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by

Harold Salwen, Principal Investigator

Final Report

For the period May 16, 1979 - August 15, 1980

Prepared for the
National Aeronautics and Space Administration
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Hampton, Virginia 23665

Under
Research Grant NSF 1619
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STEVENS INSTITUTE OF TECHNOLOGY
CASTLE POINT STATION
HOBOKEN, NEW JERSEY

DEPARTMENT OF PHYSICS AND ENGINEERING PHYSICS
STEVENS INSTITUTE OF TECHNOLOGY
CASTLE POINT STATION
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SUMMARY

The research conducted under this project has been directed toward the double objectives of providing (1) a rational foundation for the application of the linear stability theory of parallel shear flows to transition prediction and (2) an explicit method for performing the necessary calculations.

The fundamental discovery upon which our subsequent work is based was that the solutions of the linearized, three-dimensional, incompressible Navier-Stokes equations $\vec{u}, p$ and the adjoint solutions $\vec{\tilde{u}}, \tilde{p}$ satisfy a "continuity" equation

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot \vec{J} = 0 \qquad (1)$$

where $\hat{\rho}$ is a pseudo-energy density (the dot product of $\vec{\tilde{u}}^*$ and $\vec{u}$) and $\vec{J}$ is a pseudo-current. This result is derived and discussed in detail in Appendix A.

We next considered (see Appendix B) the expansion of an arbitrary, two-dimensional solution of the linearized stream function equation in terms of the discrete and continuum eigenfunctions of the Orr-Sommerfeld equation in the half-space, $y \in [0,\infty)$: that is, we considered boundary-layer, wake, jet or free-shear layer flows. We used equation (1) to derive a biorthogonality relation between the solutions of the linearized stream function equation and the solutions of the adjoint problem. This is the biorthogonality relation for the mixed initial-boundary value problem.

For the case of temporal stability, we used equation (1) to derive the formal solution of the initial value problem as a sum over the discrete modes plus an integral over the continuum functions and showed that this expansion is complete. We found that the vorticity distribution at the initial time is sufficient information to determine the expansion coefficients and gave explicit formulas to calculate these coefficients.

For the spatial stability problem, we showed that the continuum has four branches. We used equation (1) to derive the spatial biorthogonality relation and the formal solution to the boundary value problem. We have (see Appendix C) also derived the Fourier (in t), Laplace (in x) transform solution of the spatial stability problem and used it to show that our spatial expansion is complete.

The boundary conditions for the spatial problem are the Fourier transforms, in time, of the stream function and its first three partial derivatives with respect to x, evaluated at $x = 0$. As it stands, this formal solution will not give a physically acceptable solution because, given an arbitrary variation with y and t at $x = 0$ of the stream function and its first three partial derivatives with respect to x, disturbances which lie on all four branches of the continuum will be excited. Therefore, as we show in Appendix B, the spatial wave packet will contain, in addition to waves propagating toward $x = \infty$, waves propagating upstream from $x = \infty$ and standing waves whose amplitude increases towards $x = \infty$.

A condition must be imposed that, for $x > 0$, all propagating disturbances are traveling in the positive x-direction and all standing waves have amplitudes which decay in the positive x-direction. It appears that this should be done by requiring that the stream function and its first three partial derivatives with respect to x, evaluated at $x = 0$, be orthogonal, using the spatial inner product, to all eigenfunctions on branches 2 and 4 of the continuous spectrum.

It is easy to see that these two orthogonality conditions reduce the number of boundary conditions at $x = 0$ from four to two. This means that, for the spatial stability problem, the proper boundary conditions at $x = 0$ are the specification of the temporal Fourier transforms of the velocity components u and v, for all y. Although these boundary conditions were derived from consideration of the continuum eigenfunctions, they apply as well to the discrete, Tollmien-Schlicting modes. We have not yet carried out a detailed investigation of the implications of imposing this orthogonality requirement on the boundary conditions; however, the immediate result that the boundary conditions at $x = 0$ are the specification of the temporal Fourier transforms of u and v for all y appears, on physical grounds, to be correct.

We have presented preliminary numerical results of the application of this expansion method at the Fifteenth International Conference on Theoretical and Applied Mechanics (Appendix D). We considered the temporal stability problem and a simple initial disturbance. We assumed that at $t = 0$ the vorticity ζ was given by

$$\zeta = \zeta_o e^{i\alpha_o x} \delta(y - y_o) \qquad (2)$$

a periodic layer of vorticity at a distance y_o from the boundary.

The stream function is then given by equation (55) of Appendix B, and it is easily seen that the expansion coefficients are [from equations (56a, b) of Appendix B]:

$$A_n(\alpha) = \zeta_o \tilde{\phi}_n^*(y_o)\ \delta(\alpha - \alpha_o) \qquad (3a)$$

$$A_k(\alpha) = \zeta_o \tilde{\phi}_k^*(y_o)\ \delta(\alpha - \alpha_o) \qquad (3b)$$

The solution of this simple problem, which is in effect the Greens function in y of the initial value problem, shows that the amplitudes of discrete, Tollmien-Schlicting modes and the continuum functions are the products of the magnitudes of the corresponding adjoint functions, evaluated at y_o, the height of the initial disturbance from the boundary and the vortex strengths.

We applied this result to two different flows. The first is a slip flow past a bounding plane at $y = 0$. Although the base flow velocity does not vanish at the boundary, we required that the disturbance velocity vanish at $y = 0$. We found (Appendix B) that, because of the simple form of the base flow, all the calculations could be carried out analytically and the stream function could be expressed as a finite sum of exponentials and error functions. We found that the disturbance retains its identity as a periodic array of vortices for all time, but as time increases it diffuses, the vortex strength decays, and the centers of the vortices drift away from the boundary.

The second flow we considered is the Blasius boundary layer. The velocity scale was taken to be the free-stream speed U_o and the length scale was $\sqrt{\nu x/U_o}$. We chose $\alpha = 0.179$ and $R = 580.0$. At this α and R, there are seven discrete Tollmien-Schlicting modes, one of which is unstable. We numerically calculated the seven eigenfunctions and adjoint eigenfunctions and normalized them so that

$$< \tilde{\phi}_n, \phi_m > = \delta_{nm} \tag{4}$$

Plots of the amplitude and phase of the normalized eigenfunction and adjoint eigenfunction of the seven modes as a function of y, the dimensionless distance from the boundary, are given in Appendix D. These modes are numbered in order of increasing stability with mode 1 the unstable mode and mode 7 the most damped mode.

The amplitude of a mode, say ϕ_n, excited by the vortex sheet at $y = y_o$, is proportional to the amplitude of $\tilde{\phi}_n$ evaluated at y_o. It is clear from an examination of these figures that when the vortex layer at $t = 0$ is in the inner portion of the boundary layer, say $y \lesssim 2.0$ (the top of the boundary layer is at $y = 5.02$), there will be a relatively strong excitation of the discrete Tollmien-Schlicting waves. Modes 1, 2, and 3 will have the largest amplitudes, and the higher modes will have substantially smaller amplitudes. It is also quite clear that, when the initial disturbance is more than about four boundary-layer thicknesses from the wall at $t = 0$, the discrete Tollmien-Schlicting modes excited by the disturbance will have extremely small amplitudes. We believe that this result is a theoretical explanation of the experimental observation of Kachanov, Kozlov, and Levchenko (1978) that vorticity disturbances passing above a boundary layer are very inefficient generators of Tollmien-Schlicting waves in the boundary layer.

CONCLUSIONS

We believe that we have created a rational foundation for the application of the linear stability theory of parallel shear flows to transition prediction and given an explicit method to carry out the necessary calculations. We have shown that these expansions are complete. We have also carried out some sample calculations which show that a typical boundary layer is very sensitive to vorticity disturbances in the inner boundary layer, near the critical layer; vorticity disturbances three or four boundary-layer thicknesses above the boundary are nearly uncoupled from the boundary layer in that the amplitudes of the discrete Tollmien-Schlicting waves are an extremely small fraction of the amplitude of the disturbance.

After the completion of this grant we intend to continue these calculations. We will continue the calculations of temporal disturbances in typical boundary layers and begin calculation of spatial disturbances.

LITERATURE CITED

Kachanov, Yer S.; Kozlov, V.V.; and Levchenko, V. Ya.: Occurrence of Tollmien-Schlicting Waves in the Boundary Layer Under the Effect of External Perturbations. Izvest. Akad. Nauk SSSR; Mekhan. Zhid. Gaza, Vol. 5, 1978, p. 85.

APPENDIX A

EXPANSIONS IN SPATIAL OR TEMPORAL EIGENMODES OF THE LINEARIZED NAVIER-STOKES EQUATION

Submitted to Journal of Fluid Dynamics

Expansions in Spatial or Temporal Eigenmodes of the Linearized Navier-Stokes Equation

by Harold Salwen
Department of Physics and Engineering Physics
Stevens Institute of Technology, Hoboken, N. J. 07030

The expansion of an arbitrary flow field in terms of the temporal or spatial eigenmodes of the linearized Navier-Stokes (LNS) equations for an incompressible fluid is developed from a unified perspective. It is shown that, for $(\vec{v},p)$ a solution of the LNS equations for a given base flow and $(\vec{\tilde{u}},\tilde{q})$ a solution of the corresponding adjoint equations, a scalar "density", $\mathcal{E}(\vec{\tilde{u}},\vec{v})$, and a vector "flux", $\vec{\Gamma}(\vec{\tilde{u}},\tilde{q},\vec{v},p)$, may be defined such that $\mathcal{E}$ and $\vec{\Gamma}$ are bilinear in $(\vec{\tilde{u}}^*,\tilde{q}^*)$ and $(\vec{v},p)$ and satisfy the "continuity" equation, $\partial\mathcal{E}/\partial t + \vec{\nabla}\cdot\vec{\Gamma} = 0$. This equation is then used to derive biorthogonality relations between the eigenfunctions and adjoint eigenfunctions of the LNS equations for a general translationally-invariant problem. In the temporal case, the inner product is $\iiint\mathcal{E}d\tau = \iiint\frac{1}{2}\rho\vec{\tilde{u}}^*\cdot\vec{v}\,d\tau$ which is the natural extension of Schensted's inner product for two-dimensional disturbances and satisfies the requirements for an inner product in a Hilbert space. In the spatial case, the "inner product" is $\iiint \Gamma_x\,dydzdt$ which is not positive definite. The formal solution of the LNS equations is derived, in terms of the eigenfunctions and the initial or boundary conditions, for the temporal and spatial cases. It takes the form of the evolution of a three- or six-dimensional vector — (v_x, v_y, v_z) in the temporal case or $(v_x, v_y, v_z, \partial v_y/\partial x, \partial v_z/\partial x, p)$ in the spatial case.

1. Introduction

A few years ago, Grosch and I, after showing that the Orr-Sommerfeld equation for unbounded flows such as the Blasius boundary layer possesses both temporal and spatial continuous spectra (Grosch and Salwen, 1975, 1978), set out to find the form of the wave-packet expansion for the temporal or spatial evolution of the stream function of an arbitrary two-dimensional "infinitesimal" disturbance in terms of the corresponding temporal or spatial eigenfunctions.

We sought to prove a biorthogonality relation between the eigenfunctions of the Orr-Sommerfeld equation and of its adjoint and, thereby, to solve for the coefficients of the expansion in terms of the inner products of the adjoint eigenfunctions with the stream function at the initial time or position. This worked out easily in the temporal case, with an inner product equivalent to Schensted's (1960) and only minor complications due to the infinite domain and continuous spectrum. In the spatial case, on the other hand, we found that we didn't know the appropriate inner product and we couldn't find any papers dealing with the problem. I therefore undertook the spatial expansion problem and, eventually, was rewarded with the result reported here—a unified treatment of the spatial and temporal expansion problems for solutions of the linearized Navier-Stokes (LNS) equations for an incompressible fluid.

Section 2 is devoted to the derivation of a "continuity" equation which is used, in Section 3, in the definition of the inner products and the derivation of the biorthogonality relations. These, in turn, are used in Sections 4 and 5 to derive the formal solutions of the (temporal) initial value problem and the (spatial) boundary value problem, respectively. The application of these results to two-dimensional disturbances of a boundary layer has been presented in a separate paper (Salwen and Grosch, 1980).

In order for the formal solutions derived in Sections 4 and 5 to be actual solutions of initial and boundary value problems, the eigenfunctions used in the expansions must form complete sets. Not all the eigenfunction sets one might want to use have been proven to be complete but there are, by now, proofs of completeness for large classes of temporal eigenfunctions for bounded flows (Yudovich, 1965 and DiPrima and Habetler, 1969) and temporal (Salwen and Grosch, 1980) and spatial (Salwen, Kelly, and Grosch, 1980) eigenfunctions for unbounded flows.

2. "Continuity" equation

I start with the LNS equations for an incompressible fluid with a base flow $\vec{U}$,

$$\vec{\nabla}\cdot\vec{v} = 0 \tag{1a}$$

$$\rho\left[\frac{\partial v_i}{\partial t} + \vec{U}\cdot\vec{\nabla}\, v_i + \vec{v}\cdot\vec{\nabla}\, U_i\right]$$

$$= \mu\,\nabla^2 v_i - \frac{\partial p}{\partial x_i}\,,\quad i = 1,2,3, \tag{1b}$$

and the corresponding adjoint equations†,

$$\vec{\nabla}\cdot\tilde{\vec{u}} = 0 \tag{2a}$$

$$\rho\left[-\frac{\partial \tilde{u}_i}{\partial t} - \vec{\nabla}\cdot(\vec{U}^*\tilde{u}_i) + \frac{\partial \vec{U}^*}{\partial x_i}\cdot\tilde{\vec{u}}\right]$$

$$= \mu\,\nabla^2\tilde{u}_i + \frac{\partial \tilde{q}}{\partial x_i}\,,\quad i = 1,2,3. \tag{2b}$$

---

† The complex conjugate, $\vec{U}^*$, is used here in order to obtain the correct formal expressions. In most applications, $\vec{U}$ will be real, so $\vec{U}^* = \vec{U}$.

For any solutions $(\vec{v},p)$ of (1) and $(\tilde{\vec{u}},\tilde{q})$ of (2), define*

$$\mathcal{E}(\tilde{\vec{u}},\vec{v}) = \frac{1}{2}\rho\, \tilde{\vec{u}}^* \cdot \vec{v} \,. \tag{3}$$

Then

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{1}{2}\rho \sum_{i=1}^{3} \left[\tilde{u}_i^* \frac{\partial v_i}{\partial t} + \frac{\partial \tilde{u}_i^*}{\partial t} v_i \right]$$

$$= \frac{1}{2} \sum_{i=1}^{3} \Big\{ -\rho \left[(\vec{U}\, \tilde{u}_i^*) \cdot \vec{\nabla} v_i + v_i\, \vec{\nabla} \cdot (\tilde{\vec{U}}\, \tilde{u}_i^*) \right]$$

$$- \rho \left[(\tilde{u}_i^* \vec{v}) \cdot \vec{\nabla} U_i - \frac{\partial \vec{U}}{\partial x_i} \cdot (\tilde{\vec{u}}^* v_i) \right]$$

$$+ \mu \left[\tilde{u}_i^* \nabla^2 v_i - v_i \nabla^2 \tilde{u}_i^* \right] - \left[\tilde{u}_i^* \frac{\partial p}{\partial x_i} + \frac{\partial \tilde{q}^*}{\partial x_i} v_i \right] \Big\} \tag{4}$$

so, with

$$\vec{\Gamma}(\tilde{\vec{u}},q,\vec{v},p) = \left(\frac{1}{2}\rho \tilde{\vec{u}}^* \cdot \vec{v}\right) \vec{U} + \frac{1}{2}\mu \sum_{i=1}^{3} \left[(\vec{\nabla}\, \tilde{u}_i^*)\, v_i \right.$$

$$\left. - \tilde{u}_i^* \vec{\nabla} v_i \right] + \frac{1}{2}\left[\tilde{\vec{u}}^* p + \tilde{q}^* \vec{v} \right] , \tag{5}$$

we get

$$\frac{\partial}{\partial t} \mathcal{E}(\tilde{\vec{u}},\vec{v}) + \vec{\nabla} \cdot \vec{\Gamma}(\tilde{\vec{u}},q,\vec{v},p) = 0 \ , \tag{6}$$

---

* The constant factor, $\frac{1}{2}\rho$, is included in order to emphasize the relation between $\mathcal{E}$ and the energy density, $\frac{1}{2}\rho v^2$.

which has the form of a continuity equation relating the time derivative of the "density", $\mathcal{E}$, to the divergence of the "flux", $\vec{\Gamma}$. For any fixed volume V bounded by a surface S , the "continuity" equation (6) may be put into integral form,

$$\frac{d}{dt} \iiint_V \mathcal{E}(\tilde{\vec{u}},\vec{v})\, d\tau + \iint_S \hat{n}\cdot\vec{\Gamma}\,(\tilde{\vec{u}},\tilde{q},\vec{v},p)\, dS = 0 . \qquad (7)$$

3. <u>Application to a steady, translationally-invariant base flow. Biorthogonality relations</u>

In this section, Equation (7) will be applied to the case in which the base flow and boundary conditions are independent of x and t . For all x and t, the base flow $\vec{U}(y,z)$, disturbance velocity and pressure $(\vec{v},p)$ and adjoint velocity and pressure $(\tilde{\vec{u}},q)$ are assumed to be defined in a closed, bounded area, A , of the y,z plane and to satisfy the boundary conditions

$$\vec{v}(x,y,z,t) = 0 , \quad \tilde{\vec{u}}(x,y,z,t) = 0 \quad \text{for} \quad (y,z)\epsilon C , \qquad (8a,b)$$

on the boundary, C, of A. In this case, the temporal and spatial eigenfunctions discussed below will form discrete sets. (The extension to an unbounded area is not too difficult (see, e.g., Salwen and Grosch, 1980) but it requires the relaxation of the boundary condition (8) and the consideration of continuum as well as discrete modes.)

Because of the choice of base flow and boundary conditions, (1) and (2) are now invariant with respect to translations in x and t and, therefore, possess solutions of the form

$$\vec{v}(x,y,z,t) = \vec{v}_0(y,z)e^{i(\alpha x - \omega t)}, \qquad (9a)$$

$$p(x,y,z,t) = p_0(y,z)e^{i(\alpha x - \omega t)}, \tag{9b}$$

$$\tilde{\vec{u}}(x,y,z,t) = \tilde{\vec{u}}_0(y,z)e^{i(\beta x - \nu t)}, \tag{10a}$$

$$\tilde{q}(x,y,z,t) = \tilde{q}_0(y,z)e^{i(\beta x - \nu t)} \tag{10b}$$

Because of (8), $\Gamma_y = \Gamma_z = 0$ on S. Then evaluation of (7) over a thin slab perpendicular to the x-axis for functions of the form (9) and (10) gives

$$i(\nu^* - \omega) \iint_A \mathcal{E}(\tilde{\vec{u}},\vec{v})\, dydz = \frac{\partial}{\partial t} \iint_A \mathcal{E}(\tilde{\vec{u}},\vec{v})\, dydz$$

$$= - \frac{\partial}{\partial x} \iint \Gamma_x\, (\tilde{\vec{u}},\tilde{q},\vec{v},p)\, dydz$$

$$= i(\beta^* - \alpha) \iint \Gamma_x\, (\tilde{\vec{u}},\tilde{q},\vec{v},p)\, dydz\ , \tag{11}$$

which will be used here to prove biorthogonality relations for the spatial and temporal eigenmodes.

The temporal eigenfunctions are the solutions of (1) and (8a) having the form (9) with α real. These may be denoted by $(\vec{V}_{\alpha n},\ p_{\alpha n})$, corresponding to the x,t variation $e^{i(\alpha x - \omega_n(\alpha)t)}$. For each such solution, there is an adjoint eigenfunction $(\tilde{\vec{u}}_{\alpha n},\ \tilde{q}_{\alpha n})$ which is a solution of (2) and (8b) having the variation $e^{i(\alpha x - \nu_n(\alpha)t)}$ with $\nu_n(\alpha) = \omega_n^*(\alpha)$. Application of (11) to these functions gives

$$(\omega_m(\alpha) - \omega_n(\alpha)) \iint_A \mathcal{E}(\tilde{\vec{u}}_{\alpha m}\ ,\ \vec{v}_{\alpha n})\, dydz = 0\ , \tag{12}$$

so that the integral vanishes when $\omega_m(\alpha) \neq \omega_n(\alpha)$ and, with appropriate normalization,

$$\iint_A (\tilde{\vec{u}}_{\alpha m}, \vec{v}_{\alpha n})\, dydz = \delta_{mn}/2\pi \;. \tag{13}$$

(The expression is a function of x and t but is constant because the exponentials in the two factors cancel.) This biorthogonality relation for fixed α leads to the result for the full set of temporal eigenfunctions,

$$\langle \tilde{\vec{u}}_{\alpha m}, v_{\beta n} \rangle \equiv \int_{-\infty}^{\infty} \iint_A \mathcal{E}(\tilde{\vec{u}}_{\alpha m}, \vec{v}_{\beta n})\, dydz\, dx$$

$$= \int_{-\infty}^{\infty} e^{i(\beta-\alpha)x}\, dx \iint_A \mathcal{E}(\tilde{\vec{u}}_{\alpha m}, \vec{v}_{\beta n})\Big|_{x=0} dydz$$

$$= 2\pi\delta(\alpha-\beta) \iint_A \mathcal{E}(\tilde{\vec{u}}_{\alpha m}, v_{\alpha n})\Big|_{x=0} dydz$$

$$= \delta(\alpha-\beta)\, \delta_{mn} \quad \text{for all } t \;. \tag{14}$$

The spatial eigenfunctions and adjoint eigenfunctions are the solutions of (1), (2), and (8) having the forms (9) and (10) with ω and ν real. These may be denoted by $\xi_{\omega n} = (\vec{v}_{\omega n}, p_{\omega n})$, with the variation $e^{i(\alpha_n(\omega)x - \omega t)}$, and $\tilde{\xi}_{\nu m} = (\tilde{\vec{u}}_{\nu m}, q_{\nu m})$, with the variation $e^{i(\beta_m(\nu)x - \nu t)}$. As in the temporal case, the eigenfunctions and adjoint eigenfunctions may be paired, with $\beta_n(\omega) = \alpha_n^*(\omega)$ in this case. The analogous results to (12) and (13) are

$$(\alpha_m(\omega) - \alpha_n(\omega)) \iint_A \Gamma_x\, (\tilde{\vec{u}}_{\omega m}, \tilde{q}_{\omega n}, \vec{v}_{\omega n}, p_{\omega n})\, dydz = 0 \tag{15}$$

and

$$\iint_A \Gamma_x\, (\tilde{\vec{u}}_{\omega m}, \tilde{q}_{\omega n}, \vec{v}_{\omega n}, p_{\omega n})\, dydz = \delta_{mn}/2\pi \,, \tag{16}$$

which lead to the biorthogonality relation for the full set of spatial eigenfunctions

$$\left[\!\left[\tilde{\xi}_{\omega m}, \xi_{\nu n} \right]\!\right] \equiv \int_{-\infty}^{\infty} \iint_A \Gamma_x \left(\tilde{\vec{u}}_{\omega m}, \tilde{q}_{\omega m}, \vec{v}_{\nu n}, p_{\nu n}\right) \, dydz \, dt$$

$$= \int_{-\infty}^{\infty} e^{i(\omega-\nu)t} dt \iint_A \Gamma_x (\tilde{\vec{u}}_{\omega m}, \tilde{q}_{\omega m}, \vec{v}_{\nu n}, p_{\nu n}) \Big|_{t=0} \, dydz$$

$$= 2\pi\delta(\omega-\nu) \iint_A \Gamma_x \left(\tilde{\vec{u}}_{\omega m}, \tilde{q}_{\omega m}, \vec{v}_{\omega n}, p_{\omega n}\right) \Big|_{t=0} dydz$$

$$= \delta(\omega-\nu) \, \delta_{mn} \quad \text{for all} \quad x \, . \qquad (17)$$

We thus see that, with appropriate "inner products" $\langle , \rangle$ and $[\![\, , \,]\!]$, the temporal and spatial eigenfunctions satisfy biorthogonality relations with the related adjoint functions. The temporal inner product $\langle , \rangle$ satisfies all the conditions ordinarily required of an inner product. The spatial inner product $[\![\, , \,]\!]$, on the other hand, is not positive definite. This is related to the fact that disturbances can propagate in both the downstream (+x) and upstream (-x) directions.

4. Temporal expansion of an arbitrary solution of the LNS equations

The temporal inner product introduced in (14),

$$\langle \vec{u}, \vec{v} \rangle \equiv \int_{-\infty}^{\infty} \iint_A \vec{u}^{*}(x,y,z) \cdot \vec{v}\,(x,y,z) \, dydz \, dx \, , \qquad (18)$$

is defined for any pair of ordinary vector functions of position. In particular, when applied to a solution $(\vec{v}, p)$ of (1), it involves the velocity, $\vec{v}$, but not the pressure, p. It is natural then, in seeking an expansion solution in terms of the temporal eigenfunctions, to expand $\vec{v}$ alone in terms of the velocity part $\{\vec{v}_{\alpha n}\}$ of the eigenfunctions.

Let $(\vec{v},p)$ be a solution of (1), satisfying the boundary conditions (8a). Assume that $\vec{v}$ can be expanded in the form

$$\vec{v}(x,y,z,t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} c_{\alpha n}(t)\, \vec{v}_{\alpha n}(x,y,z,t)\, d\alpha \, . \tag{19}$$

Then, by (14), the coefficients are

$$c_{\alpha n}(t) = \langle \tilde{\vec{u}}_{\alpha n}, \vec{v} \rangle \, , \tag{20}$$

so that (using (7))

$$\frac{dc_{\alpha n}}{dt} = \int_{-\infty}^{\infty} \iint_A \frac{\partial}{\partial t} \mathcal{E} (\tilde{\vec{u}}_{\alpha n}, \vec{v})\, dydz\, dx$$

$$= - \int_{-\infty}^{\infty} \iint_A \vec{\nabla}\cdot\vec{\Gamma}\, (\tilde{\vec{u}}_{\alpha n}, \tilde{q}_{\alpha n}, \vec{v}, p)\, dydz\, dx = 0 \tag{21}$$

and

$$c_{\alpha n}(t) = c_{\alpha n}(0) = \langle \tilde{\vec{u}}_{\alpha n}, \vec{v} \rangle \Big|_{t=0} \, . \tag{22}$$

The result is

$$\vec{v}(x,y,z,t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \langle \tilde{\vec{u}}_{\alpha n}, \vec{v} \rangle \Big|_{t=0} \vec{v}_{\alpha n}(x,y,z,t)\, d\alpha \,. \qquad (23)$$

On the assumption that the expansion can be differentiated term-by-term,

$$\frac{\partial p}{\partial x_i} = \mu \nabla^2 v_i - \rho \left[\frac{\partial v_i}{\partial t} + \vec{U} \cdot \vec{\nabla} v_i + \vec{v} \cdot \vec{\nabla} U_i \right]$$

$$= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \langle \tilde{\vec{u}}_{\alpha n}, \vec{v} \rangle \Big|_{t=0} \{ \mu \nabla^2 v_{\alpha n_i}$$

$$- \rho \left[\frac{\partial v_{\alpha n_i}}{\partial t} + \vec{U} \cdot \vec{\nabla} v_{\alpha n_i} + \vec{v}_{\alpha n} \cdot \vec{\nabla} U_i \right] \}. \, d\alpha$$

$$= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \langle \tilde{\vec{u}}, \vec{v} \rangle \Big|_{t=0} \frac{\partial p_{\alpha n}}{\partial x_i}\, d\alpha \,, \qquad (24)$$

so that, except for an additive function of t only,

$$p(x,y,z,t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \langle \tilde{\vec{u}}, \vec{v} \rangle \Big|_{t=0} p_{\alpha n}(x,y,z,t)\, d\alpha \,. \qquad (25)$$

Equations (23) and (25) are the formal solutions for $\vec{v}$ and p in terms of the initial velocity, $\vec{v}(x,y,z,0)$.

5. Spatial expansion of an arbitrary solution

The spatial inner product, $[\![\ ,\ ]\!]$, introduced in (16), cannot be evaluated in terms of the values of $\tilde{\vec{u}}$, $\tilde{q}$, $\vec{v}$, and p at a fixed x because Γ_x involves x-derivatives associated with the second derivatives in (1b) and (2b). To get around this problem, one can regard the flow field at a given x as a 6-vector and make use of the fact that the velocities under consideration have vanishing divergence ((1a) and (2a)).

Let ξ and η be 6-vectors with components

$$\xi_1 = v_x,\ \xi_2 = v_y,\ \xi_3 = v_z,\ \xi_4 = v_y' \equiv \frac{\partial v_y}{\partial x},\ \xi_5 = v_z' \equiv \frac{\partial v_z}{\partial x},\ \xi_6 = p\ , \tag{26}$$

$$\eta_1 = u_x,\ \eta_2 = u_y,\ \eta_3 = u_z,\ \eta_4 = u_y' \equiv \frac{\partial u_y}{\partial x},\ \eta_5 = u_z' = \frac{\partial u_z}{\partial x},\ \eta_6 = q \tag{27}$$

and let $\vec{\nabla} \cdot \vec{u} = \vec{\nabla} \cdot \vec{v} = 0$. Then, in terms of these components,

$$\begin{aligned}\Gamma_x(\vec{u},q,\vec{v},p) &= \tfrac{1}{2}\, \rho(u_x{*}v_x + u_y{*}v_y + u_z{*}v_z)\, U_x \\ &+ \tfrac{1}{2}\, \mu \Big[u_x{*}\big(\frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\big) - v_x\, \big(\frac{\partial u_y{*}}{\partial y} + \frac{\partial u_z{*}}{\partial z}\big) \\ &+ (v_y u_y'{*} - u_y{*}v_y') + (v_z u_z'{*} - u_z{*}v_z') \Big] \\ &+ \tfrac{1}{2}\, (u_x{*}p + q{*}v_x)\end{aligned} \tag{28}$$

so that

$$[\![\eta, \xi]\!] = \int_{-\infty}^{\infty} \iint_A \Gamma_x\ (\vec{u},q,\vec{v},p)\ dydz\ dt \tag{29}$$

may be evaluated in terms of the components of ξ and η at fixed x . This choice of coordinates also eliminates second derivatives from (1), which becomes

$$\frac{\partial v_x}{\partial x} = - \frac{\partial v_y}{\partial y} - \frac{\partial v_z}{\partial z} \tag{30a}$$

$$\frac{\partial v_y}{\partial x} = v_y' \tag{30b}$$

$$\frac{\partial v_z}{\partial x} = v_z' \tag{30c}$$

$$\frac{\partial v_y'}{\partial x} = \frac{\rho}{\mu}\frac{\partial U_y}{\partial x} v_x + \Big[\frac{\rho}{\mu}\Big(\frac{\partial v_y}{\partial t} + U_y \frac{\partial v_y}{\partial y} + U_z \frac{\partial v_y}{\partial z} + \frac{\partial U_y}{\partial y} v_y\Big)$$

$$- \Big(\frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\Big)\Big] + \frac{\rho}{\mu}\frac{\partial U_y}{\partial z} v_z + \frac{\rho}{\mu} U_x v_y' + \frac{1}{\mu}\frac{\partial p}{\partial y} \tag{30d}$$

$$\frac{\partial v_z'}{\partial x} = \frac{\rho}{\mu}\frac{\partial U_z}{\partial x} v_x + \frac{\rho}{\mu}\frac{\partial U_z}{\partial y} v_y + \Big[\frac{\rho}{\mu}\Big(\frac{\partial v_z}{\partial t} + U_y \frac{\partial v_z}{\partial y} + U_z \frac{\partial v_z}{\partial z} + \frac{\partial U_z}{\partial z} v_z\Big)$$

$$- \Big(\frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\Big)\Big] + \frac{\rho}{\mu} U_x v_z' + \frac{1}{\mu}\frac{\partial p}{\partial z} \tag{30e}$$

$$\frac{\partial p}{\partial x} = \Big[\mu\Big(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\Big) - \rho\Big(\frac{\partial v_x}{\partial t} + U_y \frac{\partial v_x}{\partial y} + U_z \frac{\partial v_x}{\partial z} + \frac{\partial U_x}{\partial x} v_x\Big)\Big]$$

$$+ \rho\Big(U_x \frac{\partial v_y}{\partial y} - \frac{\partial U_x}{\partial y} v_y\Big) + \rho\Big(U_x \frac{\partial v_z}{\partial z} - \frac{\partial U_x}{\partial z} v_z\Big) - \mu\Big(\frac{\partial v_y'}{\partial y} + \frac{\partial v_z'}{\partial z}\Big) . \tag{30f}$$

It is now straightforward to carry out the formal solution for the spatial expansion. The expansion is

$$\xi(x,y,z,t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} c_{\omega n}(x)\, \xi_{\omega n}(x,y,z,t)\, d\omega , \tag{31}$$

with coefficients

$$c_{\omega n}(x) = [\![\tilde{\eta}_{\omega n}, \xi]\!] . \tag{32}$$

Then

$$\frac{dc_{\omega n}}{dx} = \int_{-\infty}^{\infty} \iint_A \frac{\partial}{\partial x} \Gamma_x (\tilde{\vec{u}}_{\omega n}, \tilde{q}_{\omega n}, \vec{v}, p) \, dydz \, dt$$

$$= - \int_{-\infty}^{\infty} \iint_A \Big[\frac{\partial}{\partial y} \Gamma_y (\tilde{\vec{u}}_{\omega n}, \tilde{q}_{\omega n}, \vec{v}, p)$$

$$+ \frac{\partial}{\partial z} \Gamma_z (\tilde{\vec{u}}_{\omega n}, \tilde{q}_{\omega n}, \vec{v}, p) + \frac{\partial}{\partial t} \mathcal{E} (\tilde{\vec{u}}_{\omega n}, v) \Big] \, dydz \, dt$$

$$= 0 \tag{33}$$

so

$$c_{\omega n}(x) = c_{\omega n}(0) = [\![\tilde{n}_{\omega n}, \xi]\!] \Big|_{x=0} \; . \tag{34}$$

The solution is then

$$\xi(x,y,z,t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} [\![\tilde{n}_{\omega n}, \xi]\!] \Big|_{x=0} \, \xi_{\omega n}(x,y,z,t) \, d\omega \; . \tag{35}$$

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APPENDIX B

THE CONTINUOUS SPECTRUM OF THE ORR-SOMMERFELD EQUATION
PART 2. EIGENFUNCTION EXPANSIONS

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The Continuous Spectrum Of The Orr-Sommerfeld Equation

Part 2. Eigenfunction Expansions.

By

Harold Salwen

Department of Physics and Engineering Physics,

Stevens Institute of Technology, Hoboken, N.J., 07030

and

Chester E. Grosch

Department of Oceanography and Department of Mathematics,

Old Dominion University, Norfolk, Va., 23508

(Revised 4/26/80)

ABSTRACT

The expansion of an arbitrary two-dimensional solution of the linearized stream function equation in terms of the discrete and continuum eigenfunctions of the Orr-Sommerfeld equation is discussed for flows in the half-space, $y \in [0, \infty)$. A recent result of Salwen is used to derive a biorthogonality relation between the solution of the linearized equation for the stream function and the solutions of the adjoint problem.

For the case of temporal stability, the orthogonality relation obtained is equivalent to that of Schensted for bounded flows. This relationship is used to carry out the formal solution of the initial value problem for temporal stability. It is found that the vorticity of the disturbance at $t = 0$ is the proper initial condition for the temporal stability problem. Finally, it is shown that the set consisting of the discrete eigenmodes and continuum eigenfunctions is complete.

For the spatial stability problem, it is shown that the continuous spectrum of the Orr-Sommerfeld equation contains four branches. The biorthogonality relation is used to derive the formal solution to the boundary value problem of spatial stability. It is shown that the boundary value problem of spatial stability requires the stream function and its first three partial derivatives with respect to x be specified at $x = 0$ for all t. To be applicable to practical problems, this solution will require modification to eliminate disturbances originating at $x = \infty$ and travelling upstream to $x = 0$.

For the special case of a constant base flow, the method is used to calculate the evolution in time of a particular initial disturbance.

1. Introduction

Recent calculations of the discrete eigenmodes of the Orr-Sommerfeld equation (Jordinson, 1971; Mack, 1976; Houston, Cornor, and Ross, 1976; Murdoch and Stewartson, 1977) have indicated that, for a given Reynolds number and wave number (frequency), the Orr-Sommerfeld equation for Blasius flow has only a finite number of discrete temporal (spatial) eigenfunctions. Since a finite set of functions cannot be complete, these calculations raised the question of how to expand the stream function of an arbitrary disturbance in terms of the normal modes. These authors suggested that in addition to the finite discrete spectrum, which they found, there is a continuous spectrum.

In Part 1 (Grosch and Salwen, 1978a), we dealt with the existence of the continuous spectrum and the form of the related eigenfunctions for both the temporal and spatial problems. We showed that the Orr-Sommerfeld equation, for any mean shear flow approaching a constant velocity in the far field, possesses a continuous spectrum; we gave formulae for the location of the temporal and spatial continua in the complex wave-speed plane; and we calculated the temporal continuum eigenfunctions for some particular cases. In this paper, we turn our attention to the use of the discrete and continuum eigenfunctions of the Orr-Sommerfeld equation to calculate the temporal or spatial evolution of an arbitrary solution of the linear disturbance equations.

In a recent critique of the application of stability theory to the prediction of transition, Berger and Aroesty (1977) point out that, on the basis of the limited experimental evidence that is available, the coupling of free stream disturbances to disturbances in the boundary layer appears to be extraordinarily weak and extremely selective in frequency and wavenumber. Mack (1977) makes the same point in a different way. He points out that "if there were no disturbances [inside the boundary layer], there would be no transition and the boundary layer would remain laminar. Consequently, it is futile to talk about transition without in some way bringing in the disturbances which cause it ... ". Mack adds, "... the precise mechanism by which, say, free stream turbulence, sound, and different types of roughness cause transition remains to be discovered."

The most detailed discussion of this problem appears to be that of Obremski, Morkovin, and Landahl (1969). They consider various possible mechanisms by which sound or vorticity waves in the free stream might interact with the boundary layer and cause transition. On the basis of the available experimental evidence, they conclude that only a small portion of the external disturbance field excites Tollmien-Schlichting (T-S) waves in the boundary layer and a significant portion appears to travel within the boundary layer with little or no interaction. The (unstated) conclusion seems to be that the mechanism which couples free stream disturbances to a boundary layer and, thereby, initiates transition is unknown.

The central problem here is the solution of the general initial and boundary value problems for disturbances to boundary layer flow— how, given the form of the disturbance at a time t = 0, to find its variation with time and how, given the form of the disturbance at all times on a plane, x = 0, perpendicular to the boundary layer, to find out how it propagates downstream. In this paper, we approach these problems, in the approximation obtained by assuming parallel flow and linearizing with respect to the disturbances, by expressing the solution as a sum over the discrete normal modes plus an integral over the continuum eigenfunctions of the Orr-Sommerfeld equation. If the (discrete plus continuum) eigenfunctions form a complete set, this approach will yield a valid solution of the problem.

Starting with Haupt (1912), a number of authors have dealt with the completeness of the set of temporal eigenfunctions in a bounded domain. Haupt showed that the eigenfunctions for two-dimensional disturbances to plane Couette flow form a complete set and Schensted (1960) proved completeness for the eigenfunctions for two-dimensional disturbances to plane Poiseuille flow and for axi-symmetric disturbances to Poiseuille flow in a circular pipe. Yudovich (1965) and DiPrima and Habetler (1969) have proven completeness of the eigenmodes for a large class of bounded flows. We are unaware of any work on the completeness of the spatial eigenfunctions or, previous to this paper, on the completeness of the temporal eigenfunctions in an unbounded domain.

In Section 2 we formulate the stability problem for two-dimensional disturbances to a parallel shear flow, $U(y)$, $0 \leq y < \infty$, in terms of the linearized equation for the stream function and boundary conditions. We next formulate the adjoint problem. A new result of Salwen (1979) is then used to derive a pseudocontinuity relation involving solutions of the linearized equation for the stream function and the adjoint solutions. This relation is then used to find the general biorthogonality condition for wave-like disturbances to the flow. The biorthogonality relation is specialized to the cases of temporal and spatial stability. The orthogonality relation for the temporal stability problem is that derived by Schensted (1960) and discussed by Reid (1965).

The temporal stability problem is considered in detail in Section 3. The solution is Fourier analyzed with respect to x. Then the formal solution of the initial value problem for the temporal stability of a two-dimensional disturbance to a parallel shear flow is expressed as an expansion in terms of the eigenfunctions. The expansion coefficients are determined by inner products between the initial disturbance and the eigenfunctions of the adjoint equation. We show that the disturbance vorticity at $t = 0$ is the proper initial condition for the temporal stability problem.

In Section 4 we examine the question of the completeness of the set of expansion functions for the temporal stability problem. Very recently, Gustavsson (1979) has treated the temporal initial value problem by using Fourier-Laplace transforms. He finds poles in the transform plane which correspond to the discrete T-S modes and a branch cut which corresponds to the continuous spectrum. We show in this section that the Fourier-Laplace transform solution of Gustavsson is identical to our Fourier transform, eigenfunction expansion solution for the initial value problem of temporal stability. We therefore conclude that our expansion set is complete.

The spatial stability problem is considered in detail in Section 5. The solution is Fourier analyzed in t. The formulae for the four branches of the continuous spectrum of the spatial stability problem are derived and discussed. The formal solution of the boundary value problem for the spatial stability of a two-dimensional disturbance to a parallel shear flow is expressed as an expansion in terms of the spatial eigenfunctions. The expansion coefficients are determined by inner products between the boundary conditions at $x = 0$ and the eigenfunctions of the adjoint equation. The boundary conditions at $x = 0$ are discussed. We have not yet been able to prove completeness for the set of expansion functions of the spatial stability problem.

In Section 6, we apply the results of Section 3 to the simple case of a constant base flow. In this case, we find the eigenfunctions and calculate and discuss the temporal evolution of a particular initial disturbance.

2. <u>The linearized, two dimensional Navier-Stokes equations: the biorthogonality relation.</u>

2.1 Formulation of the problem

The basic flow under consideration is a parallel shear flow, $U(y)$, in the semi-infinite region, $y \geq 0$. We are concerned with the temporal or spatial development of an "infinitesimal", two-dimensional disturbance to this flow, $(u(x, y, t), v(x, y, t), 0)$. In this case, u and v can be expressed in terms of a stream function, $\Psi(x, y, t)$, by

$$u = \frac{\partial \Psi}{\partial y}, \tag{1}$$

$$v = -\frac{\partial \Psi}{\partial x} \tag{2}$$

and the linearized Navier-Stokes equations reduce to a single partial differential equation,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) \nabla^2 \Psi - \frac{d^2U}{dy^2}\frac{\partial \Psi}{\partial x} - \frac{1}{R}\ \nabla^4 \Psi = \mathcal{L}\Psi = 0 \tag{3}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \ . \tag{4}$$

In addition, Ψ must satisfy two boundary conditions at $y = 0$,

$$\left.\frac{\partial\Psi}{\partial x}\right|_{x,0,t} = -v(x,\ 0,\ t) = 0 \tag{5}$$

and

$$\left.\frac{\partial\Psi}{\partial y}\right|_{x,0,t} = u(x,\ 0,\ t) = 0, \tag{6}$$

nd a "finiteness" condition

$$\int_0^\infty [|\frac{\partial\Psi}{\partial x}|^2 + |\frac{\partial\Psi}{\partial y}|^2]dy = \int_0^\infty [|u|^2 + |v|^2]dy < \infty. \tag{7}$$

As a consequence of eq. (7), Ψ must satisfy boundary conditions at infinity,

$$\frac{\partial\Psi}{\partial x},\ \frac{\partial\Psi}{\partial y} \to 0 \qquad \text{as} \qquad y \to +\infty. \tag{8}$$

For fixed x and t, $\Psi(x,\ y,\ t)$ belongs to a manifold, M, of functions, $\phi(y)$, satisfying

$$\phi,\ \frac{d\phi}{dy},\ \frac{d^2\phi}{dy^2},\ \frac{d^3\phi}{dy^3},\ \frac{d^4\phi}{dy^4} \text{ all defined on } [0,\ \infty), \tag{9}$$

$$\phi, \frac{d\phi}{dy}, \frac{d^2\phi}{dy^2}, \frac{d^3\phi}{dy^3}, \quad \text{continuous on } [0, \infty), \tag{10}$$

$$\phi(0) = 0 \quad \text{and} \quad \phi'(0) = 0 \tag{11}$$

and

$$\int_0^\infty |\phi(y)|^2 dy \quad \text{and} \quad \int_0^\infty \left|\frac{d\phi}{dy}\right|^2 dy \quad \text{both exist.} \tag{12}$$

The continuum eigenfunctions which will be discussed in Sections 3 and 5 do not satisfy eq. (12). Instead they belong to a manifold $M' \supset M$ of functions satisfying eqs. (9) - (11) and a weakened condition,

$$\phi(y) \quad \text{and} \quad \frac{d\phi}{dy} \quad \text{bounded in } [0, \infty). \tag{13}$$

We define an inner product,

$$(f, g) \equiv \int_0^\infty f^*(y)\; g(y) dy, \tag{14}$$

in M. The star denotes the complex conjugate. This inner product is defined for the full Hilbert space of functions satisfying eq. (12) and, in that space, has the usual properties of inner products.

2.2 The adjoint problem

For functions f, g ε M we define the adjoint, $\mathcal{L}^+$, of $\mathcal{L}$ in the usual way by

$$\iiint \{f(x,y,t)\}^* \mathcal{L}^+ g(x,y,t)\}dxdydt$$

$$= \iiint \mathcal{L} f(x,y,t)\}^* \{g(x,y,t)\}dxdydt + \text{Boundary Terms.} \qquad (15)$$

The definition of the adjoint used here yields an adjoint operator which is identical to the formal adjoint (Friedman, 1969, pp. 2,3).

An adjoint stream function, $\tilde{\Psi}(x, y, t)$, is a solution of the adjoint equation (with $U^* = U$),

$$\mathcal{L}^+\tilde{\Psi} = (\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\nabla^2\tilde{\Psi} + 2\,\frac{dU}{dy}\,\frac{\partial^2\tilde{\Psi}}{\partial x\partial y} + \frac{1}{R}\;\nabla^4\tilde{\Psi} = 0, \qquad (16)$$

with the boundary conditions at y = 0,

$$\left.\frac{\partial\tilde{\Psi}}{\partial x}\right|_{x,0,t} = \left.\frac{\partial\tilde{\Psi}}{\partial y}\right|_{x,0,t} = 0\ , \qquad (17)$$

and the finiteness condition

$$\int_0^\infty \left\{\left|\frac{\partial\tilde{\Psi}}{\partial x}\right|^2 + \left|\frac{\partial\tilde{\Psi}}{\partial y}\right|^2\right\}dy$$

$$= \int_0^\infty \{|\tilde{u}|^2 + |\tilde{v}|^2\}dy < \infty . \qquad (18)$$

As above, equation (19) implies that $\tilde{\Psi}$ must satisfy boundary conditions

$$\frac{\partial\tilde{\Psi}}{\partial x}, \frac{\partial\tilde{\Psi}}{\partial y} \to 0 \qquad \text{as} \qquad y \to \infty. \qquad (19)$$

When, as below, we look for solutions to the linearized stream function equation (3) which have a wavelike behavior in x and t, equation (3) reduces to the Orr-Sommerfeld equation and equation (16) reduces to the adjoint Orr-Sommerfeld equation. Our adjoint Orr-Sommerfeld equation is the complex conjugate of the adjoint equation derived by Schensted (1960) and quoted by Reid (1965). The reason for this difference is that we define the inner product in the usual way, (14) while Schensted's definition of the inner product (f, g) involves f instead of f*.

2.3 Biorthogonality

Salwen (1979) has shown that the solutions of the linearized, three dimensional Navier-Stokes equations, $\vec{u}$, p and the adjoint solutions $\vec{\bar{u}}$, $\bar{p}$ satisfy a "continuity" equation

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla \cdot \vec{J} = 0, \tag{20}$$

where

$$\hat{\rho} = \vec{\bar{u}}^* \cdot \vec{u}\ , \tag{21}$$

$$\vec{J} = (\vec{\bar{u}}^* \cdot \vec{u})\vec{U} + \frac{1}{R} \sum_{i=1}^{3} [(\nabla \bar{u}_i^*)u_i - \bar{u}_i^* (\nabla u_i)]$$

$$+ \vec{\bar{u}}^* p + \vec{u}\, \bar{p}^*\ , \tag{22}$$

and, as before, the star denotes a complex conjugate.

For the two dimensional disturbances considered here we will introduce two new inner products. Let Ψ be any solution of the original problem and $\bar{\Psi}$ be any solution of the adjoint problem, then define

$$\langle \bar{\Psi}, \Psi \rangle \equiv \int_0^\infty \hat{\rho}\, dy = \int_0^\infty \left(\frac{\partial \bar{\Psi}^*}{\partial x}\frac{\partial \Psi}{\partial x} + \frac{\partial \bar{\Psi}^*}{\partial y}\frac{\partial \Psi}{\partial y}\right) dy\ , \tag{23}$$

and

$$[\![\tilde{\Psi}, \Psi]\!] \equiv \int_0^\infty J_x \, dy \,, \tag{24}$$

with J_x, the x component of $\vec{J}$. Using (22) and expressing $\vec{u}$, p and $\vec{\tilde{u}}$, $\tilde{p}$ in terms of Ψ and $\tilde{\Psi}$, it can be shown that

$$\begin{aligned}
[\![\tilde{\Psi}, \Psi]\!] = \int_0^\infty \{ & \tfrac{1}{R} [\tilde{\Psi}^* \frac{\partial^3 \Psi}{\partial x^3} - \frac{\partial \tilde{\Psi}^*}{\partial x} \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \tilde{\Psi}^*}{\partial x^2} \frac{\partial \Psi}{\partial x} - \frac{\partial^3 \tilde{\Psi}^*}{\partial x^3} \Psi \\
& + 2 \frac{\partial^2 \tilde{\Psi}^*}{\partial x \partial y} \frac{\partial \Psi}{\partial y} - 2 \frac{\partial \tilde{\Psi}^*}{\partial y} \frac{\partial^2 \Psi}{\partial x \partial y}] \\
& - [\tilde{\Psi}^* \frac{\partial^2 \Psi}{\partial t \partial x} + \frac{\partial^2 \tilde{\Psi}^*}{\partial t \partial x} \Psi] \\
& - U [\tilde{\Psi}^* \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial \tilde{\Psi}^*}{\partial x} \frac{\partial \Psi}{\partial x} + \frac{\partial^2 \tilde{\Psi}^*}{\partial x^2} \Psi \\
& - 2 \frac{\partial \tilde{\Psi}^*}{\partial y} \frac{\partial \Psi}{\partial y} - \frac{\partial^2 \tilde{\Psi}^*}{\partial y^2} \Psi] \} dy.
\end{aligned} \tag{25}$$

The form of these inner products has been determined by the equations for the stream function and the adjoint stream function. However, we can use equations (23) and (25) to calculate inner products $\langle f, g \rangle$ and $[\![f, g]\!]$, evaluated at fixed x and t, of any functions f(x, y, t) and g(x, y, t).

It is straightforward to show that $\langle f, g \rangle$ is defined for the full Hilbert space of functions which satisfy equation (7) and, in that space, has the usual properties of inner products. On the other hand, $[\![f, g]\!]$ is not positive definite. This is due to the fact that it is possible to have wavelike solutions to equation (3) which propagate in either the upstream (-x) or downstream (+x) direction.

With these definitions it is easy to show that

$$\frac{\partial}{\partial t} \langle \tilde{\Psi}, \Psi \rangle + \frac{\partial}{\partial x} [\![\tilde{\Psi}, \Psi]\!] = 0 , \tag{26}$$

for any solutions of the original and adjoint problems.

If Ψ and $\tilde{\Psi}$ are wave disturbances of the form

$$\Psi_{\alpha'\omega'} = \phi_{\alpha'\omega'}(y) e^{i(\alpha' x - \omega' t)} , \tag{27}$$

$$\tilde{\Psi}_{\alpha\omega} = \tilde{\phi}_{\alpha\omega}(y) e^{i(\alpha x - \omega t)} , \tag{28}$$

equation (26) reduces to

$$(\omega' - \omega^*) \langle \tilde{\Psi}_{\alpha\omega} , \Psi_{\alpha'\omega'} \rangle = (\alpha' - \alpha^*) [\![\tilde{\Psi}_{\alpha\omega}, \Psi_{\alpha'\omega'}]\!] . \tag{30}$$

This equation may be used to derive biothogonality relations for the eigenfunctions of both the temporal and spatial stability problems.

For the temporal stability problem, α is real and given and α' equals α. The orthogonality relation for the temporal stability problem is then

$$(\omega' - \omega^*) < \tilde{\Psi}_{\alpha\omega}, \Psi_{\alpha\omega'} > = 0 , \tag{30}$$

so the solutions of the temporal stability problem and the adjoint solutions are orthogonal unless $\omega' = \omega^*$. The orthogonality condition, equation (30), can be recognized as being essentially equivalent to that derived by Schensted (1960; pg. 27, eq. (2.2.3)), and discussed by Reid (1965). The only difference is that Schensted's adjoint solution is the complex conjugate of ours.

In the case of spatial stability, ω is real and given and $\omega' = \omega$ and the orthogonality relation is

$$(\alpha' - \alpha^*) [\![\tilde{\Psi}_{\alpha\omega}, \Psi_{\alpha'\omega}]\!] = 0. \tag{31}$$

Thus, unless $\alpha' = \alpha^*$ the spatial eigenfunctions and adjoint eigenfunctions are orthogonal with the inner product defined by equation (25).

3. The Temporal Stability Problem

3.1 The Eigenvalues and Eigenfunctions

For the temporal stability problem we modify the finiteness condition, equation (7), to

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\{|\frac{\partial\Psi}{\partial x}|^2 + |\frac{\partial\Psi}{\partial y}|^2\}dxdy < \infty. \tag{7'}$$

This ensures that the Fourier integral expansion of Ψ ,

$$\Psi(x, y, t) = \int_{-\infty}^{\infty}\psi_\alpha(y, t)e^{i\alpha x}\, d\alpha , \tag{32}$$

exists. If we assume that ψ_α is of the form

$$\psi_\alpha(y, t) = \phi_\alpha(y)e^{-i\omega t} , \tag{33}$$

then ϕ_α is a solution of the Orr-Sommerfeld equation

$$\{L_\alpha^2 - i\alpha R[(U - c)L_\alpha - \frac{d^2U}{dy^2}]\}\phi_\alpha = 0 , \tag{34}$$

with

$$c = \omega/\alpha , \tag{35}$$

and

$$L_\alpha \equiv \frac{d^2}{dy^2} - \alpha^2 . \tag{36}$$

Similarly we assume that the adjoint solution, $\tilde{\Psi}$, also satisfies equation (7') thus ensuring that the Fourier integral expansion of

$$\tilde{\Psi}(x, y, t) = \int_{-\infty}^{\infty} \tilde{\psi}_\alpha(y, t)e^{i\alpha x}\, d\alpha \tag{37}$$

exists. It is assumed that $\tilde{\psi}_\alpha$ is of the form

$$\tilde{\psi}_\alpha(y, t) = \tilde{\phi}_\alpha(y)e^{-i\omega^* t} , \tag{38}$$

with $\tilde{\phi}_\alpha$ the solution of the adjoint Orr-Sommerfeld equation

$$\{L_\alpha^2 + i\alpha R[(U - c^*)L_\alpha + 2 \frac{dU}{dy}\frac{d}{dy}]\} \tilde{\phi}_\alpha = 0 , \tag{39}$$

with

$$c^* = \omega^*/\alpha. \tag{40}$$

Both ϕ_α and $\tilde{\phi}_\alpha$ satisfy the boundary conditions

$$\phi_\alpha(0) = \phi_\alpha'(0) = \tilde{\phi}_\alpha(0) = \tilde{\phi}_\alpha'(0) = 0 , \tag{41}$$

and either

$$\phi_\alpha \to \phi'_\alpha \to \tilde{\phi}_\alpha \to \tilde{\phi}'_\alpha \to 0 \qquad \text{as} \qquad y \to \infty \tag{42}$$

if ϕ_α and $\tilde{\phi}_\alpha$ are in M, or the weaker condition

$$\phi_\alpha,\ \phi'_\alpha,\ \tilde{\phi}_\alpha,\ \tilde{\phi}'_\alpha \qquad \text{bounded} \quad \text{as} \qquad y \to \infty, \tag{43}$$

if ϕ_α and $\tilde{\phi}_\alpha$ are in M'. Those eigenfunctions which belong to M will be called discrete eigenfunctions. Those which belong to M' but not M will be called continuum eigenfunctions.

It has been found (Mack, 1976, Grosch and Salwen, 1975, 1978a) that, in general, there is a finite number of discrete eigenfunctions, $\{\phi_{\alpha m}(y)\}$ with eigenvalues $\{\omega_{\alpha m}\}$ and a set, $\{\phi_{\alpha k}\}$, of continuum eigenfunctions with eigenvalues $\{\omega_{\alpha k}\}$ which depend continuously on a real parameter, k, in the range $[0, \infty)$. (Note that the k of this paper is equal to αk of Part 1.)

The number of discrete modes, which we shall denote by $N(\alpha)$, depends not only on α but also on R and on the form of $U(y)$ and can, in some cases, be zero. The adjoint eigenfunctions also include a finite set, $\{\tilde{\phi}_{\alpha m}\}$, of discrete eigenfunctions and a continuum, $\{\tilde{\phi}_{\alpha k}\}$, with eigenvalues $\{\omega^*_{\alpha m}\}$ and $\{\omega^*_{\alpha k}\}$, respectively (see discussion following (30)). For a given k, $\phi_{\alpha k}$ and $\tilde{\phi}_{\alpha k}$ vary like a linear combination of $e^{\pm iky}$ as $y \to \infty$. We therefore find that

$$\int_{k-\epsilon}^{k+\epsilon} \phi_{\alpha k'}(y)dk' \qquad \text{and} \qquad \int_{k-\epsilon}^{k+\epsilon} \tilde{\phi}_{\alpha k'}(y)dk' \ \epsilon\ M \tag{44}$$

and that, for any square-integrable f,

$$(f, \phi_{\alpha k}), \ (f, \tilde{\phi}_{\alpha k}), \ (f, \frac{d\phi_{\alpha k}}{dy}), \ (f, \frac{d\tilde{\phi}_{\alpha k}}{dy}) \text{ all exist.} \tag{45}$$

Inner products between continuum functions, such as $\langle\tilde{\phi}_{\alpha k}, \phi_{\alpha k'}\rangle$ do not exist in the ordinary sense but are definable in terms of the Dirac δ-function (Lighthill, 1960, pp. 10–21).

The discrete eigenvalues must be searched for (Mack, 1976), but the continuum eigenvalues follow from the asymptotic form ($\phi_{\alpha k}$, $\tilde{\phi}_{\alpha k}$ ~ linear combination of $e^{\pm iky}$) of the eigenfunctions as $y \to \infty$ and $U \to U_1 = U(\infty)$:

$$(-k^2 - \alpha^2)^2 - (i\alpha RU_1 - iR\omega_{\alpha k})(-k^2 - \alpha^2) = 0 \ , \tag{46a}$$

$$(-k^2 - \alpha^2)^2 + (i\alpha RU_1 - iR\omega^*_{\alpha k})(-k^2 - \alpha^2) = 0 \ , \tag{46b}$$

so that both equations yield

$$\omega_{\alpha k} = (\frac{-i}{R})(k^2 + \alpha^2 + i\alpha RU_1). \tag{47}$$

We also find that no continuum eigenvalue is also a discrete eigenvalue. Then

$$<\tilde{\phi}_{\alpha n}, \phi_{\alpha n'}> = 0 \quad \text{for} \quad \omega_{\alpha n} \neq \omega_{\alpha n'} \tag{48a}$$

$$<\tilde{\phi}_{\alpha n}, \phi_{\alpha k'}> = <\tilde{\phi}_{\alpha k}, \phi_{\alpha n'}> = 0 \ , \tag{48b}$$

and

$$<\tilde{\phi}_{\alpha k}, \int_{k_1}^{k_2} \phi_{\alpha k'} \, dk'> = 0 \quad \text{unless} \quad k_1 < k < k_2 . \tag{48c}$$

With proper labelling and normalization, it is then possible to choose the eigenfunctions in such a way that

$$<\tilde{\phi}_{\alpha n}, \phi_{\alpha n'}> = \delta_{nn'} , \tag{49a}$$

$$<\tilde{\phi}_{\alpha n}, \phi_{\alpha k'}> = <\tilde{\phi}_{\alpha k}, \phi_{\alpha n'}> = 0 , \tag{49b}$$

and

$$<\tilde{\phi}_{\alpha k}, \phi_{\alpha k'}> = \delta(k - k') . \tag{49c}$$

3.2 Expansion of an arbitrary disturbance.

If the aigenfunctions form a complete set, then, for any time, t, we may expand $\psi_\alpha(y, t)$ as a linear combination,

$$\psi_\alpha(y, t) = \sum_{n=1}^{N(\alpha)} a_n(\alpha, t)\phi_{\alpha n}(y) + \int_0^\infty a_k(\alpha, t)\phi_{\alpha k}(y)dk \tag{50}$$

of those aigenfunctions. To find the coefficients $\{a_n\}$ and $\{a_k\}$ we may make use of eq. (49) to take inner products

$$\langle\tilde{\phi}_{\alpha n}, \ \psi_\alpha(y, t)\rangle = \sum_{n'=1}^{N(\alpha)} a_{n'}(\alpha, t)\delta_{nn'} = a_n(\alpha, t), \tag{51a}$$

$$\langle\tilde{\phi}_{\alpha k}, \ \psi_\alpha(y, t)\rangle = \int_0^\infty a_{k'}(\alpha, t)\delta(k - k')dk' = a_k(\alpha, t). \tag{51b}$$

We then find that

$$\frac{\partial a_n(\alpha, t)}{\partial t} = \langle\tilde{\phi}_{\alpha n}, \frac{\partial \psi_\alpha}{\partial t}\rangle$$

$$= -i\omega_{\alpha n}\langle\tilde{\phi}_{\alpha n}, \ \psi_\alpha(y, t)\rangle = -i\omega_{\alpha n}\, a_n(\alpha, t). \tag{52a}$$

And, similarly,

$$\frac{\partial a_k(\alpha, t)}{\partial t} = \langle\tilde{\phi}_{\alpha k}, \frac{\partial \Psi_\alpha}{\partial t}\rangle = -i\omega_{\alpha k}\, a_k(\alpha, t), \tag{52b}$$

so that

$$a_n(\alpha, t) = A_n(\alpha)e^{-i\omega_{\alpha n}t} \tag{53a}$$

$$a_k(\alpha, t) = A_k(\alpha)e^{-i\omega_{\alpha k}t} \tag{53b}$$

where

$$A_n(\alpha) \equiv a_n(\alpha, 0) = \langle\tilde{\phi}_{\alpha n}, \Psi_\alpha(y, 0)\rangle \tag{54a}$$

$$A_k(\alpha) \equiv a_k(\alpha, 0) = \langle\tilde{\phi}_{\alpha k}, \Psi_\alpha(y, 0)\rangle . \tag{54b}$$

Then, referring to equations (32) and (50), we find

$$\Psi(x, y, t) = \int_{-\infty}^{\infty}\{\sum_{n=1}^{N(\alpha)} A_n(\alpha)\phi_{\alpha n}(y)e^{-i\omega_{\alpha n}t}$$

$$+ \int_0^{\infty} A_k(\alpha)\phi_{\alpha k}(y)e^{-i\omega_{\alpha k}t}dk\}e^{i\alpha x}d\alpha, \tag{55}$$

where

$$A_n(\alpha) = \langle \tilde{\phi}_{\alpha n}, \Psi_\alpha(y, 0) \rangle$$

$$= \frac{-1}{2\pi} \int_0^\infty \tilde{\phi}^*_{\alpha n}(y) \; L \int_{-\infty}^\infty \Psi(x, y, 0) e^{-i\alpha x} dx dy \tag{56a}$$

$$= \frac{-1}{2\pi} \int_0^\infty \tilde{\phi}^*_{\alpha n}(y) \int_{-\infty}^\infty [\nabla^2 \Psi]_{t=0} \; e^{-i\alpha x} dx dy,$$

and, similarly,

$$A_k(\alpha) = \frac{-1}{2\pi} \int_0^\infty \tilde{\phi}^*_{\alpha k}(y) \int_{-\infty}^\infty [\nabla^2 \Psi]_{t=0} \; e^{-i\alpha x} dx dy. \tag{56b}$$

If the discrete and continuum eigenfunctions form a complete set, then equation (55) constitutes an expansion of the stream function of an arbitrary disturbance in terms of the discrete (Tollmien-Schlichting) and continuum wave solutions,

$$\phi_{\alpha n}(y) e^{i(\alpha x - \omega_{\alpha n} t)} \quad \text{and} \quad \phi_{\alpha k}(y) e^{i(\alpha x - \omega_{\alpha k} t)} ,$$

of the disturbance equation, (3), with coefficients determined by the initial form of the disturbance $\Psi(x, y, 0)$. In the next section we will show that the discrete and continuum eigenfunctions are a complete set.

One interesting and significant result of this calculation is that the initial distribution of vorticity,

$$\zeta_0(x, y) \equiv \zeta(x, y, 0) \equiv \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]_{x,y,0} = -\nabla^2 \Psi(x, y, 0), \quad (57)$$

is sufficient information to determine the coefficients $A_n(\alpha)$ and $A_k(\alpha)$ and, therefore, the subsequent development of the disturbance.

4. Completeness of the Temporal Expansion Functions

Gustavsson (1979) has carried out a formal solution of the initial value problem of temporal stability for three dimensional disturbances. He uses the same coordinate system as we do with the addition of the z coordinate in the cross stream direction. The formal solution is obtained by taking Fourier transforms in both x and z and a Laplace transform in t, formally solving the Orr-Sommerfeld equation in the transform space, and formally inverting the transforms. If we eliminate the z-variation of Gustavsson's solution and his Fourier transform in z (replacing his k by $|\alpha|$) the two solutions should be identical. Both Gustavsson and we express the solution in physical space as an inverse Fourier transform over α, the transform variable in the x direction. In order to show that these two methods yield identical results it is therefore necessary to show that his formal solution in Fourier space, $\hat{v}$ as given in (G13)*, is equal to the factor in curly brackets in our equation (55).

In order to do this we must first translate Gustavsson's notation into our notation. Setting $\beta = 0$, after (G3) it is easily seen that we have the following correspondence,

* In order to simplify reference to the equations in Gustavsson's paper we will hereafter use the prefix G. Page references are also to Gustavsson's paper.

This paper	Gustavsson
Ψ	iv/α
$\lvert\alpha\rvert$	k
ω	is
k	σ
U_1	1

in (G3) and thereafter.

Gustavsson gives the formal solution in Fourier space in equation (G18). It consists of a sum of the residue values at the poles plus a contour integral along a branch cut. Using the definitions of W as the Wronskian, [redacted] the D_j, given after (G6), and the ϕ_j, equation (G7), it is quite straightforward to show that the residue, R_ν, at a pole s_ν is

$$R_\nu \equiv (e^{s_\nu t}/W) \lim_{s \to s_\nu} \{(s-s_\nu)\,[a_1(s)\phi_1(y,s) + a_2(s)\phi_2(y,s)]\}. \qquad (58)$$

Therefore the residue consists of a linear combination of ϕ_1 and ϕ_2 the solutions of the Orr-Sommerfeld equation that approach zero as $y \to \infty$, i.e. they satisfy (G4) and (42). At $s = s_\nu$, ϕ_1 and ϕ_2 satisfy the usual eigenvalue condition at $y = 0$ for the discrete modes of the Orr-Sommerfeld equation, condition (Gb) (at the bottom of page 1603). This linear combination thus satisfies (41). Therefore the residue at s_ν is proportional to our discrete eigenfunction $\phi_{\alpha\nu}(y)$ with eigenvalue $\omega_{\alpha\nu}$, and

$$e^{s_\nu t} \equiv e^{-i\omega_{\alpha\nu} t} .$$

It is well known (Coddington and Levinson, 1955, p. 101, problem 19) that $[D_j/W]^*$, the complex conjugates of the functions used in (G6), are solutions of the adjoint equation (39). It can be seen from the form of (G11) and the definition of our inner product (23) that

$$a_j = \langle D_j^*, \phi_o \rangle \qquad j = 3,4, \tag{59}$$

so that the coefficient of $\phi_{\alpha\nu}(y)$ in the residue is the inner product of <u>some</u> solution of the adjoint equation with $\phi(y,0)$. Finally, some straightforward, but tedious, algebra shows that the particular linear combination of the D_j^* involved satisfies the boundary conditions (41) and (42) and therefore is a multiple of our $\tilde{\phi}_{\alpha\nu}$. We thus find that the residue at s_ν is

$$R_\nu = d_{\alpha\nu}\, A_\nu(\alpha)\, \phi_{\alpha\nu}\, e^{-i\omega_{\alpha\nu} t} , \tag{60}$$

with $d_{\alpha\nu}$ independent of y and $A_\nu(\alpha)$ given by (54a). Before determining $d_{\alpha\nu}$, we turn to the contribution of the branch cut.

Using the fact that our $\omega = is$, it is clear from (G14) that the branch cut in the complex s plane is our continuous spectrum in the complex ω (or c) plane and that the branch point, $\mu = 0$, corresponds to the limit point of our continuous spectrum at $c = U_1 - i\,\alpha^2/R$, with $U_1 = 1$. The function $F(\alpha,k;\, y)$ in (G18) is, by (G17)and (G19) a linear combination of the solutions of the Orr-Sommerfeld equation which are, as $y \to \infty$, asymptotic to $e^{-\alpha y}$, e^{-iky}, and e^{+iky}. It can be shown, using (G19), (G20), (G21), and (G22), that

$$F(\alpha,k;0) = \left(\frac{dF}{dy}\right)_{y=0} = 0 , \tag{61}$$

and so $F(\alpha,k;\ y)$ is some multiple of our continuum eigenfunction $\phi_{\alpha k}(y)$. Further, it is obvious that, in (55),

$$e^{-i\omega_{\alpha k}t} = e^{-i\alpha U_1 t}\ e^{-(\alpha^2+k^2)t/R} \tag{62}$$

in (G18) with $U_1 = 1$.

Just as for the discrete modes, the $\{a_\nu^I\}$, $\nu = 2,3,4$ in (G21) are the inner product of some solutions of the adjoint equation with ϕ_0. Using the definition of the E_{mn} in (G22) and the definitions of the Ψ_m as given in the next to last paragraph on page 1604, some algebra shows that the particular linear combination satisfies the boundary conditions at $y = 0$ and so the inner product in (G20) is a multiple of the inner product of our continuum adjoint, $\tilde{\phi}_{\alpha k}(y)$, with the initial condition. Therefore, the integral term in (G18) is

$$I = \int_0^\infty d_{\alpha k}\ A_k(\alpha)\ \phi_{\alpha k}(y)\ e^{-i\omega_{\alpha k}t}\ dk\ , \tag{63}$$

with $A_k(\alpha)$ given by (54b) and $d_{\alpha k}$ independent of y. Gustavsson's result (G18) thus takes the form (in our notation)

$$\psi_\alpha\ (y,t) = \sum_{\nu=1}^{N(\alpha)} d_{\alpha\nu}\ A_\nu(\alpha)\ \phi_{\alpha\nu}(y)\ e^{-i\omega_{\alpha\nu}t}$$

$$+ \int_0^\infty d_{\alpha k}\ A_k(\alpha)\ \phi_{\alpha k}(y)\ e^{-i\omega_{\alpha k}t}\ dk\ . \tag{64}$$

Both Gustavsson and we may choose our initial condition arbitrarily, provided that the various integrals of this function with the adjoint functions exist. If we choose the initial condition that $\psi_\alpha(y,0)$ is one of the discrete eigenfunctions, say $\phi_{\alpha m}(y)$, then in (55)

$$A_n(\alpha) = \delta_{nm} \tag{65a}$$

$$A_k(\alpha) = 0 \ . \tag{65b}$$

In Gustavsson's formulation (63) we have

$$d_{\alpha\nu} \ A_\nu(\alpha) = \delta_{\nu m} \tag{66a}$$

$$A_k(\alpha) \equiv 0 \ . \tag{66b}$$

We thus see that

$$d_{\alpha\nu} \equiv 1 \ . \tag{67}$$

If we then choose the initial condition

$$\psi_\alpha(y,0) = \int_{k-\varepsilon}^{k+\varepsilon} \phi_{\alpha k'}(y) \ dk' \ , \tag{68}$$

a similar argument shows that

$$d_{\alpha k} \equiv 1 \ . \tag{69}$$

Substitution of $d_{\alpha\nu} = d_{\alpha k} = 1$, (67) and (69), makes eq. (64), derived from Gustavsson's solution, identical to the curly bracket in our expansion solution (55). We have thus shown that the formal solution obtained by Gustavsson from the Fourier-Laplace transform is identical, term by term, to our formal expansion solution.

Since any square-integrable solution possesses a Fourier-Laplace expansion, we have shown that our expansion (55) is complete whenever it is valid to separate the Fourier-Laplace transform solution into a sum over the poles plus an integral over the branch cut — that is, whenever the sum over the poles (discrete eigenvalues) converges. This is, of course, also the condition for the validity of Gustavsson's solution.

For the Blasius boundary layer, the numerical evidence (Mack 1976) indicates that, at a given R and α , the number of discrete modes is finite, so that the sum over the poles is a finite sum. If this is so, then the above condition is certainly satisfied and our expansion functions form a complete set.

We have shown that the Fourier-Laplace transform result and the eigenfunction expansion result are different forms of the same solution of the initial value problem to be chosen according to convenience in a particular case. The eigenfunction expansion formulation gives explicit formulae (54 a,b) to calculate the expansion coefficients. This allows one to calculate the amplitudes of the discrete modes (TS modes) and the continuum functions, given the initial distribution of vorticity.

5. The spatial stability problem

5.1 The eigenvalues and eigenfunctions

The finiteness condition, equation (7), is modified for the spatial stability problem to

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\{|\frac{\partial\Psi}{\partial x}|^2 + |\frac{\partial\Psi}{\partial y}|^2\}dtdy < \infty. \qquad (7'')$$

This ensures that the Fourier integral expansion of Ψ,

$$\Psi(x, y, t) = \int_{-\infty}^{\infty}\psi_{\omega}(x, y)e^{-i\omega t}d\omega \qquad (70)$$

exists. If we assume that ψ_{ω} is of the form

$$\psi_{\omega}(x, y) = \phi_{\omega}(y)e^{i\alpha x}, \qquad (71)$$

then ϕ_{ω} is the solution of the Orr-Sommerfeld equation

$$\{L_{\alpha}^2 - iR[(\alpha U - \omega)L_{\alpha} - \alpha\frac{d^2U}{dy^2}]\}\phi_{\omega} = 0, \qquad (72)$$

with L_{α} given by (36).

Similarly, we assume that the adjoint solution, $\tilde{\Psi}$, also satisfies equation (7") thus ensuring that

$$\tilde{\Psi}(x, y, t) = \int_{-\infty}^{\infty} \tilde{\Psi}_\omega(x, y) e^{-i\omega t} d\omega \tag{73}$$

exists. We assume that

$$\tilde{\Psi}_\omega(x, y) = \tilde{\phi}_\omega(y) e^{i\alpha^* x} . \tag{74}$$

Then $\tilde{\phi}_\omega$ is the solution of the adjoint Orr-Sommerfeld equation

$$\{L^2_{\alpha^*} + iR[(\alpha^* U - \omega) L_{\alpha^*} + 2\alpha^* \frac{dU}{dy} \frac{d}{dy}]\} \tilde{\phi}_\omega = 0 , \tag{75}$$

The boundary conditions are

$$\phi_\omega(0) = \phi'_\omega(0) = \tilde{\phi}_\omega(0) = \tilde{\phi}'_\omega(0) = 0 , \tag{76}$$

and

$$\phi_\omega \to \phi'_\omega \to \tilde{\phi}_\omega \to \tilde{\phi}'_\omega \to 0 \quad \text{as} \quad y \to \infty , \tag{77}$$

if ϕ_ω and $\tilde{\phi}_\omega$ are in M, or

$$\phi_\omega,\ \phi_\omega',\ \tilde{\phi}_\omega,\ \tilde{\phi}_\omega' \quad \text{bounded as} \quad y \to \infty , \tag{78}$$

if ϕ_ω and $\tilde{\phi}_\omega$ are in M'. As above, the eigenfunctions which belong to M are the discrete eigenfunctions and those that belong to M' but not M are the continuum eigenfunctions.

Jordinson (1971), Corner, Houston, and Ross (1976), and Murdock and Stewartson (1977) have shown that there is only a finite set of discrete eigenfunctions, $\{\phi_{\omega n}(y)\}$, with eigenvalues $\{\alpha_{\omega n}\}$. The set of discrete adjoint eigenfunctions, $\{\tilde{\phi}_{\omega n}\}$, with eigenvalues $\{\alpha^*_{\omega n}\}$ is also finite. The number of discrete modes, $N(\omega)$ depends on R as well as ω and can be zero.

In part 1 we showed that, in an unbounded domain, the spatial stability problem always has a continuous spectrum. Since then we have discovered (Grosch and Salwen, 1978b), that the spatial continuum of Part 1 is only one branch of a four branched spatial continuum. It is quite easy to show the existence of the four branches of the spatial continuum. We look for solutions to equations (72) and (75), $\phi_{\omega k}(y)$ and $\tilde{\phi}_{\omega k}(y)$, for a given real k, which vary like $e^{\pm iky}$ as $y \to \infty$ (the k used in discussing the spatial continuum in Part 1 is $2/R$ times the k used here). Noting that, as $y \to \infty$, $U \to U_1$, a constant, and U', $U'' \to 0$, we have

$$(-\alpha^2 - k^2)(-\alpha^2 - k^2 - i\alpha RU_1 + i\omega R) = 0 , \tag{79a}$$

and

$$(-\alpha^{*2} - k^2)(-\alpha^{*2} - k^2 + i\alpha^* RU_1 - i\omega R) = 0. \tag{79b}$$

(Note that equations (79a and b) are complex conjugates.)

It is obvious that there are four roots, $\{\alpha_j\}$, $j = 1, \ldots, 4$ with α_1 and α_2 the roots of

$$\alpha_j^2 + iRU_1\alpha_j + k^2 - i\omega R = 0 \tag{80}$$

and

$$\alpha_3 = ik \ , \ \alpha_4 = -ik \tag{81a,b}$$

The eigenvalue α_1, the root of equation (80) with positive real part, is the continuum eigenvalue discussed in Part 1. As was discussed in Part 1, the eigenfunctions of this branch of the spatial continuum are waves propagating in the downstream (+x) direction and decaying in amplitude as they travel. In the same way it can be shown that α_2 is the eigenvalue of a continuum eigenfunction which is a wave traveling in the upstream (-x) direction and decaying as it travels.

The free stream speed, U_1, can be taken to be unity for a boundary layer, wake, or free shear flow. In most cases of interest $\omega/R << 1$. It is easy to see that, with $U_1 = 1$, and $\omega/R << 1$,

$$\alpha_{1,2} = \pm(\omega/\gamma) - iR[\tfrac{1}{2}(1 \mp \gamma) \mp (\omega/R)^2/\gamma^3] + O(\frac{\omega^3}{R^2}) \tag{82}$$

with

$$\gamma = (1 + 4k^2/R^2)^{1/2} . \tag{83}$$

Define, as usual, the phase speed c_j by

$$c_j = \alpha_j^* \omega / |\alpha_j|^2 . \tag{84}$$

Then as $k \to 0$,

$$\alpha_1 \simeq \omega + iR[(\omega^2 + k^2)/R^2], \tag{85a}$$

$$\alpha_2 \simeq -\omega - iR[1 + (\omega^2 + k^2)/R^2], \tag{85b}$$

$$c_1 \simeq 1 - i[(\omega^2 + k^2)/R^2]/(\omega/R), \tag{86a}$$

$$c_2 \simeq -(\omega/R)^2 + i(\omega/R)[1 + (\omega^2 + k^2)/R^2]. \tag{86b}$$

While as $k \to \infty$,

$$\alpha_1 \simeq (\omega R/2k) + ik , \tag{87a}$$

$$\alpha_2 \simeq -\alpha_1 , \tag{87b}$$

$$c_1 \simeq \omega^2 R/2k^3 - i\omega/k , \tag{88a}$$

$$c_2 \simeq -c_1 . \tag{88b}$$

The damping rate, for the spatial eigenfunctions, is $Im(\alpha)$ and the phase speed is $Re(c)$. The equations given above show that the eigenfunctions on branch two of the spatial continuum, for boundary layers, wakes, and free shear flows, always have both a very large damping rate and a very small phase speed. This is in marked contrast to those of branch one, which, as was shown in Part 1, or can be seen from the above results, contains lightly damped eigenfunctions some of which have a very slow phase speed and some of which have a phase speed nearly equal to the free stream speed.

The spatial continuum eigenfunctions of branches 3 and 4 are standing waves in x because they vary like

$$e^{i\alpha_3 x} = e^{-kx}, \qquad e^{i\alpha_4 x} = e^{+kx}. \tag{89a,b}$$

As in the temporal case, the inner products between the spatial continuum eigenfunctions do not exist in the ordinary sense but can be defined as δ functions. Then, with proper labeling and normalization, it is possible to choose the eigenfunctions such that (with the superscript i or j indicating the branch of the continuum)

$$[\![\tilde{\phi}_{\omega n}, \phi_{\omega n'}]\!] = \delta_{nn'} , \tag{90a}$$

$$[\![\tilde{\phi}_{\omega n}, \phi_{\omega k}^{(i)}]\!] = [\![\tilde{\phi}_{\omega k}^{(i)}, \phi_{\omega n}]\!] = 0 , \tag{90b}$$

$$[\![\tilde{\phi}_{\omega k}^{(i)}, \phi_{\omega k'}^{(j)}]\!] = \delta(k - k')\delta_{ij} , \tag{90c}$$

where

$$
\begin{aligned}
[\![\tilde{\phi}_{\omega n}, \phi_{\omega n'}]\!] \equiv \int_0^\infty \{ [& \frac{-i(\alpha_{\omega n'} + \alpha_{\omega n})}{R}][(\alpha^2_{\omega n'} \\
& + \alpha_{\omega n}{}^2 - i\omega R)\tilde{\phi}^*_{\omega n}\, \phi_{\omega n'} + 2\, \frac{d\tilde{\phi}^*_{\omega n}}{dy}\, \frac{d\phi_{\omega n'}}{dy}] \\
& + U[(\alpha^2_{\omega n'} + \alpha_{\omega n'}\, \alpha_{\omega n} + \alpha^2_{\omega n})\tilde{\phi}^*_{\omega n}\, \phi_{\omega n'} \\
& + 2\, \frac{d\tilde{\phi}^*_{\omega n}}{dy}\, \frac{d\phi_{\omega n'}}{dy} + \frac{d^2\tilde{\phi}^*_{\omega n}}{dy}\, \phi_{\omega n'}] \} dy \;, \qquad (91)
\end{aligned}
$$

and there are analogous expressions for the inner products in (90b) and (90c).

5.2 Expansion of an Arbitrary Disturbance.

If the spatial eigenfunctions form a complete set, then, for any x, we may expand $\Psi_\omega(x, y)$ as

$$\Psi_\omega(x, y) = \sum_{n=1}^{N(\omega)} a_n(\omega, x)\, \phi_{\omega n}(y) + \sum_{i=1}^{4} \int_0^\infty a_k^{(i)}(\omega, x)\phi_{\omega k}^{(i)}(y)\,dk. \quad (92)$$

In order to find the coefficients $\{a_n(\omega, x)\}$ and $\{a_k^{(i)}(\omega, x)\}$ we use equation (90) to take the inner products

$$[\![\tilde{\phi}_{\omega n}, \Psi_\omega(x, y)]\!] = \sum_{n'=1}^{N(\omega)} a_{n'}(\omega, x)\delta_{nn'} = a_n(\omega, x), \quad (93a)$$

$$[\![\tilde{\phi}_{\omega k}^{(i)}, \Psi_\omega(x, y)]\!] = \int_0^\infty a_{k'}^{(j)}(\omega, x)\delta(k - k')\delta_{ij}\,dk' = a_k^{(i)}(\omega, x). \quad (93b)$$

Then

$$\frac{\partial a_n(\omega, x)}{\partial x} = [\![\tilde{\phi}_{\omega n}, \frac{\partial \Psi_\omega}{\partial x}]\!] = i\alpha_{\omega n} [\![\tilde{\phi}_{\omega n}, \Psi_\omega]\!]$$

$$= i\alpha_{\omega n}\, a_n(\omega, x), \quad (94a)$$

and

$$\frac{\partial a_k^{(i)}(\omega, x)}{\partial x} = [\![\tilde{\phi}_{\omega k}, \frac{\partial \Psi_\omega}{\partial x}]\!] = i\alpha_{\omega k}^{(i)} [\![\tilde{\phi}_{\omega k}, \Psi_\omega]\!]$$

$$= i\alpha_{\omega k}^{(i)} \; a_k^{(i)}(\omega, x), \tag{94b}$$

so that

$$a_n(\omega, x) = A_n(\omega) e^{i\alpha_{\omega n} x} , \tag{95a}$$

$$a_k^{(i)}(\omega, x) = A_k^{(i)}(\omega) e^{i\alpha_{\omega k}^{(i)} x} , \tag{95b}$$

where

$$A_n(\omega) \equiv a_n(\omega, 0) = [\![\tilde{\phi}_{\omega n}, \quad \Psi_\omega(0, y)]\!] , \tag{96a}$$

$$A_k^{(i)}(\omega) \equiv a_k^{(i)}(\omega, 0) = [\![\tilde{\phi}_{\omega k}^{(i)}, \quad \Psi_\omega(0, y)]\!] . \tag{96b}$$

From equations (73), (92), and (95), we have the formal solution to the spatial stability problem for the two dimensional, linearized Navier-Stokes equations

$$\Psi(x, y, t) = \int_{-\infty}^{\infty} \{ \sum_{n=1}^{N(\omega)} A_n(\omega) \phi_{\omega n}(y) e^{i\alpha_{\omega n} x}$$

$$+ \sum_{i=1}^{4} \int_0^{\infty} A_k^{(i)}(\omega) \phi_{\omega k}^{(i)}(y) e^{i\alpha_{\omega k}^{(i)} x} \, dk \} e^{-i\omega t} \, d\omega . \tag{97}$$

Define

$$\psi_\omega^{(0)}(y) \equiv \psi_\omega(0,\ y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(0,\ y,\ t)e^{i\omega t}dt \ , \tag{98a}$$

$$\psi_\omega^{(1)}(y) \equiv \left(\frac{\partial\psi_\omega(x,\ y)}{\partial x}\right)_{x=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\partial\Psi(x,\ y,\ t)}{\partial x}\right)_{x=0} e^{i\omega t}\ dt, \tag{98b}$$

$$\psi_\omega^{(2)}(y) \equiv \left(\frac{\partial^2\psi_\omega(x,\ y)}{\partial x^2}\right)_{x=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\partial^2\Psi(x,\ y,\ t)}{\partial x^2}\right)_{x=0} e^{i\omega t}\ dt, \tag{98c}$$

$$\psi_\omega^{(3)}(y) \equiv \left(\frac{\partial^3\psi_\omega(x,\ y)}{\partial x^3}\right)_{x=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\partial^3\Psi(x,\ y,\ t)}{\partial x^3}\right)_{x=0} e^{i\omega t}\ dt. \tag{98d}$$

Then

$$\begin{aligned}
A_n(\omega) &= [\![\tilde\phi_{\omega n},\ \psi_\omega(0,\ y)]\!] \\
&= \int_0^{\infty} \frac{1}{R}[\tilde\phi^*_{\omega n}\ \psi_\omega^{(3)} + i\alpha_{\omega n}\ \tilde\phi^*_{\omega n}\ \psi_\omega^{(2)} - \alpha_{\omega n}^2\ \tilde\phi^*_{\omega n}\ \psi_\omega^{(1)} \\
&\quad - i\alpha_{\omega n}^3\ \tilde\phi^*_{\omega n}\ \psi_\omega^{(0)} - i2\alpha_{\omega n}\ \frac{d\tilde\phi^*_{\omega n}}{dy}\ \frac{\partial\psi_\omega^{(0)}}{\partial y} \\
&\quad - 2\ \frac{d\tilde\phi^*_{\omega n}}{dy}\ \frac{d\psi_\omega^{(1)}}{dy}\ ] - i\omega[\tilde\phi^*_{\omega n}\psi_\omega^{(1)} - i\alpha\ \ \tilde\phi^*_\omega\ \psi_{\omega n}^{(0)}\ ] \\
&\quad - U[\tilde\phi^*_{\omega n}\ \psi_\omega^{(2)} + i\alpha_{\omega n}\tilde\phi^*_{\omega n}\psi_\omega^{(1)} - \alpha_{\omega n}^2\tilde\phi^*_{\omega n}\ \ \psi_\omega^{(0)} \\
&\quad - 2\ \frac{d\tilde\phi^*_{\omega n}}{dy}\ \ \frac{d\psi_\omega^{(0)}}{dy} - \frac{d^2\tilde\phi^*_{\omega n}}{dy^2}\ \psi_\omega^{(0)}]]\ dy,
\end{aligned} \tag{99}$$

and there is a similar expression for $A_k^{(i)}(\omega)$.

This is the formal solution of the spatial stability problem for an arbitrarily imposed boundary condition at $x = 0$. The boundary conditions which must be specified are the Fourier transforms in time, of the stream function and its first three partial derivatives with respect to x, evaluated at $x = 0$.

As it stands, this formal solution will not give a physically acceptable solution because, given an arbitrary $\Psi(0, y, t)$ and derivatives, disturbances which lie on all four branches of the continuum will be excited. Therefore the solution will contain, in addition to the waves propagating towards $x = \infty$ and the standing waves whose amplitude decays towards $x = \infty$, waves propagating upstream from $x = \infty$ and standing waves whose amplitude increases towards $x = \infty$.

A condition must be imposed that, for $x > 0$, all propagating disturbances are traveling in the positive x direction and all standing waves have amplitudes which decay in the positive x direction. It appears that this should be done by requiring that $\Psi(0, y, t)$ and its first three partial derivatives with respect to x be orthogonal to all eigenfunctions on branches 2 and 4 of the continuous spectrum but we have not yet investigated the implications of imposing this condition on the disturbance stream function at $x = 0$.

6. Application to the Temporal Development of a Model Flow

In this section, we apply the results of section 3. to the simple base flow,

$$U(y) = U_1 = \text{constant}, \qquad y \geq 0 , \tag{100}$$

which is a slip flow past a bounding plane at $y = 0$. Though the base flow velocity does not vanish at the boundary, we still require the disturbance velocity to be zero at $y = 0$. Because of the simplicity of the base flow, the expansion functions are elementary functions. In 6.1, we find the expansion functions. In this case, there are no discrete eigenmodes; all of the eigenfunctions are continuum functions.

In Subsection 6.2, we solve for the time-development of a particular initial disturbance by expanding in terms of these eigenfunctions. The initial disturbance chosen is a periodic layer of vorticity confined to a plane parallel to the ($y = 0$) boundary. Because of the simple form of the initial disturbance and the simplicity of the base flow, it is possible to obtain the solution in closed form in terms of error functions.

6.1 The Eigenfunctions

For the base flow of Equation (100) the differential equation, (34), for the expansion functions becomes

$$\left(\frac{d^2}{dy^2} - \alpha^2 - i\alpha R\,(U_1 - c)\right)\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi = 0 , \tag{101}$$

with the general solution (for $\alpha \neq 0$, $c \neq U_1$),

$$\phi = Ae^{-|\alpha|y} + Be^{|\alpha|y} + Ce^{py} + De^{-py} , \tag{102}$$

where

$$p^2 = \alpha^2 + i\alpha R(U_1 - c). \tag{103}$$

(In this case of constant U, $\tilde{\phi}$ must satisfy the <u>same</u> differential equation). In addition, ϕ must satisfy Eqs. (11) and (13); i.e., ϕ and ϕ' must vanish at $y = 0$ and be <u>bounded in</u> $[0, \infty)$. Since $e^{|\alpha|y}$ is unbounded, $B = 0$. To satisfy the boundary condition at the origin, we must then have

$$\phi = A[e^{-|\alpha|y} - \cosh py + \frac{|\alpha|}{p} \sinh py] , \tag{104}$$

which is unbounded as $y \to \infty$ unless p is purely imaginary. The solutions are then given by

$$p = ik; \qquad 0 < k < \infty , \tag{105}$$

$$\omega_{\alpha k} = -i(\alpha^2 + k^2 + i\alpha RU_1)/R , \tag{47}$$

$$\phi_{\alpha k}(y) = \tilde{\phi}_{\alpha k}(y) = A_{\alpha k}[(e^{-|\alpha|y} - \cos ky) + \frac{|\alpha|}{k} \sin ky], \tag{106}$$

where the normalization constant,

$$A_{\alpha k} = \frac{k}{k^2 + \alpha^2} \sqrt{\frac{2}{\pi}} \quad , \tag{107}$$

is determined by the condition

$$<\tilde{\phi}_{\alpha k} , \phi_{\alpha k'}> = \delta(k - k'). \tag{108}$$

In this case, where the $\phi_{\alpha k}$ and $\tilde{\phi}_{\alpha k}$ are known explicitly, one may show directly that, for $F(y)$ any continuous, differentiable, square-integrable function in $[0,\infty)$,

$$\int_0^\infty <\tilde{\phi}_{\alpha k}, F> \phi_{\alpha k}(y)dk = F(y) - e^{-|\alpha|y} F(0) \tag{109}$$

thus confirming that the set of $\{\phi_{\alpha k}\}$ is complete for functions in M, with $F(0) = 0$.

6.2 The Temporal Evolution of An Initial Disturbance

In order to demonstrate the application of this expansion technique, we consider the particular initial disturbance

$$-\nabla^2 \Psi(x, y, 0) = \Upsilon(x, y, 0) = \Upsilon_0 \, e^{i\alpha_0 x} \, \delta(y - y_0), \tag{110}$$

a periodic layer of vorticity at a distance y_0 from the boundary. Following section 3.2, we find that the stream function at any time will be given by

$$\Psi(x, y, t) = \int_{-\infty}^{\infty} \int_{0}^{\infty} A_k(\alpha)\phi_{\alpha k}(y) \, e^{-i\omega_{\alpha k} t} \, dk \, e^{i\alpha x} \, d\alpha \, , \tag{111}$$

where

$$A_k(\alpha) = \frac{1}{2\pi} \int_0^{\infty} \tilde{\phi}_{\alpha k}(y) \int_{-\infty}^{\infty} \Upsilon(x, y, 0) e^{-i\alpha x} \, dxdy. \tag{112}$$

It is easily seen, by substituting Eq. (110) into Eq. (112), that

$$A_k(\alpha) = \phi_{\alpha k}(y_0) \, \delta(\alpha - \alpha_0), \tag{113}$$

so that

$$\Psi(x, y, t) = \int_{-\infty}^{\infty}\int_0^{\infty} \phi_{\alpha k}(y_0)\delta(\alpha - \alpha_0)\phi_{\alpha k}(y)\ e^{-i\omega_{\alpha k}t}\ dk\ e^{i\alpha x}\ d\alpha$$

$$= e^{i\alpha_0(x - U_1 t)}\ e^{-\alpha_0^2 t/R} \int_0^{\infty} \phi_{\alpha_0 k}(y_0)\ \phi_{\alpha_0 k}(y)\ e^{-k^2 t/R}\ dk. \qquad (114)$$

After using Eqs. (106) and (107), for $\phi_{\alpha_0 k}$, we find that each term in the integral is expressible as sums of error functions. The results are given in an Appendix. From these results, it can be shown that, for $t \to \infty$ with y fixed,

$$\Psi \sim \frac{1}{\sqrt{t}}\ e^{-\alpha_0^2 t/R} \cos \alpha_0(x - U_1 t) \cdot (\text{function of } y) \qquad (115)$$

and, for $y \to \infty$ with t fixed,

$$\Psi \sim\ e^{-\alpha_0(y + y_0)} \cos \alpha_0(x - U_1 t) \cdot (\text{function of } t). \qquad (116)$$

It is clear that, even though the individual eigenfunctions used in the expansion oscillate with constant amplitude as $y \to \infty$, the <u>wave packet</u> behaves like $e^{-\alpha_0 y}$ as $y \to \infty$.

Figure 1 shows contour plots of the stream function for the disturbance, in a frame of reference moving with the free stream velocity, at six different times. We have chosen $\alpha_0 = 1.0$ and $y_0 = 1.0$ for the example shown here. Contours of the disturbance stream function have also been calculated for other combinations of values of α_0 and y_0 and, for these other values, the evolution of the disturbance in time is quite similar to that shown in figure 1.

In figure 1 the (+) and (-) indicate the position of the maximum and minimum values of the stream function. These maximum and minimum values are given in the caption to the figure. The flow is counter-clockwise around a maximum (+) and clockwise around a minimum (-).

It is clear from this figure that the disturbance, which is a periodic vortex sheet at $t = 0$, retains its identity as a periodic array for all time, but as time increases it diffuses, the strength decays, and the centers of the vortices drift away from the boundary at $y = 0$.

We could, of course, generalize this model problem by considering an initial vorticty distribution in the y direction. We have not carried out this calculation because our intent in solving this model problem was to illustrate the expansion procedure and we do not think that it warrants further elaboration.

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We wish to thank Kenneth M. Case, who referred this paper and suggested the approach which we used to prove completeness of the temporal eigenfunctions, and George F. Carrier, who convinced us that we could carry out that proof.

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Figure Caption

Figure 1: Contours of the disturbance stream function for the model problem in a frame of reference moving with the free stream velocity at six different times. In this example $\gamma_0 = 1.0$, $\alpha_0 = 1.0$, and $y_0 = 1.0$. There are twenty contour lines on each plot. The values of Ψ on these contours are 0.95 Ψmax, 0.85 Ψmax, ... , -0.95 Ψmax. The (+) and (-) indicate the positions where $\Psi = \Psi_{max}$ and Ψ_{min}. Note that $\Psi_{min} = -\Psi_{max}$.

(a) $t/R = 10^{-3}$, $\Psi_{max} = 0.425$

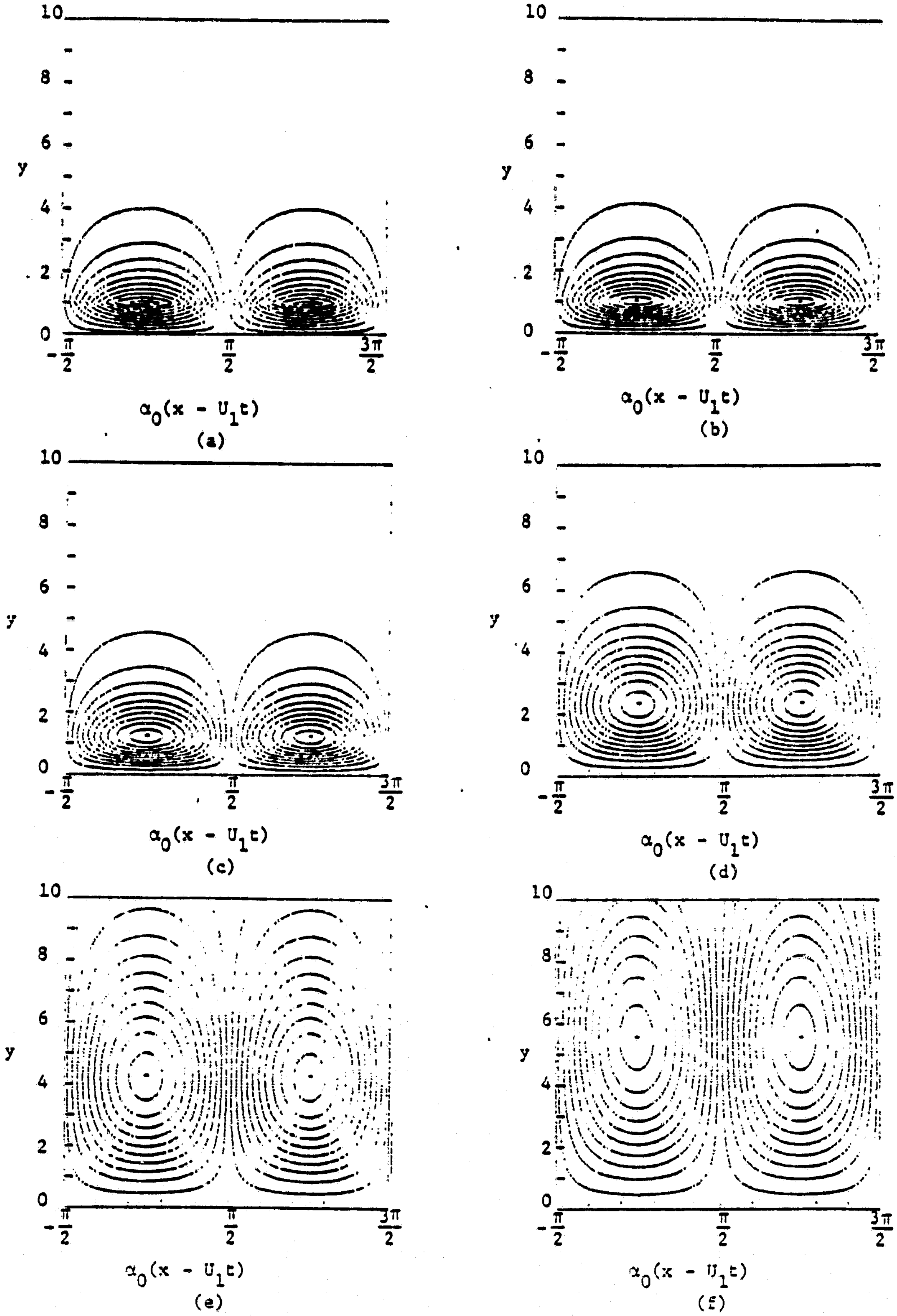
(b) $t/R = 10^{-2}$, $\Psi_{max} = 0.359$

(c) $t/R = 10^{-1}$, $\Psi_{max} = 0.212$

(d) $t/R = 1.0$, $\Psi_{max} = 0.228 \times 10^{-1}$

(e) $t/R = 5.0$, $\Psi_{max} = 0.108 \times 10^{-3}$

(f) $t/R = 10.0$, $\Psi_{max} = 0.383 \times 10^{-6}$



10
8
6
y
4
2
0
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$\frac{3\pi}{2}$
$\alpha_0(x - U_1t)$
(a)
(b)
(c)
(d)
(e)
(f)

Appendix: Solution of the Model Problem

In section 6 we showed that the stream function for the model problem is, equation (114),

$$\Psi(x, y, t) = e^{i\alpha_0(x - U_1 t)} e^{-\alpha_0^2 t/R} \int_0^\infty e^{-k^2 t/R} \phi_{\alpha_0 k}(y_0)\phi_{\alpha_0 k}(y)\,dk,$$

where $\phi_{\alpha_0 k}(y)$ is given by (106) and (107). Substituting for $\phi_{\alpha_0 k}(y)$ and $\phi_{\alpha_0 k}(y_0)$ in this integral it is straightforward to show that, with $\tau = t/R$,

$$\Psi(x, y, t) = e^{i\alpha_0(x - U_1 t)}\{I_1(\alpha_0, \tau) - e^{\alpha_0 y} I_2(\alpha_0, \tau, y_0)$$

$$+ \alpha_0 e^{-\alpha_0 y} I_3(\alpha_0, \tau, y_0) - e^{-\alpha_0 y_0} I_2(\alpha_0, \tau, y)$$

$$+ \frac{1}{2} I_2(\alpha_0, \tau, y + y_0) + \frac{1}{2} I_2(\alpha_0, \tau, y - y_0)$$

$$- \alpha_0 I_3(\alpha_0, \tau, y + y_0) + \alpha_0 e^{-\alpha_0 y_0} I_3(\alpha_0, \tau, y)$$

$$+ \frac{1}{2} \alpha_0^2 I_4(\alpha_0, \tau, y - y_0) - \frac{1}{2} \alpha_0^2 I_4(\alpha_0, \tau, y + y_0)\}, \qquad \text{(A1)}$$

where the functions I_j are given by

$$I_1(\alpha,\tau) = \frac{2}{\pi} e^{-\alpha^2\tau} \int_0^\infty e^{-k^2\tau} \left[\frac{k^2}{(k^2+\alpha^2)^2}\right]dk$$

$$= \alpha^{-1}\left[\left(\frac{1}{2}+\alpha^2\tau\right) \text{ erfc } (\alpha\tau^{1/2}) - \frac{\alpha\tau^{1/2}}{\sqrt{\pi}} e^{-\alpha^2\tau}\right], \qquad \text{(A2)}$$

$$I_2(\alpha, \tau, Z) \equiv \frac{2}{\pi} e^{-\alpha^2\tau} \int_0^\infty e^{-k^2\tau}\left[\frac{k^2}{(k^2+\alpha^2)^2}\right] \cos kZdk$$

$$= \frac{1}{4\alpha}\left[(1+2\alpha^2\tau-\alpha Z)e^{-\alpha Z} \text{ erfc } (\alpha\tau^{1/2} - \alpha Z/2\alpha\tau^{1/2})\right.$$

$$+ (1+2\alpha^2\tau+\alpha Z)e^{\alpha Z} \text{ erfc } (\alpha\tau^{1/2} + \alpha Z/2\alpha\tau^{1/2})$$

$$\left. - \frac{4\alpha\tau^{1/2}}{\sqrt{\pi}} e^{-\alpha^2\tau} e^{-\alpha^2 Z^2/4\alpha^2\tau}\right], \qquad \text{(A3)}$$

$$I_3(\alpha, \tau, Z) \equiv \frac{2}{\pi} e^{-\alpha^2\tau} \int_0^\infty e^{-k^2\tau}\left[\frac{k}{(k^2+\alpha^2)^2}\right] \sin kZdk$$

$$= \frac{1}{4\alpha^2}\left[(2\alpha^2\tau+\alpha Z)e^{\alpha Z} \text{ erfc } (\alpha\tau^{1/2} + \alpha Z/2\alpha\tau^{1/2})\right.$$

$$\left. - (2\alpha^2\tau-\alpha Z)e^{-\alpha Z} \text{ erfc } (\alpha\tau^{1/2} - \alpha Z/2\alpha\tau^{1/2})\right], \qquad \text{(A4)}$$

$$I_4(\alpha,\ \tau,\ Z) \equiv \frac{2}{\pi}\, e^{-\alpha^2\tau} \int_0^\infty e^{-k^2\tau} \left[\frac{1}{(k^2+\alpha^2)^2}\right] \cos kZ dk$$

$$= \frac{1}{4\alpha^3} \left[(1 - 2\alpha^2\tau + \alpha Z) e^{-\alpha Z} \operatorname{erfc}\,(\alpha\tau^{\frac{1}{2}} - \alpha Z/2\alpha\tau^{\frac{1}{2}})\right.$$

$$+ (1 - 2\alpha^2\tau - \alpha Z) e^{\alpha Z} \operatorname{erfc}\,(\alpha\tau^{\frac{1}{2}} + \alpha Z/2\alpha\tau^{\frac{1}{2}})\ .$$

$$\left. + \frac{4\alpha\tau^{\frac{1}{2}}}{\sqrt{\pi}}\, e^{-\alpha^2\tau}\, e^{-\alpha^2 Z^2/4\alpha^2\tau}\right] \qquad (A5)$$

and, as usual

$$\operatorname{erfc}(Z) \equiv \frac{2}{\sqrt{\pi}} \int_Z^\infty e^{-\xi^2}\, d\xi. \qquad (A6)$$

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APPENDIX C

COMPLETENESS OF SPATIAL EIGENFUNCTIONS FOR THE BOUNDARY LAYER

To be presented at the annual meeting of the Division of Fluid Dynamics of the American Physical Society (Ithaca, N.Y.), Nov. 24-26, 1980.

Abstract submitted
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Physics and Astronomy
Classification Scheme
Number 47

Suggested title of session in
which paper should be placed
Stability

Completeness of Spatial Eigenfunctions of the Boundary Layer, H. SALWEN*, K.A. KELLY, Stevens Inst. of Tech. and C.E. GROSCH*, Old Dominion U.--This is a continuation of earlier work[1] in which completeness of the temporal eigenfunctions was demonstrated by showing that the Fourier-Laplace transform solution[2] for the temporal evolution is equivalent to the eigenfunction expansion. Here, we take a Fourier transform in t and a Laplace transform in x and solve for the spatial evolution of the Fourier components. As in the temporal case, there are pole contributions equivalent to a sum over the discrete spatial modes and branch cut contributions equivalent to integrals over the spatial continua. The two methods are therefore equivalent.
*Partially supported by NASA grants NSG 1618 and 1619.
[1]H. Salwen and C.E. Grosch, J. Fluid Mech., to appear.
[2]L.H. Gustavsson, Phys. Fluids 22, 1602 (1979).

(✓) Prefer standard session

Submitted by:

Harold Salwen
Dept. of Physics/Eng. Physics
Stevens Inst. of Tech.
Hoboken, N.J. 07030

APPENDIX D

EIGENFUNCTION EXPANSIONS AND BOUNDARY-LAYER RECEPTIVITY IN THE THEORY OF HYDRODYNAMIC STABILITY

Presented at the 15th International Congress of Theoretical and Applied Mechanics (Toronto, Canada), Aug. 17-23, 1980

EIGENFUNCTION EXPANSIONS AND BOUNDARY-LAYER RECEPTIVITY IN THE THEORY OF HYDRODYNAMIC STABILITY*

Chester E. Grosch
Department of Oceanography and Department of Mathematics
Old Dominion University
Norfolk, V.A. 23508

and

Harold Salwen
Department of Physics and Engineering Physics
Stevens Institute of Technology
Hoboken, N.J. 07030

ABSTRACT

In this paper we give the solution of the boundary-layer receptivity problem: that of determining the amplitudes of the Tollmien-Schlichting modes and continuum eigenfunctions of a boundary layer given the form of the velocity profile and the disturbance, within the context of incompressible, linear stability theory for a parallel shear flow. We give the formal solution to the initial value problem for temporal stability and give the proper initial condition for this problem. The formal solution of the spatial stability problem is also given and the proper boundary conditions at $x = 0$ and radiation conditions at $x = \infty$ are discussed. We give examples of the application of this method to the calculation of the temporal evolution of a particular disturbance in two flows, a constant base flow and the Blasius boundary layer.

*This work was supported, in part, by the National Aeronautics and Space Administration under Grants NSG 1618 and 1619.

EIGENFUNCTION EXPANSIONS AND BOUNDARY-LAYER RECEPTIVITY IN THE THEORY OF HYDRODYNAMIC STABILITY

Ch)ster E. Grosch
Department of Oceanography and Department of Mathematics
Old Dominion University
Norfolk, Va. 23508

and

Harold Salwen
Department of Physics and Engineering Physics
Stevens Institute of Technology
Hoboken, N.J. 07030

SUMMARY

The last ten years has seen an increasing use of the theory of hydrodynamic stability to predict transition in boundary layers. Mack (1977) gives an excellent, up to date review of various transition prediction methods. All of these methods include at least one unknown parameter A_0, the initial amplitude of the disturbance in the boundary layer. There are numerous discussions of the boundary-layer receptivity problem, that is, the problem of determining A_0 given the velocity profile of the boundary layer and the disturbance (Obremski, Morkovin, and Landahl, 1969; Mack, 1977; Berger and Aroesty, 1977). All of these authors conclude that the mechanism by which free-stream vorticity and sound disturbances generate Tollmien-Schlichting waves in a boundary layer is unknown.

In this paper we give the solution of the boundary-layer receptivity problem within the context of incompressible, linear stability theory for a parallel shear flow. The expansion of an arbitrary two-dimensional solution of the linearized stream function equation in terms of the discrete and continuum eigenfunctions of the Orr-Sommerfeld equation is discussed for flows in the half-space, $y \varepsilon [0, \infty)$. A recent result of Salwen is used to derive a biorthogonality relation between the solution of the linearized equation for the stream function and the solution of the adjoint problem.

For the case of temporal stability, the orthogonality relation obtained is equivalent to that of Schensted (1960) for bounded flows. This relationship is used to carry out the formal solution of the initial value problem for temporal stability. It is shown that the vorticity of the disturbance at $t = 0$ is the proper initial condition for the temporal stability problem.

For the spatial stability problem it is shown that the continuous spectrum of the Orr-Sommerfeld equation contains four branches. The modes on these brances are (1) waves propagating downstream, (2) waves

propagating upstream, (3) standing waves whose amplitudes decrease downstream, and (4) standing waves whose amplitudes decrease in the upstream direction. The biorthogonality relation is used to derive the formal solution to the boundary value problem of spatial stability. We show that the boundary value problem of spatial stability requires the stream function and its first three partial derivatives with respect to x be specified at $x = 0$ for all time. The imposition of a radiation condition downstream, i.e. at $x = \infty$, eliminates disturbances which originate at $x = \infty$ and travel upstream to $x = 0$. The imposition of this radiation condition reduces the number of independent boundary conditions at $x = 0$ from four to two.

We give two examples of the application of this method to calculate the temporal receptivity of boundary layers to a disturbance. We specify the disturbance at $t = 0$ to be a vortex sheet parallel to the boundary and sinusoidal in the streamwise direction. We then calculate the evolution in time of this disturbance in (1) a constant base flow, for which the calculation can be carried out analytically and (2) in the Blasius boundary layer for which we calculate the amplitudes of the discrete Tollmien-Schlichting waves and of the continuum eigenfunctions numerically.

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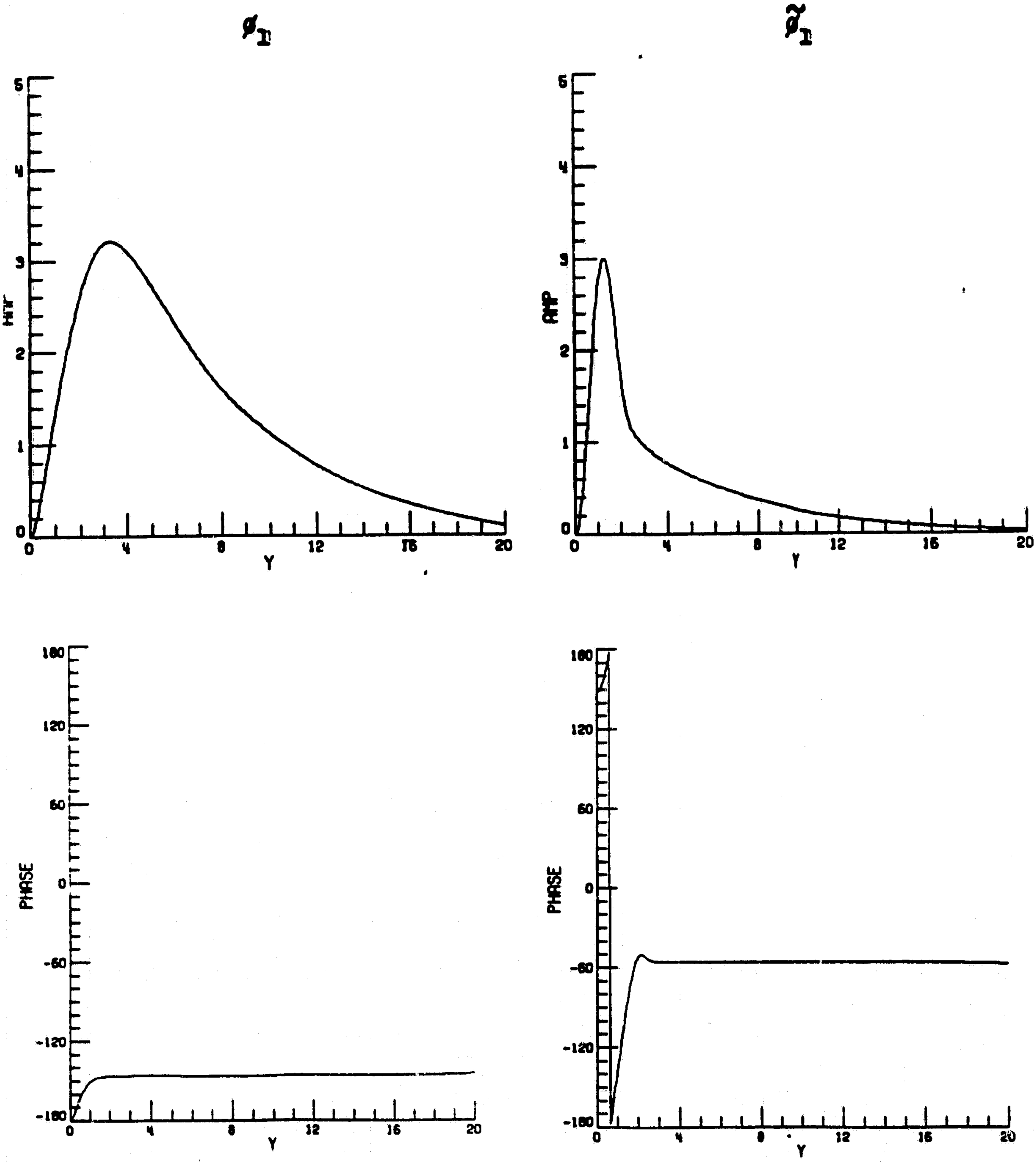
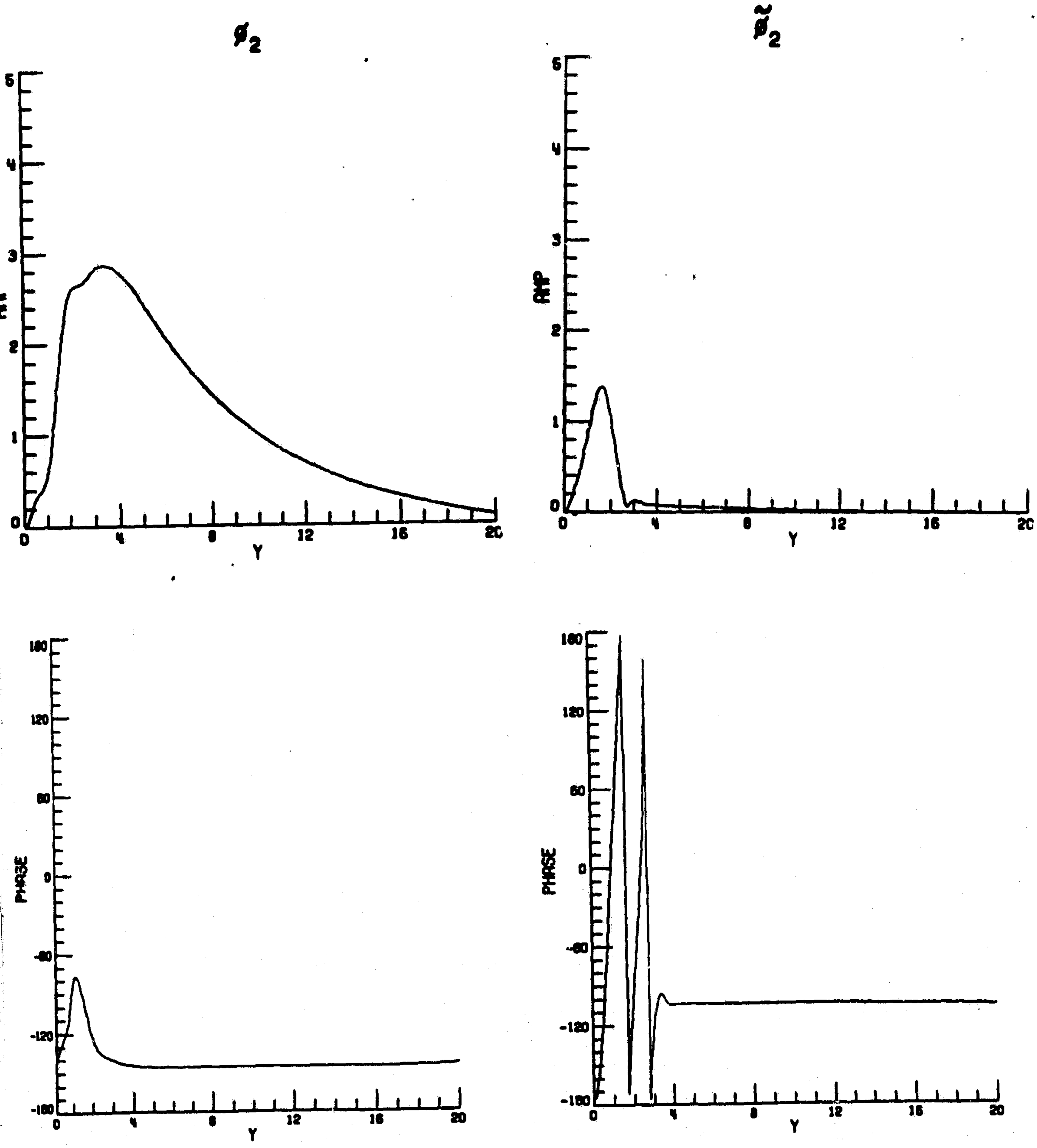


$\alpha = 0.179$ $R = 580.0$

$C = 0.36433 + i\ 0.00771$



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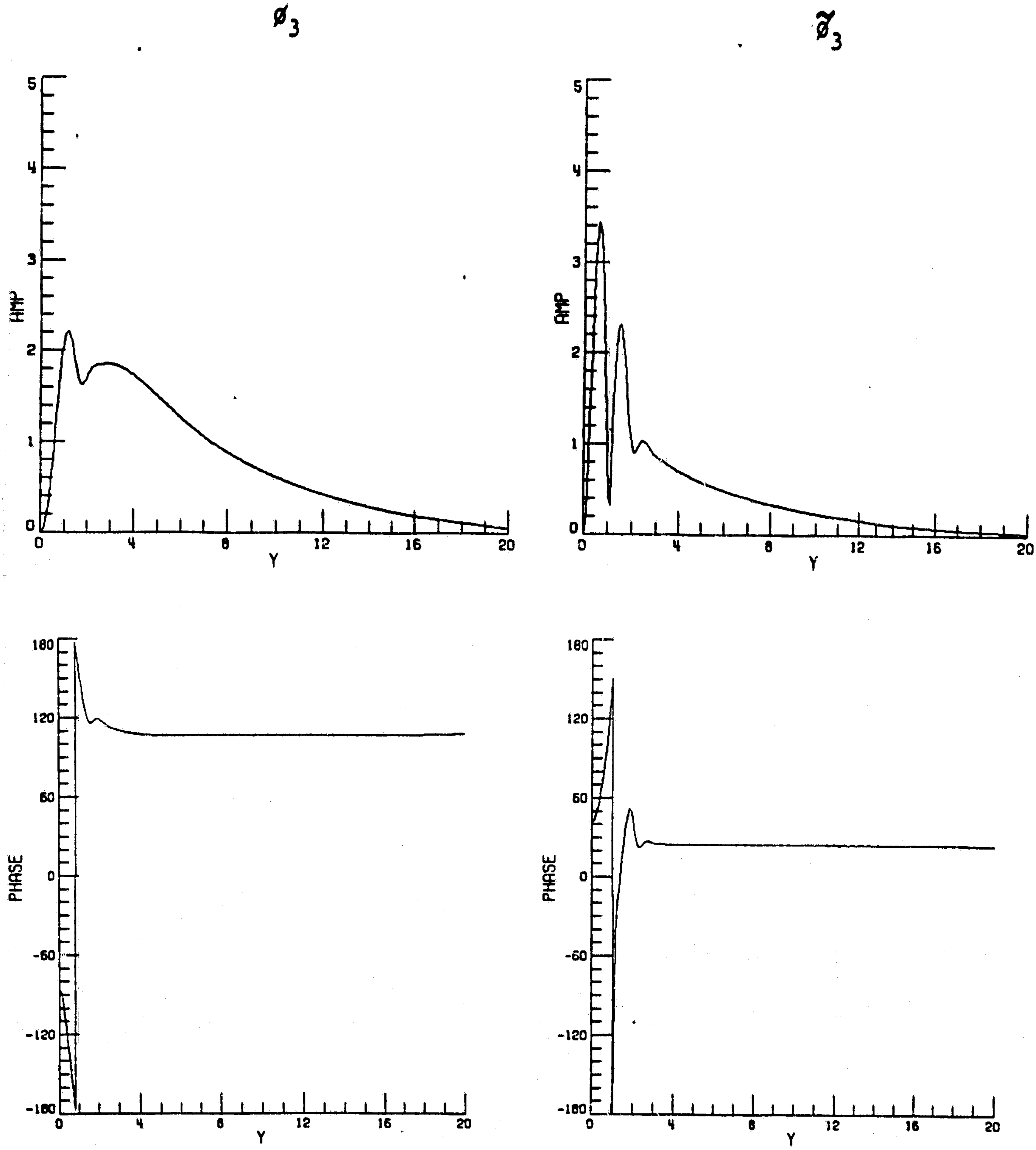

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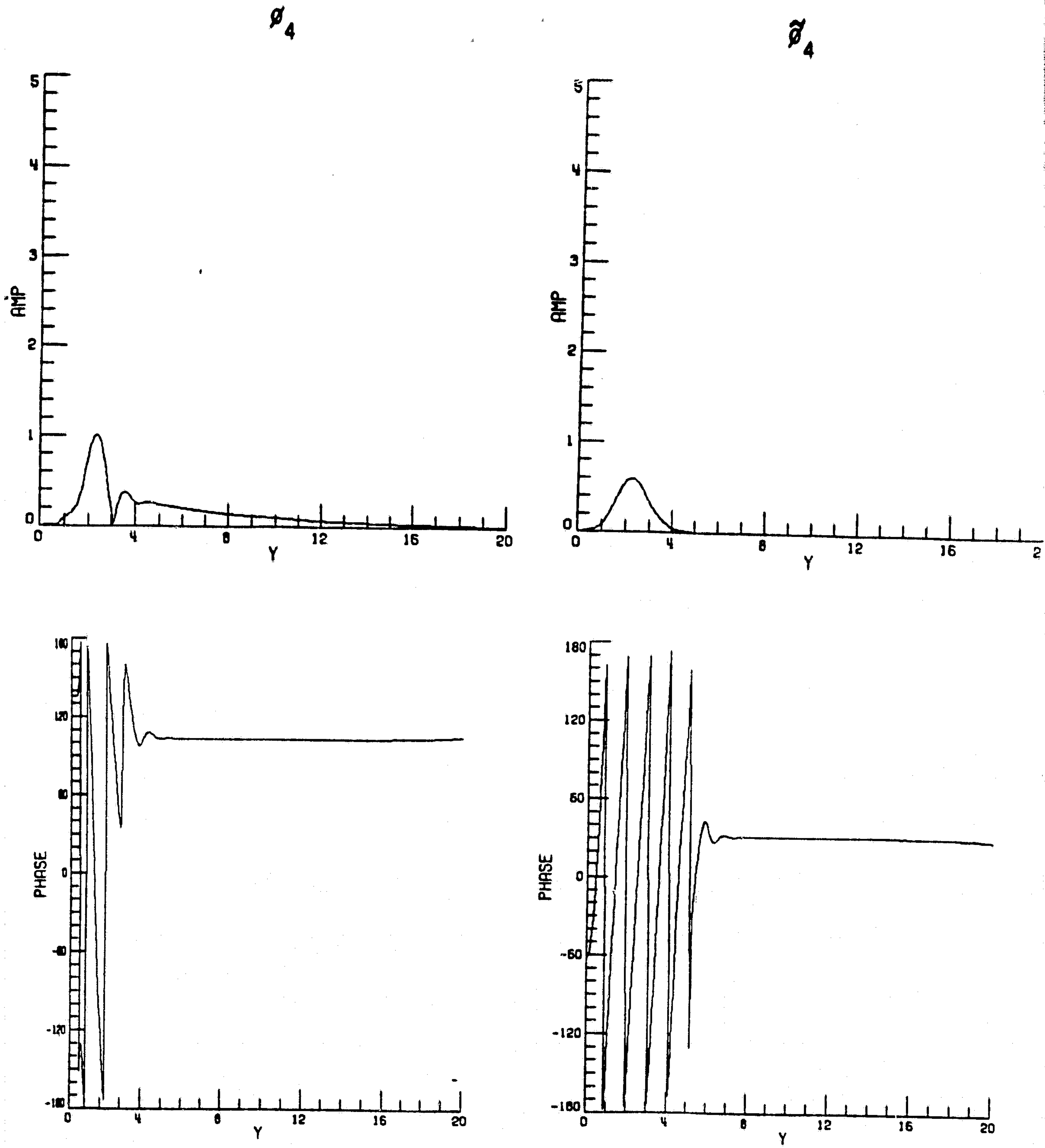

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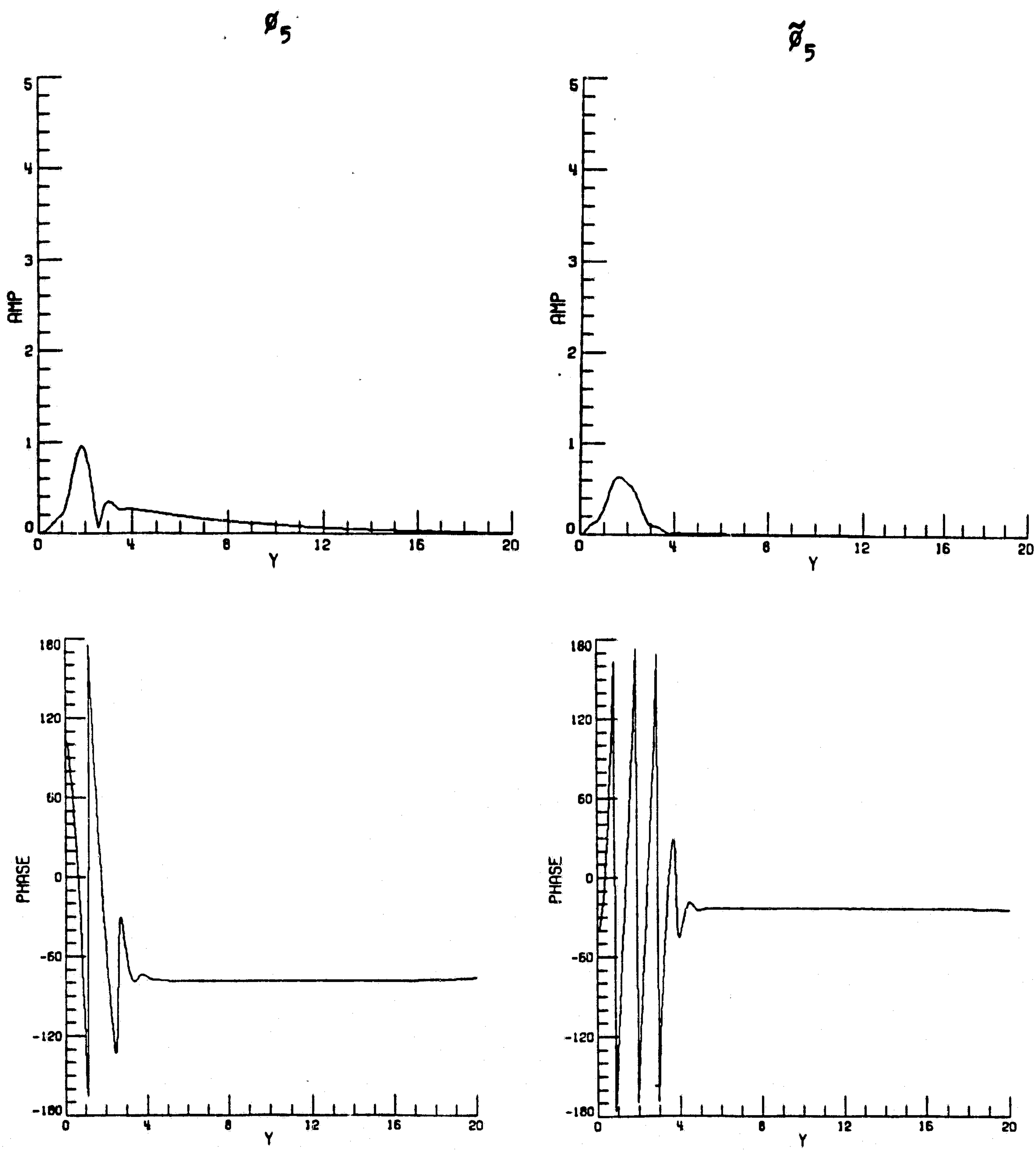
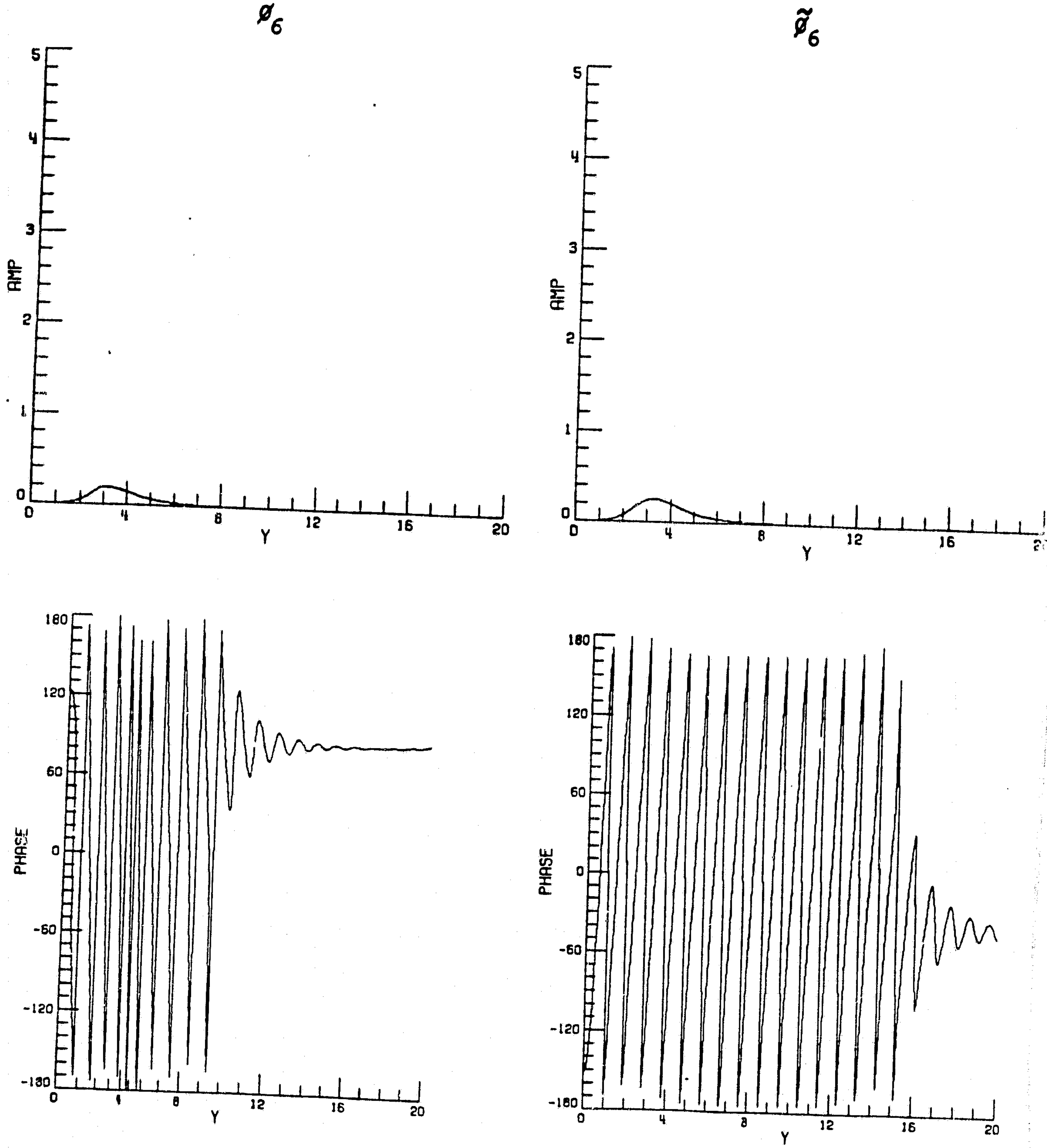


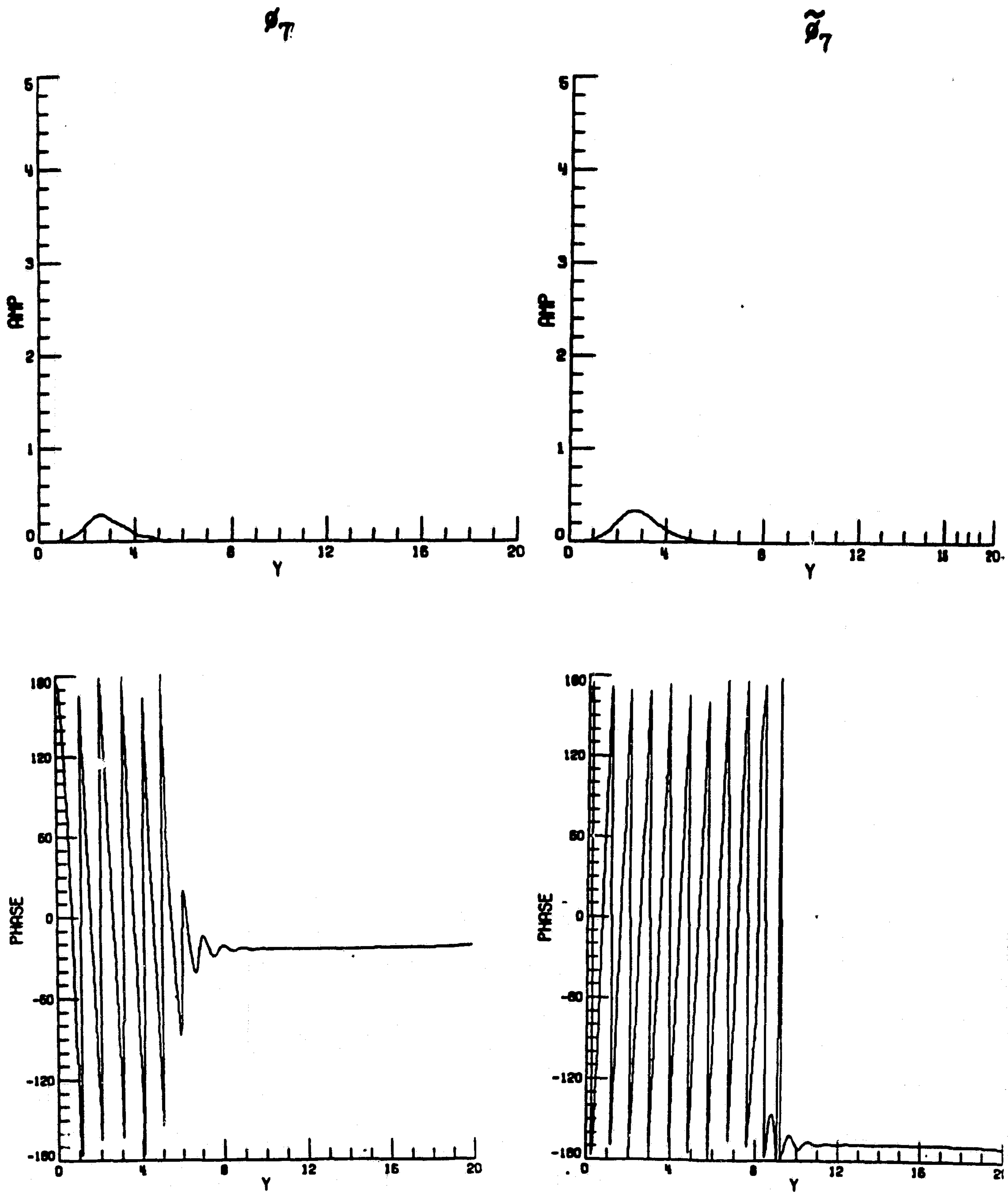
$\alpha = 0.179 \qquad R = 580.0$

$C = 0.55717 - i\ 0.36551$



$\alpha = 0.179 \qquad R = 580.0$

$C = 0.88680 - i\,0.43431$



$\alpha = 0.179$ $R = 580.0$

$C = 0.79367 - i\,0.43431$