

## An Analytical Transformation Technique for Generating Uniformly Spaced Computational Mesh

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Abstract

An analytical transformation method which can map arbitrary physical coordinate grid distribution into desired computational coordinate with uniform grid distribution is derived. The transformation function and its higher derivatives are differentiable. Salient features include; 1) precise control of grid sizes, 2) more than one location of clustered grids, 3) exact positioning of particular computational nodes in the physical plane, 4) ensuring several patches of uniformly spaced grids in the physical plane for the higher accuracies (such as at the boundaries), etc., while keeping the variation of grid spacing continuous to avoid numerical instability.

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## TRANSFORMATION FUNCTIONS

$y$  = Coordinate in physical plane.

$\eta$  = Coordinate in computational plane.  
Nodes are uniformly spaced.

<u>Parameters Specified</u>	<u>Program</u>	<u>Computed</u>
$N$ = Number of pivot		
$\eta_{p_i}$ ( $i=1, 2, \dots, N$ )	→ "FIXED" →	$\eta, y, \frac{dy}{d\eta}, \frac{d^2y}{d\eta^2}$
$\alpha_i$ ( $i=1, 2, \dots, N$ )		
$\beta_i$ ( $i=1, 2, \dots, N$ )		

$$f(\eta) = \frac{\partial y}{\partial \eta} = s_0 + \frac{1}{2} \sum_{i=1}^N \left[ \beta_i \operatorname{erfc} \left\{ \frac{y}{\alpha_i} (\eta - \eta_{p_i}) \right\} - \left\{ \operatorname{sign}(\alpha_i) 1 + 1 \right\} s_i \right]$$

$$y = \sum_{i=1}^N \frac{\beta_i \alpha_i}{2y} \left[ - \frac{y}{\alpha_i} (\eta_{\min} - \eta_{p_i}) \operatorname{erfc} \left\{ \frac{y}{\alpha_i} (\eta_{\min} - \eta_{p_i}) \right\} + \frac{1}{\sqrt{\pi}} \right. \\ \left. \cdot e^{- \left\{ \frac{y}{\alpha_i} (\eta_{\min} - \eta_{p_i}) \right\}^2} + \frac{y}{\alpha_i} (\eta - \eta_{p_i}) \operatorname{erfc} \left\{ \frac{y}{\alpha_i} (\eta - \eta_{p_i}) \right\} \right. \\ \left. - \frac{1}{\sqrt{\pi}} e^{- \left\{ \frac{y}{\alpha_i} (\eta - \eta_{p_i}) \right\}^2} \right] + \left[ \beta_0 - \frac{1}{2} \sum_{i=1}^N \left\{ \operatorname{sign}(\alpha_i) 1 + 1 \right\} s_i \right] \\ \cdot (\eta - \eta_{\min}) + y_{\min}$$

$$\frac{\partial^2 y}{\partial \eta^2} = - \sum_{i=1}^N \frac{\beta_i}{\sqrt{\pi}} \frac{y}{\alpha_i} e^{- \left\{ \frac{y}{\alpha_i} (\eta - \eta_{p_i}) \right\}^2}$$

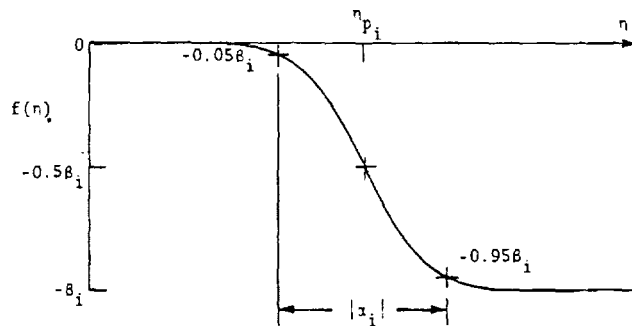
## BUILDING BLOCKS OF TRANSFORMATION FUNCTION $f(n)$

$\eta_{p_i}$  ( $i=1, 2, \dots, N$ ) = "Pivot points," particular specifiable values of  $n$  where  $f'(n)$  assumes local maxima.

$\gamma$  = 2.326, a convenient constant for the scaling  $\alpha_i$ .

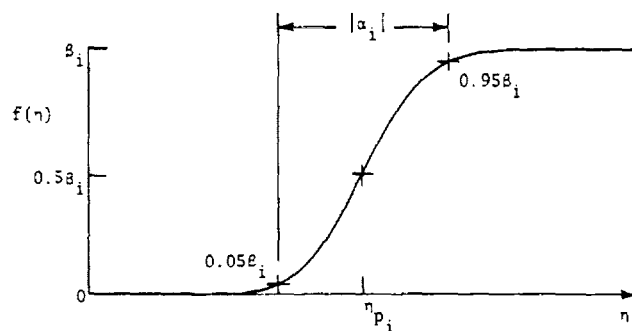
$\alpha_i$  ( $i=1, 2, \dots, N$ ) = "Width parameter," specifies width in  $n$  in which 90 percent of grid size variation takes place around the pivot  $\eta_{p_i}$ .  $\alpha_i < 0$  specifies increasing grid sizes, and  $\alpha_i > 0$  specifies decreasing grid size at pivot  $\eta_{p_i}$ .

$\beta_i$  ( $i=0, 1, 2, \dots, N$ ) = step heights for the pivots, which decides the ratio between the sizes of node spacings in  $y_i$  on both sides of the pivot.



$$f(n) = \frac{\beta_i}{2} \operatorname{erfc} \left\{ \frac{2.326}{\alpha_i} (n - \eta_{p_i}) \right\} - \beta_i, \text{ when } \alpha_i > 0$$

(a)



$$f(n) = \frac{\beta_i}{2} \operatorname{erfc} \left\{ \frac{2.326}{\alpha_i} (n - \eta_{p_i}) \right\}, \text{ when } \alpha_i < 0$$

(b)

THE EFFECTS OF CONTROL PARAMETERS, EXAMPLE 1

Grids are clustered only at  $\eta_{min}$  with a single pivot.

Specified Parameters

Total node points = 51

Number of pivots = 1

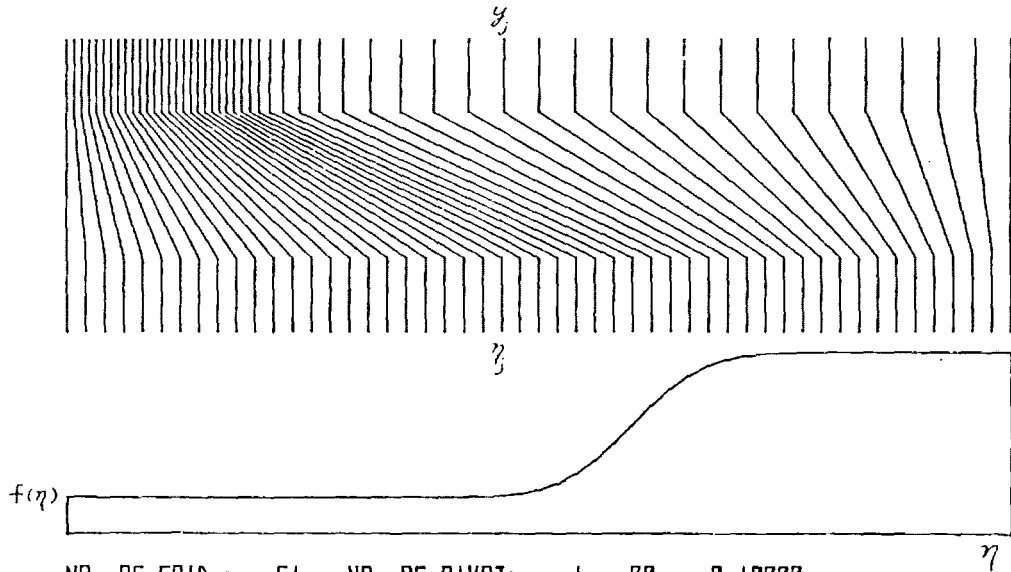
$$\eta_{p1} = 0.6$$

$$\alpha_1 = -0.2$$

$$\beta_0 = 0.1$$

$$\beta_1 = 0.4$$

Both ends of physical plane nodes have constant spacing.



NO. OF GRID = 51 NO. OF PIVOT = 1  $\beta_0 = 0.10000$   
S(J) = 0.600  
A(J) = -0.200  
B(J) = 0.400

## EFFECTS OF CONTROL PARAMETERS, EXAMPLE 2

Grids are clustered only at  $\eta_{\max}$  with a single pivot.

### Specified Parameters

Total node points = 51

Number of pivots = 1

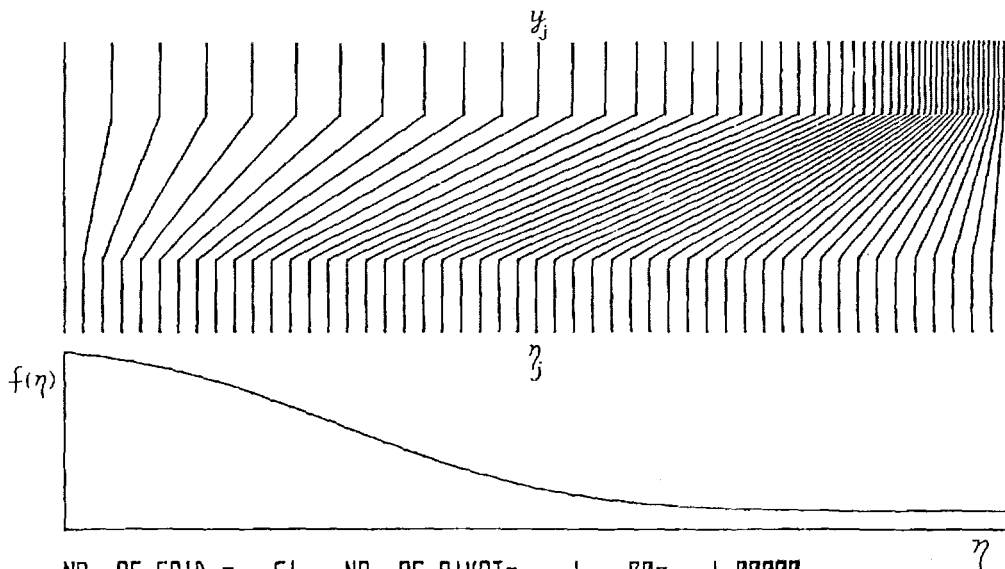
$$\eta_{p1} = 0.3$$

$$\alpha_1 = 0.6$$

$$\beta_0 = 1.0$$

$$\beta_1 = 0.9$$

Grids near  $y_{\max}$  are uniform but not those near  $y_{\min}$ .



NO. OF GRID = 51    NO. OF PIVOT = 1     $\beta_0 = 1.00000$   
 $S(j) = 0.300$   
 $R(j) = 0.600$   
 $B(j) = 0.900$

EFFECTS OF CONTROL PARAMETERS, EXAMPLE 3

Grids are clustered only at an interior point with a pair of pivots.

Specified Parameters

Total node points = 51

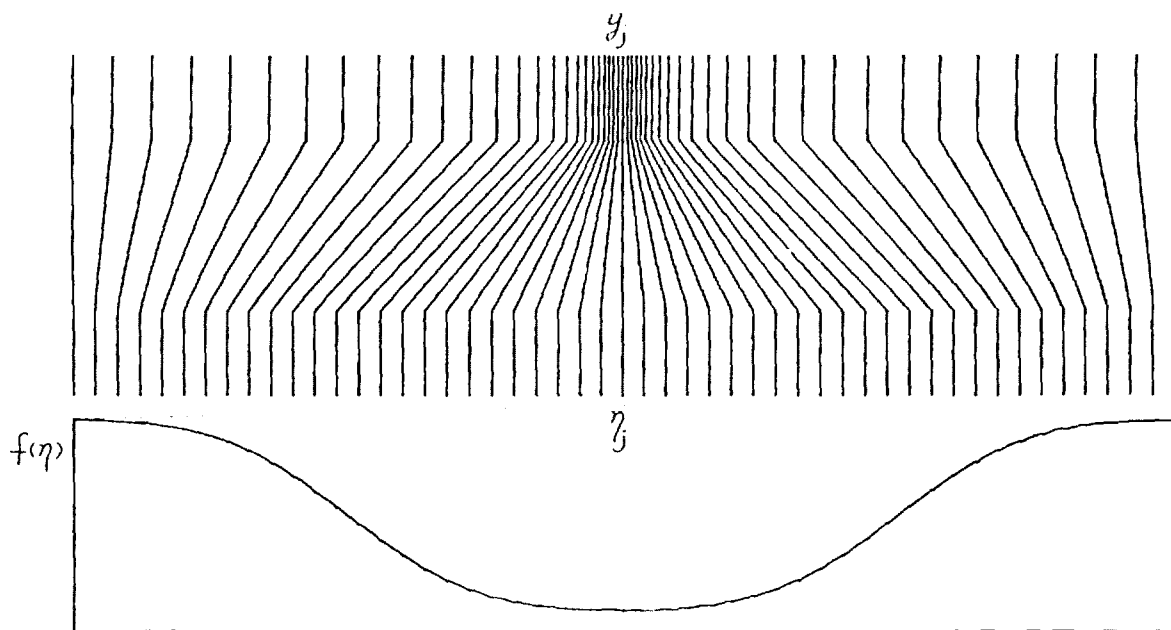
Number of pivots = 2

$$\beta_0 = 1.0$$

$$\eta_{p_1} = 0.25, \quad \alpha_1 = 0.3, \quad \beta_1 = 0.9$$

$$\eta_{p_2} = 0.75, \quad \alpha_2 = -0.3, \quad \beta_2 = 0.9$$

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NO. OF GRID = 51    NO. OF PIVOT = 2     $\beta_0 = 1.00000$     ?  
 $S(j) = 0.250 \quad 0.750$   
 $R(j) = 0.300 \quad -0.300$   
 $B(j) = 0.900 \quad 0.900$

## EFFECTS OF CONTROL PARAMETERS, EXAMPLE 4

Figure demonstrates smooth variation of grid sizes (large  $|\alpha|$  compare to pivot spacing) with multi-pivots.

### Specified Parameters

Total node points = 51

Number of pivots = 7

$\beta_0 = 0.05$

$\eta_{p_1} = 0.1, \quad \alpha_1 = -0.14, \quad \beta_1 = 1.$

$\eta_{p_2} = 0.14, \quad \alpha_2 = 0.14, \quad \beta_2 = 1.$

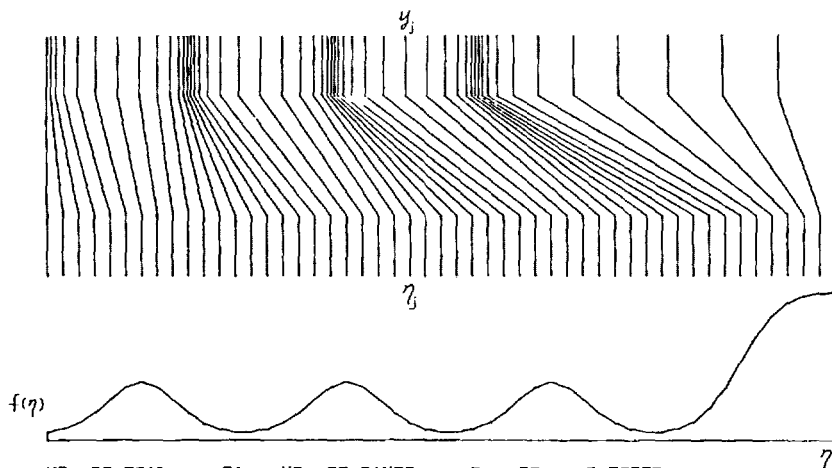
$\eta_{p_3} = 0.36, \quad \alpha_3 = -0.14, \quad \beta_3 = 1.$

$\eta_{p_4} = 0.4, \quad \alpha_4 = 0.14, \quad \beta_4 = 1.$

$\eta_{p_5} = 0.62, \quad \alpha_5 = -0.14, \quad \beta_5 = 1.$

$\eta_{p_6} = 0.66, \quad \alpha_6 = 0.14, \quad \beta_6 = 1.$

$\eta_{p_7} = 0.88, \quad \alpha_7 = -0.14, \quad \beta_7 = 1.$



NO. OF GRID = 51    NO. OF PIVOT = 7     $\beta_0 = 0.05000$   
 $S(J) = 0.100 \ 0.140 \ 0.360 \ 0.400 \ 0.620 \ 0.660 \ 0.880$   
 $R(J) = -0.140 \ 0.140 \ -0.140 \ 0.140 \ -0.140 \ 0.140 \ -0.140$   
 $B(J) = 1.000 \ 1.000 \ 1.000 \ 1.000 \ 1.000 \ 1.000 \ 1.000$

EFFECTS OF CONTROL PARAMETERS, EXAMPLE 5

Figure demonstrates discontinuous variation of grid sizes (small  $|\alpha|$  compare to pivot spacing) with multi-pivots.

Specified Parameters

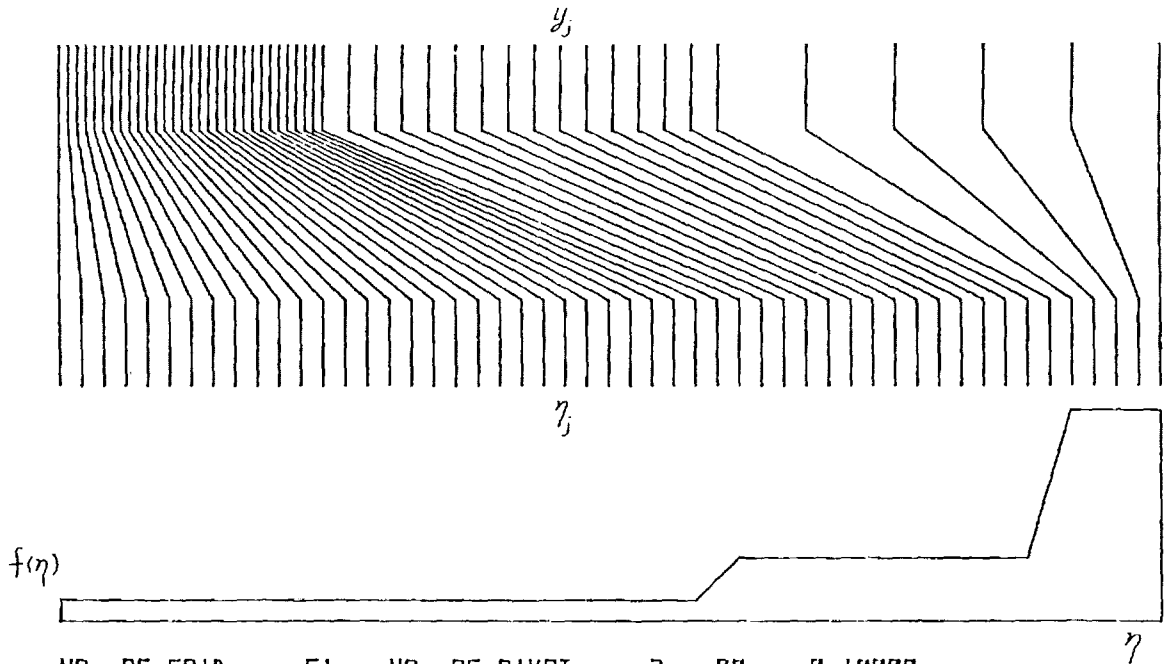
Total node points = 51

Number of pivots = 2

$\beta_0 = 0.1$

$\eta_{p1} = 0.6, \alpha_1 = -0.001, \beta_1 = 0.2$

$\eta_{p2} = 0.9, \alpha_2 = -0.001, \beta_2 = 0.7$



NO. OF GRID = 51    NO. OF PIVOT = 2     $\beta_0 = 0.10000$   
 $S(j) = 0.600 \ 0.900$   
 $A(j) = -0.001 \ -0.001$   
 $B(j) = 0.200 \ 0.700$



CONSTITUTIVE RELATIONS FOR THE  
DETERMINATION OF  $\beta$ 'S

$\eta_{c_k}$  ( $k=0, 1, 2, \dots, K$ ) = "Cluster Points," specifiable values of  $\eta$  which must coincide with particular predetermined values of  $y_{c_k}$  ( $k=0, 1, 2, \dots, K$ ) of corresponding  $k$ 's. The values  $\eta_{c_k}$  (hence  $y_{c_k}$  also) may or may not necessarily be node points.  $c_k$

$\rho_k$  ( $k=0, 1, 2, \dots, K$ ) = Ratio between the immediately neighboring  $\Delta y$  minima and maxima at the cluster point  $k$ .

2K simultaneous linear algebraic equations for 2K unknowns,  $\beta_0^*$ ,  $\beta_2$ ,  $\beta_3 \dots \beta_{2K}$  are solved by the program "FIXBY".

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\*  
$$\beta_1 = \frac{1 - \rho_0}{\rho_0} \beta_0$$

$$(1 - \rho_k) \left[ \left\{ 1 - \text{sign}(\alpha_1) \frac{1 - \rho_0}{\rho_0} \right\} \beta_0 - \sum_{i=2}^{2k-1} \left\{ \text{sign}(\alpha_i) \right\} \beta_i \right]$$

$$- \text{sign}(\alpha_{2k}) \beta_{2k} = 0, \quad (k = 1, 2, 3, \dots, K)$$

$$\left\{ (\eta_{c_k} - \eta_{\min}) + \frac{1 - \rho_0}{\rho_0} \frac{1}{2} \left[ \frac{\alpha_1}{Y} \left\{ \Theta_1(\eta_{c_k}) - \Theta_1(\eta_{\min}) \right\} - \left\{ \text{sign}(\alpha_1) \right\} \right] \right.$$

$$\left. + 1 \left\{ (\eta_{c_k} - \eta_{\min}) \right\} \right\} \beta_0 + \sum_{i=2}^{2k} \frac{1}{2} \left[ \frac{\alpha_i}{Y} \left\{ \Theta_i(\eta_{c_k}) - \Theta_i(\eta_{\min}) \right\} \right.$$

$$\left. - \left\{ \text{sign}(\alpha_i) \right\} + 1 \right\} (\eta_{c_k} - \eta_{\min}) \beta_i$$

$$= y_{c_k} - y_{\min} \quad (k = 1, 2, 3, \dots, K)$$

## APPLICATION, EXAMPLE 1

- The objectives were
- (1) to transform the physical coordinate  $y$  in the range of  $0 \leq y < 2.5$  to computational coordinate  $\eta$  in the range of  $0 \leq \eta \leq 1.0$ ;
  - (2) to obtain grids clustered at  $y=0$  and  $y=1$  with the same order of minimum grid sizes and the same rates of increase of grid spacing  $\Delta y$  away from  $y=0$  and  $y=1.0$ ;
  - (3) to use a total of 41 nodes where  $y=0, 1.0; 2.5$  which are nodes in the physical as well as in the computational plane (i.e., cluster points);
  - (4) to have grids near  $y=2.5$  which are fairly uniform

Program "FIXBY" is used to compute  $\beta_0, \beta_1, \beta_2, \beta_3$ , with the specified input values;

$$\eta_{p_i} = 0.2, 0.425, 0.825$$

$$\alpha_i = -0.17, 0.17, -0.17 \quad \text{for } i = 1, 2, 3$$

$$\eta_{c_k} = 0., 0.625$$

$$y_{c_k} = 0., 1.0$$

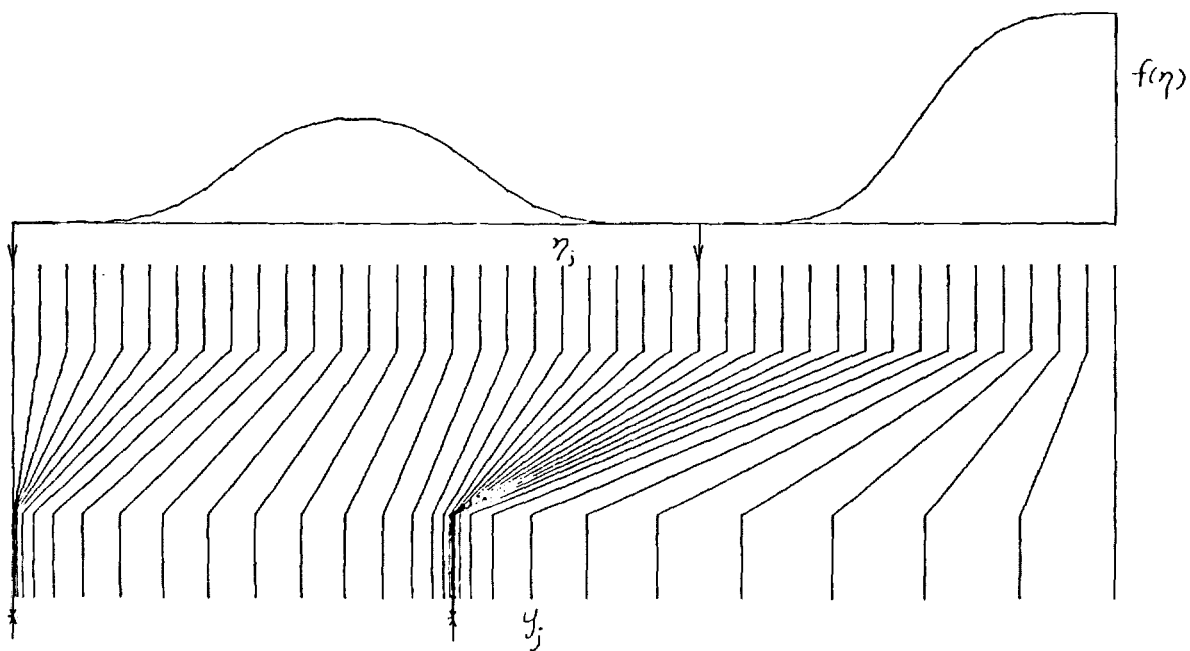
$$\rho_k = 0.0025, 0.0015 \quad \text{for } k = 1, 2$$

The choice of  $\eta_{p_i}$  decides the way in which the number of nodes are partitioned into different regions and the choice of  $\alpha_i$  decides the ratio with which the grid size varies.

Computed  $\beta$ 's are

$$\beta_i = 0.011071747, 4.417627, 4.4220557, 8.5569778 \quad (i=0, 1, 2, 3)$$

NO. GRID = 41    NO. CLUSTER = 2    NO. PIVOT = 3    B0 = 0.011071747  
 S(J) =    0.200000000    0.425000000    0.825000000  
 R(J) =    -0.170000000    0.170000000    -0.170000000  
 B(J) =    4.417627001    4.422055699    8.556977781



## APPLICATION, EXAMPLE 2

- The objectives were to have
- (1) a total of 41 nodes;
  - (2) variable ranges of  $0 \leq y \leq 8.0$  and  $0 \leq \eta \leq 1.0$ , respectively;
  - (3) one interior clustering point at  $y=0$ ; and
  - (4) grids near  $y=0$  and  $y=8.0$  which are fairly uniform.

Inputs to program FIXBY were;

$$\eta p_i = 0.3, 0.7$$

$$\alpha_i = 0.17, -0.17 \quad (i=1,2)$$

$$\eta c_1 = 0.5$$

$$y c_1 = 3$$

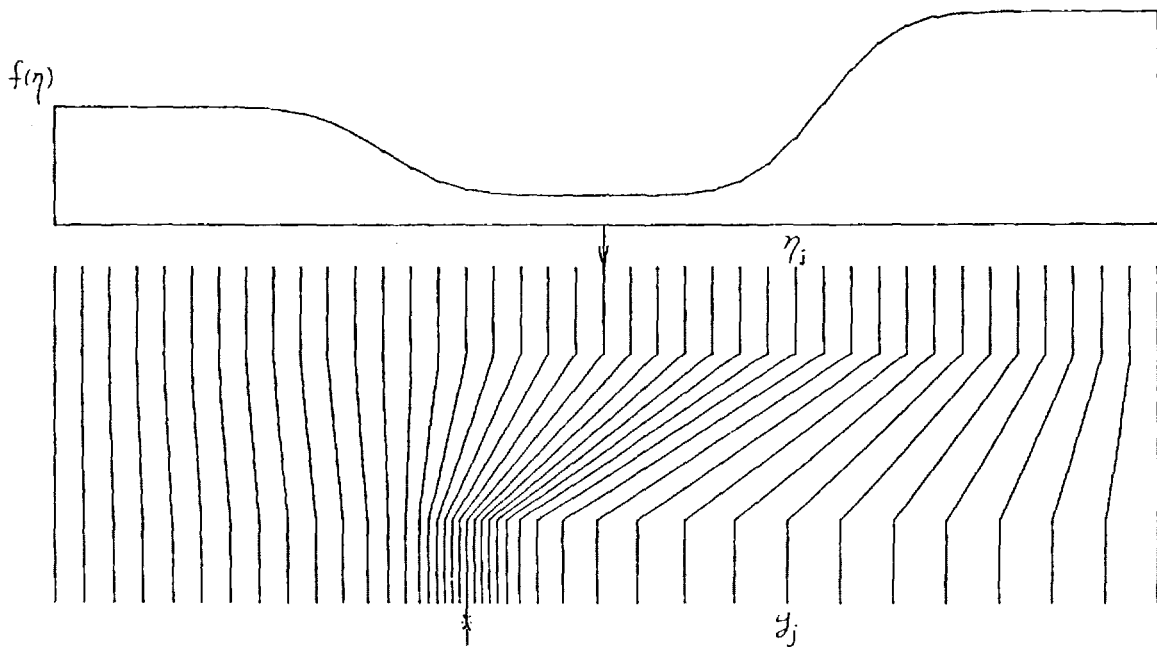
$$\rho_1 = 0.25$$

Computed  $\beta$ 's were;

$$\beta_i = 8.5714166, 6.4285625, 13.095257 \quad (i=0, 1, 2)$$

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NO. GRID = 41 NO. CLUSTER= 1 NO. PIVOT= 2 B0= 0.571416639  
S(J)= 0.3000000000 0.7000000000  
R(J)= 0.1700000000 -0.1700000000  
B(J)= 6.428562400 13.095256990



## FLOATING GRID CLUSTER REGION

When pivots are sufficiently separated so that there will be a region of " $f(\eta) = \text{constant}$ " between every pivot (small  $|\alpha|$  case) is a particular case. A separate program "FLOAT" is programmed for this case. Floating the location of clustering and varying the degree of clustering as well as locating new clustering regions during the computation (of a finite difference solution method) can be conveniently accomplished with this code. Figure shows a transformation of "FLOAT" code with a particular set of input parameters. "FLOAT" redistributed the uniform computational grids  $\eta$  in the physical plane from  $y_j^{(1)}$  to  $y_j^{(2)}$ .

