USE OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS TO GENERATE BODY FITTED COORDINATES

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Abstract

Interpreting previous work, hyperbolic grid generation procedures are formulated in the style of the elliptic partial differential equation schemes used to form body fitted meshes. For problems in which the outer boundary is not constrained, the hyperbolic scheme can be used to efficiently generate smoothly varying grids with good step size control near the body. Although only two dimensional applications are presented, the basic concepts are shown to extend to three dimensions.

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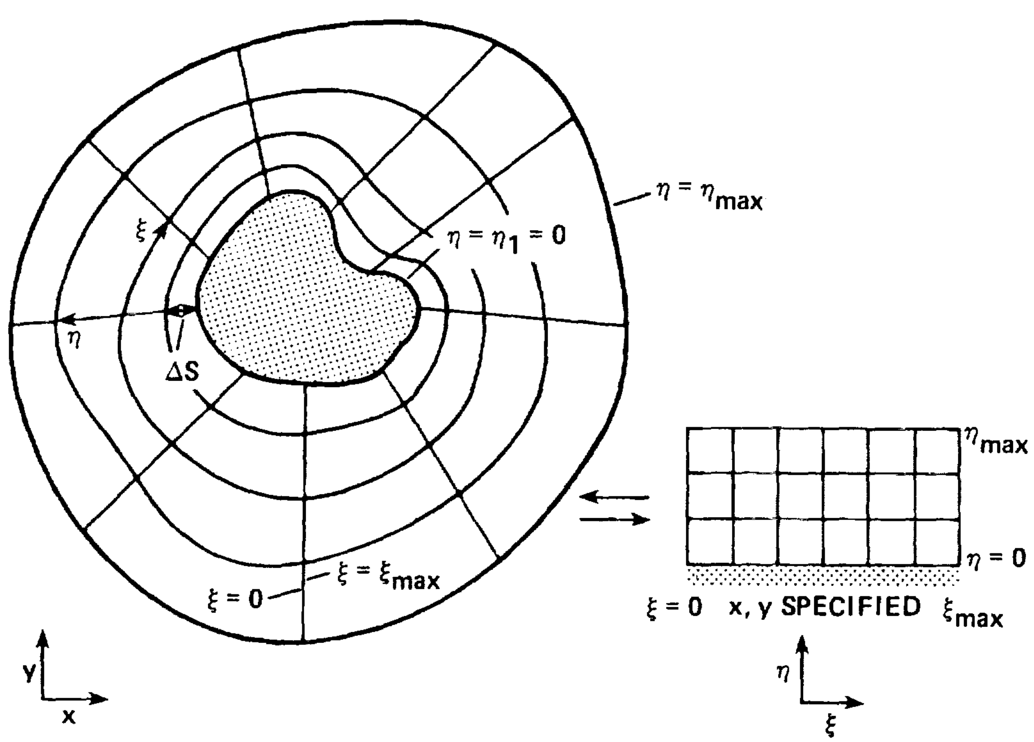
The task of generating the exterior mesh about an arbitrary closed body as indicated in this slide is undertaken. The location of the outer boundary is not specified; it only need be far removed from the inner boundary. Such a grid generation problem is encountered in external flow aerodynamics.

We seek a grid composed of constant ξ and η lines as indicated in this slide, given initial x,y data along ξ at $\eta = 0$. The grid generation equations, just as the flow field equations, are solved in the uniform transform plane.

SKETCH OF PHYSICAL AND COMPUTATIONAL PLANE



Partial differential equations are sought to generate a smoothly varying mesh such that grid lines of the same family do not interect or coalesce. Two systems of nonlinear hyperbolic partial differential equations have been considered for the given initial data sketched in the previous slide. As indicated in this slide, these systems each use the condition of orthogonality and a geometric constraint.

HYPERBOLIC GRID GENERATION EQUATIONS

ARC LENGTH-ORTHOGONALITY SCHEME

$$x_\xi^2 + y_\xi^2 + x_\eta^2 + y_\eta^2 = (\Delta s)^2$$

$$x_\xi x_\eta + y_\xi y_\eta = 0$$

VOLUME-ORTHOGONALITY SCHEME

$$x_\xi y_\eta - x_\eta y_\xi = V$$

$$x_\xi x_\eta + y_\xi y_\eta = 0$$

IN BOTH CASES $\Delta\xi = \Delta\eta = 1$

The previously described nonlinear partial differential equations must be shown to be properly posed for the given initial value data. As a first step, the equations are cast in a locally linearized form so that they can be analysed as a system of two first order partial differential equations. For the locally linearized form to be meaningful, the equations are expanded about a nearby known solution or state.

LOCALLY LINEARIZED FORM

$$x_\xi x_\eta + y_\xi y_\eta = 0$$

$$x_\xi y_\eta - x_\eta y_\xi = V$$

EXPAND x AND y ABOUT KNOW STATE $\tilde{x}, \tilde{y}$

E.G. $$x_\xi y_\eta = (\tilde{x} + x - \tilde{x})_\xi (\tilde{y} + y - \tilde{y})_\eta$$

$$= \tilde{x}_\xi \tilde{y}_\eta + (x_\xi - \tilde{x}_\xi)\tilde{y}_\eta + (y_\eta - \tilde{y}_\eta)\tilde{x}_\xi + 0(\Delta^2)$$

$$= \tilde{y}_\eta x_\xi + \tilde{x}_\xi y_\eta - \tilde{x}_\xi \tilde{y}_\eta + 0(\Delta^2)$$

OBTAIN LOCALLY LINEARIZED FORM

$$\begin{pmatrix} \tilde{x}_\eta & \tilde{y}_\eta \\ \tilde{y}_\eta & -\tilde{x}_\eta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_\xi + \begin{pmatrix} \tilde{x}_\xi & \tilde{y}_\xi \\ -\tilde{y}_\xi & \tilde{x}_\xi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_\eta = \begin{pmatrix} 0 \\ V+\tilde{V} \end{pmatrix}$$

Analysis of the locally linearized partial differential equations indicates that the equations can be marched in η provided that $x_\xi^2 + y_\xi^2 \neq 0$. That is, the grid spacing in ξ cannot be of zero length. The fact that $B^{-1}A$ is a symmetric matrix ensures that it has real, distinct eigenvalues. This then means that the system is hyperbolic if η is used as the marching or time-like direction.

HYPERBOLICITY

LOCALLY LINEARIZED VOLUME-ORTHOGONALITY EQUATION

$$\tilde{A}\vec{r}_\xi + \tilde{B}\vec{r}_\eta = \vec{f}$$

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} x_\eta & y_\eta \\ y_\eta & -x_\eta \end{pmatrix}, \quad B = \begin{pmatrix} x_\xi & y_\xi \\ -y_\xi & x_\xi \end{pmatrix}$$

FIND:

a) B^{-1} EXISTS IF $x_\xi^2 + y_\xi^2 \neq 0$

b) $B^{-1}A$ IS SYMMETRIC

THEREFORE LINEARIZED EQUATIONS ARE HYPERBOLIC

The grid generation equations can be solved using standard numerical techniques for first order systems of hyperbolic partial differential equations. In our case we have used a noniterative implicit finite difference procedure. An unconditionally stable implicit scheme was selected so that an arbitrary mesh step size can be specified in the marching direction. The same kind of numerical procedure is used to solve the flow field equations.

NUMERICAL SOLUTION OF VOLUME-ORTHOGONALITY EQUATIONS

USES IMPLICIT FINITE DIFFERENCE SCHEME FOR

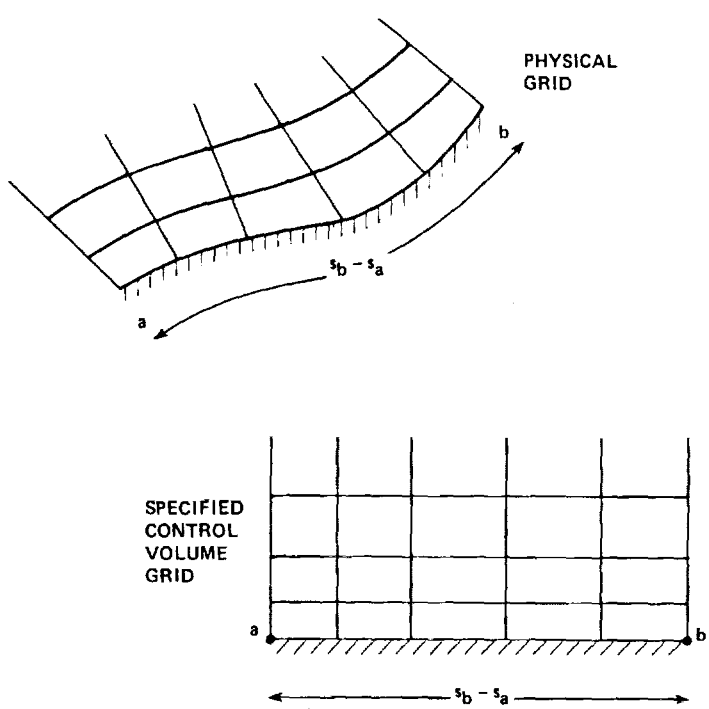
$$x_\xi y_\eta - x_\eta y_\xi = V$$

$$x_\xi x_\eta + y_\xi y_\eta = 0$$

SCHEME IS: a) UNCONDITIONALLY STABLE
b) NONITERATIVE
c) SECOND ORDER IN ξ, FIRST ORDER IN η
d) REQUIRES A BLOCK TRIDIAGONAL INVERSION

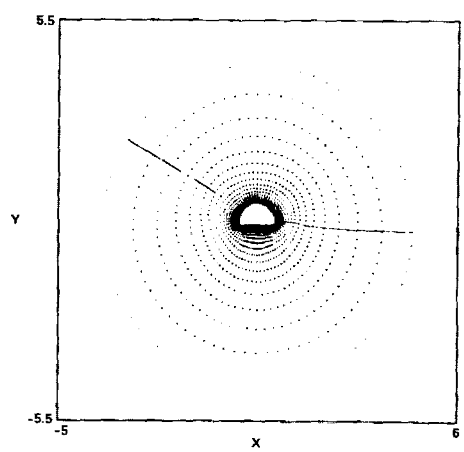
The volume orthogonality scheme requires that the user specify the volume (area in 2-D) of each mesh cell. The quality of the grid will, to a large extent, be determined by the user's cleverness in specifying these volumes. To specify these volumes, we currently define a simple geometric shape (e.g. circle or straight line) which has exactly the same arc length as the body we wish to grid. An algebraically clustered grid is then created by the user for the simple geometric shape. The volumes of this simple grid, the control volume grid, are then used directly on a point by point basis in the hyperbolic grid generation equations.

SELECTION OF VOLUMES

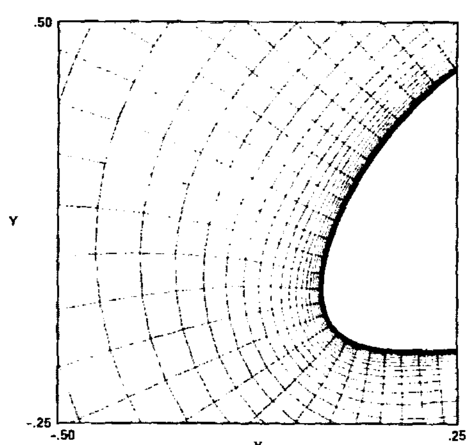


These slides show views of a viscous grid generated for a typical cross section of an aircraft fuselage. The control volumes were specified about a circle whose arc length matches that of the fuselage. Initially these control volumes are proportional to the grid-point arc-length spacing around the fuselage. Ultimately, however, control volumes that are uniformly spaced in the circumferential (i.e. θ) direction are specified. Thus in far field a polar coordinate system is formed.

VISCOUS GRID GENERATED ABOUT TYPICAL
AIRCRAFT-FUSELAGE CROSS SECTION
OVERVIEW

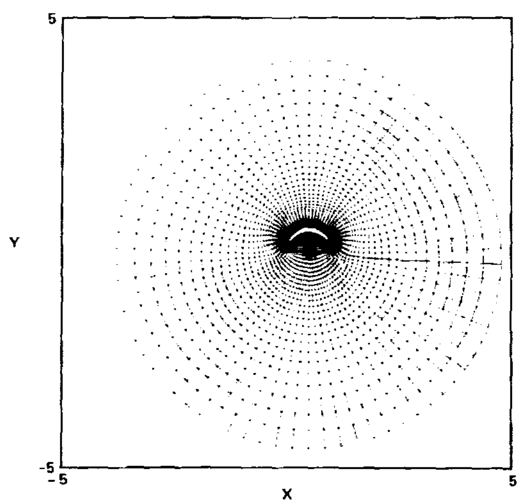


GRID DETAIL NEAR CORNER

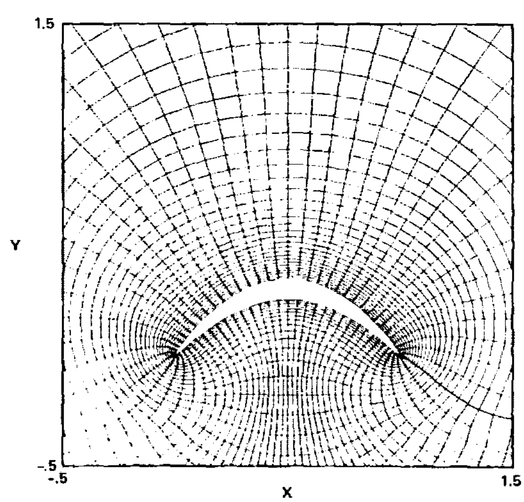


This slide shows views of an inviscid grid generated about a highly cambered profile. The same type of control volume used previously is employed. A uniform grid spacing was specified in the direction away from the body as is clear from the view showing grid detail near the body.

INVISCID GRID GENERATED ABOUT HIGHLY CAMBERED AIRFOIL
OVERVIEW



GRID DETAIL OF BLADE

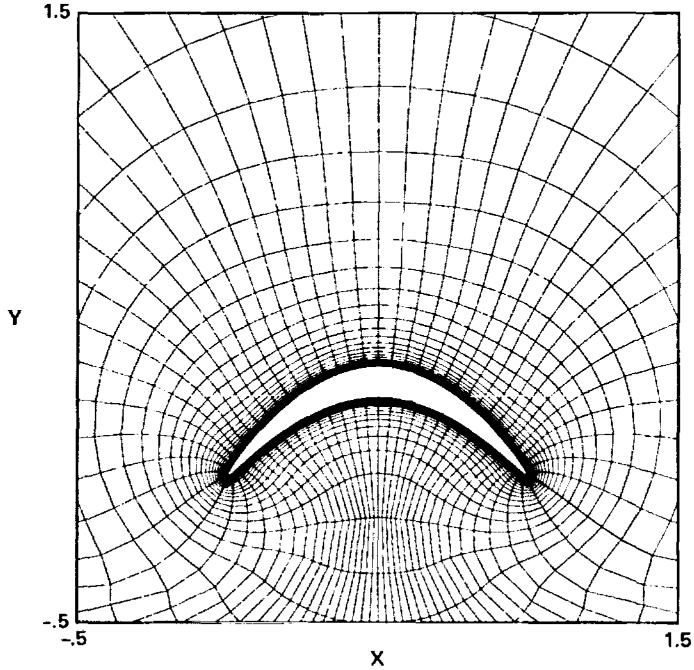


ORIGINAL PAGE IS OF POOR QUALITY

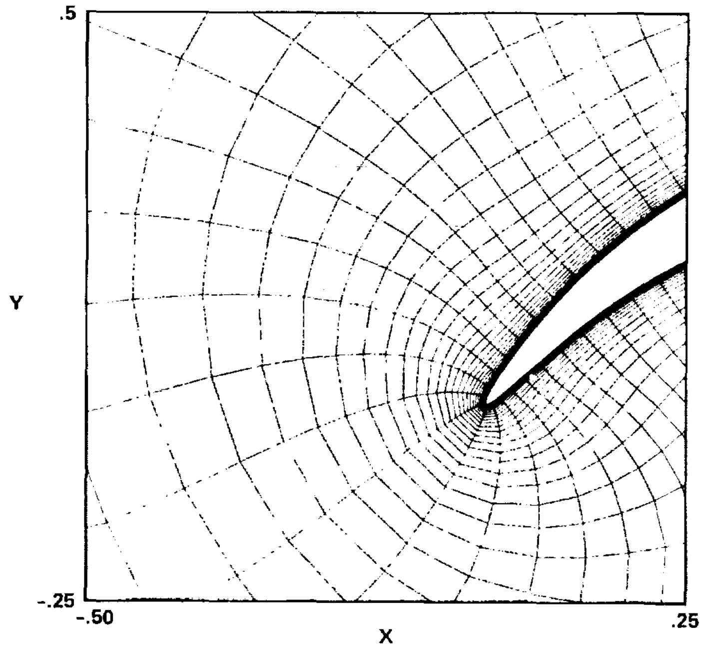
In this case a viscous grid is generated about the cambered profile. The normal grid spacing at the body is 0.01% of the chord length. Note that because volume is specified the grid spacing grows in the marching direction so as to prevent the circumferential spacing from vanishing. For a profile with twice the camber, however, this process breaks down and grid lines do coelesce. In these cases a more sophiticated means of specifying the volumes is needed.

VISCOUS GRID GENERATED ABOUT HIGHLY CAMBERED AIRFOIL

GRID DETAIL NEAR BODY

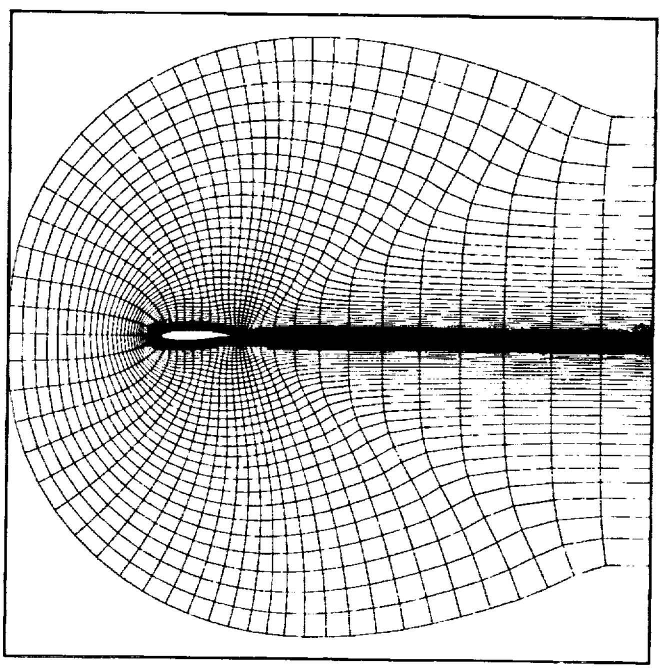


GRID DETAIL NEAR LEADING EDGE

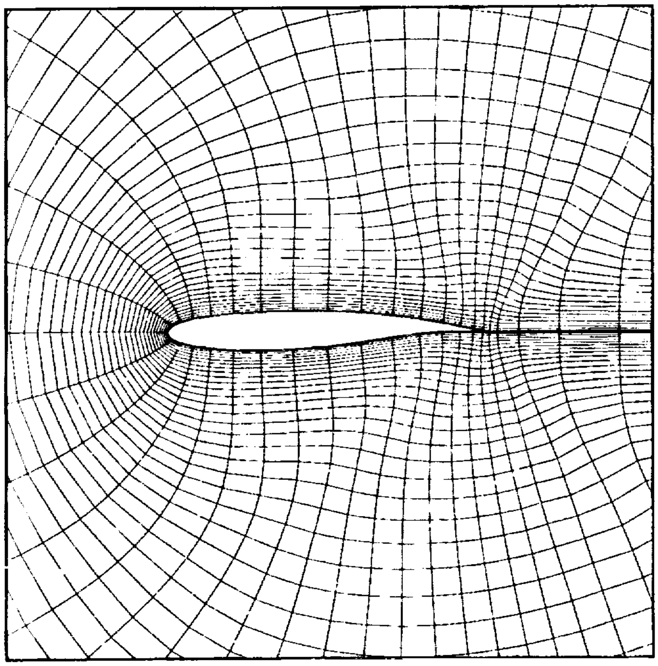


These views show the hyperbolic grid generation scheme applied to generating a "C-grid" about a cambered airfoil. Here the control volume grid is generated about a straight line, that is, it is nothing more than a clustered rectangular grid. It is clear from the view at the trailing edge that some adjustments are needed to the current numerical treatment of discontinuous boundary data.

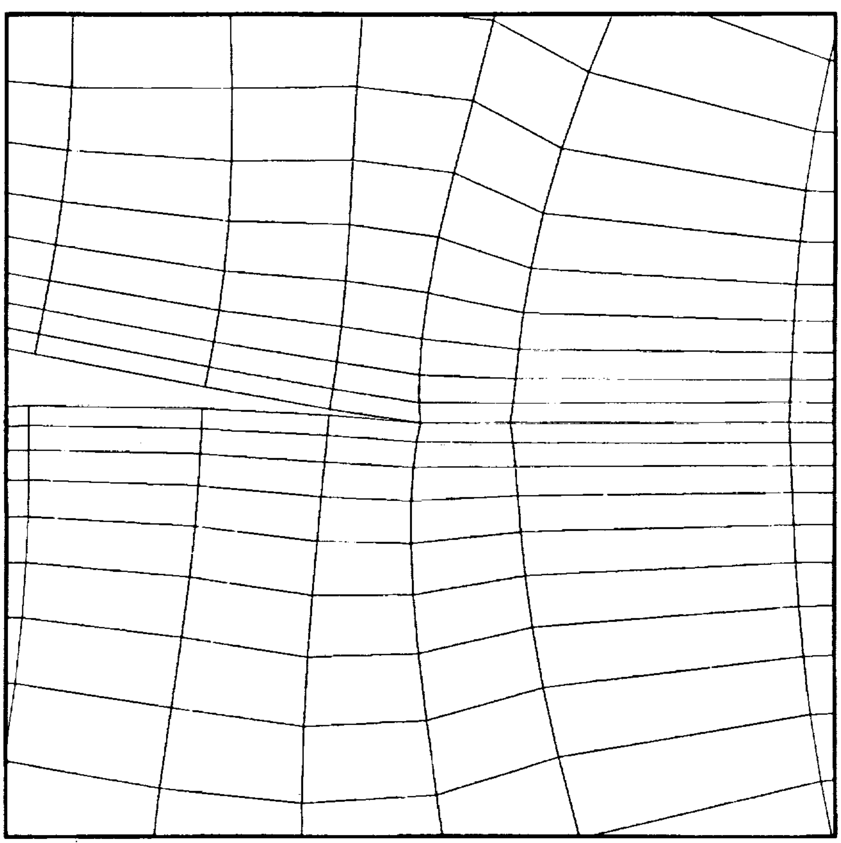
GENERATION OF C-GRID ABOUT
CAMBERED AIRFOIL
OVERVIEW



GRID DETAIL NEAR BODY



GRID DETAIL AT TRAILING EDGE



HYPERBOLIC GRID GENERATION ADVANTAGES

- SMOOTHLY VARYING GRID IS FOUND
- GOOD USER CONTROL OF CLUSTERING NEAR BOUNDARY
- FAST GRID GENERATION
- ORTHOGONAL OR NEARLY ORTHOGONAL
- AUTOMATICALLY TREATS COMPLEX SHAPES

HYPERBOLIC GRID GENERATION DISADVANTAGES

- OUTER BOUNDARY CANNOT BE SPECIFIED (UNLESS ITERATIVE SHOOTING METHOD DEVISED)
- SCHEME TENDS TO PROPAGATE DISCONTINUOUS BOUNDARY DATA
- POORLY SPECIFIED BOUNDARY DATA AND CONTROL VOLUMES CAN RESULT IN "SHOCK-WAVE" LIKE BREAKDOWN

The hyperbolic grid generation scheme can also be formulated in three dimensions. With volume specified as one constraint, orthogonality can only be enforced in two of the coordinate directions. The three partial differential equations shown form a hyperbolic system for marching in ζ. Proof that the equations are hyperbolic was quite tedious, required considerable insight, and was carried out by Dennis Jespersen of Oregon State University.

EXTENSION TO THREE DIMENSIONS – VOLUME AND TWO ORTHOGONALITY

$$\frac{d\vec{r}}{d\xi} \cdot \frac{d\vec{r}}{d\zeta} = 0$$

$$\frac{d\vec{r}}{d\eta} \cdot \frac{d\vec{r}}{d\zeta} = 0$$

$$\left| \frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)} \right| = V$$

SYSTEM IS FOUND TO BE HYPERBOLIC

The three constraints of orthogonality do not form a hyperbolic system of partial differential equations. Neither are the equations of elliptic type. In fact, their classification and what if any type of boundary data makes them unique is unknown to the authors.

EXTENSION TO THREE DIMENSIONS –

THREE ORTHOGONALITY

$$\frac{d\vec{r}}{d\xi} \cdot \frac{d\vec{r}}{d\zeta} = 0$$

$$\frac{d\vec{r}}{d\xi} \cdot \frac{d\vec{r}}{d\eta} = 0$$

$$\frac{d\vec{r}}{d\eta} \cdot \frac{d\vec{r}}{d\zeta} = 0$$

SYSTEM CANNOT BE MARCHED AND IS NOT ELLIPTIC