NASA Technical Memorandum 81614

Acoustic Transmission Matrix of a Variable Area Duct or Nozzle Carrying a Compressible Subsonic Flow

J. H. Miles
Lewis Research Center
Cleveland, Ohio

Prepared for the
One-hundredth Meeting of the Acoustical Society of America
Los Angeles, California, November 17-21, 1980

NASA

ACOUSTIC TRANSMISSION MATRIX OF A VARIABLE AREA DUCT OR NOZZLE CARRYING A COMPRESSIBLE SUBSONIC FLOW

by J. H. Miles

National Aeronautics and Space Administration

Lewis Research Center

Cleveland, Ohio

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ABSTRACT

The differential equations governing the propagation of sound in a variable area duct or nozzle carrying a one-dimensional subsonic compressible fluid flow are derived and put in state variable form using acoustic pressure and particle velocity as the state variables. The duct or nozzle is divided into a number of regions. The region size is selected so that in each region the Mach number can be assumed constant and the area variation can be approximated by an exponential area variation. Consequently, the state variable equation in each region has constant coefficients. The transmission matrix for each region is obtained by solving the constant coefficient acoustic state variable differential equation. The transmission matrix for the duct or nozzle is the product of the individual transmission matrices of each region. Solutions are presented for several geometries with and without mean flow.

NOMENCLATAURE

$\mathcal{A}$	matrix
A	area
B,C,D	matrix
b,c	constants
c	isentropic speed of sound
f	frequency, Hz

I	unit matrix
i	$(-1)^{1/2}$
k_o	propagation wave number, ω/c_o
M_o	Mach number, u_o/c_o
(MW)	molecular weight
m	$d \ln(A)/dx$
P	transformation matrix
p	pressure
R	gas constant
s	entropy of gas
T	transmission matrix
t	time
u	fluid velocity
W	mass flow
x	Cartesian coordinate
$\vec{Y}$	acoustic state vector
Z	acoustic impedance
Δx	region length
γ	specific heat ratio of gas
ε	$m/2ik_o$
ξ	$k_o/(1 - M_o^2)$
λ	eigenvalue
θ	temperature
ρ_o	gas density
ω	angular frequency, radians/sec

Superscripts and subscripts:

$(\vec{\ })$	vector quantity

$(\bar{\ })$	instantaneous quantity
$(\ )_0$	reference state quantity
$(\ )_1$	perturbed quantity

INTRODUCTION

The use of the transmission matrix approach provides considerable simplification to the algebraic complexities involved in the analysis of acoustic systems which can be modeled as a series of simple regions. The elements of a region transmission matrix are a set of coefficients which depend only on the properties of the region. Furthermore, this set of coefficients relate the acoustic pressure and particle velocity at the inlet of the region to those quantities at the exit of the region. The algebraic complexities are reduced in this case because the transmission matrix of the entire system is the product of the constituent transmission matrices. Consequently, the properties of a system can be calculated by straightforward multiplication using a building block approach based on defining the input-output relationship of each of the regions of a system.

As part of a combustion noise research program conducted at the NASA Lewis Research Center, internal pressure measurements are made in aircraft engines. Because the frequency spectrum of the combustion noise peaks below 1000 Hz and the combustion noise generally propagates in a small area annulus, the plane wave mode of propagation is extremely important. Plane wave combustion noise propagation is a ducted combustion system with flow has been studied by Miles and Raftopoulos (refs. 1 and 2). Miles and Raftopoulos constructed a system model using a transmission matrix approach to the relate acoustic pressure and particle velocity at one point in the system to those quantities at another point. In constructing the model, it was assumed that the duct system consisted of constant area duct regions which could be matched at area discontinuities. Then transmission matrices were calculated for both the constant area duct regions and the area discontinuities using the systems acoustic differential equations.

This approach of using transformation matrices should also be applicable to the study of the propagation of plane wave combustion noise in aircraft engine ducts and nozzles. However, aircraft engines do not have only constant area ducts. Also, by definition nozzles have varying area. Thus, as a first step in constructing an aircraft engine nozzle or duct system model for combustion noise propagation, this paper derives the input-output relationship in the form of a transmission matrix for the propagation of sound in a variable area duct or nozzle carrying a one-dimensional subsonic compressible fluid flow.

BACKGROUND

Many studies of acoustic problems related to the acoustics of aircraft engine-duct systems have been made. Much of the previous work was related to fan noise and this type of work is reviewed in reference 3. While the acoustics of variable area ducts is among the topics reviewed in reference 3, the transmission matrix approach is not discussed in reference 3. This topic was probably omitted because the transmission matrix approach is used to study the plane wave propagation of low frequency noise and is not used for fan noise. However, the application of transmission matrices to ducts without flow is discussed in references 4 to 11. Among these references, note that a brief but comprehensive introduction to the fundamentals of the transmission matrix representation is given by Lampton (ref. 9). As mentioned previously, transmission matrices can be used to model duct systems with flow if the systems consist of different diameter constant area ducts by taking into account the area discontinuities. Methods for dealing with area discontinuities are discussed in references 1 and 12 to 16.

To the author's knowledge, transmission matrices have not been used to model systems containing variable area regions with mean flow. However, if there is no mean flow in a variable area duct, the acoustic equations reduce to the Webster Horn equation for which solutions may be obtained for some specific duct shapes (refs. 27 to 21). Consequently, it is possible to determine exact transmission matrices for these cases.

With a mean flow in a variable area duct, the problem of finding a transmission matrix becomes more complex, since the velocity, density, and Mach number are then dependent on the duct area. Consequently, the acoustic differential equations have variable coefficients. An exact solution to the differential equations governing the propagation of sound in a variable area duct with one-dimensional flow was obtained by Eisenberg and Kao (ref. 22). However, this solution is not applicable to the present problem since it was for specific duct shapes which are semi-infinite and diverge from a sonic throat. Furthermore, the equations are not in transmission matrix form. For ducts having more general shapes, numerical methods can be used to solve the variable-coefficient equations. Solutions of the one-dimensional acoustic differential equations with variable coefficients have been obtained by Davis and Johnson (ref. 23), King and Karamcheti (ref. 24), and Lumsdaine and Ragab (ref. 25) using numerical methods. Again none of these solutions is in transmission matrix form.

Another numerical technique for calculating the behavior of sound is an approximate technique developed by Alfredson (ref. 26) which uses many small discontinuities to define the duct shape. Each discontinuity is then treated using the procedure described in reference 12. The technique described herein also uses many small discontinuities to describe the duct shape. However, the method is based on the infinite exponential horn solution (described in refs. 17 and 18) and includes mean flow effects.

The governing equations used herein have been used by H. S. Tsien (ref. 27), L. Crocco and S. Cheng (ref. 28) to study oscillations in rocket nozzles, and S. Candel (ref. 29) to study entropy noise produced by a nozzle.

In this paper, first, the governing equations are derived by linearization and put into state variable form using acoustic pressure and particle velocity as the state variables. Next, in order to solve the resulting differential equation, the duct or nozzle is divided into regions so that the coefficients of the differential equation can be made constant. The transmission matrix is obtained for each region by solving the acoustic state variable differential equation using methods described by Ogata (ref. 30). The transmission matrix for the variable area duct or nozzle is the product of the transmission matrices for each segment. Last, solutions are presented for several geometries with and without mean flow.

DERIVATION OF STATE VARIABLE EQUATION

The following assumptions are made. The duct or nozzle has an area profile $A(x)$. The flow in the duct or nozzle of an ideal nonviscous, nonconducting perfect gas is described by the following equations of continuity, momentum, energy, and state:

$$\partial(\bar{\rho}A)/\partial t + \partial(\bar{\rho}\bar{u}A)/\partial x = 0 \tag{1}$$

$$\bar{\rho}(\partial\bar{u}/\partial t + \bar{u}\partial\bar{u}/\partial x) + \partial\bar{p}/\partial x = 0 \tag{2}$$

$$(\partial/\partial t + \bar{u}\partial/\partial x)\bar{s} = 0 \tag{3}$$

$$\bar{s} - s_0 = c_v \ln(\bar{p}/\bar{\rho}^{\gamma}) \tag{4}$$

where $\bar{p}$, $\bar{\rho}$, $\bar{u}$, and $\bar{s}$ are, respectively, the instantaneous pressure, density, velocity, and entropy.

The fluctuations in the duct are assumed so small that the flow in the duct is only slightly perturbed. Consequently, the instantaneous quantities can be written in terms of a unperturbed stationary flow quantity designated by o and a small perturbed quantity designated by 1 as follows:

$$\bar{p} = p_o(x) + p_1(t,x) \tag{5}$$

$$\bar{\rho} = \rho_o(x) + \rho_1(t,x) \tag{6}$$

$$\bar{u} = u_o(x) + u_1(t,x) \tag{7}$$

$$\bar{s} = s_o(x) + s_1(t,x) \tag{8}$$

Substituting equations (5) to (8) into equations (1) to (4) yields the following zeroth order system of equations:

$$d(\rho_o u_o A)/dx = 0 \tag{9}$$

$$\rho_o u_o \; du_o dx + dp_o/dx = 0 \tag{10}$$

$$s_o = c_v \; \ln(p_o/\rho_o^\gamma) = \text{constant} \tag{11}$$

and the first order system of equations:

$$A \; \partial\rho_1/\partial t + u_o A \; \partial\rho_1/\partial x + A\rho_o \; \partial u_1/\partial x + \rho_1(u_o \; dA/dx + A \; du_o/dx)$$
$$+ u_1(\rho_o \; dA/dx + A \; d\rho_o/dx) = 0 \tag{12}$$

$$\rho_o \; \partial_1/\partial t + \rho_o u_1 \; du_o/dx + \rho_o u_o \; \partial u_1/\partial x + \rho_1 u_o \; du_o/dx + \partial p_1/\partial x = 0 \tag{13}$$

$$(\partial/\partial t + u_o \; \partial/\partial x)s_1/c_v = (\partial/\partial t + u_o \; \partial/\partial x)(p_1/p_o - \gamma\rho_1/\rho_o) = 0 \tag{14}$$

For the system under consideration the gas equation is

$$\bar{p} = \bar{\rho}\mathcal{R}\bar{\theta}/(MW) \tag{15}$$

and the isentropic speed of sound is

$$c_o^2 = \gamma\mathcal{R}\theta_o/(MW) = \gamma p_o/\rho_o \tag{16}$$

From equation (14) the density perturbation is related to a pressure perturbation by

$$\rho_1 = \rho_o p_1/\gamma p_o = p_1/c_o^2 \tag{17}$$

Substituting equation (17) into equations (12) and (13) yields the following equations in matrix form:

$$\begin{bmatrix} 1/\rho_o c_o^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \partial p_1/\partial t \\ \partial u_1/\partial t \end{pmatrix} + \begin{bmatrix} u_o/\rho_o c_o^2 & 1 \\ 1/\rho_o & u_o \end{bmatrix} \begin{pmatrix} \partial p_1/\partial x \\ \partial u_1/\partial x \end{pmatrix}$$

$$\begin{bmatrix} \dfrac{u_o}{\rho_o c_o^2}\left(\dfrac{d\ \ln(A)}{dx} + \dfrac{d\ \ln(u_o)}{dx}\right) & \left(\dfrac{d\ \ln(A)}{dx} + \dfrac{d\ \ln(\rho_o)}{dx}\right) \\ \dfrac{u_o^2}{\rho_o c_o^2}\ \dfrac{d\ \ln(u_o)}{dx} & \dfrac{du_o}{dx} \end{bmatrix} \begin{pmatrix} p_1 \\ u_1 \end{pmatrix} = 0 \qquad (18)$$

Taking the Fourier transform of equation (18) and solving for dY/dx yields

$$d\vec{Y}/dx = B\vec{Y} \qquad (19)$$

Using the zeroth order system of equations (9) to (11) and equation (16) to relate the gradients of the logarithmical variation of velocity and density to the gradient of the logarithmical variation of area yields

$$\frac{d\ \ln(u_o)}{dx} = -\frac{1}{(1 - M_o^2)}\frac{d\ \ln(A)}{dx} \qquad (20)$$

$$\frac{d\ \ln(\rho_o)}{dx} = \frac{M_o^2}{(1 - M_o^2)}\frac{d\ \ln(A)}{dx} \qquad (21)$$

Using equations (20) and (21) the elements of the B matrix in equation (19) are

$$B_{11} = \frac{M_o^2}{(1 - M_o^2)} \frac{d\ \ln(A)}{dx} + \frac{M_o(-ik_o)}{(1 - M_o^2)} \tag{22}$$

$$B_{12} = \frac{\rho_o c_o(ik_o)}{(1 - M_o^2)} + \frac{\rho_o c_o M_o^2}{\left(1 - M_o^2\right)^2} \frac{d\ \ln(A)}{dx} \tag{23}$$

$$B_{21} = \frac{ik_o}{\rho_o c_o(1 - M_o^2)} \tag{24}$$

$$B_{22} = -\frac{(1 + M_o^2)}{\left(1 - M_o^2\right)^2} \frac{d\ \ln(A)}{dx} + \frac{M_o(-ik_o)}{(1 - M_o^2)} \tag{25}$$

The acoustic state variable differential equation is given by equation (19). This equation will now be solved to obtain the duct transmission matrix.

METHOD OF SOLUTION

In the previous section a general acoustic state variable differential equation was derived (eq. (19)). In this section an approximate solution to equation (19) is obtained. The first step in solving equation (19) is to divide the duct or nozzle into a number of regions or subsections. The region size is selected so that the Mach number can be assumed constant in each region and the area variation in each region can be approximated by an exponential area variation. Consequently, in any given region

$$M_o = \text{constant} \tag{26}$$

$$A = A_o e^{mx} \tag{27}$$

With these assumptions the Mach number factors in the B matrix in this region are constant. Also, the value of $d(\ln A)/dx$ is given by

$$\frac{d(\ln(A))}{dx} = m \tag{28}$$

The B matrix is now independent of x in the region where it is to be evaluated. Thus, the solution to equation (19) for the j^{th} region is

$$\vec{Y}(x_j + \Delta x) = (e^{B\Delta x})_j Y(x_j) \tag{29}$$

where Δx is the length of the j^{th} region and $\exp(B\Delta x)$ is known as the matrix exponential. The value of $\exp(B\Delta x)$ is defined by

$$e^{B\Delta x} = I + (B\Delta x) + \frac{(B\Delta x)^2}{2!} + \frac{(B\Delta x)^3}{3!} + \ldots \tag{30a}$$

The matrix $\exp(B\Delta x)$ is the transmission matrix of the j^{th} region, $(T)_j$, so that

$$(T)_j = e^{(B\Delta x)_j} = \sum_{\ell=0}^{\infty} \frac{(B\Delta x)_j^{\ell}}{\ell!} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_j \tag{30b}$$

Consequently, the transmission matrix, T, for the variable area duct is found from

$$\vec{Y}(x)\Big|_{x=L} = \prod_{j=1}^{N} (e^{B\Delta x})_j \vec{Y}(x)\Big|_{x=0} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \vec{Y}(x)\Big|_{x=0} = T\vec{Y}(x)\Big|_{x=0} \tag{31}$$

where

$$L = \sum_{j=1}^{N} (\Delta x)_j \tag{32}$$

and N is the number of duct regions.

In this section an approximate solution to the state differential equation was obtained. In the next section computational techniques useful for obtaining analytical and numerical solutions will be discussed.

EVALUTION OF MATRIX EXPONENTIAL

The solution of the matrix space differential equation thus involves the matrix exponential $\exp(B\Delta x)$. It is necessary to express $\exp(B\Delta x)$ as a matrix in order to obtain numerical values for the transmission matrix. Methods for evaluating a matrix exponential are discussed by Ogata in chapter 6 of reference 30. These same methods can be used for evaluating $\exp(B\Delta x)$. The first method discussed produces a solution which can be evaluated on a digital computer using the built-in complex exponential function. It is also useful in generating analytical solutions in terms of exponential and trigonometric functions. The second method that will be discussed uses equation (30) to numerically obtain a value of (T) by evaluating the exponential power series in $(B\Delta x)$.

The first method is to transform the B matrix into a diagonal matrix D using a transformation matrix P. If the eigenvectors of the B matrix are distinct, then

$$e^{B\Delta x} = C^{-1}Pe^{(P^{-1}CBC^{-1}P)\Delta x}P^{-1}C \tag{33}$$

where

$$C = \begin{bmatrix} 1 & 0 \\ 0 & \rho_0 c_0 \end{bmatrix} \tag{34}$$

This method is implemented by first calculating

$$\mathcal{A} = CBC^{-1} \tag{35}$$

so that

$$\begin{aligned} \mathcal{A}_{11} &= B_{11} \\ \mathcal{A}_{12} &= B_{12}/\rho_o c_o \\ \mathcal{A}_{21} &= B_{21}\rho_o c_o \\ \mathcal{A}_{22} &= B_{22} \end{aligned} \tag{36}$$

Next, the eigenvalues of the $\mathcal{A}$ matrix, λ, are found by setting the determinant of the matrix $(\mathcal{A} - \lambda I)$ equal to zero. Consequently, the eigenvalues are the roots of the equation

$$\lambda^2 + b\lambda + c = 0 \tag{37}$$

where

$$b = -(\mathcal{A}_{11} + \mathcal{A}_{22}) \tag{38}$$

and

$$c = \mathcal{A}_{11}\mathcal{A}_{22} - \mathcal{A}_{12}\mathcal{A}_{21} \tag{39}$$

Next, the P transformation matrix is calculated using

$$P = \begin{bmatrix} \mathcal{A}_{22} - \lambda_1 & \mathcal{A}_{22} - \lambda_2 \\ -\mathcal{A}_{21} & -\mathcal{A}_{21} \end{bmatrix} \tag{40}$$

or

$$P = \begin{bmatrix} -\mathscr{A}_{21} & -\mathscr{A}_{21} \\ \mathscr{A}_{11} - \lambda_1 & \mathscr{A}_{11} - \lambda_2 \end{bmatrix} \tag{41}$$

Note that if the P matrix defined by either equation (40) or (41) is multiplied by a constant the result is also a P matrix. Consequently, it is possible to significantly simplify the P matrix generated by equation (40) or (44). The diagonal D matrix is then

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P^{-1}\mathscr{A}P \tag{42}$$

Consequently, the square matrix, exp(DΔx) is given by

$$e^{D\Delta x} = \begin{bmatrix} e^{\lambda_1 \Delta x} & 0 \\ 0 & e^{\lambda_2 \Delta x} \end{bmatrix} \tag{43}$$

The value of the square matrix exp(BΔx) is then given by

$$T = e^{B\Delta x} = C^{-1}Pe^{D\Delta x}P^{-1}C = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{44}$$

Note that the elements of the matrix contains only complex exponential functions from equation (43).

As mentioned previously, the second method for evaluating exp(BΔx) is based on expanding exp(BΔx) in the exponential power series in (BΔx) given by equation (30). Equation (30) is also important in obtaining analytical solutions. For instance, using equation (30) it is possible to prove equation (43). The virtue of equation (30) is its simplicity. However, in writing a digital computer program based on equation (30) convergence requirements must be specified and this may increase the running time of a computer program evaluating $\exp(B\Delta x)$.

In this section two procedures for implementing the solution were presented. Some analytical solutions will be presented next. These solutions are presented to clarify the computational techniques and to provide some insight into the numerical results.

ANALYTICAL SOLUTIONS

In this section analytical solutions for the input-output relationship of the acoustic pressure and the particle velocity in the form of a transmission matrix will be obtained using the methods described in the previous section. The solutions are exact since only cases where the Mach number is zero or constant are considered. The geometries considered are a duct with constant area and a duct with an exponential area variation. These solutions are presented to clarify the numerical results presented in the next section and the methods used to obtain them.

Constant Area Duct Without Flow

First, consider a constant area duct of length L with no flow. For this case $M = 0$ and $d\ln(A)/dx = 0$. Consequently, substituting these values of M and $d\ln(A)/dx$ into equations (22) to (25) yields the following B matrix

$$B = \begin{bmatrix} 0 & \rho_o c_o i k_o \\ i k_o/\rho_o c_o & 0 \end{bmatrix} \tag{45}$$

Substituting equations (45) and (32) into equation (31) yields

$$T = \prod_J (e^{B\Delta x})_J = e^{B\sum_J (\Delta x)_J} = e^{BL} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{46}$$

since the B matrix is independent of the Mach number. Using equation (30) to evaluate equation (46) yields

$$T = \begin{bmatrix} \left\{1 - \frac{(k_oL)^2}{2!} + \frac{(k_oL)^4}{4!} \ldots\right\} & i\rho_o c_o \left\{(k_oL) - \frac{(k_oL)^3}{3!} + \frac{(k_oL)^5}{5!} \ldots\right\} \\ \frac{i}{\rho_o c_o}\left\{(k_oL) - \frac{(k_oL)^3}{3!} + \frac{(k_oL)^5}{5!} \ldots\right\} & \left\{1 - \frac{(k_oL)^2}{2!} + \frac{(k_oL)^4}{4!} \ldots\right\} \end{bmatrix} \tag{47}$$

Substituting the series definitions of the trigonometric functions, it follows that

$$T = \begin{bmatrix} \cos(k_oL) & i\rho_o c_o \sin(k_oL) \\ \frac{i \sin(k_oL)}{\rho_o c_o} & \cos(k_oL) \end{bmatrix} \tag{48}$$

Constant Area Duct With Mean Flow

Next, consider a constant area duct of length L with mean flow. Since the duct area is constant again $d \ln(A)/dx = 0$. Using equations (22) to (25) the B matrix is given by

$$B = \begin{bmatrix} \frac{M_o}{1 - M_o^2}(-ik_o) & \frac{\rho_o c_o (ik_o)}{1 - M_o^2} \\ \frac{(ik_o)}{\rho_o c_o (1 - M_o^2)} & \frac{M_o(-ik_o)}{1 - M_o^2} \end{bmatrix} \tag{49}$$

Substituting equation (49) into equation (31) shows that the transmission matrix is again given by equation (46). For this case the value for exp(BL) will be evaluated using equation (33).

First, the $\mathscr{A}$ matrix is calculated from equation (35). Next, the eigenvalues are found using equations (37) to (39). Thus,

$$\left. \begin{aligned} \lambda_1 &= (ik_o)/(1 + M_o) \\ \lambda_2 &= (-ik_o)/(1 - M_o) \end{aligned} \right\} \tag{50}$$

Based on equation (40) a satisfactory P matrix is

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{51}$$

The diagonal D matrix calculated by substituting equations (34), (35), (40), and (51) into equation (42) is

$$D = \begin{bmatrix} \frac{ik_o L}{1 + M_o} & 0 \\ 0 & -\frac{ik_o L}{1 - M_o} \end{bmatrix} \tag{52}$$

The transmission matrix calculated by substituting equations (34), (51), and (52) into equation (44) is

$$T = \begin{bmatrix} \frac{\left(e^{\lambda_1 L} + e^{\lambda_2 L}\right)}{2} & \frac{\left(e^{\lambda_1 L} - e^{\lambda_2 L}\right)}{2} \\ \frac{\left(e^{\lambda_1 L} - e^{\lambda_2 L}\right)}{2} & \frac{\left(e^{\lambda_1 L} + e^{\lambda_2 L}\right)}{2} \end{bmatrix} \quad (53)$$

or using the complex definition of the sine and cosine functions

$$T = e^{-i\xi M_o L} \begin{bmatrix} \cos(\xi L) & i\rho_o c_o \sin(\xi L) \\ \frac{i \sin(\xi L)}{\rho_o c_o} & \cos(\xi L) \end{bmatrix} \quad (54)$$

where $\xi = k_o/(1 - M_o^2)$.

Note that for $M = 0$, the transmission matrix given by equation (54) reduces to that given by equation (48).

Variable Area Duct Without Flow

The second geometry discussed in this section is a duct of length L having an exponential area variation

$$A_L = A_o e^{mL} \quad (55)$$

First, the zero frequency limit transmission matrix for no flow will be calculated. For this case $M = 0$, $k = 0$, and $d \ln(A)/dx = m$. Consequently, from equations (22) to (25) the B matrix is given by

$$B = \begin{bmatrix} 0 & 0 \\ 0 & -m \end{bmatrix} \tag{56}$$

Substituting equation (56) into equation (31) again yields equation (46). For this case, equation (46) can be evaluated using equation (30). Again substituting for the power series obtained yields a closed form solution as follows

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{-mL} \end{bmatrix} \tag{57}$$

Substitution of equation (55) into equation (57) yields

$$T = \begin{bmatrix} 1 & 0 \\ 0 & A_o/A_L \end{bmatrix} \tag{58}$$

The corresponding set of simultaneous equation is then

$$p(i\omega,L) = p(i\omega,o) \tag{59}$$

$$A_L u(i\omega,L) = A_o u(i\omega,o) \tag{60}$$

Consequently, at low frequencies for no flow the acoustic pressure is the same at both ends of the duct and the acoustic volume flow is constant.

Next, the general transmission matrix for the exponential duct without flow will be calculated. For this case $d\ \ln(A)/dx = m$. Consequently, from equations (22) to (25) the B matrix is given by

$$B = \begin{bmatrix} 0 & ik_o\rho_o c_o \\ \frac{ik_o}{\rho_o c_o} & -m \end{bmatrix} \tag{61}$$

Substituting equation (61) into equation (31) shows that the transmission matrix is given by equation (46). For this case, the value of exp(BL) will be evaluated using equation (33). The eigenvalues found using equations (37) to (39) are

$$\left.\begin{aligned} \lambda_1 &= ik_o\left(-\varepsilon + \sqrt{\varepsilon^2 + 1}\right) \\ \lambda_2 &= -ik_o\left(\varepsilon + \sqrt{\varepsilon^2 + 1}\right) \end{aligned}\right\} \tag{62}$$

where

$$\varepsilon = m/2ik_o \tag{63}$$

Based on equation (40) a $\mathscr{P}$ matrix is given by

$$P = \begin{bmatrix} \varepsilon + \sqrt{\varepsilon^2 + 1} & \varepsilon - \sqrt{\varepsilon^2 + 1} \\ 1 & 1 \end{bmatrix} \tag{64}$$

The diagonal D matrix calculated by substituting equation (64) into equation (42) and using equation (35) to determine the $\mathscr{A}$ matrix is

$$D = \begin{bmatrix} ik_o\left(-\varepsilon + \sqrt{\varepsilon^2 + 1}\right) & 0 \\ 0 & -ik\left(\varepsilon + \sqrt{\varepsilon^2 + 1}\right) \end{bmatrix} \tag{65}$$

The transmission matrix calculated by substituting equations (34), (63), and (65) into equation (44) is

$$T = \begin{bmatrix} \dfrac{\epsilon\left(e^{\lambda_1 L} - e^{\lambda_2 L}\right) + \sqrt{1+\epsilon^2}\left(e^{\lambda_1 L} + e^{\lambda_2 L}\right)}{2\sqrt{1+\epsilon^2}} & \dfrac{\rho_0 c_0\left(e^{\lambda_1 L} - e^{\lambda_2 L}\right)}{2\sqrt{1+\epsilon^2}} \\ \dfrac{\left(e^{\lambda_1 L} - e^{\lambda_2 L}\right)}{2\rho_0 c_0\sqrt{1+\epsilon^2}} & \dfrac{-\epsilon\left(e^{\lambda_1 L} - e^{\lambda_2 L}\right) + \sqrt{1+\epsilon^2}\left(e^{\lambda_1 L} + e^{\lambda_2 L}\right)}{2\sqrt{1+\epsilon^2}} \end{bmatrix} \tag{66}$$

or

$$T = e^{-ik_0\epsilon L} \begin{bmatrix} \left\{\dfrac{i\epsilon \sin\left(k_0 L\sqrt{1+\epsilon^2}\right)}{\sqrt{1+\epsilon^2}} + \cos\left(k_0 L\sqrt{1+\epsilon^2}\right)\right\} & \dfrac{\rho_0 c_0 i \sin\left(k_0 L\sqrt{1+\epsilon^2}\right)}{\sqrt{1+\epsilon^2}} \\ \dfrac{i \sin\left(k_0 L\sqrt{1+\epsilon^2}\right)}{\rho_0 c_0\sqrt{1+\epsilon^2}} & \left\{\dfrac{-i\epsilon \sin\left(k_0 L\sqrt{1+\epsilon^2}\right)}{\sqrt{1+\epsilon^2}} + \cos\left(k_0 L\sqrt{1+\epsilon^2}\right)\right\} \end{bmatrix} \tag{67}$$

Note that for a constant area duct $m = 0$ and thus $\varepsilon = 0$. The transmission matrix given by equation (67) for this case reduces to the transmission matrix given by equation (48).

Also note that for the low frequency case, the eigenvalues are given by

$$\left.\begin{aligned} \lim_{k \to 0} \lambda_1 &= 0 \\ \lim_{k \to 0} \lambda_2 &= -m \end{aligned}\right\} \qquad (68)$$

Substituting these eignevalues into the transmission matrix given by equation (66) and letting k go to zero, yields the transmission matrix given by equation (57).

In this section analytical solutions to equation (31) were found for the case where the Mach number in equations (22) to (25) is zero or constant and equation (31) yields an exact solution. It was shown that using equation (33) to obtain a solution for $\exp(BL)$ yields a solution in terms of exponential or trigonometric functions. It was also demonstrated that while equation (30) can be used to obtain an analytical solution, it is only useful in the most simple cases where the resulting power series recognizable. The next section discusses numerical solutions.

COMPUTATIONAL EXAMPLES

Numerical calculations of transmission matrices for several duct configurations with and without flow were made by evaluating equation (31). The calculations yield exact solutions for cases where the Mach number is zero or constant. Thus for any of the cases discussed above the numerical solutions are identical to the corresponding analytical solution.

When the Mach number is not constant or zero the numerical calculations yield an approximate solution. The approximation is improved as the duct is divided into smaller subsections. Typical results obtained by dividing the duct configuration into 500 equal length subsections are presented next. For all solutions, the zeroth order system of equations is solved first to evaluate p_0, u_0, θ_0, and ρ_0 at the start of each subsection using a stagnation pressure of one atmosphere and a stagnation temperature of 295 K. The ratio of gas specific heats was 1.4.

The numerical results are presented as plots of the magnitude and phase angle of each of the matrix elements versus frequency.

Constant Area Duct

Results obtained for a one-meter-long cylindrical duct will be discussed first. Since in a cylindrical duct the Mach number is zero or constant, the numerical solution is identical to the analytical solution. Transmission matrices for the one-meter-long constant area duct calculated for $M = 0$, $M = 0.25$, and $M = 0.6$ are shown in figure 1. The solid curves in figure 1 correspond to solutions obtained using equation (48). The dashed curves correspond to solutions obtained using equation (54). The curves are functions of $\xi L = (2\pi f/c_0)L/(1 - M_0^2)$. Consequently, the curves appear to contract with increasing Mach number. This is because the frequency location of corresponding peaks and dips which occur at constant ξL values must shift to lower frequencies as the Mach number is increased. In addition, as the Mach number increases so does the phase factor $\exp(-i\xi M_0 L)$ which multiplies each transmission matrix element.

The ratio of the pressure at $x = 1.0$ to the pressure at $x = 0$ with zero acoustic particle velocity at $x = 0$, T_{11}, is equal to $\exp(-i\xi M_0 L)\cos(\xi L)$ which is plotted in figure 1(a). The ratio of the acoustic particle velocity at $x = 1.0$ to the acoustic particle velocity at $x = 0$ with zero acoustic pressure at $x = 0$, T_{22}, is shown in figure 1(d) and is also equal to $\exp(-i\xi M_0 L)\cos(\xi L)$.

The ratio of the pressure at $x = 1.0$ to the acoustic particle velocity at $x = 0$ with zero acoustic pressure at $x = 0$, T_{12}, is $\rho_0 c_0 \exp(i(\pi/2 - \xi M_0 L))\sin(\xi L)$ which is shown in figure 1(b). The ratio of the acoustic particle velocity at $x = 1.0$ to the acoustic pressure at $x = 0$ with zero acoustic particle velocity at $x = 0$, T_{21}, is $\exp(i(\pi/2 - \xi M_0 L)\sin(\xi L)/\rho_0 c_0$ which is shown in figure 1(c).

Variable Area Ducts

Results obtained for one-meter-long ducts having exponential area variation will be discussed next. Two examples are presented. One example which is identified as the long exponential nozzle (LEN) has an area which decreases exponentially from 0.02 to 0.01 m ($d \ln(A)/dx = m_{LEN} = \ln(1/2)$). The transmission matrix for this geometry calculated for mass flows of 0 and 2.0 kg/sec are shown in figure 2. The second example which is identified as the long exponential horn (LEH) has an area which increases exponentially from 0.01 to 0.02 m ($d \ln(A)/dx = m_{LEH} = \ln(2)$). The transmission matrix for this geometry calculated for mass flows of 0 and 2.0 kg/sec is shown in figure 3.

Without flow. - First, the no-flow cases will be discussed. At low frequencies, the analytical solution given by equation (58) for the long exponential nozzle transmission matrix, T_{LEN}, is

$$T_{LEN}(f = 0) = \begin{bmatrix} 1 & 0 \\ 0 & (A_0/A_L)_{LEN} \end{bmatrix} \qquad (69)$$

where $\exp(-m_{LEN}L) = (A_0/A_L)_{LEN} = 2.0$. Furthermore, at low frequencies, the analytical solution given by equation (58) for the long exponential horn transmission matrix, T, is

$$T_{LEH}\ (f = 0) = \begin{bmatrix} 1 & 0 \\ 0 & (A_0/A_H)_{LEH} \end{bmatrix} \tag{70}$$

where $\exp(-m_{LEH}L) = (A_0/A_L)_{LEH} = 1/2$.

In addition, at frequencies greater than $|mc_0/4\pi|$ the analytical solution given by equation (67) for T_{LEN} is

$$T_{LEN} = \left(\frac{A_0}{A_L}\right)^{1/2}_{LEN} \begin{bmatrix} \cos(k_0L) & i\rho_0c_0 \sin(k_0L) \\ \dfrac{i \sin(k_0L)}{\rho_0c_0} & \cos(k_0L) \end{bmatrix} \tag{71}$$

where $(A_0/A_L)_{LEN} = 2$. Moreover, at frequencies greater than $|mc_0/4\pi|$ the analytical solution given by equation (67) for T_{LEH} is

$$T_{LEH} = \left(\frac{A_0}{A_L}\right)^{1/2}_{LEH} \begin{bmatrix} \cos(k_0L) & i\rho_0c_0 \sin(k_0L) \\ \dfrac{i \sin(k_0L)}{\rho_0c_0} & \cos(k_0L) \end{bmatrix} \tag{72}$$

where $(A_0/A_L)_{LEH} = 1/2$. The results of the numerical calculations shown in figures 2 and 3, for the case without flow are identical with these analytic results. These few examples show that the numerical calculations are in agreement with the analytical solutions.

With flow. - The numerically calculated transmission matrix for the long exponential nozzle and horn with a mass flow of 2.0 kg/sec shown in figures 2 and 3 will be discussed next.

As expected from the results presented in figure 1 the frequency location of the peaks and dips shifted to lower frequencies. Also as expected, the phase angle is significantly different for this case. However, the types of changes in the magnitude and phase angle of the elements of the long exponential nozzle and horn transmission matrix is unexpected. The interaction of the flow and the gradient of the logarithmical area variation change the transition matrix elements significantly. For instance, the low frequency value of $(T_{22})_{LEN}$ shifts from 2 with no flow to 3.3 with a mass flow of 2 kg/sec. Also, the low frequency value of $(T_{22})_{LEH}$ shifts from 1/2 with no flow to 0.295 with a flow of 2.0 kg/sec.

The numerical method can be applied to any type of area variation. The transmission matrix for a short exponentially contracting duct and a linearally contracting duct, both 20 cm long, are compared in figure 4. Below 50 Hz the transmission matrices are similar. Above 50 Hz the magnitudes do not agree. However, the phase angle differences become significant only above 400 Hz.

CONCLUDING REMARKS

A numerical method for determining the transmission matrix of a variable area duct or nozzle carrying a compressible subsonic mean flow was presented. The method yields an exact solution for the zero mean flow case and an approximate solution for the nonzero mean flow case. However, in using the numerical method the duct or nozzle is divided into regions in a way that makes the method more accurate as the number of regions increases and the region size decreases. Consequently, the accuracy of the method is only limited by the region size.

Furthermore, for the no-flow case not only does the numerical solution become exact, but it also becomes possible to use the equations which describe it to generate an analytical solution in terms of exponential and trigonometric functions.

The method was applied to several geometries with and without flow, including both expanding and contracting duct sections. For the cases without flow analytical solutions were obtained.

The numerical method may also be useful in studying sound propagation in ducts without a mean flow having shapes for which the Webster Horn equation does not have an exact solution. Furthermore, the method is easy to apply. Therefore, even for ducts with shapes for which the Webster Horn equation does have an exact solution, the method may be useful where a numerical solution is adequate.

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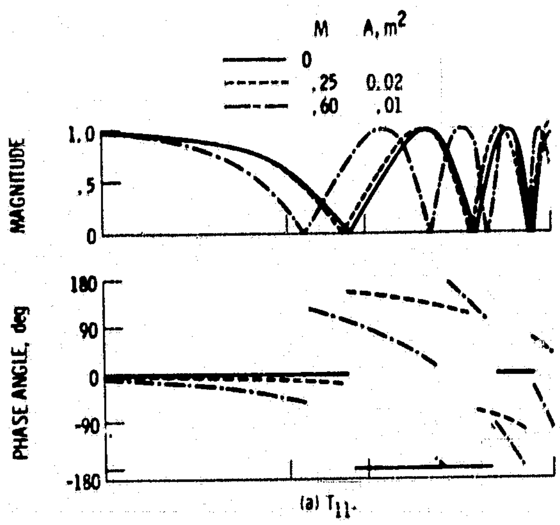

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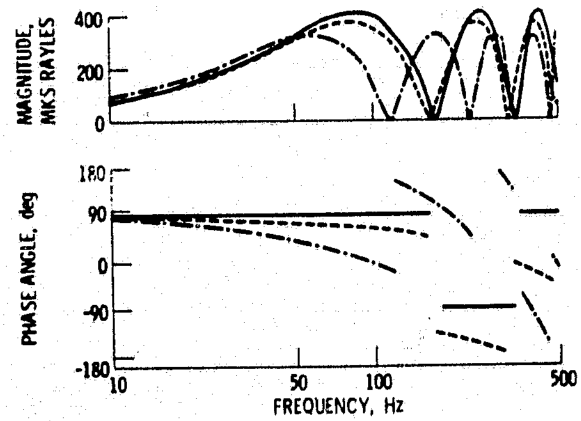
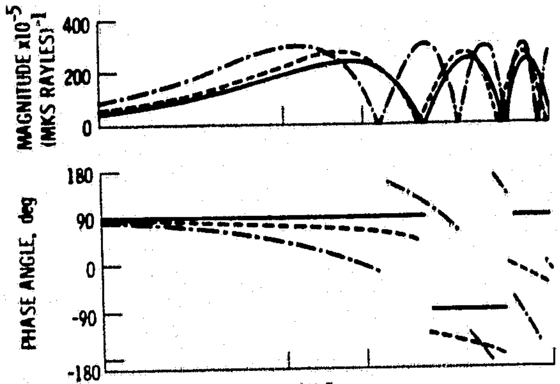
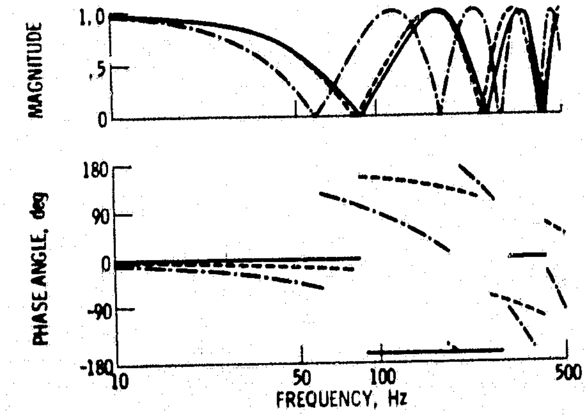


(b) T_{12}.

Figure 1. - Cylindrical duct transmission matrix, T, where $\tilde{Y}(x = 1.0) = T\tilde{Y}(x = 0.0)$ (L = 1 m, T = 295 K, W = 2 kg/sec).



(c) T_{21}.



(d) T_{22}.

Figure 1. - Concluded.

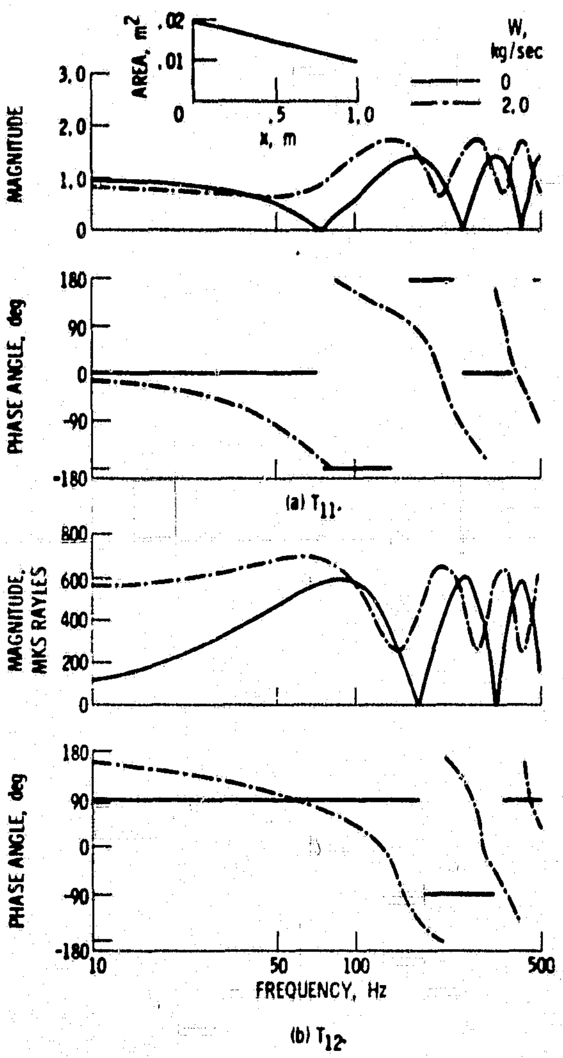
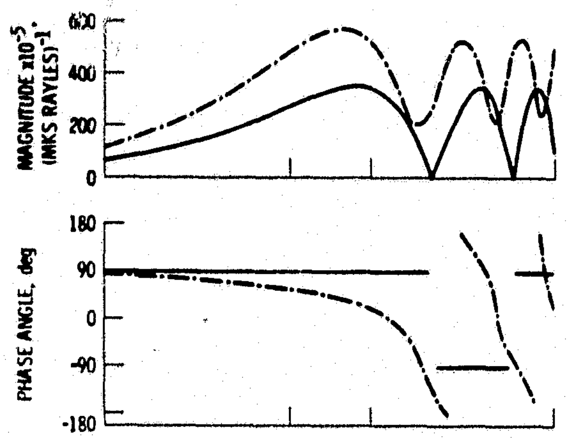


Figure 2. - Long exponential nozzle transmission matrix, T, where $\tilde{Y}(x = 1.0) = T\tilde{Y}(x = 0.0)$.



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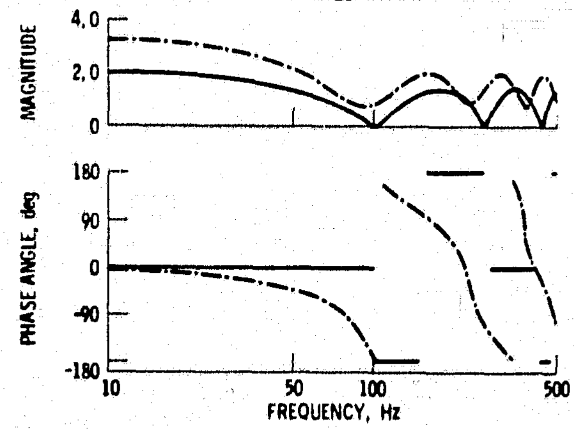

(d) T_{22}.

Figure 2. - Concluded.

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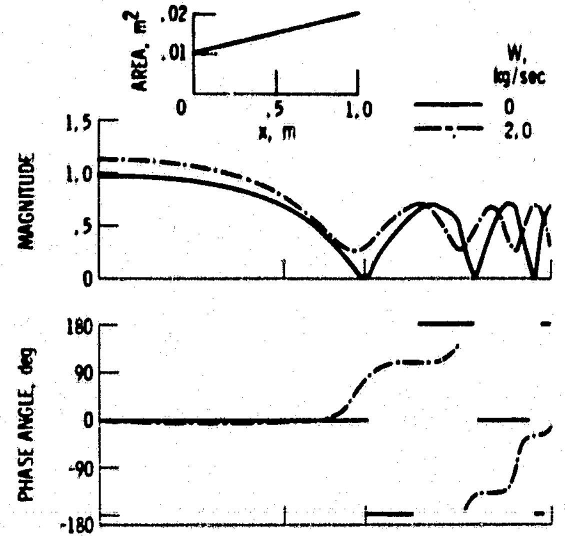

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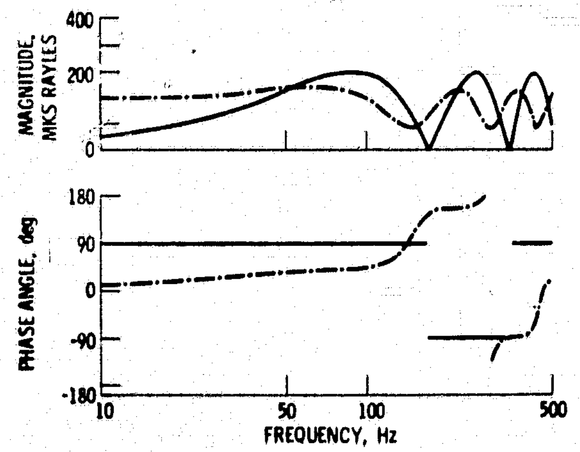
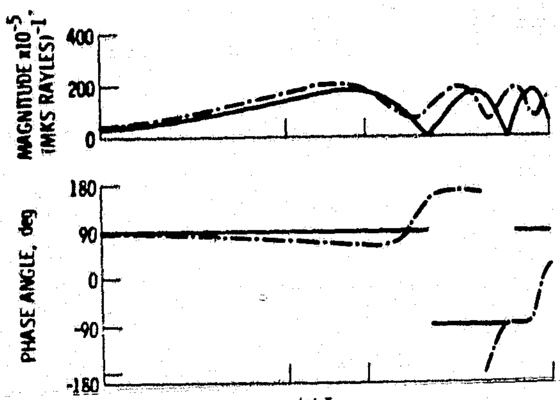
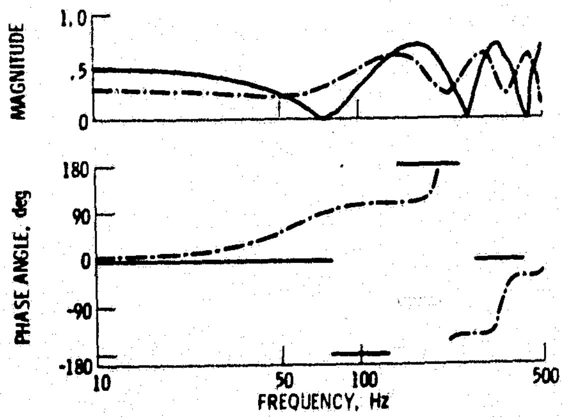


(b) T_{12}.

Figure 3. - Long exponential horn transmission matrix, T, where $\tilde{Y}(x = 1.0) = T\tilde{Y}(x = 0.0)$.



(c) T_{21}.



(d) T_{22}.

Figure 3. - Concluded.

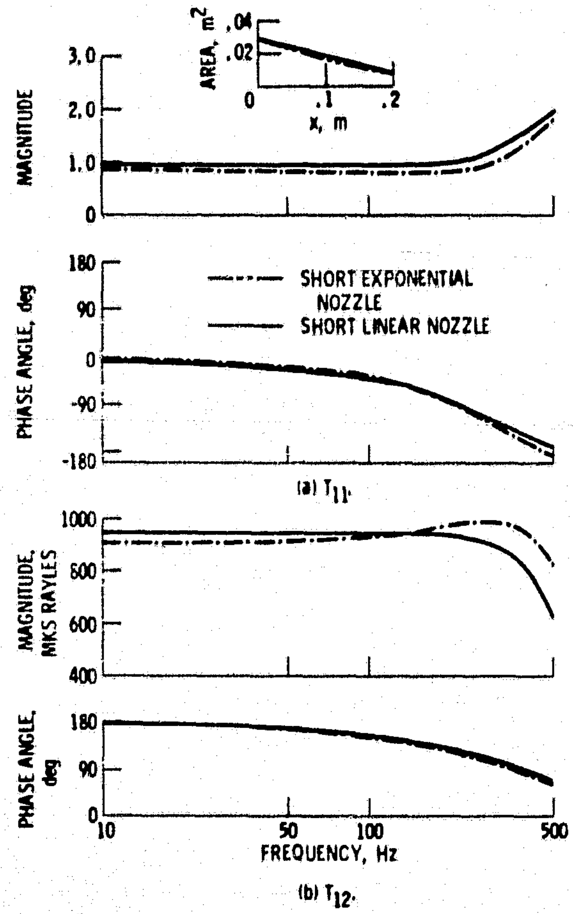
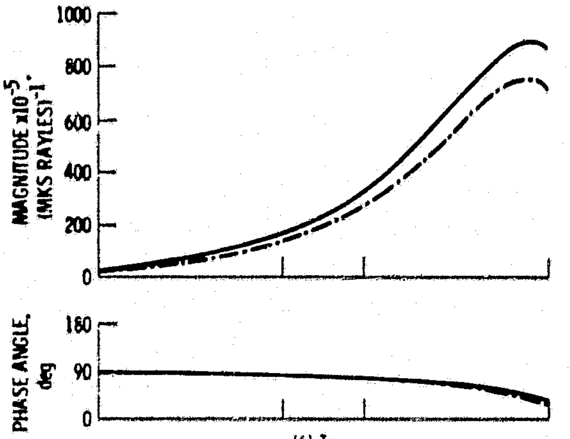


Figure 4. - Comparison of short exponential nozzle and short linear nozzle transmission matrices, T, where $\tilde{Y}(x = 0.2) = T\tilde{Y}(x = 0.0)$ for W = 2.0.



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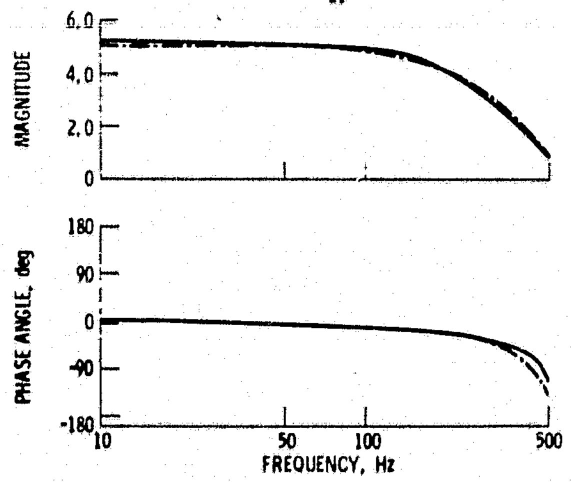

(d) T_{22}.

Figure 4. - Concluded.

1. Report No	2. Government Accession No	3. Recipient's Catalog No
NASA TM-81614		

4. Title and Subtitle
ACOUSTIC TRANSMISSION MATRIX OF A VARIABLE AREA DUCT OR NOZZLE CARRYING A COMPRESSIBLE SUBSONIC FLOW

5. Report Date

6. Performing Organization Code

7. Author(s)
J. H. Miles

8. Performing Organization Report No.
E-613

9. Performing Organization Name and Address
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio 44135

10. Work Unit No.

11. Contract or Grant No.

12. Sponsoring Agency Name and Address
National Aeronautics and Space Administration
Washington, D.C. 20546

13. Type of Report and Period Covered
Technical Memorandum

14. Sponsoring Agency Code

15. Supplementary Notes
Prepared for the One-hundredth Meeting of the Acoustical Society of Los Angeles, California, November 17-21, 1980.

16. Abstract
The differential equations governing the propagation of sound in a variable area duct or nozzle carrying a one-dimensional subsonic compressible fluid flow are derived and put in state variable form using acoustic pressure and particle velocity as the state variables. The duct or nozzle is divided into a number of regions. The region size is selected so that in each region the Mach number can be assumed constant and the area variation can be approximated by an exponential area variation. Consequently, the state variable equation in each region has constant coefficients. The transmission matrix for each region is obtained by solving the constant coefficient acoustic state variable differential equation. The transmission matrix for the duct or nozzle is the product of the individual transmission matrices of each region. Solutions are presented for several geometries with and without mean flow.

17. Key Words (Suggested by Author(s))
Acoustics; Sound; Noise; Propagation; Ducts; Acoustic ducts; Transmission matrix; Matrices; Nozzle; Acoustic propagation

18. Distribution Statement
Unclassified - unlimited
STAR Category 71

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified		

* For sale by the National Technical Information Service, Springfield, Virginia 22161