An Adaptive Computation Mesh for the Solution of Singular Perturbation Problems

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An adaptive mesh for singular perturbation problems in two and three dimensions is reported. The adaptive mesh is generated by the solution of potential equations which are derived by minimizing the integral I, written,

$$I = \int_D [\{(\nabla\xi)^2+(\nabla\eta)^2\}+\lambda_v\{wJ\}+\lambda_o\{(\nabla\xi\cdot\nabla\eta)^2\}]dV \qquad (1)$$

where $x(\xi,\eta), y(\xi,\eta)$ represent a mapping from a parameter space P, $0 \le \xi \le I$, $0 \le \eta \le J$, where $w(x,y) > 0$ is given, λ_v and λ_o are nonnegative constants, and J, the Jacobian, is written,

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} \quad . \qquad (2)$$

In the usual manner, the mesh is constructed by joining points in (x,y) corresponding to integer values of ξ and η by straight lines to form a net of arbitrarily shaped, quadrilateral cells (1).

The variational formulation is equivalent to Winslow's method (2) when λ_o and λ_v are zero. The Euler equations are those given by Winslow, and their solution maximizes the smoothness of the mapping. The additional terms modify other attributes of the mapping in a similar way. When $\lambda_o > 0$, the mapping is modified to make it more orthogonal. When $\lambda_v > 0$, the mapping is modified to make wJ^2 more nearly constant over the mesh. By choosing w(x,y) appropriately, and with λ_v, $\lambda_o > 0$, the zone size variation and skewness can be controlled.

In singular perturbation problems, control of zone size variation can affect the effort required to obtain accurate, numerical solutions of finite difference equations. Consider a simple, difference approximation,

$$\left\langle \frac{df}{dx} \right\rangle = \frac{f_{i+1} - f_{i-1}}{x_{i+1} - x_{i-1}} \quad , \qquad (3)$$

where i corresponds to ξ, $0 \le i \le I$. In the usual manner, the truncation error is written,

$$\varepsilon = \frac{1}{2}\frac{d^2f}{dx^2} x_{\xi\xi} + \frac{1}{6}\frac{d^3f}{dx^3}(x_\xi)^2 + \ldots \quad . \tag{4}$$

When f is sufficiently smooth, ε is least when $x_{\xi\xi} = 0$. However, when f is given, for example,

$$f = (1 + \exp(x/\delta))^{-1} \quad , \tag{5}$$

for which $d^nf/dx^n = O(\delta^{-n})$ is not finite for $\delta = 0$, the error ε is bounded only if $(x_\xi/\delta) < 1$ in the interval $-\delta < x < \delta$. An equally spaced mesh with x_ξ sufficiently small satisfies this requirement, but one with $x_\xi \lesssim \delta$ only when $-\delta < x < \delta$, and larger everywhere else satisfies it with fewer mesh points. With w(x) given by,

$$w(x) = \left| \frac{1}{f}\frac{df}{dx} \right| \quad , \tag{6}$$

minimizing the integral in Eq. 1 causes the mesh spacing to approach that given by the equation,

$$\left| \frac{1}{f}\frac{df}{dx} \right| x_\xi = \text{const.} \quad , \tag{7}$$

as λ_V increases. As a result, where w is largest the mesh spacing is smallest, and vice versa.

Numerical results for a singular perturbation problem in two dimensions are shown in the accompanying figures. A steady solution to the equation,

$$\frac{\partial\phi}{\partial t} + \nabla\cdot(\underset{\sim}{u}\phi) = \kappa\nabla^2\phi \quad , \tag{8}$$

is sought for small values of κ with u given. Such solutions are obtained when the diffusion and convective transport are in balance everywhere,

$$\underset{\sim}{u} = \frac{\kappa}{\phi}\nabla\phi \quad . \tag{9}$$

When $\underset{\sim}{u}$ is given by,

$$\underset{\sim}{u} = -\frac{1}{\kappa}(1+\exp((r-r_o)/\kappa))^{-1}(1+\exp(-(r-r_o)/\kappa))^{-1} \, , \tag{10}$$

the convective transport term is significantly different from zero in an annulus of width κ with radius r_o. When $w(r,\theta)$ is given by,

$$w(r,\theta) = \underset{\sim}{u} \cdot \underset{\sim}{u} \quad , \tag{11}$$

the zones of the computation mesh are made smaller where $|\underset{\sim}{u}|$ is largest as shown in Fig. 1.

In Fig. 2, the error in the numerical solution of Eq. 8 on the adaptive mesh as measured by the maximum norm,

$$\varepsilon_{max} = \underset{i,j}{\mathrm{Max}} \left| \underset{\sim}{u} - \langle \frac{1}{6} \nabla \phi \rangle \right| \quad , \tag{12}$$

is compared with the error on a fixed rectilinear mesh. The accuracy obtained by adapting the mesh can be obtained, in most cases, only by a threefold refinement of the regular mesh in each coordinate direction.

The adaptive mesh has been used in calculations of resistive magnetohydrodynamic flow in two dimensions with the weight function,

$$w = (\nabla \times \underset{\sim}{B}/\underset{\sim}{B})^2 \quad .$$

The results indicate a significant increase in the maximum, representable magnetic Reynolds number. The adaptive mesh can be applied easily to other fluid flow problems with the appropriate choice of weight function.

The use of the adaptive mesh in time dependent flow problems will be discussed, and results will be presented.

1. Joe F. Thompson, Frank C. Thames and C. Wayne Mastin, J. Comp. Phys. 15, 299 (1974).

2. A. M. Winslow, J. Comp. Phys. 1, 149 (1966).

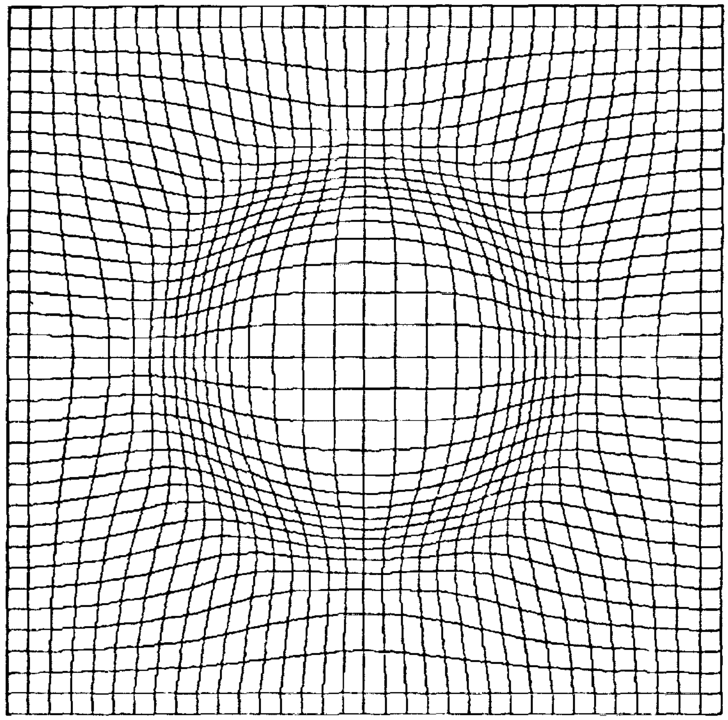

Figure 1.- An adaptive mesh with r_0 equal to 1/4 and κ equal to 1/40 the mesh width. The cells are concentrated in the region of maximum gradient.

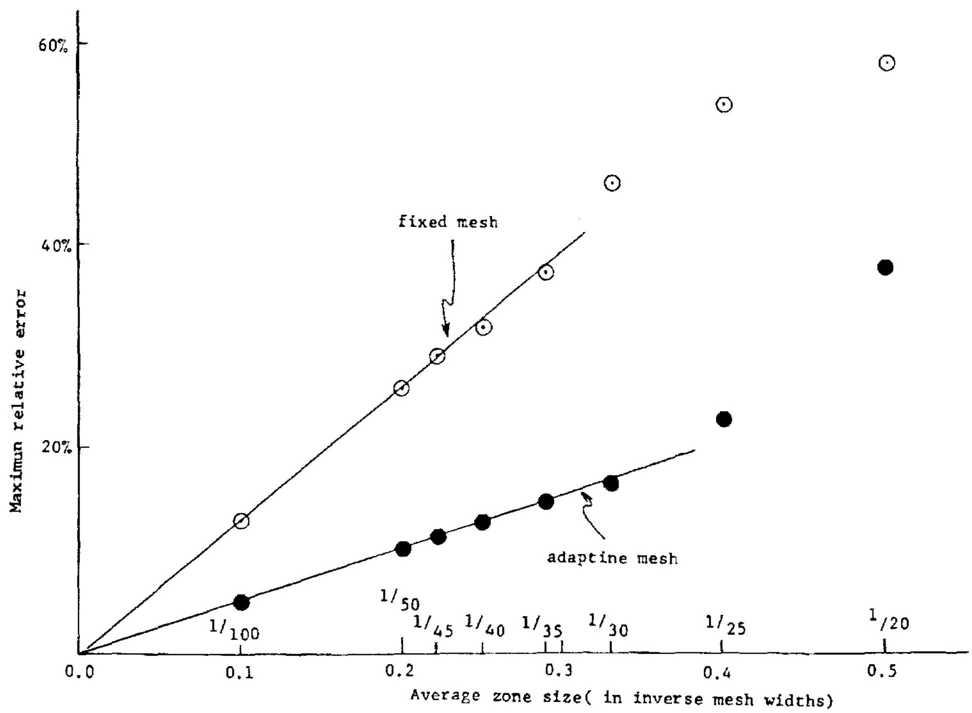

Figure 2.- Similar scaling of the maximum relative error with zone size obtained for both adaptive meshes (like the one shown in Fig. 1) and fixed meshes, but many fewer zones are required with an adaptive mesh.