STUDY OF SOLUTION PROCEDURES FOR NONLINEAR STRUCTURAL EQUATIONS

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SUMMARY

A method for the reduction of the cost of solution of large nonlinear structural equations was developed. Verification was made using the MARC-STRUC structure finite element program with test cases involving single and multiple degrees-of-freedom for static geometric nonlinearities. The method developed was designed to exist within the envelope of accuracy and convergence characteristic of the particular finite element methodology used.

INTRODUCTION

At present the finite element codes in conjunction with the large, high-speed computers available are capable of producing reasonable solutions to practically all static problems conceivable in structural analysis. In addition, well-behaved problems such as those involving small elastic deformations are solved relatively inexpensively and accurately. Computational difficulties do not arise until the stiffness of the structure becomes a function of displacement and/or displacement history. An opinion widely held is that when this does occur an implicit solution scheme is necessary for accuracy. All implicit schemes require an iterative solution where there is an attempt to reduce some error term to zero at each iteration. Therefore, a nonlinear problem is more expensive to solve and can become astronomically so depending upon the degree of nonlinearity and the convergence criteria used.

In the solution of nonlinear structural equations the reformulation of the stiffness matrix is a first order contribution to the cost. The first logical step in attempting to reduce the cost would be to seek a less expensive way to update the stiffness matrix. This of course has been done with some success and is apparently still being researched. Looking at only the most recent developments or evaluations we see that Mondkar and Powell [1] have used the constant alpha technique to try updating the stiffness matrix for the modified Newton-Raphson approach. Matthies and Strang [2,3] have taken similar approaches born from a paper by Dennis and Moré [4] on Quasi-Newton methods. The basic premise was that the stiffness matrix could be updated without going through the full process of reformulation and decomposition or inversion. The most popular approach was to update the stiffness matrix by a matrix of rank two. This is known as the BFGS (Broyden-Fletcher-Goldfarb-Shanno) update. Crisfield [5] used a method similar to a BFGS update of rank one. All of these papers show conclusive evidence of cost reduction for certain problems. The last by Crisfield is closest in form to the method developed here.

A second logical step is to reduce the number of iterations required to satisfy the convergence criteria. This can be done by determining an estimated displacement as accurately as possible. The development which follows shows how to do this. All forces and loads are of an incremental nature.

DEVELOPMENT

In general, the iterative methods of solution using the stiffness formulation will by some logical means calculate a generalized displacement for a given generalized load. Returning then to the elemental level, the elemental stiffness matrices are altered to reflect this change in shape and the total resistance of the structure to the applied load is determined. If the structure is to be considered in equilibrium, the applied load must be exactly balanced by the resulting resistance of the structure. Any imbalance is termed a residual force and must be considered as an error. An attempt is made to reduce this error by altering the estimated displacement. The rate of convergence depends on the manner of estimated displacement selection.

The vast majority of implicit schemes available utilize only the most recent residual and thereby ignore any possible trend determination. Felippa [6] recognized this and proposed a viable method for determining the displacement that would yield the least residual within specified limitations. This approach required the determination of a weighting matrix that was dependent upon the elements chosen and the applied loads. The development in this paper is independent of the physical characteristics of the elements.

A key element in the success of the approach developed is the finite element method used. As mentioned before the MARC-STRUC structure program was used but the variational formulation of the structural equations was performed according to the method of Jones [7]. It is most important to have the most accurate method possible for the determination of the residuals.

Considering the solution form, in Figure 1 a graph of force versus displacement is shown. The curve represents the calculated resistance of the structure. The original stiffness matrix, $\tilde{K}_o$, assumes linearly elastic deformation and yields the displacement, $\bar{u}_o$, and the residual, $\bar{R}_o$, for the applied load, $\bar{F}$. The displacement, $\bar{u}_o$, and residual, $\bar{R}_o$, are then used in a Quasi-Newton fashion to update the stiffness matrix to $\bar{K}_1$ and a new displacement, $\bar{u}_1$, and consequently a new residual, $\bar{R}_1$, are calculated. Highly accurate answers may result, but they are clearly expensive to obtain.

The extrapolation method presented in this paper is clearly exemplified by the triangle, ACE, shown in Figure 2. The method used was identical to the direct iteration method (shown in Figure 1 earlier) up to and through the calculation of $\bar{u}_o$, $\bar{R}_o$ and $\bar{u}_1$, $\bar{R}_1$. It was the determination of the new estimated displacement, $\bar{u}_2$, that was performed differently. In a one dimensional sense the residuals $\bar{R}_o$, $\bar{R}_1$ and the distance between them, $\bar{d}$, were used to calculate a scalar, ω, that predicted the displacement at which equilibrium supposedly existed under the load, $\bar{F}$. Of course, this was not the equilibrium position and a new residual, $\bar{R}_2$, was determined. The residuals, $\bar{R}_1$, $\bar{R}_2$, and the distance between them, $\omega\bar{d} - \bar{d}$, were then used to predict a new equilibrium position. The process continued until convergence was satisfied.

A major difficulty encountered was the determination of the scalar, ω. In a one dimensional case it was easy enough to see that

$$\omega = \frac{\bar{R}_o}{\bar{R}_o - \bar{R}_1} \qquad (1)$$

However, since in general the vectors $\bar{R}_o$, $\bar{R}_1$ and $\bar{d}$ are heterogeneous in their components' units, a division as mentioned above is impossible even when using vector lengths. The solution to this difficulty came about by considering the units of work. In fact, this extrapolation process may be symbolically thought of as minimizing the work done by the residuals. In this light it was then decided that equating the area of the trapezoid, ABDE, plus the area of the triangle, BCD, to the area of the triangle, ACE, would result in an equation with only one unknown. Simplifying and rearranging, the following was obtained.

$$\omega = \frac{\bar{R}_o \cdot \bar{d}}{(\bar{R}_o - \bar{R}_1) \cdot \bar{d}} \qquad (2)$$

At this point it was decided to implement the theory and test for a one degree-of-freedom case and follow that with a more complex case.

VERIFICATION

In an attempt to determine the validity of the aforementioned extrapolation method it was determined that a one dimensional buckling problem would be appropriate as a first test case. The bar-spring problem of Jones was selected.

Bar-Spring Problem

In Figure 3 the dimensions used on the problem may clearly be seen. The length of the spring was unimportant as long as nonlinear effects did not enter the calculations for the spring's deflection. The bar was modeled so as to allow only a change in length and no bending deformation, hence the absence of an EI term. A load was applied at the end of the bar and spring in the direction of deformation to render the problem one of a purely single dimensional case. The buckling load was at 2.7 kg. (6 lb.) with the results tabularized in Table I. The exact deformation was calculated and plotted in Figure 4 to show the high degree of nonlinearity of the problem.

In analyzing the results (see Table I) it was decided that a comparison of the values calculated against the exact values as well as a comparison of the number of iterations required for each method would be of use. The raised numbers beside the calculated displacements in Table I represent the number of iterations required above the original estimate to reduce the quotient of the calculated displacement and the estimated displacement to the tolerance indicated at the column heading.

It should be noted that at the buckling load the tolerance required to obtain two significant digits accuracy, 1.001, led to a 5 vs. 26 advantage in iterations for the new method. However, the new method was edged by the old in the post-buckled region by a consistent 4 vs. 7 margin. The reason for this was apparently that the linear extrapolation did not follow the changing stiffness of the structure very well. If so, a better approximation would be obtained with a parabolic extrapolation.

On the whole this extrapolation showed promise in this case but not of a clearly decisive nature. Therefore, the motivation for a more complex example was established.

Ring Buckling Problem

This problem was to determine the deflection of a ring under a uniformly loaded external pressure of varying values. The ring was modeled through 90 degrees as shown in Figure 5. The 90 degree arch was broken into two substructure. The degrees of freedom per node were

1. Z
2. R
3. dZ/ds
4. dR/ds

with the rotations positive as shown by Θ in Figure 5. The ring was modeled with a modulus of elasticity of 2.1×10^6 kg/cm^2 (30×10^6 psi) and a radius of 51 cm (20 in.). Finally, a kicker force was applied at node 1 of substructure 1 in the negative R direction with a magnitude of 1.5×10^{-6} kg. (3.4×10^{-6} lb). Obviously, this was simply to force the ring into a buckled mode without altering the deflections due to the pressure loading.

As there was no exact solution other than the known collapse load, the tolerance chosen, 1.001, was that which gave two significant digits accuracy for the bar-spring problem. The results obtained are shown in Table II. The point at which the structure would "collapse" was 4.22 kg/cm^2 (60 psi). As can be seen, the results were quite remarkable as the structure became softer. At 4.18 kg/cm^2 (59.5 psi) the number of iterations reached by the old method were not enough yet to satisfy the tolerance requirement of 1.001. The authors suspect that another 50 to 100 iterations would have been required.

CONCLUDING REMARKS

The problem discussed in this paper was the cost of solution of large nonlinear structural equations. This difficulty has been and is being researched; however, the direction of most present research is apparently concerned with the second partial of the strain energy expression (stiffness matrix). This paper implies and subsequent research by the authors supports the supposition that the first partial of the strain energy expression (resisting force) is not being fully utilized in the determination of the new estimated displacement needed for implicit methods. It may well be determined that updating and/or reformulation of the stiffness matrix is occurring far too often in present solution techniques.

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TABLE 1

BAR-SPRING PROBLEM

Load kg (lb)	Method	Exact (cm)	1.1	1.01	1.001	1.0001	1.00001
1.4 (3.0	Old	0.59781	0.5915^{1}	0.5964^{2}	0.5978^{4}	0.5978^{6}	0.5978^{7}
	New		0.5915^{1}	0.5964^{3}	0.5978^{5}	0.5978^{6}	0.5978^{7}
2.7 (6.0)	Old	2.5400	2.0177^{3}	2.4545^{13}	2.5320^{26}	2.5392^{39}	2.5400^{51}
	New		2.2329^{2}	2.5018^{4}	2.5373^{5}	2.5400^{6}	2.5400^{6}
4.1 (9.0)	Old	4.4821	4.4933^{1}	4.4806^{1}	4.4821^{1}	4.4821^{1}	4.4821^{1}
	New		4.4788^{1}	4.4816^{1}	4.4821^{1}	4.4821^{1}	4.4821^{3}
5.4 (12.0)	Old	5.0800	5.0792^{2}	5.0800^{3}	5.0800^{4}	5.0800^{4}	5.0800^{5}
	New		5.0828^{3}	5.0803^{5}	5.0800^{7}	5.0800^{8}	5.0800^{9}
6.8 (15.0)	Old	5.4907	5.4943^{1}	5.4907^{3}	5.4907^{4}	5.4907^{6}	5.4907^{6}
	New		5.4948^{1}	5.4910^{5}	5.4907^{7}	5.4907^{9}	5.4907^{10}
8.2 (18.0)	Old	5.8143	5.8176^{1}	5.8146^{3}	5.8143^{4}	5.8143^{6}	5.8143^{7}
	New		5.8176^{1}	5.8148^{5}	5.8146^{7}	5.8143^{9}	5.8143^{11}

*Buckling Load

Note: When the number of iterations is less than (2), there is NO difference between the new and old methods.

TABLE II

RING PROBLEM (1.001)

Load kg/cm² (psi)	Method	Iterations	Substructure 1 (cm) Node 1 D.O.F. 2	Substructure 2 (cm) Node 1 D.O.F. 2
.5	Old	2	-.78547 E-03	-.41397 E-03
(7)	New	4	-.78555 E-03	-.41397 E-03
1.5	Old	5	-.25537 E-02	-.10468 E-02
(21)	New	4	-.25545 E-02	-.10459 E-02
2.5	Old	9	-.49439 E-02	-.10607 E-02
(35)	New	4	-.49472 E-02	-.10576 E-02
3.5	Old	23	-.10230 E-01	.18172 E-02
(49)	New	3	-.10237 E-01	.18243 E-02
3.9	Old	54	-.22451 E-01	.12827 E-01
(56)	New	3	-.22597 E-01	.12972 E-01
4.2	Old	149*	-.88354 E-01*	.77978 E-01*
(59.5)	New	4	-.95669 E-01	.85268 E-01

*Maximum number of iterations allowed.
Convergence not yet satisfied.

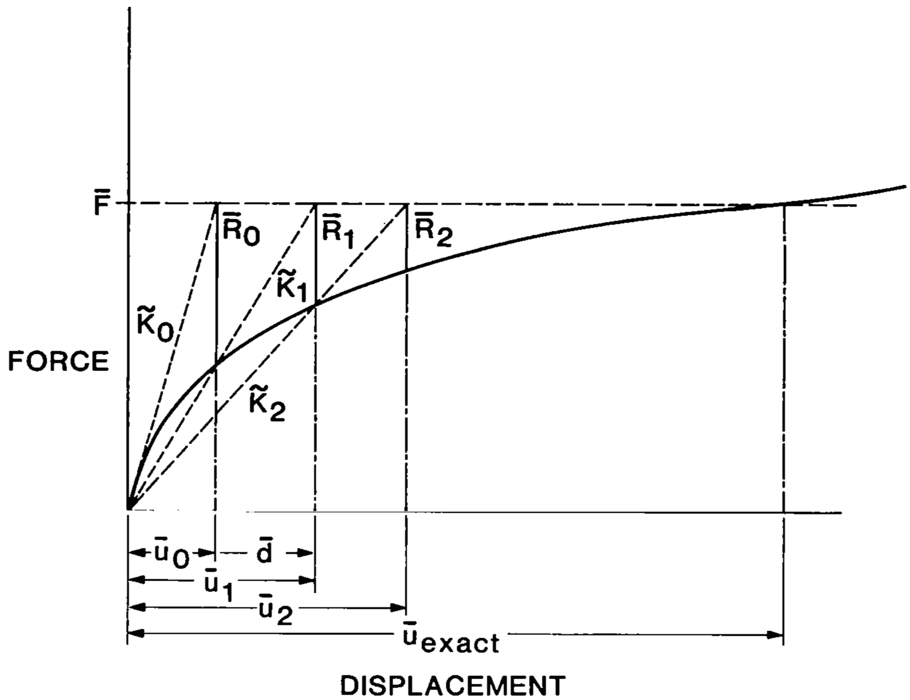

Figure 1.- Direct iteration method.

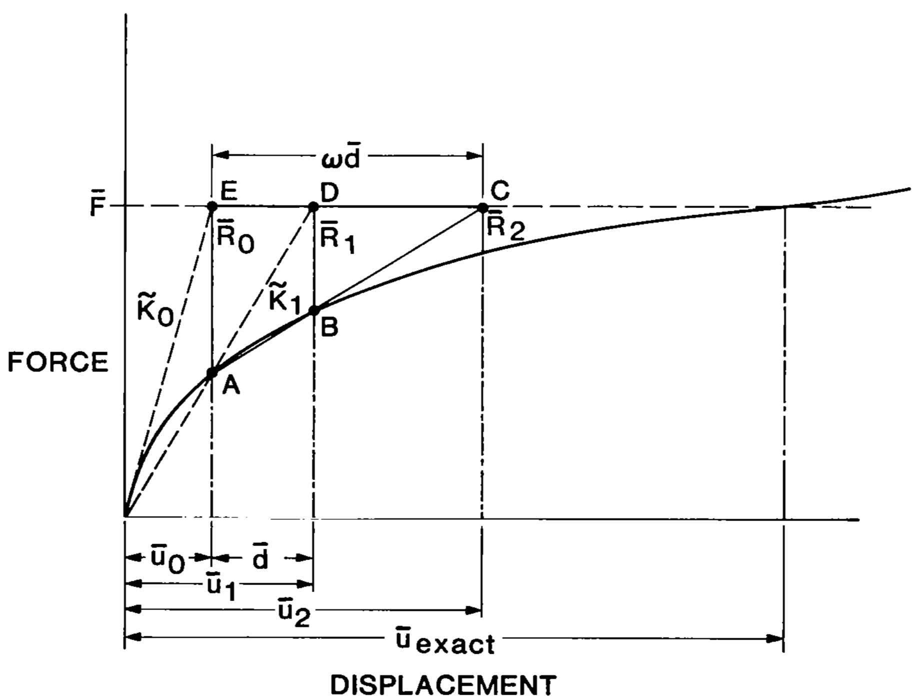

Figure 2.- Direct iteration method with extrapolation modification.

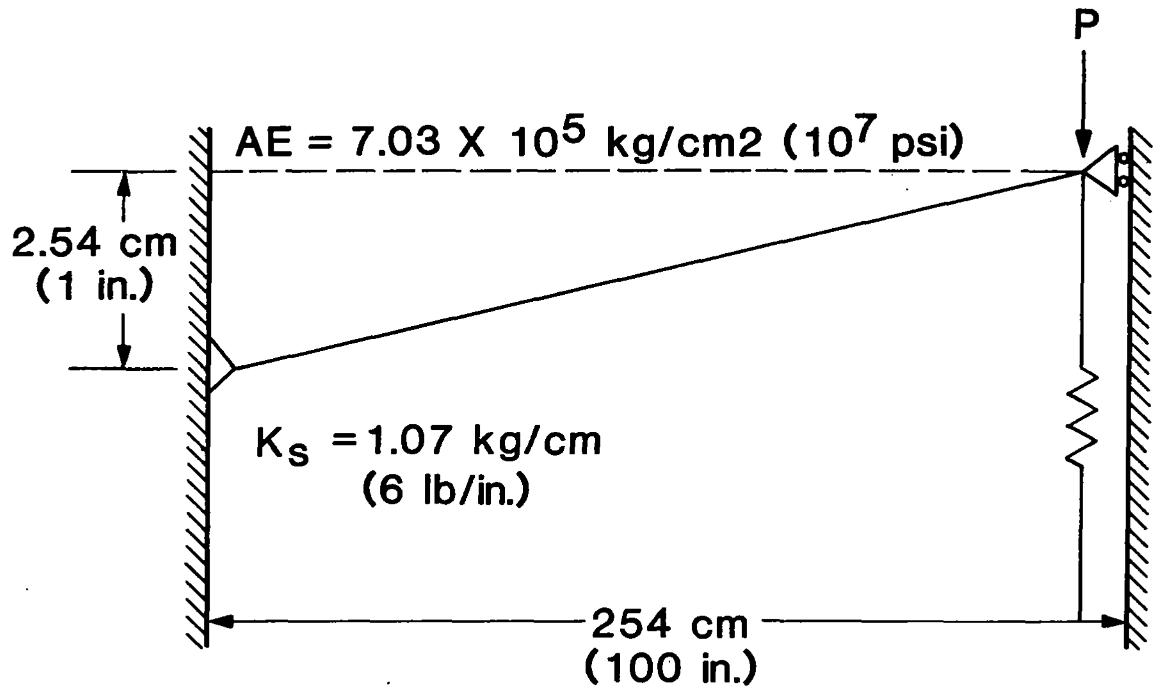

Figure 3.- Bar-spring problem.

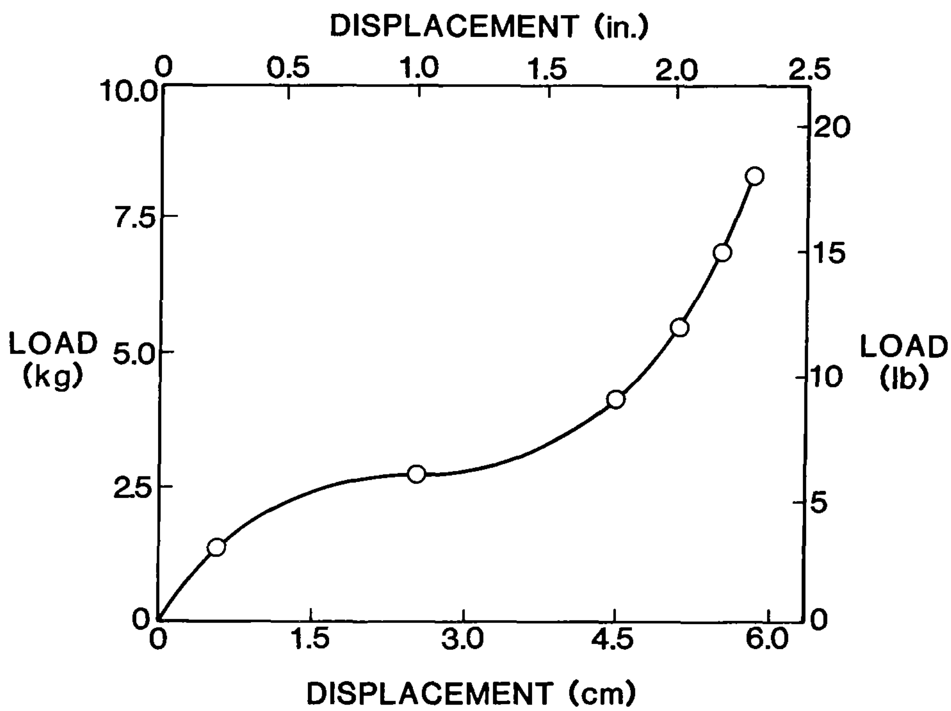

Figure 4.- Exact solution to bar-spring problem.

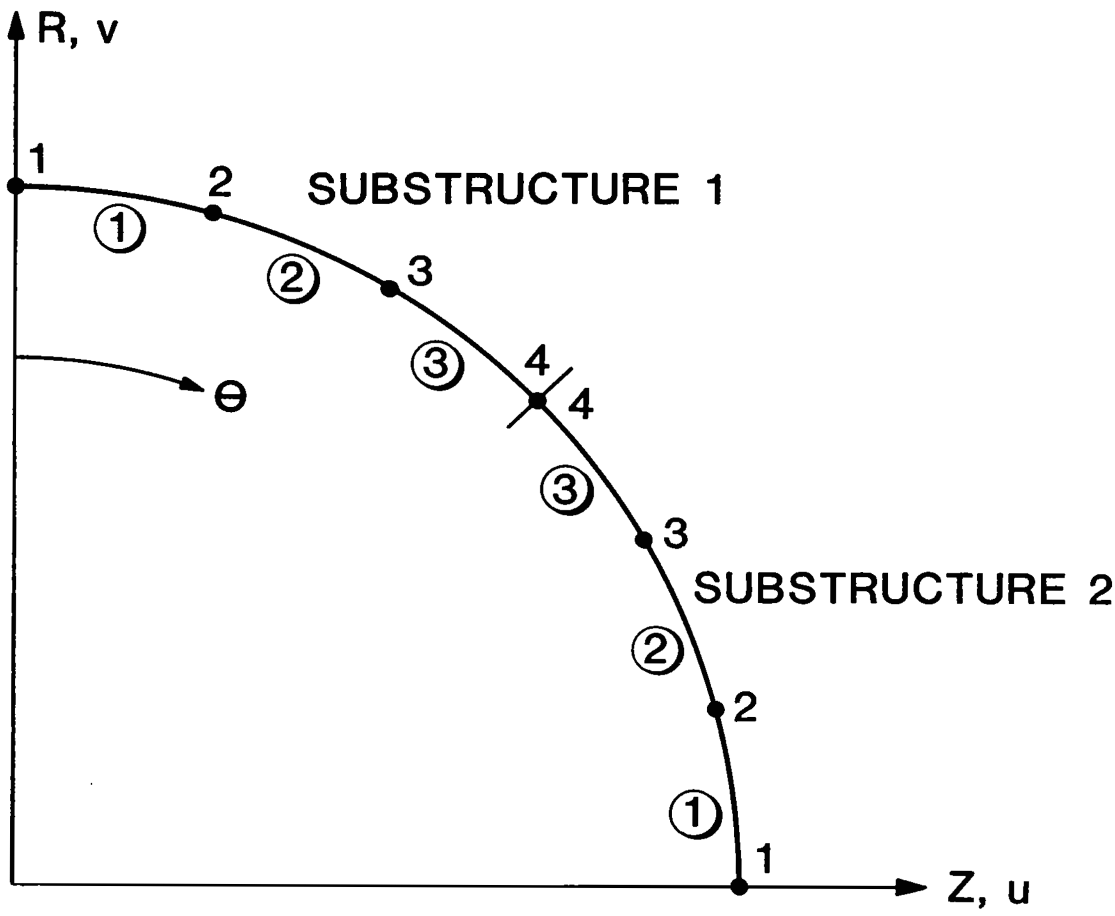

Figure 5.- Ring problem.