OPTIMAL LARGE ANGLE MANEUVERS WITH SIMULTANEOUS SHAPE CONTROL/VIBRATION ARREST

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ABSTRACT

A relaxation method is demonstrated which reliably solves the nonlinear two-point-boundary-value problem which arises when optimal control theory is applied to determination of large angle maneuvers of flexible spacecraft. The basic ideas are summarized and several idealized maneuvers are determined. The emphasis is upon demonstrating the basic ideas and practical aspects of the methodology. References are cited, particularly Turner's dissertation which presents detailed formulations and more general applications.

Discussion of Figures

With reference to Figure 1, we employ the method of assumed modes to obtain a set of ordinary differential equations which govern deflections and rotations. The form of the equations of motion are given in Figure 2. Note the high dimensionality and the variability of the coefficient matrix. Note that solution for the acceleration coordinates is required in order to integrate motion as a function of time, and in order to apply optimal control theory.

Figure 3 displays a partitioned algorithm which efficiently determines the inverse of the high-dimensioned, configuration-variable coefficient matrix. Consistent with this partitioning algorithm, we consider in Figure 4 an algorithm for obtaining partial derivatives of the inverted coefficient matrix with respect to deflection coordinates (required in the optimal control algorithm).

Figure 5 summarizes the state and co-state differential equations which follow from Pontryagin's principle as the necessary conditions satisfied by optimal (minimum quadratic cost) maneuvers. Observe that the initial and final states are generally known, but the initial and final co-states are usually unknown. Thus, as usual, a nonlinear two point boundary value problem (TPBVP) has resulted. Notice the quadratic angular velocity nonlinearity due to "rotational stiffness."

In Figure 6, we summarize an imbedding/relaxation approach which has proven a reliable approach for solving TPBVP's of the above structure. In essence, a one parameter (α) family of problems is constructed that one special member ($\alpha = 0$) has an analytical solution, while another member ($\alpha = 1$) is the true problem of interest. By relaxing α through a sequence of increasing values $0 \leqslant \alpha_i \leqslant 1$, we can extrapolate arbitrarily good initial or final co-state estimates (by adjusting the α-increment) from previous converged solutions, thereby allowing efficient differential corrections to isolate accurate co-states corresponding to each α. Typically, only 4 or 5 α_i values are actually required to reach the desired $\alpha = 1$ solution. This method and related methods are developed and applied to several examples in Reference 3.

Considering now a specific configuration, we refer to Figure 7. The four identical cantilevered appendages are mounted in the same plane to the rigid central hub. We neglect the hub radius in any equation in which it appears divided by the appendage length. Referring to Figure 8, we restrict attention to pure spin rotations and antisymmetric deflections, consistent with spin-up, spin-down, and rest-to-rest maneuvers with the configuration initially and finally undeformed. We consider only the case of torques applied to the hub.

Table 1 describes seven maneuver calculations, corresponding to three sets of maneuver boundary conditions and four different dynamical models. These cases are selected to demonstrate the effects of rotational stiffening and to show that the relaxation method can handle both high dimensionality and nonlinearities.

Figures 9a - c display the angle of rotation, angular rate and torque for the case 1 maneuver (rigid appendages). For comparison, Figures 10a - c display the same variables for cases 2L and 2N of flexible appendages, assuming a 1 mode expansion. It is of interest to note that the flexibility

effects are large indeed. The flexible case torque oscillates anti-symmetrically about the rigid case torque, the desired final angle and angular rate are achieved and the modal amplitude (and its derivative) are simultaneously driven to zero. It is interesting that the linear and nonlinear solutions were identical, to graphical accuracy, due to the small deflections and velocities of this particular maneuver.

Figure 11a – d and 12a – d display angle of rotation, torque history, and amplitudes of the first two modes for cases 3L and 3N, respectively. The maneuver is an extremely rapid spinup from rest to 0.5 rad/sec in 60 sec. The linear (3L) and nonlinear (3N) solutions differ significantly, but the linear solution retains the general shape and amplitudes differ by less than 10% throughout most of the motion.

Figure 13a – g display the angle of rotation, angular rate, torque, and the first four modal amplitudes for case 4L (a rest-to-rest maneuver through a 360° rotation). These results simply show that, indeed, the large rigid rotations and vibration suppression of several degrees of freedom are determined.

We offer the following significant conclusions:

- An Optimal Control Formulation is Presented for General 3 Dimensional Maneuvers of a Class of Flexible Satellites
- A Partitioning Method is Introduced to Invert the Rotational-Vibrational Equations of Motion for Acceleration Coordinates and to Obtain the Adjoint Equations
- An Imbedding/Relaxation Process if Demonstrated for Solution of the Two-Point-Boundary-Value Problem.
- Numerical Studies Indicate that Practical Algorithms Result from these Developments

References

1. Junkins, J.L. and J.D. Turner, "Optimal Continuous Torque Attitude Maneuvers," AIAA preprint #78-1400, presented to the AIAA/AAS Astrodynamics Conference, Palo Alto, CA., August 1978; also, to appear, AIAA Journal of Guidance and Control.

2. Turner, J.D. and J.L. Junkins, "Optimal Large Angle Single Axis Rotational Maneuvers of Flexible Spacecraft," presented to the 2nd AIAA/VPI&SU Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, VA., June 1979, proceedings in press; also, to appear in AIAA Journal of Guidance and Control.

3. Turner, J.D. "Optimal Large Angle Maneuvers for Large Flexible Space Structures," Ph.D. Dissertation, Virginia Polytechnic Institute and State University, Blacksburg, VA., in press.

THE METHOD OF ASSUMED MODES

The deflection of the jth flexible member is modeled as

$$\left.\begin{aligned} u_j(x,y,z,t) &= \sum_{i=1}^{L_j} \alpha_{ji}(t)\ U_{ji}(x,y,z) = \underline{\alpha}_j^T\ \underline{U}_j \\ v_j(x,y,z,t) &= \sum_{i=1}^{M_j} \beta_{ji}(t)\ V_{ji}(x,y,z) = \underline{\beta}_j^T\ \underline{V}_j \\ w_j(x,y,z,t) &= \sum_{i=1}^{N_j} \gamma_{ji}(t)\ W_{ji}(x,y,z) = \underline{\gamma}_j^T\ \underline{W}_j \end{aligned}\right\} j=1,2,\ldots,n \quad (1)$$

The sets of spatial "assumed modes"

$$\left\{ U_{11}(x,y,z) \ \ldots\ U_{1i}(x,y,z) \right\} \ldots \left\{ U_{n1}(x,y,z) \ \ldots\ U_{ni}(x,y,z) \right\}$$

$$\left\{ V_{11}(x,y,z) \ \ldots\ V_{1i}(x,y,z) \right\} \ldots \left\{ V_{n1}(x,y,z) \ \ldots\ V_{ni}(x,y,z) \right\}$$

$$\left\{ W_{11}(x,y,z) \ \ldots\ W_{1i}(x,y,z) \right\} \ldots \left\{ W_{n1}(x,y,z) \ \ldots\ W_{ni}(x,y,z) \right\}$$

are <u>prescribed</u>. As minimum requirements, they must

- be linearly independent
- satisfy u, v, w's geometric boundary conditions

The amplitude functions constitute the *configuration vector* $\underline{\eta}(t)$

$$\underline{\eta}(t) = \left\{ \underline{\alpha}_1^T,\ \underline{\beta}_1^T,\ \underline{\gamma}_1;\ \ldots\ \underline{\alpha}_n^T,\ \underline{\beta}_n^T,\ \underline{\gamma}_n^T \right\}^T$$

The amplitude's play the role of discrete generalized coordinates.

Figure 1

ROTATIONAL/DEFLECTIONAL EQNS OF MOTION

DYNAMICS OF FLEXIBLE SATELLITES

$$\{\dot{\theta}\} = [F(\theta)]\{\omega\} \tag{1}$$

$$[J(\eta)]\{\dot{\omega}\} = -[H^T]\{\ddot{\eta}\} + \{f(\theta,\omega,\eta,\dot{\eta},t)\} + \{u\} \tag{2}$$

$$[M]\{\ddot{\eta}\} = -[H]\{\dot{\omega}\} + \{g(\theta,\omega,\eta,\dot{\eta},t)\} \tag{3}$$

Combine (2) & (3)

$$\begin{pmatrix} J(\eta) & \vdots & H^T \\ \cdots & \vdots & \cdots \\ H & \vdots & M \end{pmatrix} \begin{Bmatrix} \dot{\omega} \\ \cdots \\ \ddot{\eta} \end{Bmatrix} = \begin{Bmatrix} f + u \\ \cdots \\ g \end{Bmatrix} \tag{4}$$

Note

[H] & [M] are constant

$[J(\eta)] = [J_o] + [J_D(\eta)]$, $||J_D|| << ||J_o||$ (typically)

$[J_o]$	$[J_D(\eta)]$
Inertia of undeformed vehicle	Inertia variations due to deformations

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A problem:

We need eqns of motion in the state space form $\dot{\underline{x}} = \underline{F}(\underline{x},\underline{u},t)$,

but

(i) The coefficient matrix of (4) is variable

(ii) Its dimensions may be several hundred

Figure 2

PARTITIONED/PERTURBATION INVERSION OF THE COEFFICIENT MATRIX

Name the submatrices:

$$\begin{pmatrix} C_{11} & \vdots & C_{21}^T \\ \cdots & \vdots & \cdots \\ C_{21} & \vdots & C_{22} \end{pmatrix} \equiv \begin{pmatrix} J & \vdots & H^T \\ \cdots & \vdots & \cdots \\ H & \vdots & M \end{pmatrix}^{-1}$$

The C_{ij} can be expressed directly as a function of J, M, H as:

	Form 1	Form 2
C_{11}	$J^{-1} - J^{-1} H^T C_{21}$	$(J - H^T M^{-1} H)^{-1}$
C_{22}	$(M - H J^{-1} H^T)^{-1}$	$M^{-1} - M^{-1} H C_{21}^T$
C_{21}	$-C_{22} H J^{-1}$	$-M^{-1} H C_{11}$

For direct numerical calculations, Form 2 is preferred since

(i) $(J - H^T M^{-1} H)$ is a 3×3 matrix

(ii) M is generally diagonally dominant (an identity matrix if one first solves an eigenvalue problem - Note M is positive-definite symmetric)

Figure 3

CERTAIN REQUIRED PARTIAL DERIVATIVES & HOW TO DETERMINE THEM

Rotational/Vibrational Equations of Motion

$$\underbrace{\begin{pmatrix} \underline{\underline{J}}(\underline{\eta}) & \vdots & \underline{\underline{H}}^T \\ \cdots\cdots & \vdots & \cdots \\ \underline{\underline{H}} & \vdots & \underline{\underline{M}} \end{pmatrix}}_{\underline{\underline{M}}(\underline{\eta})} \begin{Bmatrix} \dot{\underline{\omega}} \\ \cdots \\ \ddot{\underline{\eta}} \end{Bmatrix} = \begin{Bmatrix} \underline{f}(\underline{\theta},\underline{\omega},\underline{\eta},\dot{\underline{\eta}},t) + \underline{u}(t) \\ \\ \underline{g}(\underline{\theta},\underline{\omega},\underline{\eta},\dot{\underline{\eta}},t) \end{Bmatrix}$$

or, in inverted form

$$\begin{Bmatrix} \dot{\underline{\omega}} \\ \cdots \\ \ddot{\underline{\eta}} \end{Bmatrix} = \left| \underline{\underline{M}}^{-1}(\underline{\eta}) \right| \begin{Bmatrix} \underline{f} + \underline{u} \\ \cdots\cdots \\ \underline{g} \end{Bmatrix}$$

Note

$$\underline{\eta}^T = \{\eta_1 \quad \eta_2 \quad \cdots \quad \eta_n\}$$

$$\frac{\partial}{\partial \eta_i} \begin{Bmatrix} \dot{\underline{\omega}} \\ \cdots \\ \ddot{\underline{\eta}} \end{Bmatrix} = \frac{\partial}{\partial \eta_i} \left| \underline{\underline{M}}^{-1} \right| \begin{Bmatrix} \underline{f} + \underline{u} \\ \cdots\cdots \\ \underline{g} \end{Bmatrix} + \underline{\underline{M}}^{-1} \frac{\partial}{\partial \eta_i} \begin{Bmatrix} \underline{f} + \underline{u} \\ \cdots\cdots \\ \underline{g} \end{Bmatrix}$$

To determine $\frac{\partial}{\partial \eta_i} \left| \underline{\underline{M}}^{-1} \right|$, observe

$$\underline{\underline{M}}^{-1}\underline{\underline{M}} = \underline{\underline{I}}$$

from which

$$\frac{\partial}{\partial \eta_i} \left| \underline{\underline{M}}^{-1} \right| \underline{\underline{M}} + \underline{\underline{M}}^{-1} \left[\frac{\partial \underline{\underline{M}}}{\partial \eta_i} \right] = 0$$

or

$$\boxed{\frac{\partial}{\partial \eta_i} \left| \underline{\underline{M}}^{-1} \right| = -\underline{\underline{M}}^{-1} \left[\frac{\partial \underline{\underline{M}}}{\partial \eta_i} \right] \underline{\underline{M}}^{-1}} \quad , \quad i=1,2,\ldots,n$$

where

$$\left[\frac{\partial \underline{\underline{M}}}{\partial \eta_i} \right] = \begin{bmatrix} \dfrac{\partial \underline{\underline{J}}(\underline{\eta})}{\partial \eta_i} & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix}$$

Figure 4

FORMULATION OF THE OPTIMAL CONTROL PROBLEM

STATE VARIABLES

$$\underline{x}_1 = \{\theta\}, \quad \underline{x}_2 = \{\eta\}, \quad \underline{x}_3 = \{\omega\}, \quad x_4 = \{\dot{\eta}\}$$

STATE DIFFERENTIAL EQUATIONS

$$\dot{\underline{x}}_1 = [F(x_1)]\ \{\underline{x}_3\} = \underline{F}_1(\underline{x}_1, -, \underline{x}_3, -, -, -)$$

$$\dot{\underline{x}}_2 = \underline{x}_4 = \underline{F}_2(-, -, -, \underline{x}_4, -, -)$$

$$\begin{Bmatrix} \dot{\underline{x}}_3 \\ \dot{\underline{x}}_4 \end{Bmatrix} \equiv \begin{Bmatrix} \{\dot{\omega}\} \\ \{\ddot{\eta}\} \end{Bmatrix} = \begin{Bmatrix} \underline{F}_3(\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4, \underline{u}, t) \\ \underline{F}_4(\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4, \underline{u}, t) \end{Bmatrix} = \underline{\underline{M}}^{-1}(\underline{x}_2) \begin{Bmatrix} \underline{f}(\underline{x}_i, t) + \underline{u} \\ \cdots\cdots\cdots \\ \underline{g}(\underline{x}_i, t) \end{Bmatrix}$$

Find $\underline{u}(t)$ generating a trajectory initiating at $\underline{x}_i(t_o)$, terminating at $\underline{x}_i(t_f)$, which minimizes the function

$$J = \frac{1}{2} \int_{t_o}^{t_f} (\underline{u}^T W_{uu}\, \underline{u} + \sum_{i=2}^{4} \underline{x}_i^T W_{ii}\, \underline{x}_i) dt$$

HAMILTONIAN

$$H = \frac{1}{2}(\underline{u}^T W_{uu}\, \underline{u} + \sum_{i=2}^{4} \underline{x}_i^T W_{ii}\, \underline{x}_i) + \sum_{i=1}^{4} \underline{\lambda}_i^T \underline{F}_i$$

PONTRYAGIN'S NECESSARY CONDITIONS

Co-state Equations

$$\dot{\underline{\lambda}}_i = -\frac{\partial H}{\partial \underline{x}_i} \equiv \underline{G}_i(\underline{x}_1, \ldots, \underline{x}_4; \underline{\lambda}_1, \ldots, \underline{\lambda}_4; \underline{u}, t)$$

Optimal Control

Minimize H at each instant with respect to admissible $\underline{u}(t)$, this yields $\underline{u} = \underline{U}(\underline{x}_1, \ldots, \underline{x}_4; \underline{\lambda}_1, \ldots, \underline{\lambda}_4, t)$

Figure 5

IMBEDDING/RELAXATION METHOD FOR SOLVING TWO-POINT BOUNDARY VALUE PROBLEM

Define merged vector

$$\underline{Z} \equiv \{\underline{x}^T \quad \underline{\lambda}^T\}^T$$

The coupled state and costate differential equations are then

$$\dot{\underline{Z}} = [A]\underline{Z} + \alpha\{\text{all nonlinear terms}\}$$

- Typically, we know $\underline{x}(t_0)$ and $\underline{x}(t_f)$, but not $\underline{\lambda}(t_0)$, $\underline{\lambda}(t_f)$.
- For $\alpha = 0$, we can solve for $\underline{\lambda}(t_0)$ exactly.
- By taking sufficiently small α-increments, we can use converged $\underline{\lambda}(t_0)$ from neighboring optimal solutions to initiate successive approximations with *arbitrarily good starting estimates* for the unknown $\underline{\lambda}(t_0)$.
- Typically, only 5 to 10 intermediate α-values are required a practical algorithm results.

Figure 6

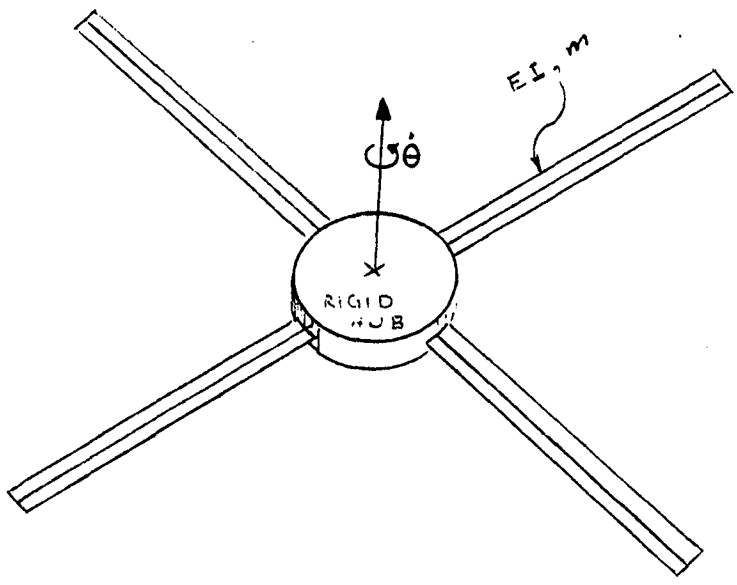

FIGURE 7 UNDEFORMED STRUCTURE

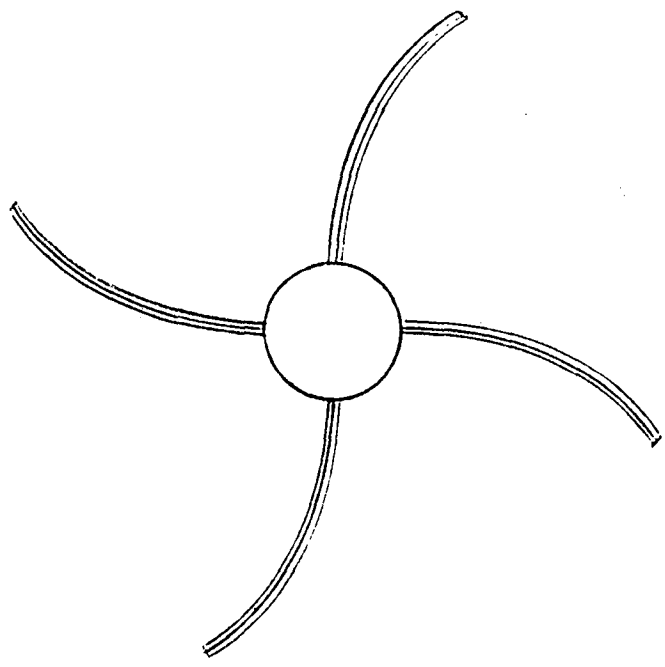

FIGURE 8 ANTISYMMETRIC DEFORMATION

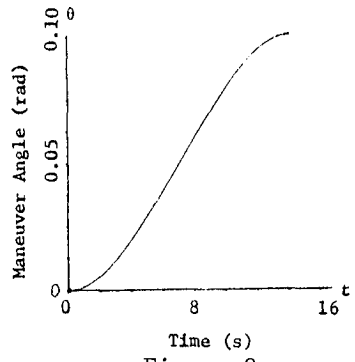

Figure 9a

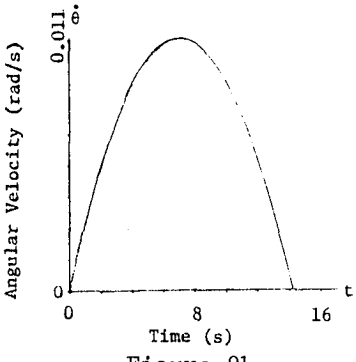

Figure 9b

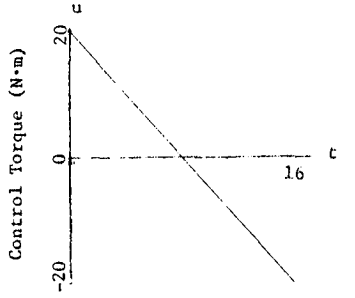

Figure 9c

Figure 9 Case 1 Rigid Body Rest-To-Rest Maneuver

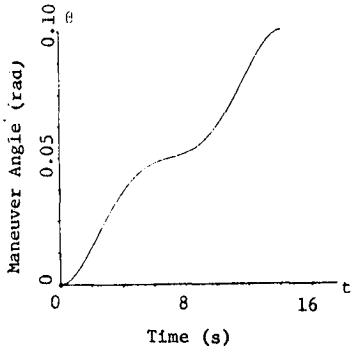

Figure 10a

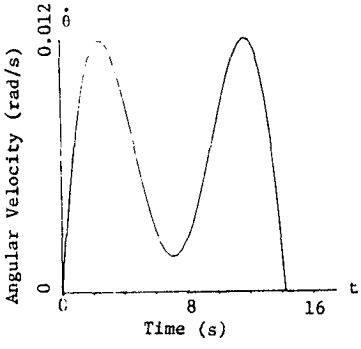

Figure 10b

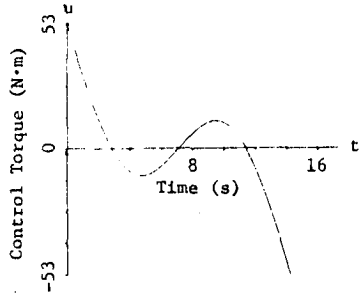

Figure 10c

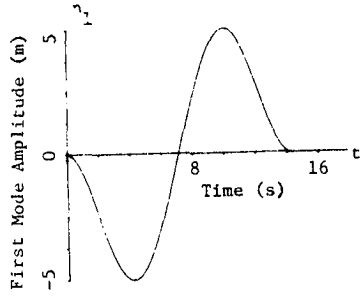

Figure 10d

Figure 10 Cases 2L, 2N Flexible Appendages Rest-To-Rest Maneuver $t_f = 2\pi/\omega_1$

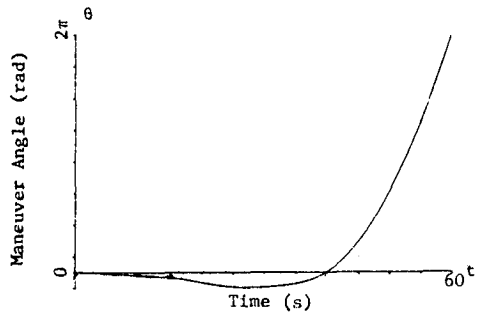

Figure 11a

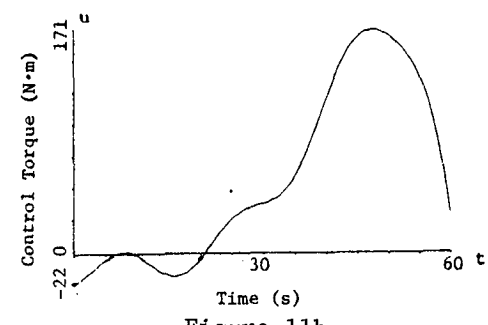

Figure 11b

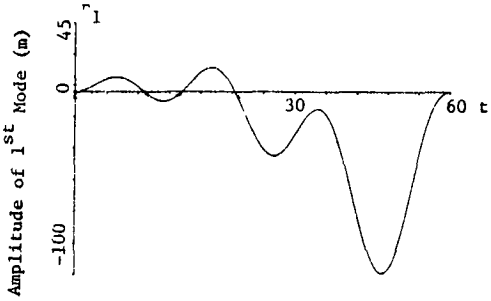

Figure 11c

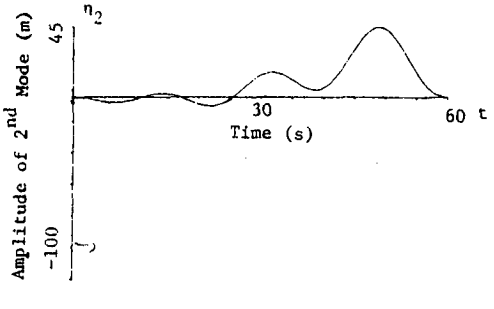

Figure 11d

Figure 11 Case 3L Linear Spinup Maneuver

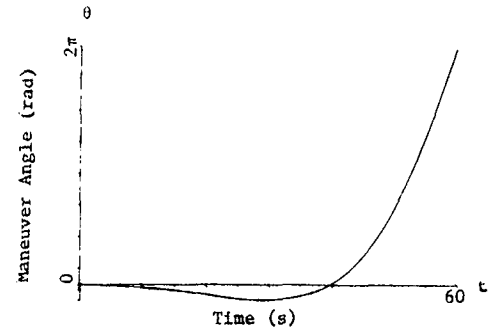

Figure 12a

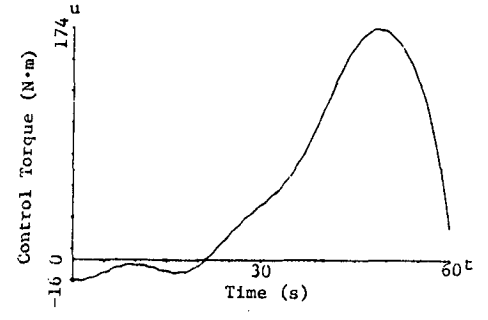

Figure 12b

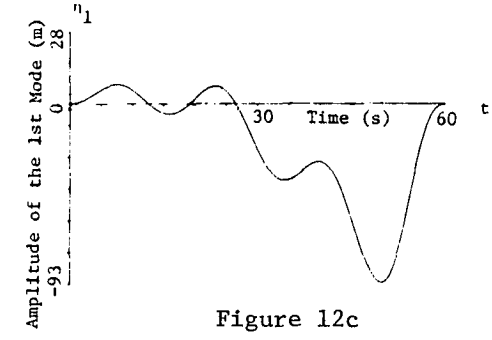

Figure 12c

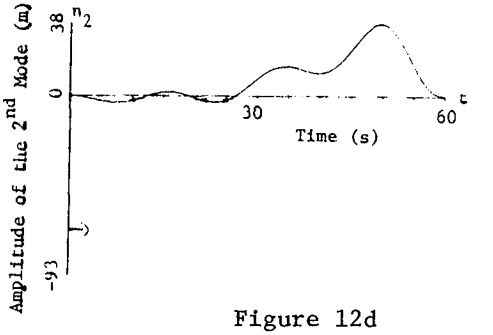

Figure 12d

Figure 12 Case 3N Nonlinear Spinup Maneuver

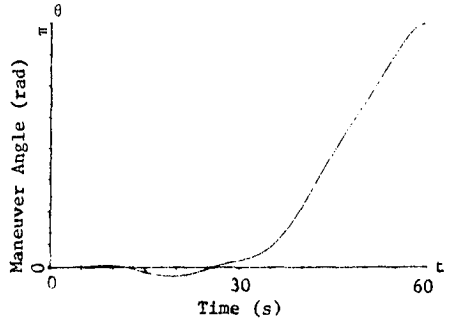

Figure 13a

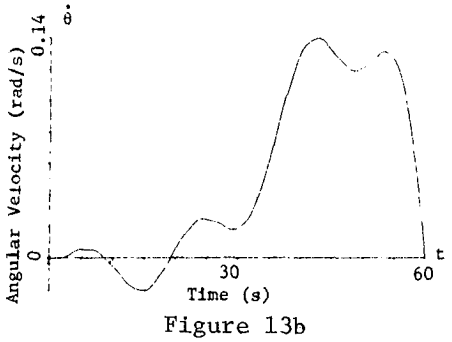

Figure 13b

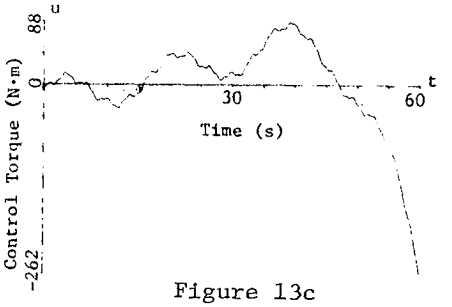

Figure 13c

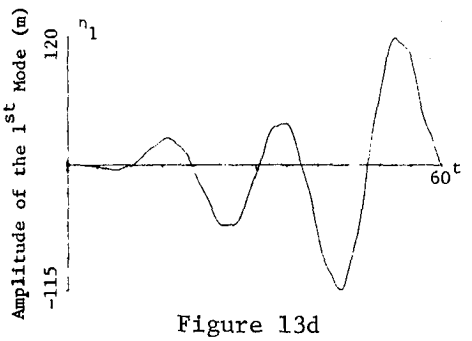

Figure 13d

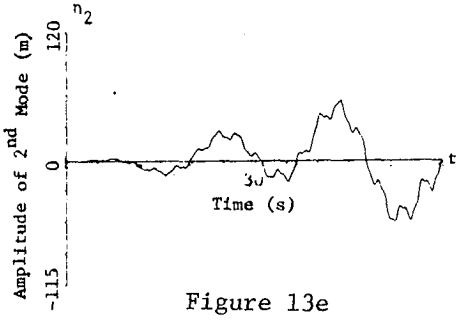

Figure 13e

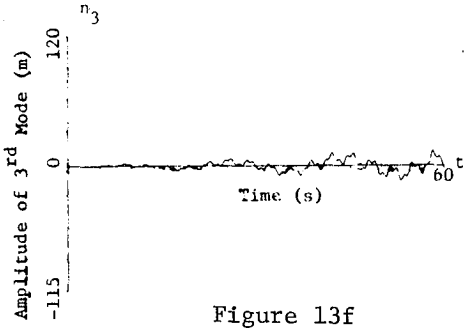

Figure 13f

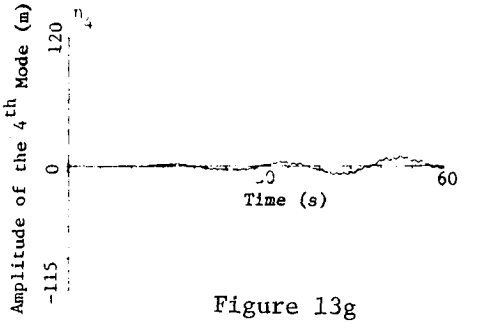

Figure 13g

Figure 13 Case 4L Rest-To-Rest Maneuver
Controlling and Arresting 4 Modes

TABLE 1 DESCRIPTION OF TEST CASE MANEUVERS

Case #	Qualitative Description	# of Modes (N)	θ_o (RAD)	$\dot{\theta}_o$ (RAD/SEC)	θ_f (RAD)	$\dot{\theta}_f$ (RAD/SEC)	W_{uu}	W_{ss}
1	Rigid Appendages Rest-to-Rest Maneuver t_f = 14.221 sec.	0	0	0	0.1	0	1.0	[0]
2L	Linear Kinematics Rest-to-Rest Maneuver $t_f = 2\pi/\omega_1$ = 14.221 sec	1	0	0	0.1	0	1.0	[I]
2N	Nonlinear Kinematics Rest-to-Rest Maneuver $t_f = 2\pi/\omega_1$ = 14.221 sec	1	0	0	0.1	0	1.0	[I]
3L	Linear Kinematics Spinup Maneuver t_f = 60 sec	2	0	0	2π	0.5	1.0	[I]
3N	Nonlinear Kinematics Spinup Maneuver t_f = 60 sec	2	0	0	2π	0.5	1.0	[I]
4L	Linear Kinematics Rest-to-Rest Maneuver t_f = 60 sec	4	0	0	π	0	1.0	[I]
4N	Nonlinear Kinematics Rest-to-Rest Maneuver t_f = 60 sec	4	0	0	π	0	1.0	[I]