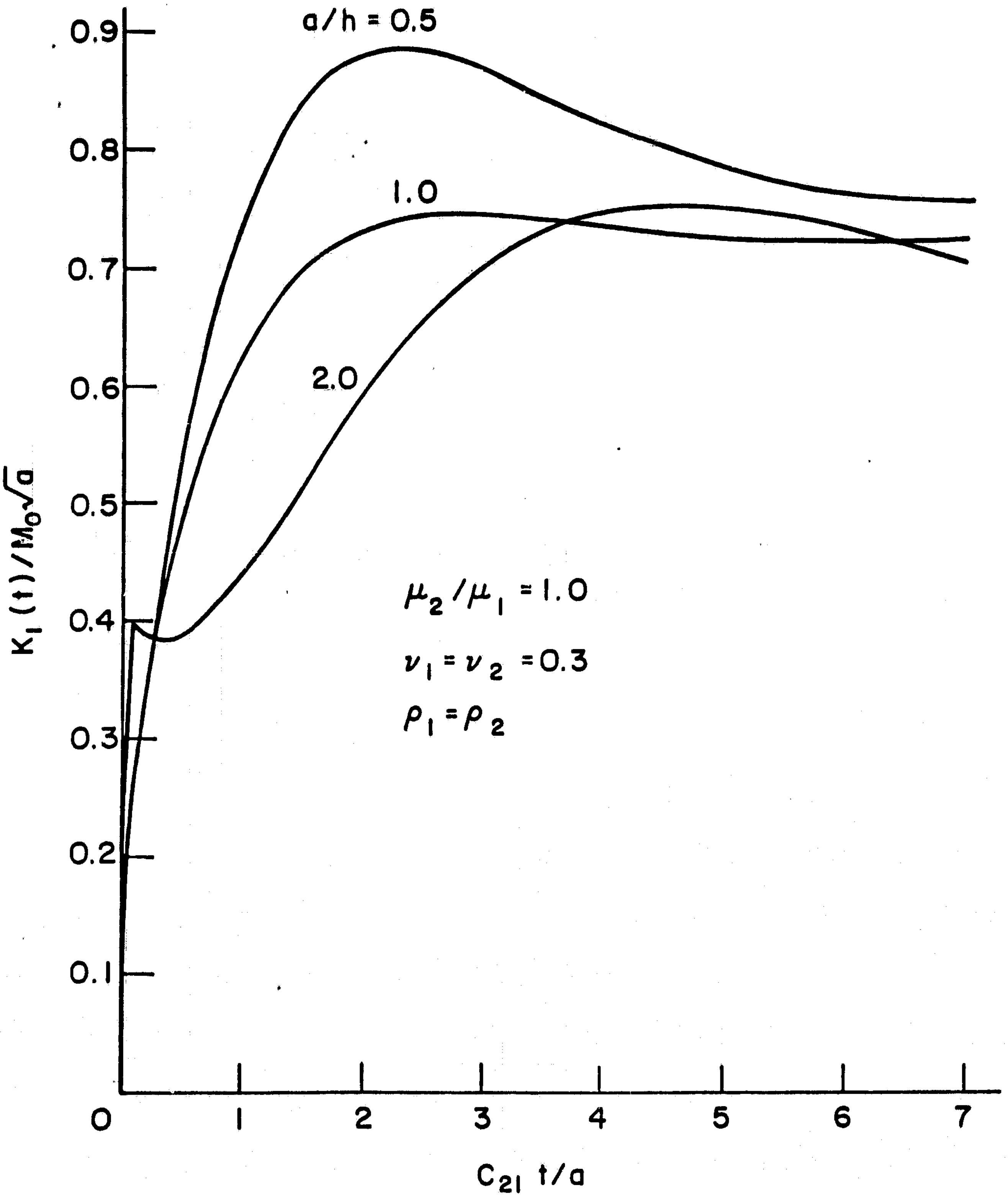

Figure 6 - Normalized moment intensity factor as a function of $c_{21}t/a$ for $\mu_2/\mu_1 = 1.0$

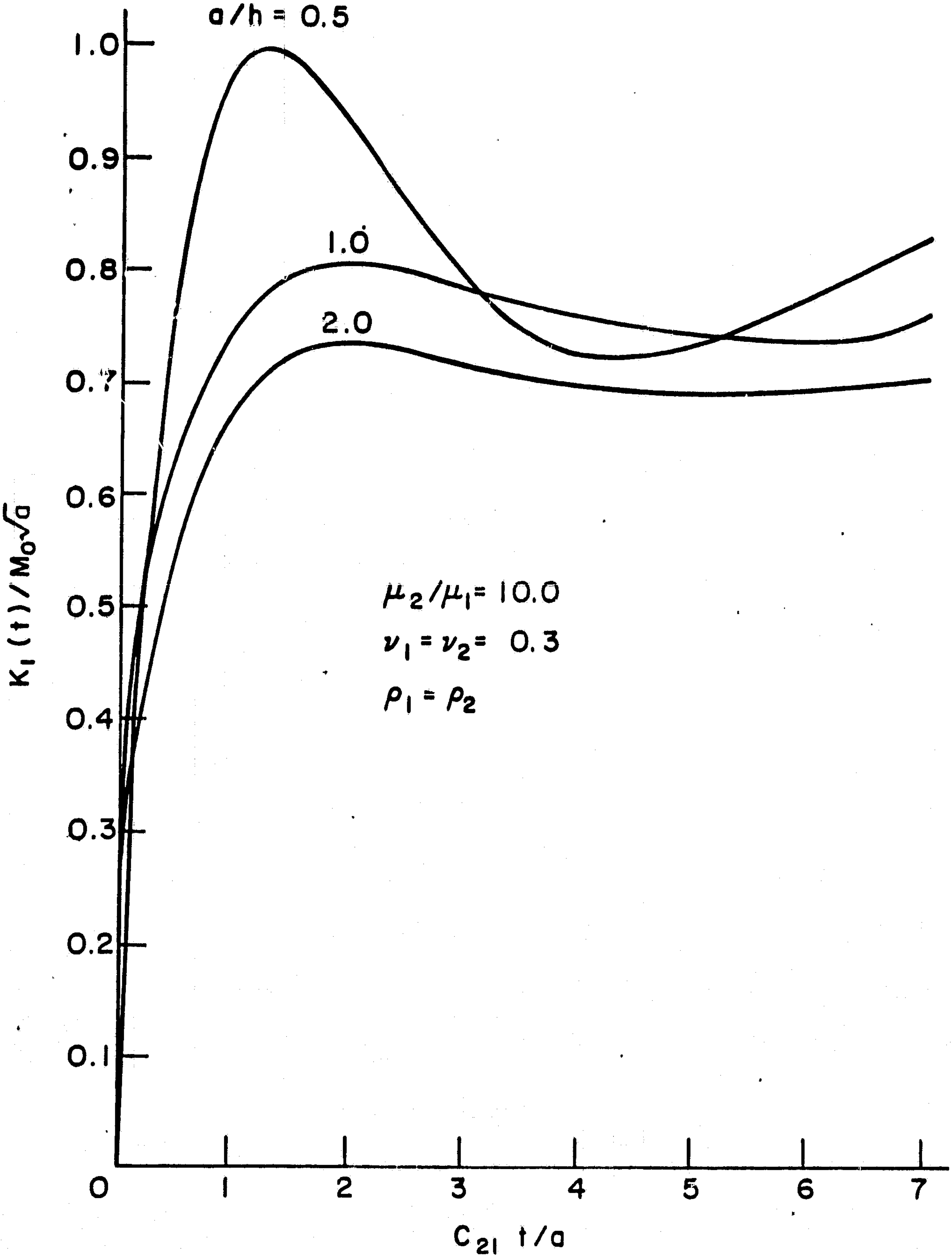

Figure 7 - Normalized moment intensity factor as a function of $c_{21}t/a$ for μ_2/μ_1 = 10.0

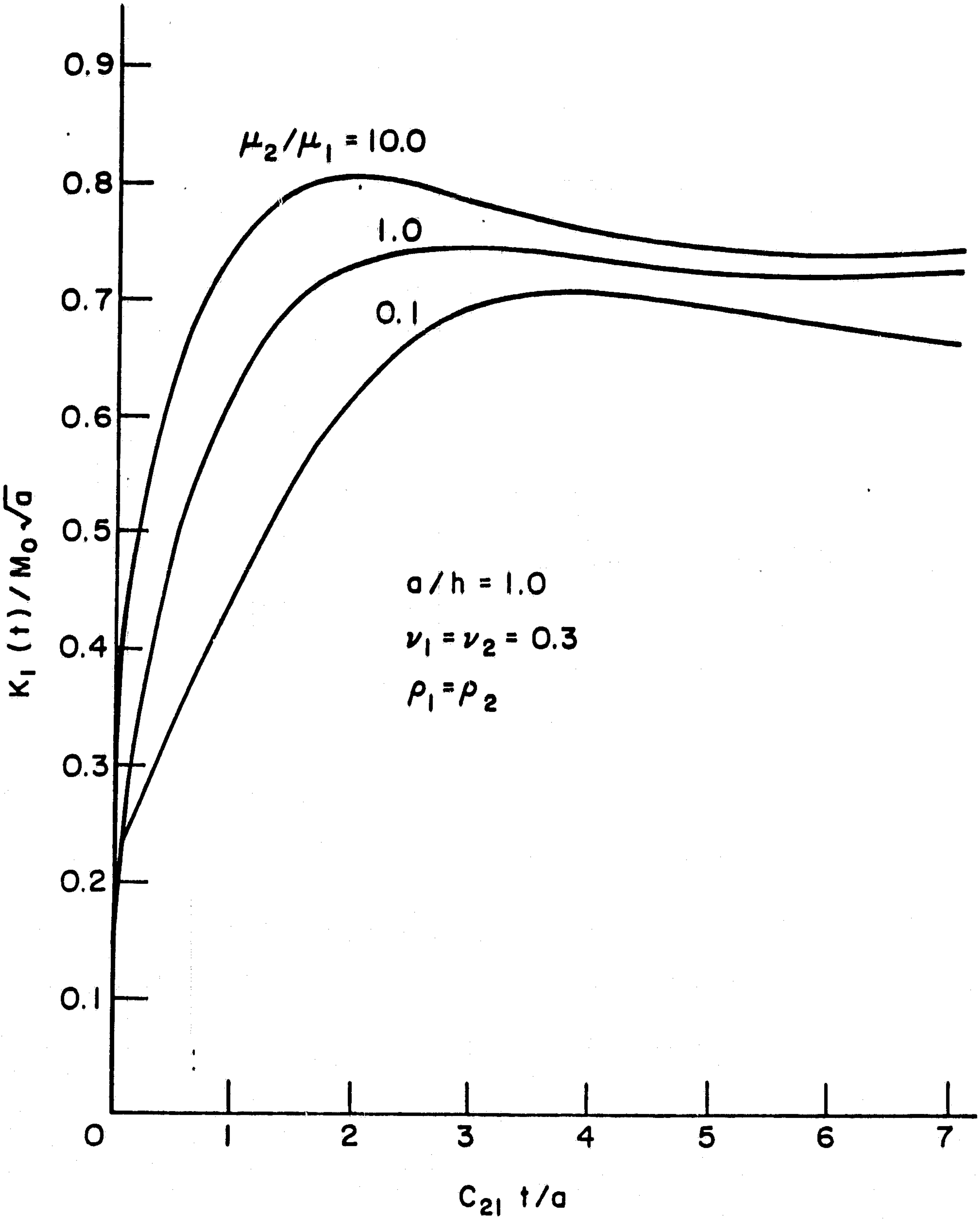

Figure 8 - Normalized moment intensity factor as a function of $c_{21}t/a$ for $a/h = 1.0$

```
      PROGRAM BETA(INPUT,OUTPUT)
      REAL NON(4),F(4,4,1),G(4,4),D(4),PT(4)
      REAL B(4),C(4)
      REAL LP(19),DTA(19)
      EQUIVALENCE (NON,B)
       COMMON K1,K2,K3,K4
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      LP(1)=0.0
      DTA(1)=0.0
      READ 2,K1,K2,K3,K4
    2 FORMAT(I2)
*   K1 = ORDER OF SYSTEM OF EQUATIONS
*   K2 = NO. OF DISTINCT KERNELS
*   K3 = NO. OF DATA POINTS
*   K4 = NO. OF DATA SETS TO BE EVALUATED
*   SET UP DATA POINTS
      AK=K3
      DO 5 N=1,K3
      AN=N
    5 PT(N)=AN/AK
*   SET UP INTEGRATION MATRIX
      M=K3-2
      N=K3-1
      A=K3
      A=1./(3.*A)
      DO 10 K=2,M,2
   10 D(K)=2.*A
      DO 15 K=1,N,2
   15 D(K)=4.*A
      D(K3)=A
*   CALCULATE NONHOMOGENEOUS TERMS
      RHS=1.0
      DO 22 I=1,K2
      PRINT 9
    9 FORMAT(1H1)
      DO 999 II=1,K4
      DO 35 N=1,K3
   35 NON(N)=RHS*SQRT(PT(N))
*   CALCULATE KERNEL MATRICES
      CALL CONST(I)
      DO 20 N=1,K3
      DO 20 M=1,K3
      F(M,N,I)=FU(I,PT(M),PT(N))
   20 CONTINUE
      CALL CHANGE(F,G,D,I)
      CALL LINEQ(G,B,C,    K3)
       DO 40 L=1,K3
       PRINT 6,PT(L),NON(L)
    6  FORMAT(5X,F8.4,F15.6)
   40  CONTINUE
       LP(II+1)=NON(K3)
       DTA(II+1)=P
  999  CONTINUE
       CALL LAPINV(DTA,LP)
   22  CONTINUE
       END
```

```
      FUNCTION SIMP(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      MXYZ=2**15
      DEL=0.25*(B-A)
      IF(DEL)40,45,50
 45   SIMP=0.0
      RETURN
 50   CONTINUE
      SA=Z(I,A)+Z(I,B)
      SB=Z(I,A+2.*DEL)
      SC=Z(I,A+DEL)+Z(I,A+3.*DEL)
      S1=(DEL/3.)*(SA+2.*SB+4.*SC)
      IF(S1.EQ.0.0) GO TO 45
      K=8
 35   SB=SB+SC
      DEL=0.5*DEL
      SC=Z(I,A+DEL)
      J=K-1
      DO 5 N=3,J,2
      AN=N
 5    SC=SC+Z(I,A+AN*DEL)
      S2=(DEL/3.)*(SA+2.*SB+4.*SC)
      DIF=ABS((S2-S1)/S1)
      ER=0.01
      IF(DIF-ER)30,25,25
 30   SIMP=S2
      RETURN
 25   K=2*K
      S1=S2
      IF(K-MXYZ)35,35,40
 40   PRINT 42,I,A,B
  42  FORMAT(5X,* INT. DOES NOT CONVERGE *,I3,2F9.4)
      PRINT 60,X,Y
 60   FORMAT(2F10.5)
      DO 70 J=1,10
      DIP=J
      DIP=DIP/10.
      W=Z(I,DIP)
      PRINT 60,W
70    CONTINUE
      CALL EXIT
      END
```

```
      SUBROUTINE LINEQ(A,B,T,N)
      REAL A(N,N),B(N),T(N)
      DO 5 I=2,N
    5 A(I,1)=A(I,1)/A(1,1)
      DO 10 K=2,N
      M=K-1
      DO 15 I=1,N
   15 T(I)=A(I,K)
      DO 20 J=1,M
      A(J,K)=T(J)
      J1=J+1
      DO 20 I=J1,N
      T(I)=T(I)-A(I,J)*A(J,K)
   20 CONTINUE
      A(K,K)=T(K)
      IF(K.EQ.N) GO TO 10
      M=K+1
      DO 25 I=M,N
   25 A(I,K)=T(I)/A(K,K)
   10 CONTINUE
*  BACK SUBSTITUTE
      DO 31 I=1,N
      T(I)=B(I)
      M=I+1
      IF(M.GT.N) GO TO 31
      DO 30 J=M,N
      B(J)=B(J)-A(J,I)*T(I)
   30 CONTINUE
    31 CONTINUE
      DO 35 I=1,N
      K=N+1-I
      B(K)=T(K)/A(K,K)
      K1=K-1
      IF(K1.EQ.0) GO TO 35
      DO 36 J1=1,K1
      J=K-J1
      T(J)=T(J)-A(J,K)*B(K)
    36 CONTINUE
   35 CONTINUE
      RETURN
      END
```

```
      FUNCTION FU(I,A,B)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      X=A
      Y=B
      IF(A*B)5,10,5
 10   FU=0.0
      RETURN
  5   SUM=SIMP(I,0.0,5.0)
      FR=0.01
      DEL =5.0
 20   UP=DEL+5.0
      ADDL=SIMP(I,DEL,UP)
      DEL =UP
      TEST=ABS(ADDL/SUM)
      SUM=SUM+ADDL
      IF(TEST-FR)15,20,20
 15   FU=SQRT(X*Y)*SUM
      RETURN
      END
```

```
      SUBROUTINE CHANGE(F,G,D,I)
      REAL F(4,4,1),G(4,4),D(4)
      COMMON K1,K2,K3,K4
      DO 10 N=1,K3
      DO 10 M=1,K3
      G(M,N)   =F(M,N,I)*D(N)
 10   CONTINUE
      DO 20 N=1,K3
 20   G(N,N)=G(N,N)+1.0
      RETURN
      END
```

```
      FUNCTION BESJO(A)
      IF(A-3.)5,5,10
  5   B=A*A/9.
      W=1.-2.2499997*B
      Z=B*B
      W=W+1.2656208*Z
      Z=Z*B
      W=W-.3163866*Z
      Z=Z*B
      W=W+.0444479*Z
      Z=Z*B
      W=W-.0039444*Z
      Z=Z*B
      BESJO=W+.00021*Z
      RETURN
 10   B=3./A
      W=.79788456-.00000077*B
      V=A-.78539816-.04166397*B
      Z=B*B
      W=W-.0055274*Z
      V=V-.00003954*Z
      Z=Z*B
      W=W-.00009512*Z
      V=V+.00262573*Z
      Z=Z*B
      W=W+.00137237*Z
      V=V-.00054125*Z
      Z=Z*B
      W=W-.00072805*Z
      V=V-.00029333*Z
      Z=Z*B
      W=W+.00014476*Z
      V=V+.00013558*Z
      BESJO=W/SQRT(A)*COS(V)
      RETURN
      END
```

```
      SUBROUTINE CONST(I)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      PK1=0.3
      PK2=0.3
      BMU=50.0
      H=1.0
      READ 2,P
    2 FORMAT(F10.5)
      HH=1./H
      PRINT 1,BMU,PK1,PK2,HH,P
    1 FORMAT(/////5X,*  MU2/MU1 =*F6.2,* NU1 =*F4.2,* NU2 =*F4.2//5X,*
     1A/H =*F4.2,* C21/PA =*F4.2//)
      RETURN
      END


        SYMBOLIC REFERENCE MAP (R=1)

ENTRY POINTS
    3  CONST

VARIABLES       SN  TYPE            RELOCATION
     4  BMU         REAL                AUX                0  H         REAL
    55  HH          REAL                                   0  I         INTEGER   *UNUSED
     1  P           REAL                AUX                2  PK1       REAL
     3  PK2         REAL                AUX                5  X         REAL
     6  Y           REAL                AUX

FILE NAMES          MODE
        INPUT       FMT                 OUTPUT     FMT

STATEMENT LABELS
    37  1       FMT                        25  2        FMT

COMMON BLOCKS   LENGTH
        AUX          7

STATISTICS
  PROGRAM LENGTH                 56B       46
  SCM LABELED COMMON LENGTH       7B        7
         47000B SCM USED
```

```
      FUNCTION Z(I,S)
      COMMON/AUX/H,P,PK1,PK2,BMU,X,Y
      COMPLEX AC,AL1,AL2,SA,SB
      COMPLEX GA,GB,BA,BB,BC,F,G
      PI=3.1415926
      PP=P*P
      R=H*H/12.
      AA=1.+7.*BMU*(1.-PK1)/(1.-PK2)
      DEO=48.*(1.-PK1)/(PP*H*H*AA)
      SS=H*H*AA/(2.*PI*PI*(1.-PK1)*(1.+BMU))
      XNUO=(PK1-PK2+AA*PK2)/AA
      AB=(R+SS)*DEO
      ZZ=(R-SS)*(R-SS)*DEO*DEO-4.*DEO
      G=CMPLX(ZZ,0.0)
      AC=CSQRT(G)
      AL1=0.5*(AB+AC)
      AL2=0.5*(AB-AC)
      AL3=2.*(R*DEO+1./SS)/(1.-XNUO)
      SA=AL2/(R*DEO+1./SS)
      SB=AL1/(R*DEO+1./SS)
      GA=CSQRT(S*S+AL1)
      GB=CSQRT(S*S+AL2)
      GC=SQRT(S*S+AL3)
      BA=2./(1.-XNUO*XNUO)/(AL1-AL2)
      BB=GA*GA-XNUO*S*S
      BC=GB*GB-XNUO*S*S
      F=BA*((1.-SA)*BB*BB/GA-(1.-SB)*BC*BC/GB-2.*S*S*GC*(1.-XNUO)*(AL1-A
     1L2)/AL3)
      Q=REAL(F)
      QA=AIMAG(F)
      IF(QA-0.0)5,10,5
   10 Z=(Q-S)*BESJ0(S*X)*BESJ0(S*Y)
      RETURN
    5 PRINT 9,P,S,F
    9 FORMAT(4F10.5)
      CALL EXIT
      END
```

```
      SUBROUTINE LAPINV(GLAM,PHI)
C     THIS PROGRAM EVALUATES THE COEFFICIENTS FOR SERIES
C     OF JACOBI POLYNOMIALS WHICH REPRESENTS A LAPLACE
C     INVERSION INTEGRAL
      REAL MUL
      DIMENSION A(50),GLAM(50),PHI(50),C(4,50)
      DIMENSION BK(101),TT(101)
      COMMON/2/TI,TF,DT,MN,BK,TT
      READ 1,NN,MN,MM
    1 FORMAT(3I2)
      READ 2,TI,TF,DT
    2 FORMAT(3F10.5)
      PRINT 99
   99 FORMAT(1H1)
      CALL SPLICF(GLAM,PHI,MM,C)
      PRINT 101
  101 FORMAT(/////5X,*    GLAM                PHI      *)
      PRINT 102,(GLAM(I),PHI(I),I=1,MM)
  102 FORMAT(5X,F10.5,5X,F10.5)
      M11=MM-1
      PRINT 99
      DO 10 I=1,NN
      READ 3,BET,DEL
    3 FORMAT(2F10.5)
      PRINT 98,BET,DEL
   98 FORMAT(/////5X,*BETA =*F5.3,* DELTA =*F5.3)
      DO 11 L=1,MN
      AL=L
      S=1./(AL+BET)/DEL
      CALL SPLINF(GLAM,PHI,MM,C,S,G)
      F=G*S
      IF(AL-2.)81,82,83
   81 A(1)=(1.+BET)*DEL*F
      GO TO 11
   82 A(2)=((2.+BET)*DEL*F-A(1))*(3.+BET)
      GO TO 11
   83 CONTINUE
      TOP=1.
      L1=L-1
      AL1=L1
      DO 12 J=1,L1
      AJ=J
      TOP=AJ*TOP
   12 CONTINUE
      L2=2*L-1
      BOT=1.
      DO 13 J=L,L2
      AJ=J
      BOT=(AJ+BET)*BOT
   13 CONTINUE
      MUL=BOT/TOP
      SUM=0.0
      DO 14 N=1,L1
      AN=N
      IF(AN-2.)85,86,87
   85 TON=1.
      GO TO 88
```

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OF POOR QUALITY

```
 86 TOD=AL1
    GO TO 88
 87 CONTINUE
    TOD=1.
    ICH=L1-(N-2)
    DO 15 J=ICH,L1
    AJ=J
    TOD=AJ*TOD
 15 CONTINUE
 88 CONTINUE
    BOD=1.
    JA=L1+N
    DO 16 J=L,JA
    AJ=J
    BOD=BOD*(AJ+BET)
 16 CONTINUE
    CO=TOD/BOD
    SUM=SUM+CO*A(N)
 14 CONTINUE
    A(L)=MUL*(DEL*F-SUM)
 11 CONTINUE
    CALL JACSER(DEL,A,BET)
 10 CONTINUE
999 CONTINUE
    RETURN
    END
    SUBROUTINE JACSER(D,C,B)
    DIMENSION C(50),SF(50),P(50)
    DIMENSION BK(101),TT(101)
    COMMON/2/TI,TF,DT,MN,BK,TT
    TT(1)=0.0
    BK(1)=0.0
    LM=1
    T=TI
 12 T=T+DT
    X=2.*EXP(-D*T)-1.
    CALL JACOBI(MN,X,B,P)
    SF(1)=C(1)*P(1)
    DO 10 L=2,MN
    L1=L-1
    AL=L
    SF(L)=SF(L1)+C(L)*P(L)
 10 CONTINUE
    LM=LM+1
    BK(LM)=SF(5)
    TT(LM)=T
    IF(T.LE.TF) GO TO 12
    PRINT 97
 97 FORMAT(/////5X,*   T           K           T           K           T           K
   1       T          K      *)
    DO 31 MY=1,25
    MA=MY+1
    MB=MA+25
    MC=MB+25
    MD=MC+25
    PRINT 96,TT(MA),BK(MA),TT(MB),BK(MB),TT(MC),BK(MC),TT(MD),BK(MD)
 96 FORMAT(5X,F5.2,3X,F7.5,3X,F5.2,3X,F7.5,3X,F5.2,3X,F7.5,3X,F5.2,3X,
   1F7.5)
 31 CONTINUE
    RETURN
    END
```

```
      SUBROUTINE JACOBI(N,X,B,PB)
C     THIS PROGRAM CALCULATES JACOBI POLYNOMIALS OF ORDER
C     K-1 WITH ARG X AND PARAMETER B GT -1
      DIMENSION PB(N)
      AN=N
      IF(AN-2.)1,2,3
    1 PB(1)=1.
      RETURN
    2 PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      RETURN
    3 BSQ=B*B
      BONE=B+1.
      PB(1)=1.
      PB(2)=X-B*(1.-X)/2.
      DO 4 K=3,N
      AK=K
      AK1=AK-1.
      AK2=AK-2.
      K1=K-1
      K2=K-2
      CO1=((2.*AK1)+B)*X
      CO1=((2.*AK2)+B)*CO1
      CO1=((2.*AK2)+BONE)*(CO1-BSQ)
      CO2=2.*AK2*(AK2+B)*((2.*AK1)+B)
      CO=2.*AK1*(AK1+B)*((2.*AK2)+B)
    4 PB(K)=(CO1*PB(K1)-CO2*PB(K2))/CO
      RETURN
      END
```

```
      SUBROUTINE SPLINE(X,Y,M,C,XINT,YINT)
      DIMENSION X(50),Y(50),C(4,50)
      IF(XINT-X(1))1,10,11
   10 YINT=Y(1)
      RETURN
   11 CONTINUE
      IF(X(M)-XINT)1,12,13
   12 YINT=Y(M)
      RETURN
   13 CONTINUE
      K=M/2
      N=M
    2 CONTINUE
      IF(X(K)-XINT)3,14,5
   14 YINT=Y(K)
      RETURN
    3 CONTINUE
      IF(XINT-X(K+1))4,15,7
   15 YINT=Y(K+1)
      RETURN
    4 CONTINUE
      YINT=(X(K+1)-XINT)*(C(1,K)*(X(K+1)-XINT)**2+C(3,K))
      YINT=YINT+(XINT-X(K))*(C(2,K)*(XINT-X(K))**2+C(4,K))
      RETURN
    5 CONTINUE
      IF(X(K-1)-XINT)6,16,17
    6 K=K-1
      GO TO 4
   16 YINT=Y(K-1)
      RETURN
   17 N=K
      K=K/2
      GO TO 2
    7 LL=K
      K=(N+K)/2
    8 CONTINUE
      IF(X(K)-XINT)3,14,18
   18 CONTINUE
      IF(X(K-1)-XINT)6,16,19
   19 N=K
      K=(LL+K)/2
      GO TO 8
    1 PRINT 101
  101 FORMAT(* OUT OF RANGE FOR INTERPOLATION  *)
      STOP
      END
```

```
      SUBROUTINE SPLICE(X,Y,M,C)
      DIMENSION X(50),Y(50),D(50),P(50),E(50),C(4,50)
      DIMENSION A(50,3),B(50),Z(50)
      MM=M-1
      DO 2 K=1,MM
      D(K)=X(K+1)-X(K)
      P(K)=D(K)/6.
    2 E(K)=(Y(K+1)-Y(K))/D(K)
      DO 3 K=2,MM
    3 B(K)=E(K)-E(K-1)
      A(1,2)=-1.-D(1)/D(2)
      A(1,3)=D(1)/D(2)
      A(2,3)=P(2)-P(1)*A(1,3)
      A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)
      A(2,3)=A(2,3)/A(2,2)
      B(2)=B(2)/A(2,2)
      DO 4 K=3,MM
      A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)
      B(K)=B(K)-P(K-1)*B(K-1)
      A(K,3)=P(K)/A(K,2)
    4 B(K)=B(K)/A(K,2)
      Q=D(M-2)/D(M-1)
      A(M,1)=1.+Q+A(M-2,3)
      A(M,2)=-Q-A(M,1)*A(M-1,3)
      B(M)=B(M-2)-A(M,1)*B(M-1)
      Z(M)=B(M)/A(M,2)
      MN=M-2
      DO 6 I=1,MN
      K=M-I
    6 Z(K)=B(K)-A(K,3)*Z(K+1)
      Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)
      DO 7 K=1,MM
      Q=1./(6.*D(K))
      C(1,K)=Z(K)*Q
      C(2,K)=Z(K+1)*Q
      C(3,K)=Y(K)/D(K)-Z(K)*P(K)
    7 C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)
      RETURN
      END
```

MU2/MU1 = 50.00 NU1 = .30 NU2 = .30

A/H =1.00 C21/PA = .02

.2500	.033865
.5000	.059829
.7500	.090175
1.0000	.257758

MU2/MU1 = 50.00 NU1 = .30 NU2 = .30

A/H =1.00 C21/PA = .04

.2500	.069941
.5000	.119396
.7500	.183013
1.0000	.352715

MU2/MU1 = 50.00 NU1 = .30 NU2 = .30

A/H =1.00 C21/PA = .06

.2500	.103272
.5000	.169658
.7500	.249217
1.0000	.412993

BETA =0.000 DELTA = .200; ($\rho_1 = \rho_2$; $\nu_1 = \nu_2 = 0.3$; $a/h = 1.0$; $\mu_2/\mu_1 = 50.0$)

T	K	T	K	T	K	T	K	GLAM	PHI
.10	.59591	2.60	.79662	5.10	.75682	7.60	.76022	0.00000	0.00000
.20	.62915	2.70	.79458	5.20	.75628	7.70	.76082	.02000	.25776
.30	.65817	2.80	.79247	5.30	.75582	7.80	.76142	.04000	.35271
.40	.68339	2.90	.79031	5.40	.75543	7.90	.76205	.06000	.41299
.50	.70522	3.00	.78813	5.50	.75511	8.00	.76268	.08000	.45535
.60	.72402	3.10	.78595	5.60	.75486	8.10	.76333	.10000	.48761
.70	.74010	3.20	.78379	5.70	.75468	8.20	.76399	.20000	.57966
.80	.75378	3.30	.78167	5.80	.75457	8.30	.76465	.30000	.62537
.90	.76530	3.40	.77959	5.90	.75451	8.40	.76533	.40000	.65346
1.00	.77492	3.50	.77757	6.00	.75451	8.50	.76601	.50000	.67240
1.10	.78286	3.60	.77562	6.10	.75457	8.60	.76669	.60000	.68644
1.20	.78930	3.70	.77374	6.20	.75468	8.70	.76738	.70000	.69714
1.30	.79443	3.80	.77194	6.30	.75484	8.80	.76808	.80000	.70558
1.40	.79841	3.90	.77023	6.40	.75504	8.90	.76877	.90000	.71262
1.50	.80139	4.00	.76860	6.50	.75529	9.00	.76947	1.00000	.71825
1.60	.80348	4.10	.76707	6.60	.75558	9.10	.77017	2.00000	.74576
1.70	.80482	4.20	.76563	6.70	.75591	9.20	.77087	3.00000	.75541
1.80	.80549	4.30	.76428	6.80	.75628	9.30	.77157	4.00000	.76103
1.90	.80561	4.40	.76303	6.90	.75667	9.40	.77227	5.00000	.76423
2.00	.80524	4.50	.76187	7.00	.75710	9.50	.77296		
2.10	.80447	4.60	.76081	7.10	.75756	9.60	.77366		
2.20	.80336	4.70	.75983	7.20	.75805	9.70	.77435		
2.30	.80197	4.80	.75895	7.30	.75856	9.80	.77503		
2.40	.80036	4.90	.75816	7.40	.75909	9.90	.77572		
2.50	.79856	5.00	.75745	7.50	.75965	10.00	.77640		

Derivation of Equation (25). Equation (25) can be derived by first expressing equation (24) in the form

$$T + V = \int_{t_o}^{t} dt \iint \left(\frac{\rho_o h^3}{24}\left[\frac{\partial\psi_x}{\partial t}\frac{\partial^2\psi_x}{\partial t^2} + \frac{\partial\psi_y}{\partial t}\frac{\partial^2\psi_y}{\partial t^2}\right] + \frac{\bar{\rho}h}{2}\frac{\partial w}{\partial t}\frac{\partial^2 w}{\partial t^2}\right)dxdy$$

$$+ \int_{t_o}^{t} dt \iint \frac{\partial\overline{W}}{\partial t} dxdy + \overline{T}_o + \overline{V}_o \tag{59}$$

in which $\partial\overline{W}/\partial t$ can be written as

$$\frac{\partial\overline{W}}{\partial t} = \frac{\partial\overline{W}}{\partial\Gamma_x}\frac{\partial\Gamma_x}{\partial t} + \frac{\partial\overline{W}}{\partial\Gamma_y}\frac{\partial\Gamma_y}{\partial t} + \cdots + \frac{\partial\overline{W}}{\partial\Gamma_{yz}}\frac{\partial\Gamma_{yz}}{\partial t}$$

$$= \left(M_x\frac{\partial}{\partial x} + H_{xy}\frac{\partial}{\partial y} + Q_x\right)\frac{\partial\psi_x}{\partial t}$$

$$+ \left(H_{xy}\frac{\partial}{\partial x} + M_y\frac{\partial}{\partial y} + Q_y\right)\frac{\partial\psi_y}{\partial t}$$

$$+ \left(Q_x\frac{\partial}{\partial x} + Q_y\frac{\partial}{\partial y}\right)\frac{\partial w}{\partial t} \tag{60}$$

Denoting n and s as the normal and tangential direction, equation (60) may be integrated to yield

$$\iint \frac{\partial\overline{W}}{\partial t} dxdy = \oint \left(\frac{\partial\psi_n}{\partial t}M_n + \frac{\partial\psi_s}{\partial t}H_{ns} + \frac{\partial w}{\partial t}Q_n\right)ds - \iint \left[\frac{\partial\psi_x}{\partial t}\left(\frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - Q_x\right)\right.$$

$$\left. + \frac{\partial\psi_y}{\partial t}\left(\frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y\right) + \frac{\partial w}{\partial t}\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right)\right] dxdy \tag{61}$$

Putting equation (61) into (59) and observing the relations in equations (15), the expression for $\overline{T}+\overline{V}$ in equation (25) is obtained.