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Application of Functional Analysis to Perturbation Theory of Differential Equations

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SHUTTLE PROGRAM

APPLICATION OF FUNCTIONAL ANALYSIS TO PERTURBATION
THEORY OF DIFFERENTIAL EQUATIONS

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INTRODUCTION

The object of this paper is to investigate the deviation of the solution of the differential equation

$$y' = f(t, y), \quad y(0) = y_0$$

from the solution of the perturbed system

$$z' = f(t, z) + g(t, z), \quad z(0) = z_0$$

where f and g are continuous functions on $I \times \mathbb{R}^n$ into \mathbb{R}^n , where $I = (0, a)$ or $I = (0, \infty)$. These functions are assumed to satisfy the Lipschitz condition in the variable z .

The space $Lip(I)$ of all such functions with suitable norms forms a Banach space. Introducing a suitable norm in the space of continuous functions $C(I)$, one can reduce the problem to an equivalent problem in terminology of operators in such spaces.

For the sake of completeness of the presentation, a theorem on existence and uniqueness of the solution is presented by means of Banach space technique.

Norm estimates on the rate of growth of such solutions are found. As a consequence, estimates of deviation of a solution due to perturbation are obtained. Continuity of the solution on the initial data and on the perturbation is established.

As an illustration a nonlinear perturbation of the harmonic oscillator is considered. Also, a perturbation of equations of the restricted three-body problem linearized at libration point is considered.

DISCUSSION

Let Y be the \mathbb{R}^n space and $|\cdot|$ a norm in it. Readers familiar with vectorial integration in Banach spaces (ref. 1) may assume that Y is a Banach space.

Let I be a closed interval of the form $[0, a]$, where $a > 0$, or $[0, \infty)$. Let f be a continuous function from the product $I \times Y$ into Y such that

$$|f(t, y) - f(t, z)| \leq L|y - z|$$

for all $y, z \in Y$ and all $t \in I$, and moreover,

$$|f(t, 0)| \leq L$$

for all $t \in I$.

Such functions will be called Lipschitzian in the sequel. Let $\|f\|$ denote the infimum of all constants L appearing in the previous inequalities. Denote by $\text{Lip}(I)$ the set of all such Lipschitzian functions from $I \times Y$ into Y . The pair $\text{Lip}(I), \|\cdot\|$ represents a Banach space.

The object of this paper will be to establish how much the solution of the equation

$$y'(t) = f(t, y(t)) \quad \text{for } t \in I$$

satisfying the initial condition $y(0) = y_0$ will change in a suitable norm in the space of continuous functions if the function f is perturbed by a Lipschitzian function g of small norm.

Let $C_k(I)$ denote the set of all continuous functions h from the interval I into the Banach space Y such that the number

$$\sup \left\{ |h(t)| e^{-kt}; \quad t \in I \right\} = \|h\|_k$$

is finite. Notice that the pair $C_k(I), \|\cdot\|_k$ forms a Banach space for every $k \geq 0$. Moreover if the interval I is bounded; i.e., $a < \infty$, then all the spaces $C_0(I) = C_k(I)$ are equal, and the norms $\|\cdot\|_0$ and $\|\cdot\|_k$ are equivalent for any k . If the interval I is unbounded, then $C_k(I)$ is contained in $C_m(I)$ if $k < m$.

For the sake of completeness of the presentation, the following theorem shall be proved.

Theorem 1. For every Lipschitzian function $f \in \text{Lip}(I)$ and every initial value $y_0 \in Y$ there exists a constant k_0 such that the differential equation

$$y'(t) = f(t, y(t)) \quad \text{for all } t \in I \tag{1.1}$$

with the initial condition

$$y(0) = y_0 \tag{1.2}$$

has a unique solution in every space $C_k(I)$ such that $k \geq k_0$.

Proof. Using vectorial integration, it can be established that the equation (1.1) with the initial condition (eq. 1.2) is equivalent to the integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s)) ds \quad \text{for all } t \in I \quad (1.3)$$

Let F denote the operator defined by the right side of the equation and considered over the space $C_k(I)$; i.e., $h = F(y)$, where h and y are functions and $y \in C_k(I)$, if and only if,

$$h(t) = y_0 + \int_0^t f(s, y(s)) ds \quad \text{for all } t \in I$$

It can be proved that the function h is continuous for every function $y \in C_k(I)$.

Lemma. The operator F maps the space $C_k(I)$ into itself and

$$\|F(y) - F(z)\|_k \leq \|f\| \|y - z\|_k (1/k)$$

for all $y, z \in C_k(I)$ and all $k > 0$.

Proof of the Lemma. Take any $y, z \in C_k(I)$ and let $\bar{y} = F(y)$ and $\bar{z} = F(z)$. Then by definition of the operator F

$$|\bar{y}(t) - \bar{z}(t)| = \left| \int_0^t (f(s, y(s)) - f(s, z(s))) ds \right|$$

It follows from the definition of the space $Lip(I)$ that

$$|f(s, y(s)) - f(s, z(s))| \leq \|f\| |y(s) - z(s)|$$

for all $s \in I$. From the definition of the norm $\| \cdot \|_k$ we get

$$|y(s) - z(s)| \leq \|y - z\|_k e^{ks} \text{ for all } s \in I.$$

Thus we get the estimate

$$|\bar{y}(t) - \bar{z}(t)| \leq \|f\| \|y - z\|_k \int_0^t e^{ks} ds$$

for all $t \in I$. Because

$$\int_0^t e^{ks} ds = (e^{kt} - 1)/k \leq e^{kt}/k$$

for all $t \in I$, we get

$$e^{-kt} |\bar{y}(t) - \bar{z}(t)| \leq \|f\| \|y - z\|_k / k$$

for all $t \in I$. Taking the supremum over all $t \in I$ of the left side of this inequality, one gets

$$\|\bar{y} - \bar{z}\|_k \leq \|f\| \|y - z\|_k / k$$

and this proves the lemma.

Now let $k_0 = \|f\|$. Then for $k > k_0$ the operator F is a contraction of the Banach space $C_k(I)$. Thus, from Banach's fixed point theorem (ref. 1), we get the existence of a unique point $y \in C_k(I)$ such that

$$y = F(y)$$

This proves the theorem.

Theorem 2. Let y, a be two functions from the space $C_k(I)$ and $c > 0$. If

$$|y(t)| \leq |a(t)| + c \int_0^t |y(s)| ds \quad \text{for all } t \in I \quad (2.1)$$

then

$$\|y\|_k \leq \|a\|_k + (c/k)\|y\|_k \quad \text{for } k > 0 \quad (2.2)$$

Proof. From the definition of the norm $\| \cdot \|_k$ it follows that

$$\int_0^t |y(s)| ds \leq \int_0^t \|y\|_k e^{ks} ds \leq \|y\|_k e^{kt}/k$$

for all $t \in I$ and $k > 0$.

Thus, from inequality (eq. 2.1) we have

$$|y(t)| \leq |a(t)| + c\|y\|_k e^{kt}/k \quad \text{for all } t \in I$$

Multiplying both sides by e^{-kt} and using the definition of the norm $\| \cdot \|_k$ we get

$$e^{-kt}|y(t)| \leq \|a\|_k + (c/k)\|y\|_k$$

for all $t \in I$.

Hence

$$\|y\|_k \leq \|a\|_k + (c/k)\|y\|_k$$

Theorem 3. Let f be a function from the space $Lip(I)$, $y_0 \in Y$, and let $y \in C_k(I)$, where $k > \|f\|$, be the solution of the initial value problem

$$y'(t) = f(t, y(t)) \text{ for all } t \in I$$

$$y(0) = y_0$$

Then

$$\|y\|_k \leq (|y_0| + \|f\|/k)/(1 - \|f\|/k)$$

In other words,

$$|y(t)| \leq e^{kt}(|y_0| + \|f\|/k)/(1 - \|f\|/k)$$

for $t \in I$.

Proof.

The following estimate is derived from the triangle inequality and from the definition of the space $Lip(I)$:

$$\begin{aligned} |f(t, y)| &\leq |f(t, y) - f(t, 0)| + |f(t, 0)| \\ &\leq \|f\| |y - 0| + \|f\| \\ &\leq \|f\|(|y| + 1) \text{ for all } t \in I, y \in y. \end{aligned}$$

Thus, for any $y \in C_k(I)$ satisfying the initial value problem,

$$y(t) = y_0 + \int_0^t f(s, y(s)) ds \text{ for all } t \in I$$

This yields

$$|y(t)| \leq |y_0| + \left| \int_0^t f(s, y(s)) ds \right| \text{ for all } t \in I$$

From the previous estimate we get

$$|f(s, y(s))| \leq \|f\| (|y(s)| + 1) \text{ for all } s \in I$$

Thus

$$|y(t)| \leq (|y_0| + \|f\|t) + \|f\| \int_0^t |y(s)| ds$$

Using Theorem 2 and noticing that the norm of the function a , given by the formula

$$a(t) = |y_0| + \|f\|t \text{ for all } t \in I$$

satisfies the estimate

$$\|a\|_k \leq |y_0| + \|f\|/k$$

we get the required inequality.

We shall prove the following theorem because of its simplicity. It is sufficient in the investigation of continuity of solution with respect to perturbation. Finer results will be given in reference 2.

Perturbation Theorem. Let f and g be two Lipschitzian functions from the Banach space $\text{Lip}(I)$. Let $y_0, z_0 \in Y$ and let $y, z \in C_k(I)$, where $k > k_0 = \|f\| + \|g\|$, be the solutions of the initial value problems

$$y'(t) = f(t, y(t)) \quad \text{for all } t \in I, \quad y(0) = y_0$$

and

$$z'(t) = f(t, z(t)) + g(t, z(t)) \quad \text{for all } t \in I, \quad z(0) = z_0$$

Then the following inequality holds

$$\|y - z\|_k \leq \gamma$$

where

$$\gamma = (|y_0 - z_0| + \|g\|e/k) / (1 - (\|f\| + \|g\|)/k)$$

$$c = (|y_0| + 1) / (1 - \|f\|/k)$$

or equivalently

$$|y(t) - z(t)| \leq \gamma e^{kt} \quad \text{for all } t \in I$$

Proof. Using the integral equivalents of the above initial value problems we get

$$y(t) = y_0 + \int_0^t (f(s, y(s)) + g(s, y(s))) ds - \int_0^t g(s, y(s)) ds$$

$$z(t) = z_0 + \int_0^t (f(s, z(s)) + g(s, z(s))) ds$$

for all $t \in I$.

Subtracting both sides of these equalities and using the triangle inequality, one gets

$$\begin{aligned} |y(t) - z(t)| &\leq |y_0 - z_0| + \int_0^t (\|f\| + \|g\|)(|y(s) - z(s)|) ds \\ &\quad + \int_0^t \|g\|(|y(s)| + 1) ds \end{aligned}$$

for all $t \in I$.

The last term of the inequality can be estimated as follows

$$\int_0^t ||g|| (|y(s)| + 1) ds \leq ||g|| \int_0^t (||y||_k e^{ks} + e^{ks}) ds$$

$$\leq ||g|| (||y||_k + 1) e^{kt/k}$$

for all $t \in I$.

Similarly one gets the estimate

$$\int_0^t (||f|| + ||g||) |y(s) - z(s)| ds \leq (||f|| + ||g||) ||y - z||_k e^{kt/k}$$

These estimates yield

$$|y(t) - z(t)| \leq |y_0 - z_0| + (||f|| + ||g||) (||y - z||_k) e^{kt/k}$$

$$+ ||g|| (||y||_k + 1) e^{kt/k}$$

for all $t \in I$.

Multiplying both sides of this inequality by e^{-kt} and using the definition of the norm $|| \cdot ||_k$ one gets

$$||y - z||_k \leq |y_0 - z_0| + (||f|| + ||g||) (||y - z||_k) / k$$

$$+ ||g|| (||y||_k + 1) / k$$

From theorem 3 we have the estimate

$$||y||_k \leq (|y_0| + ||f||/k) / (1 - ||f||/k)$$

Using it and taking the term containing $||y - z||_k$ from the right side onto the left side in the previous inequality, one gets, after simplifications, the following inequality

$$||y - z||_k \leq \gamma$$

where

$$\gamma = (|y_0 - z_0| + \|g\|c/k) / (1 - (\|f\| + \|g\|)/k)$$

and

$$c = (|y_0| + 1) / (1 - \|f\|/k)$$

This proves the perturbation theorem.

Corollary. Let $g_n \in \text{Lip}(I)$ be a sequence such that $\|g_n\| \rightarrow 0$,

$\|g_n\| \leq b$ for all n and $z_n \in Y$ such that $|z_n - y_0| \rightarrow 0$. Let $h_n \in C_k(I)$ be the solution of the perturbed initial value problem

$$h_n'(t) = f(t, h_n(t)) + g_n(t, h_n(t)) \quad \text{for all } t \in I$$

$$h_n(0) = z_n$$

Assume that y is the solution of $y'(t) = f(t, y(t))$ for $t \in I$ with the initial condition $y(0) = y_0$. Then

$$\|y - h_n\|_k \rightarrow 0$$

when n tends to infinity for any fixed $k > \|f\| + b$.

This shows that the solution $y \in C_k(I)$ of the initial value problem depends continuously on the initial value y_0 and Lipschitzian perturbations of the function f .

Examples

Nonlinear perturbation of harmonic oscillator

Consider the following second-order differential equation

$$x'' + \omega^2 x = \epsilon(\cos x + \sin x')$$

$$x(0) = x_0, \quad x'(0) = v_0$$

(A)

Treating the right side as a perturbation, find an estimate how much the solution of the equation

$$x'' + \omega x = 0, \quad x(0) = x_0, \quad x'(0) = v_0 \quad (B)$$

differs from the solution of the equation (A).

Introducing the vector variable $y = (y_1, y_2)$ by the formulas $y_1 = x$, $y_2 = x'$ and writing the equation in the normal form one gets the system

$$y_1' = y_2$$

$$y_2' = -\omega^2 y_1 + \epsilon(\cos y_1 + \sin y_2)$$

with initial conditions

$$y_1(0) = x_0, \quad y_2(0) = v_0$$

Equivalently in vector form we get

$$y' = f(y) + \epsilon g(y), \quad y(0) = (x_0, v_0)$$

where

$$f(y) = (y_2 - \omega^2 y_1)$$

$$g(y) = (0, \cos y_1 + \sin y_2)$$

for all vectors $y = (y_1, y_2) \in \mathbb{R}^2$.

The Lipschitz norm of the function f can be estimated as follows. Introduce in \mathbb{R}^2 the norm

$$|y| = |y_1| + |y_2|$$

for all $y = (y_1, y_2)$.

Let $z = f(y)$ and $\bar{z} = f(\bar{y})$ then

$$\begin{aligned} |z - \bar{z}| &= |z_1 - \bar{z}_1| + |z_2 - \bar{z}_2| = |y_2 - \bar{y}_2| + \omega^2 |y_1 - \bar{y}_1| \\ &\leq \lambda(\omega) (|y_1 - \bar{y}_1| + |y_2 - \bar{y}_2|) \leq \lambda(\omega) |y - \bar{y}| \end{aligned}$$

where $\lambda(\omega) = \omega^2$ if $\omega \geq 1$ and $\lambda(\omega) = 1$ if $0 < \omega < 1$.

This yields the estimate

$$|f(y) - f(\bar{y})| \leq \lambda(\omega) |y - \bar{y}|$$

for all $y, \bar{y} \in \mathbb{R}^2$. Because $f(0) = 0$, we get for the Lipschitz norm of f

$$\|f\| \leq \lambda(\omega)$$

Similarly, let $z = g(y)$ and $\bar{z} = g(\bar{y})$.

Then

$$\begin{aligned} |z - \bar{z}| &= |z_1 - \bar{z}_1| + |z_2 - \bar{z}_2| \\ &= 0 + |\cos y_1 - \cos \bar{y}_1 + \sin y_2 - \sin \bar{y}_2| \end{aligned} \tag{C}$$

From the Cauchy theorem of calculus

$$\cos y_1 - \cos \bar{y}_1 = (-\sin c_1)(y_1 - \bar{y}_1)$$

$$\sin y_2 - \sin \bar{y}_2 = (\cos c_2)(y_2 - \bar{y}_2)$$

for some intermediate values c_1 and c_2 .

These values give the estimates

$$|\cos y_1 - \cos \bar{y}_1| \leq |y_1 - \bar{y}_1|$$

$$|\sin y_2 - \sin \bar{y}_2| \leq |y_2 - \bar{y}_2|$$

Thus from (C) one gets

$$|z - \bar{z}| \leq |y_1 - \bar{y}_1| + |y_2 - \bar{y}_2| = |y - \bar{y}|$$

i.e.,

$$|g(y) - g(\bar{y})| \leq |y - \bar{y}|$$

for all $y, \bar{y} \in \mathbb{R}^2$.

Because

$$|g(0)| = |(0, 1)| \leq 1$$

we get the estimate for the Lipschitz norm of g

$$\|g\| \leq 1$$

Just from the perturbation theorem one gets the estimates

$$\gamma \leq (\varepsilon c/k)/(1 - (1 + \varepsilon)/k)$$

where

$$c \leq (|x_0| + |v_0| + 1)/(1 - 1/k)$$

if $k > 1 + \varepsilon$.

To be concrete let $k = 2$, $\varepsilon = 0.5$, $\omega = 1$, $x_0 = 0$, $v_0 = 1$. Then

$$\gamma \geq (0.5 \cdot 4/2)/(1 - 1.5/2) = 4$$

Thus

$$|y(t) - z(t)| \leq 4e^{2t} \quad \text{for all } t \geq 0$$

The solution of the unperturbed system $y' = f(y)$, $y(0) = (0,1)$ is given by

$$y_1(t) = \sin(t)$$

$$y_2(t) = \cos(t)$$

for all $t \in \mathbb{R}$. Comparing it with the solution of the perturbed system $z' = f(z) + \epsilon g(z)$, $z(0) = (0,1)$, one gets for the function

$$h(t) = |y(t) - z(t)| = |\sin(t) - z_1(t)| + |\cos(t) - z_2(t)|$$

the values in table I.

Perturbation of restricted three-body problem linearized near libration point

Consider the equations of restricted three-body problem

$$\ddot{x} - 2\dot{y} = Bx + Ay$$

$$\ddot{y} - 2\dot{x} = Ax + 3By$$

and perturbed equations

$$\ddot{x} - 2\dot{y} = Bx + Ay + \epsilon k(\dot{x})$$

$$\ddot{y} - 2\dot{x} = Ax + 3By + \epsilon k(y)$$

where $k(u) = u/(7 + u^2)$ for all $u \in \mathbb{R}$.

Introducing a new vector variable $z = (z_1, z_2, z_3, z_4)$ by the formulas

$x = z_1$, $\dot{x} = z_2$, $y = z_3$, $\dot{y} = z_4$ and writing the equations in the normal form one gets

$$\dot{z} = f(z) + \epsilon g(z)$$

where

$$f(z) = z \text{ for all } z \in \mathbb{R}^4$$

and \mathcal{A} being the matrix

$$\mathcal{A} = \begin{bmatrix} 0, & 1, & 0, & 0 \\ B, & 0, & A, & 2 \\ 0, & 0, & 0, & 1 \\ A, & -2, & 3B, & 0 \end{bmatrix}$$

$$A = \frac{3\sqrt{3}}{4}(1 - 2\mu), \quad B = \frac{3}{4}, \quad \mu = 1/82.45$$

and

$$g(z) = \begin{bmatrix} 0 \\ k(z_2) \\ 0 \\ k(z_3) \end{bmatrix}$$

for all $z \in \mathbb{R}^4$. Notice that $f(0) = 0$ and $g(0) = 0$. Thus, to find an estimate of the Lipschitz norm $\|f\|$ and $\|g\|$ it is sufficient to find an estimate of the Lipschitz constant for the functions f and g . To this end take

any $z, \bar{z} \in \mathbb{R}^4$ and estimate $|f(z) - f(\bar{z})|$.

Using the norm $|z| = |z_1| + |z_2| + |z_3| + |z_4|$ for $z \in \mathbb{R}^4$, we get the estimate

$$\begin{aligned} |f(z) - f(\bar{z})| &\leq (A + B)|z_1 - \bar{z}_1| + 3|z_2 - \bar{z}_2| + (A + 3B)|z_3 - \bar{z}_3| \\ &\quad + 3|z_4 - \bar{z}_4| \end{aligned}$$

Thus, if

$$\lambda = \max \{3, A + 3B\} = A + 3B = 3.5175272$$

we get

$$|f(z) - f(\bar{z})| \leq \lambda |z - \bar{z}| \quad \text{for all } z, \bar{z} \in \mathbb{R}^4$$

This yields $\|f\| \leq \lambda$.

To estimate the Lipschitz norm of the function g let us first find the Lipschitz norm of the function k . Notice that

$$l(x) = |k'(x)| = |(1 - x^2)/(1 + x^2)^2|$$

for all x .

The function l has an absolute maximum on the interval $(-\infty, \infty)$ at $x = 0$. Thus,

$$|k'(x)| \leq l(0) = 1$$

for all $x \in (-\infty, \infty)$.

Using the Cauchy intermediate value theorem, we get

$$|k(y) - k(\bar{y})| = |k'(c)| |y - \bar{y}| \leq |y - \bar{y}|$$

for all $y, \bar{y} \in \mathbb{R}$.

Now from the definition of the function g one gets

$$\begin{aligned} |g(z) - g(\bar{z})| &\leq |k(z_2) - k(\bar{z}_2)| + |k(z_3) - k(\bar{z}_3)| \\ &\leq |z_2 - \bar{z}_2| + |z_3 - \bar{z}_3| \\ &\leq |z_1 - \bar{z}_1| + |z_2 - \bar{z}_2| + |z_3 - \bar{z}_3| + |z_4 - \bar{z}_4| \\ &= |z - \bar{z}| \end{aligned}$$

for all $z, \bar{z} \in \mathbb{R}^4$. This yields

$$\|g\| \leq 1$$

The estimates for the deviation $|y(t) - z(t)|$ one gets similarly as in the previous example.

For formulas on Lipschitz constant for the case of Euclidean norm and for finer error estimates, see reference 2.

CONCLUDING REMARKS

Though the estimates on deviation obtained in this paper are simple, the results they yield for longer periods of time will be very pessimistic. Finer estimates can be found in reference 2. That reference also contains general methods of computing Lipschitz constants.

REFERENCES

1. Liusternik, L. A.; and Sobolev, V. J.: Elements of Functional Analysis, Fredrich Ungar Publishing Co., New York (1961).
2. Bogdan, V. M.; and Bond, V. R.: A Priori Global Estimates of Deviation of Solutions of Differential Equations Due to Perturbation with Applications to Analytical and Celestial Mechanics (to be published).

TABLE I.- COMPARISON OF COMPUTED GLOBAL ERROR WITH PREDICTED BOUND

| t | h(t) | R exp(2t) |
|-----|-----------|---------------|
| 0.0 | 0.0000000 | 4.0000000 |
| .1 | .0976182 | 4.8856111 |
| .2 | .2050097 | 5.9672988 |
| .3 | .3199916 | 7.2884752 |
| .4 | .4401596 | 8.9021636 |
| .5 | .5629649 | 10.8731271 |
| .6 | .6857933 | 13.2804674 |
| .7 | .8060412 | 16.2207992 |
| .8 | .9211782 | 19.8121288 |
| .9 | 1.0287934 | 24.1985884 |
| 1.0 | 1.1266198 | 29.5562222 |
| 1.1 | 1.2125354 | 36.1000504 |
| 1.2 | 1.2845469 | 44.0927005 |
| 1.3 | 1.3407616 | 53.8549452 |
| 1.4 | 1.3793622 | 65.7785778 |
| 1.5 | 1.3985967 | 80.3421354 |
| 1.6 | 1.3968026 | 98.1301050 |
| 1.7 | 1.3724767 | 119.8563795 |
| 1.8 | 1.3244038 | 146.3929100 |
| 1.9 | 1.2518431 | 178.8047028 |
| 2.0 | 1.1547580 | 218.3925552 |
| 2.1 | 1.0340512 | 266.7452583 |
| 2.2 | 1.0904912 | 325.8033905 |
| 2.3 | 1.2169999 | 397.9371529 |
| 2.4 | 1.3288932 | 486.0415306 |
| 2.5 | 1.4217193 | 593.6524582 |
| 2.6 | 1.4916894 | 725.0887451 |
| 2.7 | 1.5360930 | 885.6253815 |
| 2.8 | 1.5535694 | 1081.7052612 |
| 2.9 | 1.5441907 | 1321.1977844 |
| 3.0 | 1.5093642 | 1613.7145996 |
| 3.1 | 1.4515959 | 1970.9954376 |
| 3.2 | 1.3741666 | 2407.3792419 |
| 3.3 | 1.2807751 | 2940.3796082 |
| 3.4 | 1.1751914 | 3591.3877258 |
| 3.5 | 1.1587874 | 4386.5308228 |
| 3.6 | 1.2601225 | 5357.7207642 |
| 3.7 | 1.3560715 | 6543.9348145 |
| 3.8 | 1.4477527 | 7992.7799683 |
| 3.9 | 1.5358592 | 9762.4033203 |
| 4.0 | 1.6206077 | 11923.8262939 |