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NUMERICAL APPROXIMATION OF BOUNDARY CONDITIONS WITH APPLICATIONS TO INVISCID EQUATIONS OF GAS DYNAMICS

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Numerical Approximation of Boundary Conditions With Applications to Inviscid Equations of Gas Dynamics

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16. Abstract

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APPLICATIONS TO INVISCID EQUATIONS OF GAS DYNAMICS

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SUMMARY

A comprehensive overview of the state of the art of well-posedness and stability analysis of difference approximations for initial boundary value problems of the hyperbolic type is presented. The applicability of recent theoretical developments to practical calculations for nonlinear gas dynamics is examined. The one-dimensional inviscid gas-dynamics equations in conservation-law-form are selected for numerical experiments. The class of implicit schemes developed from linear multistep methods in ordinary differential equations is chosen and the use of linear extrapolation as an explicit boundary scheme is emphasized. Specification of boundary data in the primitive variables and computation in terms of the conservative variables in the interior are discussed. Some numerical examples for the quasi-one-dimensional nozzle are given.

1. INTRODUCTION

The proper specification of boundary conditions which yield a well-posed problem for a partial differential equation is essential for the behavior of the solution. Overspecification of boundary data precludes the existence of smooth solutions except in very special unrealistic situations in which the exact solution is known on the boundary without error. In the development of difference approximations for mixed initial boundary value problems in the applied science field, the boundary conditions may be quite difficult to construct, and a poor choice can lead to inaccuracies and instabilities. Part of the difficulty starts with the original differential equations where the proper boundary conditions are not always known (nonlinear fluid dynamics problems, for example). The problem is compounded in the difference schemes where quite often extra boundary conditions are needed because the difference equations are of higher order than the differential equations. Therefore, in the study of how the extra boundary conditions affect the stability and accuracy of numerical schemes, we not only have to examine the difference schemes used, but we also have to first examine the well-posedness of the original differential equations (refs. 1-15). Thus a good understanding of the theory of "well-posed problems" is a necessity.

The two principal objectives of this report are to (1) present a comprehensive overview of the state of the art of well-posedness and of stability analysis of difference approximations for initial boundary value problems of the hyperbolic type, and (2) to examine the applicability of the current theory

to the inviscid (Euler) equations of gas dynamics. (We will us he terms
"inviscid gas-dynamics equations" and "Euler equations of gas d ɨmics" inter-
changeably.) Through an understanding of the theory, we can g   some insights
into how to impose the physical boundary conditions more corre   y, and we can
be guided in the construction of stable numerical schemes for   ᴈ practical
problems. In this context, "stable numerical schemes" are sᴄ .es that are
stable for the combined interior and boundary schemes. Read ᴈ who are famil-
iar with the theory and who are only interested in the appl⸗ ɔion can skip
the first four subsections of the second section and can sk⸗ the third section
altogether.

In this report, we will discuss several ways of formulating the boundary
approximation for the one-dimensional inviscid gas-dynamics equations in con-
servative form. Since in general the Euler equations have mixed positive and
negative eigenvalues, appropriate one-sided and uncentered boundary approxima-
tions are essential. Some of the methods proposed in this report combine the
theory of Gustafsson et al. (ref. 9) with the flux-vector splitting technique
of Steger and Warming (ref. 16) to study the applicability of some <u>uncondi-
tionally</u> stable schemes for the one-dimensional (1-D) linearized Euler equa-
tions to their <u>nonlinear</u> counterpart. A few detailed numerical results for
the quasi-1-D nozzle with various inflow-outflow conditions are given. The
boundary approximations being used are one-sided spatial differencing and
linear extrapolation. It was found that we can use fairly large CFL numbers
(i.e., Courant, Fredrick, and Levy condition for the stability of differences
schemes).

The review of the theory of well-posed problems and stability analysis of
difference schemes is desirable because significant progress on a general,
workable theory for the initial boundary value problem of the hyperbolic (and
parabolic) type is quite recent. Much of it begins with the work of Kreiss
(ref. 1) published in 1970. The recent research papers on this rapidly-
developing subject are principally addressed to highly-theoretical audiences,
and there is no text or basic, up-to-date review article covering the material.
A primary purpose of this report is to collect the relevant information and to
identify some of the strengths and weaknesses of the existing theory when it
is applied to physical problems. The material is presented with the needs of
applied scientists in computational fields in mind. Consequently, basic con-
cepts and practical ideas are emphasized while exact mathematical definitions
and theorems are minimized. Only initial boundary value problems of the
hyperbolic type are considered.

Section two of this report is a review of the state of the art of how to
impose boundary conditions in order to obtain a well-posed problem. Section
three is a comprehensive review of the current status of stability analysis of
difference approximations. For example, a recent result by Oliger (ref. 10)
provides a useful guide in the construction of composite stable schemes. In
the fourth section, a detailed application of these theories is given for the
one-dimensional Euler equations of gas dynamics; several numerical experiments
are included. In addition to the numbered references that are cited through-
out the text, a separate bibliography is provided. The bibliographic entries
are categorized according to their particular relevance to sections 2, 3,
and 4.

2. WELL-POSEDNESS OF INITIAL BOUNDARY VALUE PROBLEMS (IBVP) FOR HYPERBOLIC EQUATIONS

In this section, we summarize the status of well-posedness of initial boundary value problems (IBVP) for hyperbolic partial differential equations. A more detailed discussion of this subject is given in subsequent sections. Some of the related mathematical definitions and examples are described in appendixes A and B. Readers who are not familiar with the definitions should refer to appendix A.

The term "well-posed," or correctly-posed, problem appears frequently in the literature. There are many different definitions for a well-posed partial differential equation; for example, well-posedness in Hadamand's sense is different from well-posedness in Kreiss's or Petrovskii's sense. The various definitions can be found in sources contained in the first two sections of the Bibliography. In this report, we only consider well-posedness in Kreiss's sense; that is, "well-posedness in the L_2 norm." The basic requirement for a well-posed IBVP is to not overspecify or underspecify the boundary conditions with given smooth initial data. In order to mathematically define a well-posed IBVP, we have to establish the existence and uniqueness of the solution and its continuous dependence on the initial and boundary data or to establish the existence of certain a priori estimates or energy inequalities.

Well-posedness of the governing partial differential equation is a very crucial consideration commonly overlooked by investigators in the field of computations; that is, the problem is defined only when a proper set of initial and/or boundary conditions is given. We cannot expect our difference approximations to be reasonable if they approximate a problem that does not have reasonable solutions. In many instances, a good understanding of the theory of well-posed problems can guide us to exclude many boundary conditions which might look physically reasonable.

The theory for the IBVP of 1-D systems or degenerate 1-D systems (higher dimension systems that can be reduced to 1-D problems (ref. 17)) has been established for some time. For higher dimension systems (with constant coefficient problems), results are known for the strictly hyperbolic and the symmetric hyperbolic case (see "More Than One Space Dimension," p. 11, for definition). Some partial results for the multidimensional Euler equations were established by Oliger and Sundström (ref. 6).

The following sections are summary discussions of ways to impose or to check for well-posedness of IBVP for hyperbolic equations in the L_2 norm (see appendix A). We will discuss the following types of problems:

1. 1-D scalar equation
2. 1-D system of equations
3. Several space dimensions equations

with constant coefficients, variable coefficients, and quasilinear proper 3.
We assume that the problems we are considering have smooth initial data.   1
of the permissible ways of imposing boundary conditions in the subsequer
sections are necessary and sufficient conditions for well-posedness of ·   1-D
hyperbolic equations. The discussions are based on the method of chare ris-
tics. For the more-than-one-space-dimensions problem, the analogous fc ula-
tion need not be well-posed. A proper way of getting a necessary and suffi-
cient condition in this case is by the normal mode analysis (ref. 1). One way
of getting a sufficient condition is by the energy method (refs. 4, 12).

Scalar Equation

Consider the problem

$$u_t + cu_x = 0 \qquad t \geq 0 , \qquad c \text{ real constant} \tag{1a}$$

with initial condition

$$u(x,0) = f(x) \tag{1b}$$

We can divide the above problem into the following three categories.

a. <u>The Cauchy (initial value) problem ($-\infty < x < \infty$)</u>: The exact solution
is given by

$$u(x,t) = f(x - ct) \tag{2}$$

Hence the solution of (1) is constant along the characteristic lines
$x - ct =$ constant. There are no boundary conditions involved since $-\infty < x < \infty$.

b. <u>Half-space problem ($0 \leq x < \infty$)</u>:

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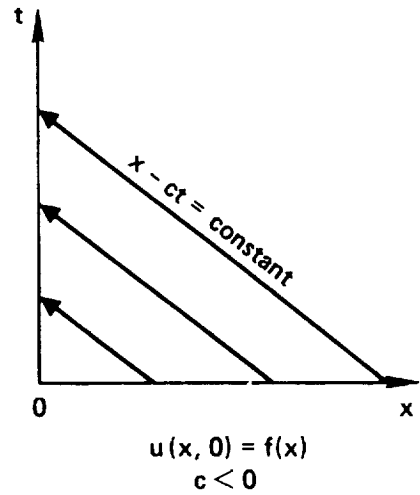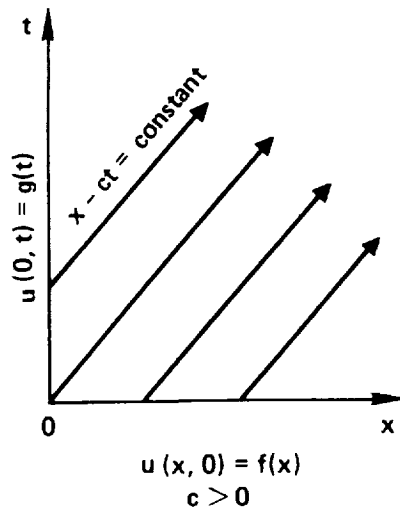
Figure 1.- Half-space problem (c > 0).   Figure 2.- Half-space problem (c < 0).

If   c > 0, then   u(x,t) is only determined in the triangular region
x - ct ≥ 0 (see fig. 1).   In this case, we need a boundary condition

$$u(x,0) = g(t) \qquad t \geq 0$$

to determine the solution for   x - ct < 0.   If   c < 0, then   u(x,t) is
uniquely determined by (2) and it is not appropriate to specify a boundary
condition at   x = 0 (see fig. 2).   Note that the solution   u(x,t) is continu-
ous in a neighborhood of   x - ct = 0 if and only if   f and g   are continuous
and satisfy the compatibility condition

$$f(0) = g(0)$$

c.   <u>Finite domain problem (0 ≤ x ≤ 1)</u>:
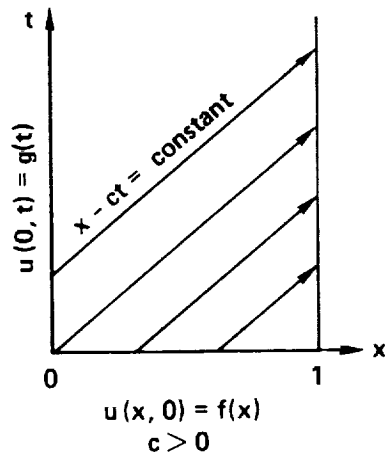
Figure 3.- Finite domain problem   Figure 4.- Finite domain problem
        (c > 0).   (c < 0).
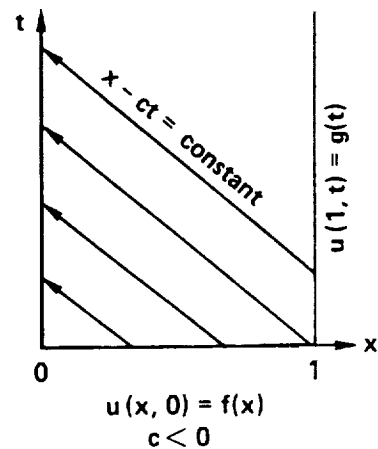
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In this case (see figs. 3 and 4), the necessary and sufficient boundary condition to produce a well-posed problem is

$$u(0,t) = g(t) \quad \text{if} \quad c > 0$$

$$u(1,t) = g(t) \quad \text{if} \quad c < 0$$

System of Equations

A system of first-order constant coefficient partial differential equations

$$u_t + Au_x = 0 \quad t \geq 0, \quad 0 \leq x \leq 1$$

is said to be hyperbolic if A is diagonalizable and with real eigenvalues. We can divide the system of equations into the following five categories:

    a. System of hyperbolic equations in diagonal form
    b. System of hyperbolic equations in coupled form
    c. Nonpositive definite systems
    d. Variable coefficients
    e. Quasi-linear systems

Each is discussed below.

System of hyperbolic equations in diagonal form:

$$u_t + \Lambda u_x = 0 \quad t \geq 0, \quad 0 \leq x \leq 1 \tag{3}$$

or

$$\begin{pmatrix} u_t^{I} \\ u_t^{II} \end{pmatrix} + \begin{pmatrix} \Lambda^{I} & 0 \\ 0 & -\Lambda^{II} \end{pmatrix} \begin{pmatrix} u_x^{I} \\ u_x^{II} \end{pmatrix} = 0$$

Here

$$u^{I} = (u^1, \ldots, u^{\ell})^T$$

$$u^{II} = (u^{\ell+1}, \ldots, u^{N})^T$$

are the dependent vector functions and

$$\Lambda^{I} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{\ell} \end{pmatrix}$$

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$$\Lambda^{II} = \begin{pmatrix} \lambda_{\ell+1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot \\ & & & \lambda_N \end{pmatrix}$$

are positive definite diagonal matrices. We can categorize the system further.

i) N decoupled equations with decoupled boundary conditions: The solution is uniquely determined if we specify initial values

$$u^I(x,0) = f^I(x)$$

$$u^{II}(x,0) = f^{II}(x)$$

and boundary conditions

$$u^I(0,t) = g^I(t)$$

$$u^{II}(1,t) = g^{II}(t)$$

In this case, we just solve N independent scalar equations.

ii) N decoupled equations with coupled boundary conditions: We can couple these equations if we replace the above boundary condition by

$$\left. \begin{array}{l} u^I(0,t) = S_I u^{II}(0,t) + g^I(t) \\[2mm] u^{II}(1,t) = S_{II} u^I(1,t) + g^{II}(t) \end{array} \quad t \geq 0 \right\} \tag{4}$$

where S_I, S_{II} are rectangular matrices with dimension $\ell \times (N - \ell)$ and $(N - \ell) \times \ell$, respectively. From the examination of how the direction of the characteristic lines (and the initial data) determine the solution of the finite domain scalar equation, we can conclude that the solution of equation (3) is again uniquely determined by conditions (4) and the initial data. Geometrically the values of u^I are transported along the characteristic to the boundary $x = 1$ (see fig. 5). Then, by the boundary conditions $u^{II}(1,t) = S_{II} u^I(1,t) + g^{II}(t)$, these values are transformed into values for u^{II}, which are then transported to the boundary $x = 0$, etc. Therefore, the number of boundary conditions for $x = 0$ is equal to the number of positive eigenvalues of the matrix Λ. And the number of boundary conditions for $x = 1$ is equal to the number of negative eigenvalues. Thus a necessary and sufficient condition for the IBVP of system (3) to be well-posed is to impose the boundary conditions in the form of (4). But the analogous formulation for problems in more than one space dimension is not necessarily well-posed. This subject is discussed briefly in the next subsection.
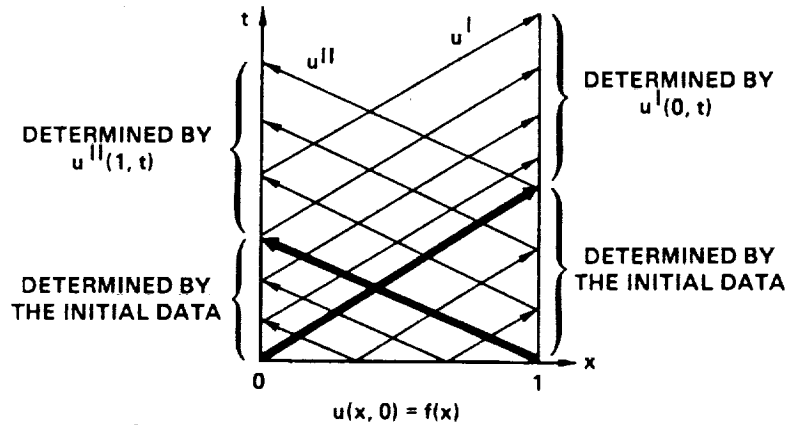
Figure 5.- Coupled boundary conditions.

System of hyperbolic equations of coupled form: In most applications, the system of differential equations is coupled

$$u_t + Au_x = 0 \qquad t \geq 0, \qquad 0 \leq x \leq 1$$

where A is assumed to be a constant matrix which can be diagonalized by a transformation matrix T.

$$T^{-1}AT = \Lambda$$

where Λ has the same form as (3) (or we can rearrange Λ in order to have the same form as (3)). For x = 0, the boundary conditions consist of ℓ linear relations among the components of u, that is, in matrix form

$$Lu(0,t) = g(t) \tag{5}$$

where L is an $\ell \times N$ matrix. Recall that ℓ is the number of positive eigenvalues of A. Let the characteristic variables be defined by

$$w = T^{-1}u$$

Then w is the solution of equation (3), and the problem is well-posed if the boundary condition

$$LTw(0,t) = g(t)$$

at x = 0, and the similar boundary condition at x = 1, can be written in the form (4). Here, the rank of L must be equal to the number of positive eigenvalues of A at x = 0. Therefore, any boundary conditions specified

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for the original system, must be transformable to boundary conditions of the form (4).

Nonpositive definite systems: If Λ^I or Λ^{II} are not positive definite, then the components $u^j(x,t)$ corresponding to $\lambda_j = 0$ must be considered as outgoing variables and will be included in u^{II} for $x = 0$ and in u^I for $x = 1$. (Variables associated with negative eigenvalues are termed as outgoing variables, and variables associated with positive eigenvalues are termed as incoming variables for the left boundary.) Since the characteristic is vertical, the solution along this characteristic is determined by the initial condition. Therefore, our discussion can assume that $\Lambda^I > 0$ for $x = 0$ and $\Lambda^{II} > 0$ for $x = 1$. That is, we should not specify the corresponding jth boundary condition with respect to $\lambda_j = 0$.

An example of a well-posed system of hyperbolic equations (from Kreiss and Oliger, ref. 2) follows. Consider

$$A = - \begin{pmatrix} c & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & c \end{pmatrix} \quad \begin{array}{l} t \geq 0 , \\ \\ 0 \leq x \leq 1 \end{array}$$

The eigenvalues, λ_j of A, are

$$\lambda_1 = -c$$

$$\lambda_2 = -(c + 1)$$

$$\lambda_3 = -(c - 1)$$

Assume $0 < c < 1$, then A has one positive and two negative eigenvalues. Therefore, we have to specify one boundary condition at $x = 0$ and two boundary conditions at $x = 1$.

Let us consider the boundary conditions

$$\left. \begin{array}{l} u_3(0,t) = 0 \\ u_1(1,t) = 0 \\ u_3(1,t) = 0 \end{array} \right\} \tag{6}$$

and check whether (6) produces a well-posed problem. That is, we have to show that these conditions are, after transformation, of the form (4). The transformation T that diagonalizes A is

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

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and (6) becomes

$$w_3(0,t) = w_2(0,t)$$

$$w_1(1,t) = 0$$

$$w_2(1,t) = w_3(1,t)$$

Here

$$w^I = w_3$$

$$w^{II} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and

$$S_I = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$S_{II} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The above equations are obviously of the form of equation (4).

Variable coefficients:

$$u_t + A(x,t)u_x = 0 \qquad t \geq 0, \qquad 0 \leq x \leq 1$$

For every fixed $t = t_0$, the form of the well-posed boundary conditions at the boundaries $x = 0$ and $x = 1$ is determined by the systems with constant coefficients

$$v_t + A(0,t_0)v_x = 0$$

$$w_t + A(1,t_0)w_x = 0$$

respectively. This is the so-called "freezing" method. Note that the theory does not cover the case when an eigenvalue of $A(0,t)$ or $A(1,t)$ changes sign in the time interval of interest. Therefore, any eigenvalue of $A(0,t)$ or $A(1,t)$ has to remain the same sign over the time interval of interest if the current theory is to be applied.

Quasilinear systems:

$$u_t + A(u,x,t)u_x = 0 \tag{7}$$

Assume $A(u,x,t)$ is a smooth function of the arguments u, x, and t, and that u can be represented as

$$u(x,t) = U(x,t) + \tilde{u}(x,t)$$

where $U(x,t)$ represents a known smooth solution and $\tilde{u}(x,t)$ a small perturbation. Linearizing equation (7) with respect to $u(x,t)$ gives us a linear system

$$\tilde{u}_t + A(U,x,t)\tilde{u}_x = B(U,x,t)\tilde{u} + F(U,x,t)$$

The boundary conditions for the well-posed problem are determined by $A(U,x,t)$ which is discussed in the variable coefficient case. The matrix $B(U,x,t)$ only affects the initial conditions and F is the nonhomogeneous part of the equations.

More Than One Space Dimension

Half-space scalar problems ($x \geq 0$, $-\infty < y < \infty$):

$$\left.\begin{array}{ll} u_t + au_x + bu_y = 0 & x \geq 0, \ -\infty < y < \infty, \ t \geq 0 \\[2mm] u(x,y,0) = f(x,y) & a, \ b \ \text{real} \end{array}\right\} \tag{8}$$

The solution of (8) is

$$u(x,y,t) = f(x - at, \ y - bt)$$

If $a < 0$, then $u(x,y,t)$ is completely determined by f. If $a > 0$ then we have to specify boundary values

$$u(0,y,t) = g(y,t)$$

Again, the sign of "a" determines whether we need to impose boundary values. The additional space dimension y does not interfere with the above boundary condition, since the y domain is $-\infty < y < \infty$.

Bounded region (scalar problem): Consider equation (8) in a closed bounded region Ω with smooth boundary $\partial\Omega$.
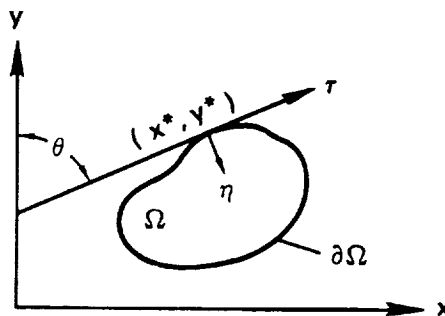
Let (x^*,y^*) be a point on $\partial\Omega$. The boundary data which should be specified at (x^*,y^*) are again determined by the direction of the characteristic at   t point. That is, we have to pointwise map the boundary onto the tangen   _ane (η,τ). This can be formalized by introducing a new coordinate system   ι origin (x^*,y^*) and axes directed as the tangent τ and the internal ɾ   ɑl η

$$\tilde{x} = (x - x^*)\cos\theta - (y - y^*)\sin\theta$$

$$\tilde{y} = (x - x^*)\sin\theta + (y - y^*)\cos\theta$$

where θ is the angle between the y and τ axis. The new transfor ed equation has the form

$$u_t + \tilde{a}u_{\tilde{x}} + \tilde{b}u_{\tilde{y}} = 0 \qquad \tilde{x} \geq 0, \; -\infty < \tilde{y} < \infty, \; t \geq 0$$

where

$$\tilde{a} = a\cos\theta - b\sin\theta$$

$$\tilde{b} = a\sin\theta + b\cos\theta$$

The sign of $\tilde{a}$ determines whether we need to specify boundary data at (x^*,y^*).

More-than-one-space-dimension system of equations: The form of the necessary and sufficient conditions for well-posed problems for the 1-D system only give necessary conditions for their multidimension counterpart. In order to obtain necessary and sufficient conditions, we have to resort to normal-mode analysis (see appendix A for definition) and Laplace transform (refs. 1-3) types of approach. Known theory by the normal mode analysis is only for strictly hyperbolic systems and symmetric hyperbolic systems.

Consider a first-order system in two space dimensions

$$u_t + Au_x + Bu_y = 0 \qquad x \geq 0, \; -\infty < y < \infty, \; t \geq 0$$

with constant coefficient matrices A and B with dimension $N \times N$. The system is hyperbolic if for all real ω_1, ω_2 with $\omega_1^2 + \omega_2^2 = 1$, there is a nonsingular transformation $T = T(\omega_1, \omega_2)$ for which both T and T^{-1} are uniformly bounded such that

$$T(\omega_1 A + \omega_2 B)T^{-1} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \cdot \\ & & & & \lambda_N \end{pmatrix}$$

and λ_j are real. If all the λ_j are distinct, the system is strictly hyperbolic. If the matrices A and B are both symmetric, then we call the system symmetric hyperbolic. We remark that the 2-D and 3-D inviscid gas-dynamics equations are not strictly hyperbolic.

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We will not discuss the normal mode analysis method here; interested readers are referred to Kreiss's original paper. Here, we want to discuss the example that Kreiss and Oliger (ref. 2) and Kreiss (ref. 3) have used to illustrate the insufficiency of the method of characteristics. Kreiss has considered the linearized shallow-water equations

$$w_t + Aw_x + Bw_y = 0 \qquad x \geq 0, \; -\infty < y < \infty, \; t \geq 0$$

where

$$A = \begin{pmatrix} u_0 & 0 & 1 \\ 0 & u_0 & 0 \\ 1 & 0 & u_0 \end{pmatrix} \qquad 0 < u_0 < 1$$

$$B = \begin{pmatrix} v_0 & 0 & 0 \\ 0 & v_0 & 1 \\ 0 & 1 & v_0 \end{pmatrix}$$

$$w = \begin{pmatrix} u \\ v \\ \phi \end{pmatrix}$$

Then the matrix A has two positive eigenvalues and two boundary conditions have to be described at $x = 0$. Kreiss used boundary conditions of the form

$$v = 0$$

$$\beta u + \alpha \phi = 0$$

Choosing different values of α, β, the system can have solutions (1) that grow arbitrarily fast with time, (2) that have too much reflection at the boundary, or (3) that are smooth and well-behaved. The following are his findings:

i) For $\alpha < -1$, $\beta = 1$, situation (1) occurs
ii) For $\beta = 0$, $\alpha = 1$, situation (2) occurs
iii) For $\beta = 1$, $\alpha = 0$, and $\alpha = \beta = 1$ situation (3) occurs

For problems in several space dimensions that have smooth boundaries and smooth coefficients, Majda and Osher (ref. 5) showed that we only need to look at the family of frozen constant-coefficient problems on half-spaces obtained by locally mapping the boundary onto the tangent plane at each point of the boundary, freezing the coefficients locally and disregarding the rest of the boundary. They showed that the original problem is well-posed if every member of this family of problems is well-posed.

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Gas-Dynamics Problems

For the inviscid systems with smooth solutions or problems with low Reynolds number, Oliger and Sunström (ref. 6) and Oliger (ref. 7) have est lished conditions for well-posedness of multidimensional problems. For s sonic inflow problems this set of admissible conditions, with a few excej ns, is of the form similar to (5) with almost full nonzero entries for L. ^ t means we have to impose a set of conditions that are linear combinations the physical variables instead of the physical variables themselves. But, ɪ physical reasons, we can only specify boundary data that are measurable. In this case, most of the admissible boundary conditions for subsonic inflow problems do not have physical significance or are not measurable quantities. For example, specifying pressure for subsonic inflow is very desirable physically, but theoretically the solution results in continued loss of smoothness globally. The most physically well-posed boundary conditions they have shown are when all of the velocity components along with either the density or the temperature are specified. By using the method of characteristics or the normal mode analysis, the 1-D inviscid gas-dynamics equations possess some features that their higher dimensional counterparts do not have; that is, there are boundary conditions that are well-posed for the 1-D Euler equation but are not well-posed for the 2-D and 3-D case. This 1-D case is discussed in more detail in the next section.

One-Dimensional Inviscid Equations of Gas Dynamics

In one spatial dimension, the inviscid equations of gas dynamics can be written in the conservative form as

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \qquad (9)$$

where

$$U = \begin{pmatrix} \rho \\ m \\ e \end{pmatrix}$$

are the conservative variables and

$$F = \begin{pmatrix} m \\ m^2/\rho + p \\ (e + p)m/\rho \end{pmatrix}$$

is the flux vector, and $m = \rho u$. The primitive variables (denoted by $\tilde{U}$) are the density ρ, the velocity u, and the pressure p. The total energy per unit volume, e, is defined as

$$e = \rho\varepsilon + \rho u^2/2$$

with ε as the internal energy per unit mass. The pressure p for a perfect gas is defined as

$$p = (\gamma - 1)[e - m^2/2\rho]$$

where γ is the ratio of specific heats. We can write equation (9) in quasi-linear nonconservative form as

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0 \tag{10}$$

with

$$A = \left(\frac{\partial F}{\partial U}\right) = \begin{pmatrix} 0 & 1 & 0 \\ (\gamma - 3)\dfrac{u^2}{2} & (3 - \gamma)u & \gamma - 1 \\ (\gamma - 1)u^3 - \dfrac{\gamma e u}{\rho} & \dfrac{\gamma e}{\rho} - \dfrac{3(\gamma - 1)u^2}{2} & \gamma u \end{pmatrix}$$

The nonconservative primitive variable form of the equation is

$$\frac{\partial \tilde{U}}{\partial t} + \tilde{A} \frac{\partial \tilde{U}}{\partial x} = 0 \tag{11}$$

where

$$\tilde{A} = M^{-1}AM$$

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{-u}{\rho} & \dfrac{1}{\rho} & 0 \\ \dfrac{(\gamma - 1)u^2}{2} & (1 - \gamma)u & (\gamma - 1) \end{pmatrix}$$

and

$$\frac{\partial \tilde{U}}{\partial t} = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_t = M^{-1}\frac{\partial U}{\partial t}$$

$$\frac{\partial \tilde{U}}{\partial x} = M^{-1}\frac{\partial U}{\partial x}$$

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We can freeze system (11) (assume constant values of $\tilde{A} = \tilde{A}_O$) (notice t... we do not have to freeze the coefficient before getting into this form; th freezing of the coefficient is for later analysis) and transform (11)

$$T^{-1} \frac{\partial \tilde{U}}{\partial t} + T^{-1}\tilde{A}_O T T^{-1} \tilde{U}_x = 0 \qquad (12)$$

where

$$T^{-1}\tilde{A}_O T = \begin{pmatrix} u_O & 0 & 0 \\ 0 & u_O + c_O & 0 \\ 0 & 0 & u_O - c_O \end{pmatrix}, \qquad c_O = \sqrt{\frac{\gamma p_O}{\rho_O}}$$

and

$$T^{-1} = \begin{pmatrix} 1 & 0 & \dfrac{-1}{c_O{}^2} \\ 0 & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}\rho_O c_O} \\ 0 & \dfrac{-1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}\rho_O c_O} \end{pmatrix}$$

with W as the characteristic variables

$$W = T^{-1}\tilde{U}$$

and u_O, c_O, and ρ_O are the "frozen coefficient" values or, numerically, the values at a given time-step and grid point. System (12) is transformed into the following characteristic form

$$\frac{\partial W}{\partial t} + \begin{pmatrix} u_O & 0 & 0 \\ 0 & u_O + c_O & 0 \\ 0 & 0 & u_O - c_O \end{pmatrix} \frac{\partial W}{\partial x} = 0$$

On the other hand, we can locally linearize system (11) into

$$\frac{\partial \bar{\bar{U}}}{\partial t} + A(U_O) \frac{\partial \bar{\bar{U}}}{\partial x} + B(U_O)\bar{\bar{U}} + F(U_O,x,t) = 0 \qquad (13)$$

where $U = U_O + \bar{\bar{U}}$, and U_O represents a smooth solution and $\bar{\bar{U}}$ is a small perturbation. This local linearization of (11) is for checking the well-posedness of boundary conditions. The boundary conditions are then determined by $A[U_O(x,t)]$; that is, the form of the boundary conditions at the boundaries, say, for fixed $t = t_O$, $0 \le x \le 1$, are determined by the systems with constant coefficients

$$\frac{\partial v}{\partial t} + A[U_O(0,t_O)] \frac{\partial v}{\partial x} = 0 \qquad \text{at} \quad x = 0 \qquad (14)$$

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$$\frac{\partial w}{\partial t} + A[U_0(1,t_0)] \frac{\partial w}{\partial x} = 0 \qquad \text{at} \quad x = 1 \qquad\qquad (15)$$

But if we know the type of inflow-outflow conditions beforehand, there is a very simple way of checking the well-posedness (instead of eqs. (14) and (15)) once a given set of "analytical boundary conditions" are proposed. (We introduce the term "analytical boundary conditions" as the boundary conditions that are required for the partial differential equations, so that the reader will not be confused with the term "numerical boundary conditions" that are required for the finite difference equations but not the differential equations.) The following is a summary of the conditions for well-posedness; refer to appendix B for a detailed derivation. In the following, we use k_{ij} and t_{ij} as the ith row and jth column of the matrices $M^{-1}T^{-1}$ and T^{-1}, respectively, where M^{-1} and T^{-1} are defined as before (with frozen coefficient). The boundary is assumed to be at the left of the domain and the flow direction is from left to right. The g_i's and $\tilde{g}_i$'s are given values.

Subsonic inflow $0 < u < c$: There are two positive eigenvalues of $\tilde{A}$. We require two analytical boundary conditions. The necessary and sufficient conditions for well-posedness are as follows.

Conservative form: Impose any pair

$$\left\{ \begin{matrix} \rho \\ m \end{matrix} \right. \ ; \ \left\{ \begin{matrix} \rho \\ e \end{matrix} \right. \ ; \ \left\{ \begin{matrix} m \\ e \end{matrix} \right.$$

or impose

$$\left\{ \begin{array}{l} k_{11}\rho + k_{12}m + k_{13}e = g_1(t) \\[2mm] k_{21}\rho + k_{22}m + k_{23}e = g_2(t) \end{array} \right.$$

Non conservative form: Must impose ρ, that is, we have to impose

$$\left\{ \begin{matrix} \rho \\ u \end{matrix} \right. \quad \text{or} \quad \left\{ \begin{matrix} \rho \\ p \end{matrix} \right. \quad \text{but not} \quad \left\{ \begin{matrix} u \\ p \end{matrix} \right.$$

or impose

$$\left\{ \begin{array}{l} t_{11}\rho + t_{12}u + t_{13}p = \tilde{g}_1(t) \\[2mm] t_{21}\rho + t_{22}u + t_{23}p = \tilde{g}_2(t) \end{array} \right.$$

Subsonic outflow $(0 > u > -c)$: There is one positive eigenvalue. We require one analytical boundary condition. The necessary and sufficient conditions for well-posedness, for either the conservative or nonconservative form, are as follows: Impose any one of the variables or impose

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$$k_{21}\rho + k_{22}m + k_{23}e = g_2(t)$$

for the conservative form, or impose

$$t_{21}\rho + t_{22}u + t_{23}p = \breve{g}_2(t)$$

for the nonconservative form.

Supersonic inflow u > c: There are three positive eigenvalues. We require three analytical boundary conditions.

Supersonic outflow u < -c: There is no positive eigenvalue. Therefore we may not impose any analytical boundary condition.

3. STABLE DIFFERENCE APPROXIMATIONS FOR HYPERBOLIC INITIAL BOUNDARY VALUE PROBLEMS (IBVP) IN A FINITE DOMAIN

There are essentially three main considerations in studying approximate solutions to the initial boundary value problems (IBVP): (1) well-posedness of the original partial differential equations (PDE); (2) the method of constructing extra boundary conditions required for the finite difference equations (FDE) but not the PDE; and (3) the stability and accuracy of the FDE. In this section, we will review some of the well-known theory on stability analysis for IBVP, and list some of the commonly used stable schemes (stable for the combined interior and boundary schemes). The subject of accuracy will not be addressed here. The reader should refer to Gustafsson (refs 18, 19), Varah (ref. 20), Sköllermo (refs. 21, 22), and Sloan (ref. 23) for more detail. The major result for accuracy analysis is due to Gustafsson (refs. 18, 19), who proved that boundary schemes can be at most one order lower than the interior schemes, without loss of global accuracy.

The treatment of difference approximations relating to Cauchy (initial value) problems of the hyperbolic type is quite well established. On the other hand, the treatment of mixed IBVP is considerably less well established. So far, the boundary conditions are quite difficult to construct and a poor choice can lead to inaccuracies and instabilities. The stability theory for difference approximations of the IBVP is really only complete for one space dimension, although this theory is essentially sufficient if the approximations are dissipative in the tangential directions (ref. 4) for multidimensional problems. For a one-space-dimension variable coefficient or quasi-linear system of hyperbolic equations with smooth solution (no shocks), the theory is well established. Care is needed to avoid exponential growth due to improper boundary extrapolation (refs. 9, 24). Recently Oliger (ref. 10) develop an easy way of constructing stable boundary schemes for the 1-D scalar prob For problems of higher dimension, little is known except for problems with smooth boundaries, constant coefficients, and strictly hyperbolic cases.

In the study of how boundary approximations affect the stability of gas-dynamics equations, rigorous stability analyses have only been applied to 1-D and 2-D scalar equations with variable coefficients or quasi-linear property, or to systems of equations with constant coefficients. Boundary approximations

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for problems with open boundaries and for viscous fluids at high Reynolds numbers have not been studied sufficiently. Boundary approximations for factored or splitted implicit methods have not been analyzed. Crandall and Majda (ref. 25) have developed a complete treatment of the stability and convergence properties for scalar conservation laws in several space variables. Their method is a conservation-form, monotone difference approximation. Many investigators have applied various boundary approximations to the nonconservative form of the nonlinear system and have compared the results with experimental data (see the Bibliography: Fluid Dynamics). Coughran (ref. 26) has devised a numerical method based on normal mode analysis (defined in appendix C) to study stable boundary schemes for the 1-D Euler equations. The following is a summary of the recent developments of currently available tools for stability analysis — concentrating on the more fundamental aspects of the subject, with a more detailed description of the theory for one space dimension. All of the initial data that we use throughout the report are assumed to be square-integrable.

Fundamental Concepts

In order to explain some of the difficulties, let us consider the differential equation

$$\left.\begin{array}{l} u_t - u_x = 0 \qquad x \geq 0, \ t \geq 0 \\[2mm] u(x,0) = f(x) \end{array}\right\} \tag{16}$$

From the well-posedness of the problem, we know that no boundary conditions should be specified for $x = 0$, $t \geq 0$. If we want to solve equation (16), using some finite difference scheme, we need information about u at the "numerical boundary" $x = 0$, unless we use appropriate one-sided spatial differencing. For convenience, we will call the imposed boundary condition the "analytical boundary condition" and the extra boundary condition needed for the difference approximation the "numerical boundary condition."

Let us say we want to solve the above problem using the leap-frog scheme

$$v_j^{n+1} = v_j^{n-1} + \frac{\Delta t}{\Delta x} (v_{j+1}^n - v_{j-1}^n) \tag{17}$$

where $v_j^n = v(j\Delta x, n\Delta t)$ denotes the numerical solution of u. We assume that $\Delta t/\Delta x < 1$; that is, equation (17) is a stable approximation for the Cauchy problem. We need an additional equation for $v(0,t)$. Let us overspecify $v(0,t)$ as

$$v(0,t) = g(t)$$

In general this overspecification will destroy the convergence. The only exception is the case in which $v(0,t) = u(0,t)$, where $u(0,t)$ denotes the solution at the boundary. But normally, we would not know about the exact solution. Kreiss and Lunqvist (ref. 27) and Gustafsson and Kreiss (ref. 17) have shown that "inexact" overspecification of boundary conditions leads to oscillatory solutions for this type of scheme (centered scheme, nondissipative).

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Therefore, one needs to be very careful when overspecifying boundary cor.
tions. The solution will look nicer if the approximation is dissipativ.
because the oscillations will be damped. However, near the boundary th   rors
are quite serious. If one considers a system of equations, this error   be
propagated into the interior of the region by the ingoing characterist   of
the coupled variables, even when dissipative approximations are used.   .e
stable way of handling v_0^n is

$$v_0^n = v_1^{n-1}$$

Another is

$$v_0^n = 2v_1^{n-1} - v_2^{n-2}$$

To illustrate another difficulty, let us consider

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x \qquad -1 \leq x \leq 1, \ t \geq 0$$

$$u(x,0) = f_1(x)$$

$$v(x,0) = f_2(x)$$

$$u(-1,t) = g_1(t)$$

$$u(1,t) = g_2(t)$$

where the f_i's and g_i's are square-integrable. From the method of charac-
teristics or normal-mode analysis, it can be shown that this problem is well-
posed. In solving the equation numerically, we generally need special differ-
ence equations to find v at both boundaries, even though analytically, the
solution is uniquely determined for the PDE. Gottlieb and Turkel (ref. 24)
have shown that if one uses the Lax-Wendroff finite difference method in the
interior and quadratic spatial extrapolation for v at the boundary, then the
resulting system is unstable. But Gustafsson et al. (ref. 9) have shown that
the same extrapolation is stable in conjunction with the Lax-Wendroff method
for scalar equations. In reference 28, Gottlieb et al. show that a straight-
forward extension of the scalar results to a system may not work. However, by
proper use of the characteristic variable at the boundaries, they demonstrate
how the results of the scalar equation can be extended to a system. They show
(in ref. 28) that by using quadratic spatial extrapolation for the appropriate
characteristic variables, the revised method is stable. This is sometimes
called the "characteristic stability theorem."

Overview and Development of Stability Theory

For a one-space-dimension linear constant coefficient system, we can divide difference methods into two classes — those that belong to the "method of lines" approach and those that do not. The method of lines uses a finite-difference approximation in space and an ODE (ordinary differential equation) solver in time. For the method of lines approach, the stability of some of the popular schemes, like the central, forward, and backward spatial differencing schemes, coupled with simple boundary approximations was analyzed by Gary (ref. 29) (the matrix method), and by Dahlquist (ref. 30) (the positive real-function approach). In appendix D, we discuss the stability analysis of Gary and Dahlquist. They only showed the stability of the method for fixed Δx; that is, they did not show stability in the usual sense. In order to satisfy the definition of stability, these methods involve the additional analysis of infinite dimension matrices.

For the approach that is not a method of lines approach, the simplest heuristic condition for stability was discussed by Trapp and Ramshaw (ref. 11). Their analysis used the interior as well as the boundary approximation to do a related Cauchy problem by the Fourier method (Von Neumann). An interior or boundary approximation is said to be Cauchy stable if it is stable for the related Cauchy problem (the related initial value problem, i.e., the domain for 1-D is $-\infty < x < \infty$). They claimed that the minimum of the related Cauchy stability bound for the interior and the boundary can be used as the stability bound of the entire problem. But this heuristic approach does not provide sufficient conditions or proper hypotheses for stability of the IBVP.

The most rigorous classical approach to the stability bound is the energy method. It is a powerful tool in dealing with certain particular equations or particular classes of equations (refs. 3, 6, 12). It can become rather complicated or tricky to apply, but it can deal effectively with boundary conditions and handles variable coefficients easily. However, it does not give necessary and sufficient conditions.

A more unified approach to stability theory is due to Kreiss (ref. 8), and to Gustafsson et al. (ref. 9). It is sometimes called the normal-mode analysis. Strikwerda (ref. 31) has applied this theory for the method of lines approach. Godunov and Ryabenkii, whose work is discussed in reference 12, first gave necessary stability conditions for 1-D problems by considering modes of the form $u_j^n \sim \kappa^n \mu^j$ (n – time step index, j – space mesh point index), where $|\mu| < 1$ and j counts mesh points away from the boundary. Kreiss (ref. 8) and Gustafsson et al. (ref. 9) have greatly refined the approach, giving only mildly stricter conditions which are necessary and sufficient for stability. However, the analysis is more complex than that for the interior (i.e., the Cauchy problem). There are some important simple cases that have been studied in detail by this method, especially for dissipative approximations. This theory is a posteriori in nature. Given a difference method, we can use this theory to determine whether the method is stable; but the stability criteria are often very difficult to verify. An example of how this theory applies to the first-order hyperbolic scalar equation with the simple well-known difference approximations can be found in appendix C. Recently, Oliger (ref. 10) gave sufficient stability conditions that are very easy to check.

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A detailed discussion of the related theory is presented in appendix E. These conditions can be used to guide us in the construction of stable method r the initial boundary value problems. In order to make the development understandable, we use (in appendix E) the case of a strictly hyperbol ys- tem with constant coefficients and coupled boundary conditions that ar ll- posed. Then we discuss how we can arrive at the point at which it is necessary to consider anything more complex than a single scalar equa n for each transformed variable. The stability analysis of this scalar eq ion in a finite domain is equivalent to the analysis of two related quarter ane problems. We then proceed to discuss the way to construct stable schemes. The main assumption of the theory for constructing stable schemes is that the interior and boundary approximations are Cauchy stable and at least one of the approximations is dissipative. A point of caution — the sufficient condition does not guarantee sharp limits for conditionally stable methods.

Some Stable Boundary Schemes (for Right Quarter Plane Problem, i.e., $x \geq 0$)

The following are some popular boundary schemes.

Extrapolation:

$$v_0^{n+1} = v_1^{n+1} \tag{18a}$$

$$v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1} \tag{18b}$$

$$v_0^{n+1} = v_1^{\,n} \tag{18c}$$

$$v_0^{n+1} = 2v_1^{\,n} - v_2^{\,n-1} \tag{18d}$$

One-sided scheme:

$$v_0^{n+1} = v_0^{\,n} + \Delta t \left(\frac{v_1^{\,n} - v_0^{\,n}}{\Delta x} \right)$$

Box scheme:

$$v_0^{n+1} + v^{n+1} - \Delta t \left(\frac{v_1^{n+1} - v_0^{n+1}}{\Delta x} \right) = v_0^{\,n} + v_1^{\,n} + \Delta t \left(\frac{v_1^{\,n} - v_0^{\,n}}{\Delta x} \right)$$

By using the normal-mode analysis (ref. 9), it can be shown that using the boundary schemes ((18a), (18b)), the one-sided scheme and the box scheme, together with the Lax-Wendroff or the Crank-Nicholson method, produces stable schemes for the right quarter-plane model problem:

$$\left. \begin{aligned} u_t - u_x &= 0 \qquad x \geq 0,\ t \geq 0 \\ u(x,0) &= f(x) \end{aligned} \right\} \tag{19}$$

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Extrapolation at the same time level ((18a), (18b)) (spatial extrapolation) is not a stable process for the leap-frog scheme. For leap-frog types of schemes, we have to use (18c) and (18d), the one-sided scheme, or the box scheme. For predictor corrector schemes, like that of Richtmyer and Morton (ref. 12), or the MacCormick scheme (for linear constant coefficient, these two methods are identical) there are intermediate steps involved; Gottlieb and Turkel (ref. 24) have studied these schemes in detail. They have shown that spatial extrapolation ((18a), (18b)) and the one-sided boundary schemes are good choices.

Now, we consider the class of interior schemes that evolves from linear multistep methods in ordinary differential equations (ref. 32). For model equation (19) with central spatial differencing, this class of schemes is of the form

$$\rho(E)u_j{}^n = -\Delta t \sigma(E)\left(\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right) \tag{20}$$

Here E is the shift operator defined by

$$Eu_j{}^n = u_j^{n+1}$$

and ρ and σ are polynomials defined by

$$\rho(E) = (1 + \xi)E^2 - (1 + 2\xi)E + \xi$$

$$\sigma(E) = \theta E^2 + (1 - \theta + \phi)E - \phi$$

The notation is consistent with that for linear multistep methods for ordinary differential equations and $\rho(E)$ should not be confused with density. Some of the well-known methods (in time) belonging to this class are listed in table 1.

TABLE 1.- PARTIAL LISTING OF LINEAR MULTISTEP
METHODS

Method	ξ	θ	ϕ
1. Backward Euler	0	1	0
2. Two-step backward Euler	-1/2	1	0
3. Trapezoidal (Crank-Nicholson)	0	1/2	0
4. Backward differentiation	1/2	1	0
5. Adams	0	3/4	-1/4
6. Lees	-1/2	1/3	-1/3
7. Two-step trapezoidal	-1/2	1/2	-1/2
8. A-contractive	-1/6	5/9	-2/9
9. Leap-frog	-1/2	0	0
10. Milne	-1/2	1/6	-1/6

The first eight methods are unconditionally stable for the Cauchy problem and the remaining ones are conditionally stable. For the class of all two-step methods that are at least second-order time accurate, the parameters (θ, ξ, ϕ) are related by (ref. 32)

$$\phi = \xi - \theta + 1/2$$

The class of all third-order methods (in time) is obtained by imposing the additional condition

$$\xi = 2\theta - 5/6$$

There is a unique fourth-order method, specified by

$$\theta = -\phi = -\xi/3 = 1/6$$

which is called Milne's method. Asume $\lambda = (\Delta t/2\Delta x)$ is chosen such that the method being discussed is stable for the related Cauchy problem. That is, equation (20) with $j = 0, \pm 1, \pm 2, \ldots$, is stable. Gustafsson and Oliger (ref. 33) have proved the following results:

I. If the boundary extrapolation ((18a) and (18b)) is used with the method (20) in table 1, then the resulting methods are all stable for the initial boundary value problem (19) except for the leap-frog and Milne methods.

II. If the boundary extrapolation (18a) and (18b) is used with the method (20) in table 1, then the resulting methods are all stable for the initial boundary value problem (19) except for the two-step backward Euler, Lees, and two-step trapezoidal methods.

All of the numerical schemes (interior + boundary) that we are going to study in the next section are mainly implicit schemes. For the model equation (19), these schemes are unconditionally stable. One of the schemes is the backward Euler method in equations (20) and (18b).

Stability Analysis of a Finite Domain

Consider the scalar hyperbolic equation

$$u_t + cu_x = 0 \qquad 0 \le x \le 1, \, t \ge 0 \tag{21}$$

with initial condition $u(x,0) = f(x)$. From the well-posedness of the problem (ref. 1), we have to specify analytically a boundary condition at the right boundary $x = 1$, when c is negative, or at the left boundary $x = 0$, when c is positive. Hence, in addition to equation (21) and the initial data, we specify boundary conditions

$$u(1,t) = g_1(t) \qquad \text{if} \quad c < 0$$

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or

$$u(0,t) = g_0(t) \quad \text{if} \quad c > 0$$

Let us assume $c < 0$. From a theorem of Gustafsson et al. (ref. 9), the stability of a difference approximation for the initial boundary value problem (21) on $0 \leq x \leq 1$ is equivalent to the stability of two related quarter-plane problems. The related right and left quarter-plane problems are defined as

$$\left. \begin{array}{l} u_t + cu_x = 0 \quad\quad 0 \leq x < \infty, \, t \geq 0 \\ \quad\quad\quad\quad\quad\quad\quad\quad c < 0 \\ u(x,0) = f(x) \end{array} \right\} \quad\quad (22)$$

$$\left. \begin{array}{l} u_t + cu_x = 0 \quad\quad -\infty < x < 1, \, t \geq 0 \\ \quad\quad\quad\quad\quad\quad\quad\quad c < 0 \\ u(x,0) = f(x) \\ \\ u(1,t) = g_1(t) \end{array} \right\} \quad\quad (23)$$

respectively. If only two- and three-point schemes are considered, then the stability analysis of the IBVP associated with (3.6) is transferred to the right quarter-plane problem (22). The stability of the left quarter-plane problem (3.8) reduces to the stability of a Cauchy problem.

4. APPLICATIONS TO THE 1-D INVISCID EQUATIONS OF GAS DYNAMICS

From the computational point of view, the unsteady inviscid gas-dynamics equations (Euler equations) in conservation law form have the following properties:

  a. They are a quasi-linear hyperbolic system.

  b. In general, the Jacobian of the flux vector consists of mixed positive and negative eigenvalues (characteristic speed).

  c. The flux vectors of the Euler equations are homogeneous functions of degree one in the dependent variables.

  d. The homogeneous properties provide a formal procedure for decomposing the flux vectors into subvectors, each of which depends on eigenvalues of the same sign (flux-vector splitting (ref. 16). Consequently, one-sided spatial difference operators can be used to construct a dissipative scheme.

There are essentially two popular forms of the Euler equations being used in the computational fluid dynamics field: the conservative form (9) and the nonconservative form (11). Mathematically they are equivalent, but from the computational point of view they produce different solutions, if the same

numerical scheme is applied on the two forms. The study of well-posedness and stability of difference approximations is easier using the nonconservative form. Most of the existing theory and applications of the theory on the above studies use the nonconservative form. Recently, the development of computational methods utilizing one-sided differencing gained popularity. Some of the one-sided differencing schemes are those of Godunov (ref. 34), Steger and Warming (ref. 16), Engquist and Osher (ref. 35), Roe (ref. 36), Carver (ref. 37), and Lax and Harten (private communication). At this time, there are no published results comparing the above one-sided differencing schemes, but some are more difficult to use than the others. The flux vector splitting method is useful for the application of a one-sided dissipative scheme on the conservative form, since the method is very simple to use and provides a proper way of handling the inflow-outflow boundary efficiently. For example, we can apply the one-sided difference operators on the split-flux subvectors over the interior and boundary points or, we can apply the one-sided difference operators on the split-flux subvector over the boundary points only.

We are going to discuss the stability of a few numerical schemes for the 1-D Euler equations. Stability analysis is based on local linearization and solutions are assumed to be smooth near the boundaries. The various methods of handling the numerical boundary will be discussed briefly, but the method of linear extrapolation in the characteristic variables will be the main topic. Some numerical solutions of the quasi-1-D nozzle problem will be used to illustrate the commonly discussed issues; for example, explicit versus implicit boundary schemes, unconditionally stable schemes, and underspecification or overspecification of boundary conditions.

Flux-Vector Splitting

As discussed earlier, the nonlinear flux vector $F(U)$ is a homogenous function of degree one in U; that is $F(\alpha U) = \alpha F(U)$. By application of Euler's theorem on homogenous functions, it follows that

$$F = AU = \frac{\partial F}{\partial U} U$$

F can be split into two parts as (ref. 16)

$$F = F^+ + F^-$$

where F^+ corresponds to the subvector associated with the positive eigenvalues λ^+ of A, and F^- corresponds to the negative eigenvalues λ^-. Therefore

$$F = F^+ + F^- = (A^+ + A^-)U$$

with

$$Q = MT , \qquad Q^{-1} = T^{-1}Q^{-1}$$

$$A^+ = Q\Lambda^+ Q^{-1} , \qquad A^- = Q\Lambda^- Q^{-1} \tag{24}$$

and matrices M^{-1} and T^{-1} are defined in section 2.

For $0 \leq u \leq c$, we have

$$F^+ = \frac{\rho}{2\gamma} \begin{pmatrix} 2\gamma u + c - u \\ 2(\gamma - 1)u^2 + (u + c)^2 \\ (\gamma - 1)u^3 + \frac{(u + c)^3}{2} + \frac{(3 - \gamma)(u + c)c^2}{2(\gamma - 1)} \end{pmatrix} \qquad (25)$$

$$F^- = \frac{\rho}{2\gamma} \begin{pmatrix} u - c \\ (u - c)^2 \\ \frac{(u - c)^3}{2} + \frac{(3 - \gamma)(u - c)c^2}{2(\gamma - 1)} \end{pmatrix} \qquad (26)$$

For $u > c$, we have

$$F^+ = F , \qquad F^- = 0$$

The diagonal matrices Λ^+, Λ^- are given by

$$\Lambda^+ = \begin{pmatrix} \frac{u + |u|}{2} & 0 & 0 \\ 0 & \frac{u + c + |u + c|}{2} & 0 \\ 0 & 0 & \frac{u - c + |u - c|}{2} \end{pmatrix} ,$$

$$\Lambda^- = \begin{pmatrix} \frac{u - |u|}{2} & 0 & 0 \\ 0 & \frac{u + c - |u + c|}{2} & 0 \\ 0 & 0 & \frac{u - c - |u - c|}{2} \end{pmatrix}$$

Difference Approximations of the Inviscid Equations of Gas Dynamics

By adopting the notation of Warming and Beam (ref. 38) and of Beam and Warming (ref. 32), the 1-D system of inviscid gas-dynamics equations can be approximated by a simple generalized three-level time differencing in the $\rho(E)$ form as

$$\left(I + \omega\Delta t \frac{\partial}{\partial x} A^n\right)\rho(E)U^n = -\Delta t[\sigma(E) - \omega\rho(E)]\left(\frac{\partial F}{\partial x}\right)^n \qquad (27)$$

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(Note: $[I + \omega\Delta t(\partial/\partial x)A^n]\rho(E)U^n = \rho(E)U^n + \omega\Delta t\{\partial[A^n\rho(E)U^n]\}/\partial x$.) The parameter $\omega = [\theta/(1+\xi)]$ is determined by the particular time-differencing approximation used. Scheme (27) includes the following well-known implicit formulas (see sec. 3):

$$\xi = 0 \ , \ \theta = \frac{1}{2} \ , \ \phi = 0 \qquad \text{trapezoidal (Crank-Nicholson)}$$

$$\xi = 0 \ , \ \theta = 1 \ , \ \phi = 0 \qquad \text{backward Euler}$$

$$\xi = \frac{1}{2} \ , \ \theta = 1 \ , \ \phi = 0 \qquad \text{backward differentiation}$$

In (27), $A^n = A(U^n)$, $(\partial F/\partial x)^n = [\partial F(U^n)/\partial x]$, and U^n is the solution at $t = n\Delta t$ with ΔT as the time step.

There are two ways to utilize the flux-vector splitting:

a. Apply one-sided approximations on the split-flux subvector throughout the entire computational domain of definition. (For example, use backward spatial differences for the "positive" subvector and forward differences for the "negative" subvector.)

b. Apply the one-sided approximations on the split-flux subvector on the first and last interior points only.

If we apply the flux-vector splitting on both the interior points and boundary points, system (27) can be expressed in the following form

$$\left[I + \omega\Delta t\left(\frac{\partial}{\partial x} A^{+n} + \frac{\partial}{\partial x} A^{-n}\right)\right]\rho(E)U^n = -\Delta t[\sigma(E) - \omega\rho(E)]\left[\left(\frac{\partial F^+}{\partial x}\right)^n + \left(\frac{\partial F^-}{\partial x}\right)^n\right] \qquad (28)$$

where A^+, A^-, F^+, F^- are defined as in equations (24)-(26). One-sided first-order backward and forward-difference operators can be used for the spatial derivatives on the left-hand side of (28):

$$\frac{\partial}{\partial x}[A^{+n}\rho(E)U^n]\Big|_j = \frac{A^{+n}_j\rho(E)U^n_j - A^{+n}_{j-1}\rho(E)U^n_{j-1}}{\Delta x} + 0(\Delta x)$$

$$\frac{\partial}{\partial x}[A^{-n}\rho(E)U^n]\Big|_j = \frac{A^{-n}_{j+1}\rho(E)U^n_{j+1} - A^{-n}_j\rho(E)U^n_j}{\Delta x} + 0(\Delta x)$$

The spatial derivatives on the right of (28) can be approximated by the first- or second-order approximations. The second-order approximations are of the form:

$$\left.\frac{\partial F^{+n}}{\partial x}\right|_j = \frac{3F^{+n}_j - 4F^{+n}_{j-1} + F^{+n}_{j-2}}{2\Delta x} + 0(\Delta x^2)$$

$$\left.\frac{\partial F^{-n}}{\partial x}\right|_j = \frac{-3F^{-n}_j + 4F^{-n}_{j+1} - F^{-n}_{j+2}}{2\Delta x} + 0(\Delta x^2)$$

The resulting algorithm (for the associated linearized Cauchy problem) is a dissipative, unconditionally stable, second-order algorithm, if we use two-step backward Euler time-differencing ($\theta = 1$, $\xi = 1/2$, $\phi = 0$). The solution of (28) requires block tridiagonal inversion. We can introduce an approximate factorization of the left-hand side of (28), and change system (28) to the product of two operators as

$$\left(I + \omega\Delta t\,\frac{\partial}{\partial x}\,A^{+n}\right)\left(I + \omega\Delta t\,\frac{\partial}{\partial x}\,A^{-n}\right)c(E)U^n = -\Delta t[\sigma(E) - \omega c(E)]\left[\left(\frac{\partial F^+}{\partial x}\right)^n - \left(\frac{\partial F^-}{\partial x}\right)^n\right]$$

(29)

The solution of (29) only requires block bidiagonal inversion. The stability of (29) is more difficult to analyze. We will only use form (28) for the quasi-1-D nozzle.

Instead of using one-sided differencing throughout, we use system (27) without splitting A and F into two parts in the interior. The spatial derivative can be approximated by central differencing. For the first and last computational points, we can use the form (28).

So far, stability analyses of variable coefficient or quasi-linear hyperbolic problems are only known for scalar equations or for systems with smooth coefficients and smooth solutions (ref. 2). For systems with nonsmooth coefficients or solutions, nonlinear instability can occur; for example, when an eigenvalue changes sign. One remedy is to use a dissipative scheme or add a dissipative term to the original differential equation. The one-sided spatial difference schemes "comes" with dissipation and frequently we have no control over it. The centered (spatial) schemes require "added" on dissipation but allow different dissipative weight treatment in different regions of the solution. Both methods are quite popular in the computational fluid dynamics field.

Stability Analysis

As we have discussed before, theory for stability analysis of difference approximations for 1-D nonlinear hyperbolic equations has been established only for schemes that are dissipative or for problems with smooth solutions. The method of analysis depends on the "freezing method." If we freeze ($\partial F/\partial U$), then there is no distinction between the conservative and the nonconservative form. For each $x = x_0$, $t = t_0$ we have a system of constant coefficient equations to analyze; that is,

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$$\frac{\partial U}{\partial t} + A[U(x_0,t_0)] \frac{\partial U}{\partial x} = 0 \qquad (30)$$

As we will see later, the analysis is very simple. System (30) can be decoupled into three scalar equations:

$$\left. \begin{array}{l} \dfrac{\partial w_1}{\partial t} + \lambda_1 \dfrac{\partial w_1}{\partial x} = 0 \\[12pt] \dfrac{\partial w_2}{\partial t} + \lambda_2 \dfrac{\partial w_2}{\partial x} = 0 \\[12pt] \dfrac{\partial w_3}{\partial t} + \lambda_3 \dfrac{\partial w_3}{\partial x} = 0 \end{array} \right\} \qquad (31)$$

with

$$\lambda_1 = u(x_0,t_0) = u_0$$

$$\lambda_2 = u(x_0,t_0) + c(x_0,t_0) = u_0 + c_0$$

$$\lambda_3 = u(x_0,t_0) - c(x_0,t_0) = u_0 - c_0$$

Thus, at each time-step, the stability analysis consists of a _pointwise_ examination of equations (31). For the higher order explicit methods, it is easier to use Oliger's method (ref. 10) than the normal-mode analysis method to check for stability. On the surface, Oliger's sufficient condition consists of two parts (assuming the combined interior and boundary schemes are stable for the model problem).

a. Apply the interior difference scheme to (30) and do Cauchy stability shecks for all x_0 that are interior points.

b. Apply the boundary difference schemes to (30) and do Cauchy stability checks for all x_0 that are boundary points.

If conditions (a) and (b) pass the stability tests at each point for every time step, what can we say about the stability of the original uncoupled nonlinear system? Stability is confirmed if at least one of the approximations is dissipative (this is a sufficient condition; that is, an "unstable" boundary scheme for the related Cauchy problem does not imply that combined — interior plus boundary — scheme is not stable) and if the solutions are smooth. In the actual case, the stability checks of part (a) and (b) involve scalar equations only. For popular numerical schemes, Cauchy stability bounds are known. The major work is the testing of the values of λ_i $i = 1, 2, 3$ at each grid point and time step. This is trivial since the λ_i are known. The method of normal mode analysis can follow the same approach, except in this case we have a necessary and sufficient condition. But higher order methods are more difficult to verify. Often, we have to resort to numerical methods of solving a set of complicated resolvent equations

(defined in appendix C). For problems with shocks, there is no guarantee that stability of the "freezing" family will imply stability of the original nonlinear problem. But, usually, it is quite promising if we use a dissipative scheme.

The Numerical Boundary Conditions

To simplify the discussion, let us assume that the spatial differencing we are going to use will be a first-order-one-sided or central-difference scheme, and denote the left and right boundary node index as 0 and J. Then the spatial differencing of (27) and (28) on the first and last computational points involves terms like

$$E_O \Delta U_O{}^n$$

$$E_J \Delta U_J{}^n$$

where E_O, E_J are some known matrices determined from the previous time step, and $\Delta U^n = U^{n+1} - U^n$. The $\Delta U_O{}^n$, $\Delta U_J{}^n$ are partially known from the analytical boundary condition, with the exception of supersonic inflow. A few of the popular methods of obtaining the expression for the numerical boundary conditions are by

    a. Extrapolating in space or space-time (refs. 28, 39).

    b. Discretizing the Riemann invariant equations (the nonlinearized form of the characteristic equations) or the characteristic equations (12) locally (refs. 40, 41, and J. Oliger (private communication)).

    c. Taking derivatives of the known condition in order to produce an extra boundary condition (refs. 19, 42, and M. Hyman (private communication)).

    d. Using nonreflecting boundary conditions (refs. 17, 43, 44).

    e. Overspecifying the boundary conditions.

For implicit schemes, methods (a)-(d) above are quite complicated to implement into a computer code. Method (e) is of limited usefulness since it requires a priori knowledge of the exact solution to the difference equation at the boundary. Method (a) has the advantage of being the easiest to use; therefore, our study concentrates on method (a). But, as we know, extrapolation procedures suffer from the disadvantage of not modeling the differential equation (or not depending on the differential equations). However, if we use spatial linear extrapolation together with the two ways of utilizing the flux-vector splitting from the preceding subsection (Stability Analysis), the spatial differencing is already tailored to the direction of the characteristic curve locally. The extra unknowns that are required at the boundaries are due to the noniterative property of the scheme and the coupling of the physical equations. Therefore, the numerical procedure for the extra unknowns at the boundaries should not be as crucial — spatial linear extrapolation appears

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to be a good choice. As before we will use the term "numerical boundary conditions" as the extra boundary conditions that are required for the FDE but not for the PDE, or as the extra unknowns at the boundaries due to the non-iterative property of the scheme (local linearization).

In implementing any methods (a)-(d), there are numerous and complicated details involved. Here, we will simply consider the spatial linear extrapolation in detail. The main point of this study is to show that the use of spatial linear extrapolation as boundary schemes for the implicit method (discussed in the subsection "Difference Approximations of the Inviscid Equations of Gas Dynamics," sec. 4), is quite successful. Other comparisons of methods and application to different types of physical problems will be reported elsewhere (ref. 45).

Spatial Linear Extrapolation for the Numerical Boundary Conditions

For physical reasons, we sometimes prefer to specify boundary data in the primitive variables and compute in terms of the conservative variables in the interior. Thus the choice of variables for the analytical boundary conditions to be imposed and numerical (or extra) boundary conditions to be extrapolated for the conservative form (9) can be divided into the following four groups:

Group	Variable (anal. B.C.)	Variable (num. B.C.)
I	Conservative	Conservative
II	Conservative	Characteristic
III	Primitive	Primitive
IV	Primitive	Characteristic

Under certain inflow-outflow combinations, not all of the above ways of imposing analytical boundary conditions are mathematically possible (or physically desirable). If possible, group I is by far the simplest to implement with the rest appearing in increasing order of complexity. Group IV, on the other hand is more physically desirable and more theoretically sound (ref. 28). Groups II and IV reduce to the scalar model hyperbolic equations for the linearized equations of (9) and (11), respectively. We can have a whole class of stable schemes to choose from, as discussed in section 3. This is also true for group I in the supersonic inflow or supersonic outflow case. Now we turn to discuss group III. For the subsonic inflow case, it has been shown by Gustafsson and Oliger (ref. 33) that all the approximations (27) with parameter values in table 1 are stable, with the following boundary conditions:

$$\rho_0{}^n \quad \text{given}$$

$$u_0{}^n \quad \text{given}$$

$$P_0{}^n = 2p_1{}^n - p_2{}^n$$

For the subsonic outflow case, Gustafsson and Oliger (ref. 33) also proved that all the approximations (27) with parameter values in table 1 (except for leap-frog and Milne) are stable, with the following ways of handling the boundary conditions:

(i)

$$u_0{}^n \quad \text{given}$$

$$\rho_0{}^n = 2\rho_1{}^n - \rho_2{}^n$$

$$p_0{}^n = 2p_1{}^n - p_2{}^n$$

(ii)

$$p_0{}^n \quad \text{given}$$

$$\rho_0{}^n = 2\rho_1{}^n \quad \rho_2{}^n$$

$$u_0{}^n = 2u_1{}^n - u_2{}^n$$

Here, we will describe the spatial linear extrapolations in the characteristic variables, that is, group II. Other groups can follow similar procedures.

The relation between the conservative and characteristic variables is

$$T^{-1}M^{-1}U_t = W_t$$

with U the vector of conservative variables, and W the vector of characteristic variables. The procedures for group II at inflow (left boundary) will be

i) Make a first-order approximation:

$$(T^{-1}M^{-1})_0{}^n \Delta U_0{}^n = \Delta W_0{}^n$$

ii) Reorder $U_0{}^n$ into subvectors $(U^I)_0{}^n$ and $(U^{II})_0{}^n$ where $(U^I)_0{}^n$ is the "analytical" boundary condition and $(U^{II})_0{}^n$ is the "numerical" boundary condition.

iii) Reorder $W_0{}^n$ into subvectors $(W^I)_0{}^n$ and $(W^{II})_0{}^n$ where $(W^I)_0{}^n$ corresponds to the subvector associated with the positive eigenvalues of $\tilde{A}$ and $(W^{II})_0{}^n$ corresponds to the negative eigenvalues of $\tilde{A}$ (for outflow right boundary, the signs of the eigenvalues are the reverse).

iv) Reorder $(T^{-1}M^{-1})_0{}^n$ and partitioned it accordingly as

$$\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}_0^n$$

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Then we have

$$\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}_o^n \begin{pmatrix} \Delta U^I \\ \Delta U^{II} \end{pmatrix}_o^n = \begin{pmatrix} \Delta W^I \\ \Delta W^{II} \end{pmatrix}_o^n$$

Note that the delta formulation (ΔU) is important for step (i) b ause of the nonlinear relation between conservative, primitive, and characteristic variables. Now spatial linear extrapolation in the characteristic variables means

$$(\Delta W^{II})_o^n = 2(\Delta W^{II})_1^n - (\Delta W^{II})_2^n$$

This implies

$$(P_3 \Delta U^I + P_4 \Delta U^{II})_o^n = 2(P_3 \Delta U^I + P_4 \Delta U^{II})_1^n - (P_3 \Delta U^I + P_4 \Delta U^{II})_2^n$$

Since P_4 should be nonsingular for a well-posed problem, we can rearrange terms and obtain

$$(\Delta U^{II})_o^n = R_o (\Delta U^I)_o^n + R_1 (\Delta U)_1^n + R_2 (\Delta U)_2^n \qquad (32)$$

where R_0, R_1, R_2 are known rectangular matrices which can be evaluated from the previous time step. (Note the mixture of dimensions in the equations.) Similarly, the outflow numerical boundaries can be expressed as

$$(\Delta U^{II})_J^n = S_o (\Delta U^I)_J^n + S_1 (\Delta U)_{J-1}^n + S_2 (\Delta U)_{J-2}^n \qquad (33)$$

A similar formula can be derived if we impose the analytical boundary condition with the <u>primitive variables</u> (group IV)

$$(\Delta \tilde{U}^{II})_o^n = \tilde{R}_o (\Delta \tilde{U}^I)_o^n + \tilde{R}_1 (\Delta U)_1^n + \tilde{R}_2 (\Delta U)_2^n$$

for the inflow boundary. By imposing primitive variables as analytical boundary conditions for the conservative system, group IV involves extra linearization and extra computations.

If instead of using linear extrapolation for the numerical boundary conditions we discretize the characteristic equation and obtain an expression for $(\Delta U^{II})_o^n$, the counterpart of R_1's will be even more complicated than the $\tilde{R}_1$'s.

There are two ways to alter the existing code by using the implicit boundary scheme:

(a) Add correction matrices like (32) and (33) onto the first and last block rows of the block tridiagonal matrix.

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(b) Use equations like (32) and (33) as extra equations in the block tri-diagonal matrix; that is, increase the dimension of the block tridiagonal matrix by $\dim(U_0{}^{II}) + \dim(U_J{}^{II}) - \dim(U_0{}^{II})$ means the dimension of $U_0{}^{II}$. A word of caution, the final form of the matrix might not be in block tridiagonal form.

Some Numerical Results

The nozzles we consider are shown in figures 6 and 7 (refs. 39, 46). We use the unsteady gas-dynamics equations to obtain the steady-state solutions for various inflow-outflow conditions. The numerical spatial derivative approximations for the quasi-1-D nozzle problem are summarized as follows in table 2. The time differencing is the backward Euler method (high in stability). The trapezoidal formula, although yielding greater accuracy for small CFL numbers, results in instabilities for large CFL numbers. Additional time-differencing approximations and numerical boundary condition procedures will be considered in a future paper (ref. 45).
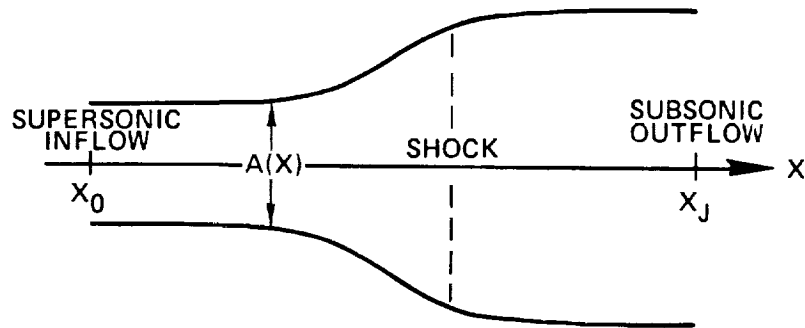
$$A(X) = 1.398 + 0.347 * \text{TANH} (0.8\ X - 4)$$

Figure 6.- Shubin nozzle, (ref. 46) for supersonic inflow, subsonic outflow study.
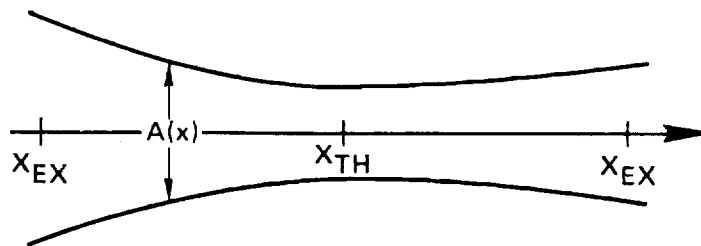
$$A(x) = 1 + (A_{EN} - 1)\ [(X_{TH} - x)\ /\ X_{TH}]^2 \qquad x \leqslant X_{TH}$$
$$A(x) = 1 + (A_{EX} - 1)\ [(x - X_{TH})\ /\ (X_{EX} - X_{TH})]^2 \qquad x > X_{TH}$$

Figure 7.- Convergent-divergent nozzle (ref. 39) for subsonic inflow, outflow study.

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TABLE 2.- NUMERICAL SCHEMES

Method	Interior	Boundary, numerical
1	Second-order one-sided[a] (flux-vector splitting)	Linear extrapolation
2	First-order one-sided (flux-vector splitting)	Linear extrapolation
3	Central[b,c] + spectral norm (equivalent to Scheme 2) (ref. 16)	Linear extrapolation
4	Central[b,c]	Linear extrapolation
5	Central[b] + one-sided at first and last computational points	Linear extrapolation

[a]Second-order for $\partial F^+/\partial x$ and $\partial F^-/\partial x$, but first-order for $\partial A^+/\partial x$ and $\partial A^-/\partial x$.

[b]Fourth-order dissipation was added for the interior scheme.

[c]Second-order dissipation was added at the boundary points.

The numerical boundary conditions are treated either explicitly (E), set to values at previous time step (replace n by n - 1 on the right-hand side of eqs. (32) and (33)), or implicitly (I), alteration of appropriate block tridiagonal matrix elements.

The numerical scheme for each numerical experiment is defined by the temporal differencing (ξ,θ,ϕ), the spatial differencing (method 1, 2, 3. 4, or 5 of table 2), the variables chosen for the boundary conditions (groups I, II, III, or IV), and the temporal treatment of the boundary conditions (E or I). These choices obviously provide a large array of combinations which we must selectively sample.

Typical steady-state solutions for three different flow conditions are shown in figures 8-10. Tables 3-8 present some of the results of numerical stability investigations. The calculations were made with a series of fixed CFL number and the numerical stability recorded.

Although not extensive at this time, several general observations can be made:

a. The results with boundary conditions I and II are very similar. Although the solutions are slightly different in the vicinity of the shock, the extrapolation of the conservative variables produces results that are comparable to those obtained when the characteristic variables are extrapolated (see tables 3-6).

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INFLOW: SUPERSONIC
OUTFLOW: SUBSONIC

— EXACT SOL
◇   METHOD 2
      CFL = 5
      400 STEP

B.C. INFLOW : ρ, u, p
        OUTFLOW : p

$\Delta X = 0.125$
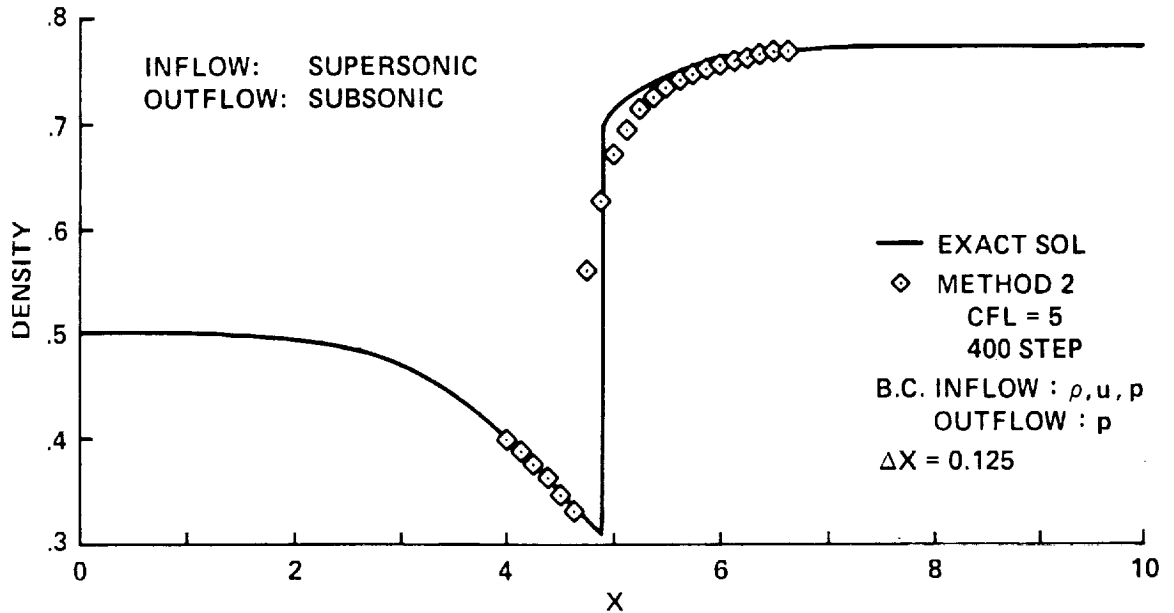
Figure 8.- Density distribution: supersonic inflow, subsonic outflow,
Shubin nozzle.

b. For some schemes (see tables 7 and 8), the explicit and implicit
treatment of the numerical boundary conditions produce similar numerical sta-
bility bounds; that is, implicit treatment of numerical boundary conditions is
not necessary for CFL > 1 (for some schemes).

c. Overspecification of exact boundary conditions causes no problems.
Figure 11 shows the supersonic inflow-outflow case.

d. Methods 2 and 3 of table 2 behave almost identically.

For the supersonic-subsonic problem, if we underspecify the boundary con-
dition at the outflow, that is, without specifying anything, the solution
diverges. Moreover, updating the boundary points via the delta form (32)
and (33), and then obtaining

$$U_o^{n+1} = \Delta U_o^n + U_o^n$$

$$U_J^{n+1} = \Delta U_J^n + U_J^n$$

instead of updating the boundary points directly through the approximation

$$(U^{II})_o^n = R_o (U^I)_o^n + R_1 U_1^n + R_2 U_2^n$$

$$(U^{II})_J^n = S_o (U^I)_J^n + S_1 U_{J-1}^n + S_1 U_{J-2}^n$$

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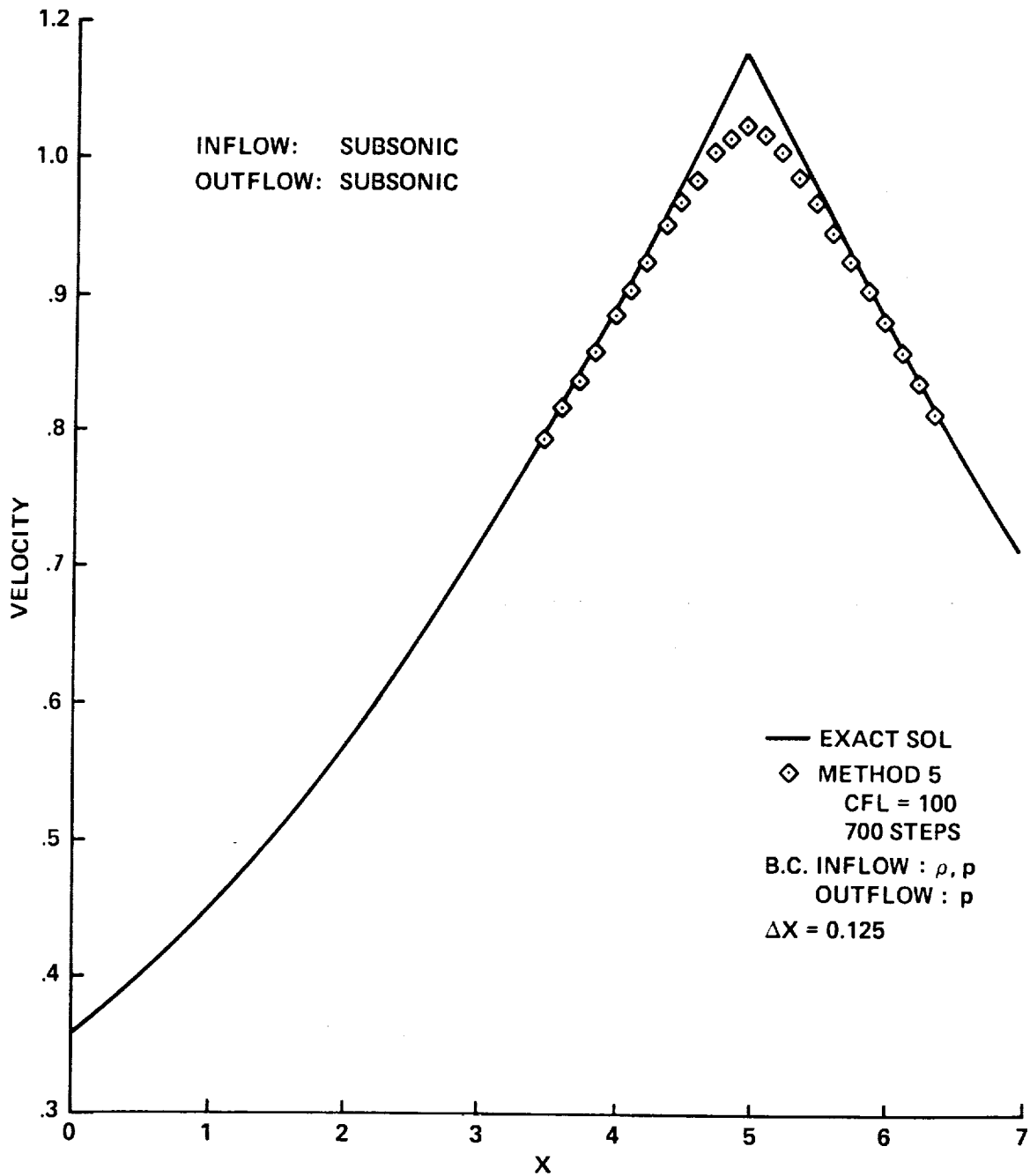
Figure 9.- Velocity distribution: subsonic inflow, subsonic outflow, convergent-divergent nozzle, area ratio 2:1.16.

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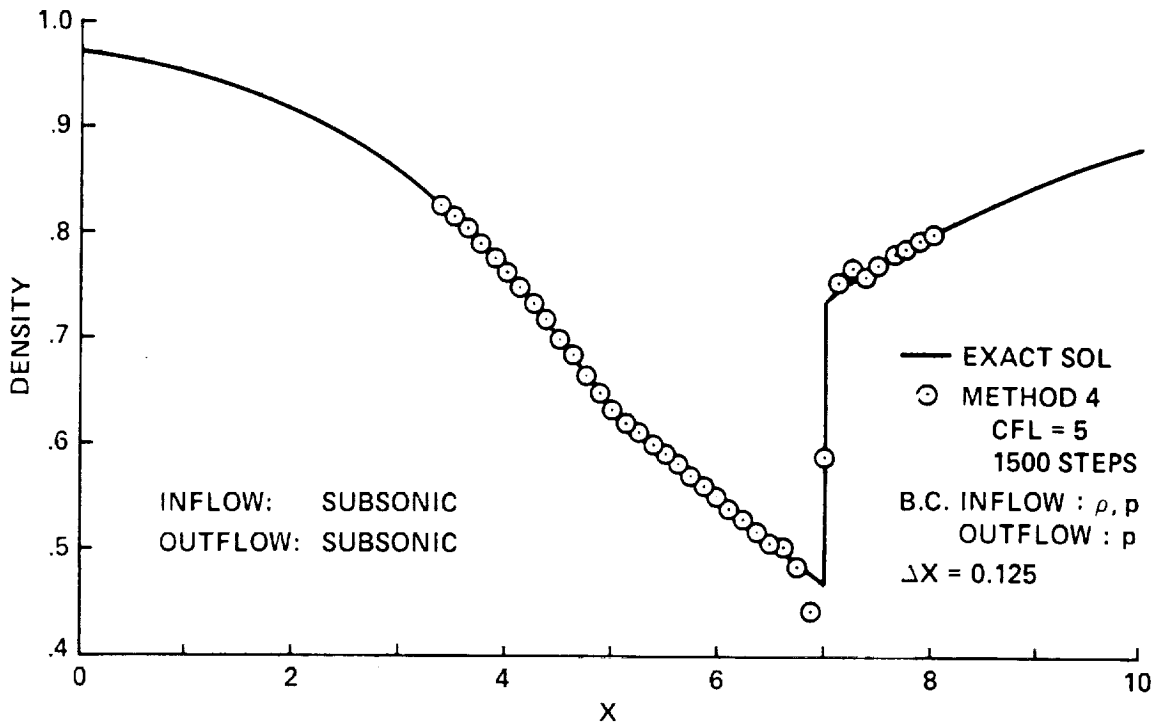
Figure 10.- Density distribution: subsonic inflow, subsonic outflow, convergent-divergent nozzle, area ratio 2.5:1.5.

TABLE 3.- NUMERICAL STABILITY CHART:
BOUNDARY SCHEME I, SHUBIN NOZZLE
(Boundary conditions: inflow = c,m,e; outflow = p)

CFL	Method 2	Method 4	Method 5
1	(I) Yes (E) Yes	(I) Yes (E) No	(I) Yes (E) Yes
5	(I) Yes (E) Yes	(I) Yes ---	(I) Yes (E) Yes
10	(I) Yes (E) Yes	(I) Yes ---	(I) Yes (E) Yes
100	(I) No (E) No	(I) No ---	(I) No (E) No

Notes: Supersonic inflow; subsonic outflow.
I = implicit numerical boundary condition
E = explicit numerical boundary condition

TABLE 4.- NUMERICAL STABILITY CHART:
BOUNDARY SCHEME I, SHUBIN NOZZLE
(Boundary conditions: inflow = ρ,m,e; outflow =

CFL	Method 2	Method 4	Method 5
1	(I) Yes (E) Yes	(I) Yes (E) Yes	(I) Yes (E) Yes
5	(I) Yes (E) Yes	(I) Yes (E) No	(I) Yes (E) Yes
10	(I) Yes (E) Yes	(I) Yes ---	(I) Yes (E) Yes
100	(I) No (E) No	(I) No ---	(I) No (E) No

Notes: Supersonic inflow; subsonic outflow.
I = implicit numerical boundary condition
E = explicit numerical boundary condition

TABLE 5.- NUMERICAL STABILITY CHART:
BOUNDARY SCHEME II, SHUBIN NOZZLE
(Boundary conditions: inflow = ρ,m,e; outflow = ρ)

CFL	Method 2	Method 4	Method 5
1	(I) Yes (E) Yes	(I) Yes (E) No	(I) Yes (E) Yes
5	(I) Yes (E) Yes	(I) Yes ---	(I) Yes (E) Yes
10	(I) Yes (E) Yes	(I) Yes ---	(I) Yes (E) Yes
100	(I) No (E) No	(I) No ---	(I) No (E) No

Notes: Supersonic inflow; subsonic outflow.
I = implicit numerical boundary condition
E = explicit numerical boundary condition

TABLE 6.- NUMERICAL STABILITY CHART:
BOUNDARY SCHEME II, SHUBIN NOZZLE
(Boundary conditions: inflow = ρ,m,e; outflow = m)

CFL	Method 2	Method 4	Method 5
1	(I) Yes (E) Yes	(I) Yes (E) No	(I) Yes (E) Yes
5	(I) Yes (E) Yes	(I) Yes ---	(I) Yes (E) Yes
10	(I) Yes (E) Yes	(I) Yes ---	(I) Yes (E) Yes
100	(E) Yes (E) No	(I) No ---	(I) No (E) No

Notes: Supersonic inflow; subsonic outflow.
I = implicit numerical boundary condition
E = explicit numerical boundary condition

TABLE 7.- NUMERICAL STABILITY CHART:
BOUNDARY SCHEME IV, SHUBIN NOZZLE
(Boundary conditions: inflow = ρ,u,p; outflow = p)

CFL	Method 1	Method 2	Method 4	Method 5
1	(I) Yes (E) Yes	(I) Yes (E) Yes	(I) Yes (E) No	(I) Yes (E) Yes
5	(I) No (E) No	(I) Yes (E) Yes	(I) Yes (E) No	(I) Yes (E) Yes
10	---	(I) Yes (E) Yes	(I) Yes (E) No	(I) Yes (E) Yes
100	---	(I) No (E) No	(I) No ---	(I) No (E) No

Notes: Supersonic inflow; subsonic outflow.
I = implicit numerical boundary condition
E = explicit numerical boundary condition

TABLE 8.- NUMERICAL STABILITY CHART: BOUNDARY SCHEME IV,
CONVERGENT-DIVERGENT NOZZLE
(Boundary condition: inflow = ρ, p; outflow = p)

CFL	Method 1	Method 2	Method 4	Method 5
1	(I) Yes (E) Yes		(I) Yes (E) No	(I) Yes (E) Yes
10	(I) No (E) No		(I) Yes (E) No	(I) Yes (E) No
20	---	Accuracy problem	(I) Yes (E) No	(I) Yes (E) No
10^2	---		(I) Yes ---	(I) Yes ---
10^3	---		(I) Yes ---	(I) No ---

Notes: Subsonic inflow; subsonic outflow;
area ratio 2:1.16. No shock.
I = implicit numerical boundary condition;
E = explicit numerical boundary condition.
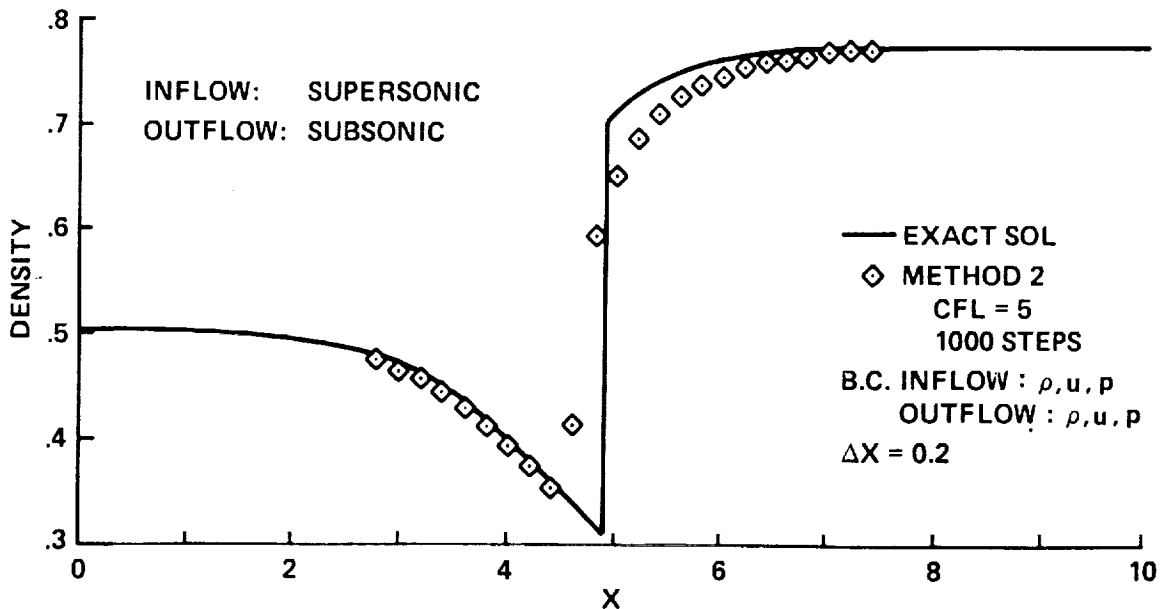

INFLOW: SUPERSONIC
OUTFLOW: SUBSONIC

EXACT SOL
◇ METHOD 2
CFL = 5
1000 STEPS
B.C. INFLOW : ρ, u, p
OUTFLOW : ρ, u, p
$\Delta X = 0.2$

Figure 11.- Density distribution: overspecify at outflow,
Shubin nozzle.

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where the R_i's and S_i's are the same as in equations (32) and (33), produces a solution that is not as smooth near the boundary.

The smoothing parameter for the fourth-order dissipation term for methods (3)-(5) of table 3 are 0.5. No study has been made for varying the smoothing parameters for different solution behavior zones.

CONCLUSIONS

A comprehensive overview of the state of the art of well-posedness and stability analysis of FDE for IBVP of the hyperbolic type was presented. The "freezing" theory was used as a guide to construct boundary schemes for the 1-D inviscid gas-dynamics equations. The use of primitive variables as the analytical boundary conditions for the conservative form of the 1-D inviscid gas-dynamics equations was formulated and then applied to the quasi-1-D nozzle problem.

Spatial linear extrapolation as a boundary scheme can produce reasonable steady-state solutions. It is scheme-independent, and thus provides a compact form for computer code implementation. Added dissipation terms, the linearization of the $(\partial F/\partial U)$ matrix and ways of updating the boundary points can affect the stability and accuracy of the solution. Future work in this area is needed.

APPENDIX A

DEFINITIONS AND EXAMPLES OF WELL-POSED HYPERBOLIC DIFFERENTIAL EQUATIONS IN "L_2 NORM"

Well-Posedness of Cauchy Problem

Definition A.1: The L_2 norm of a vector function $u(x)$ with $-\infty < x < \infty$ is defined as

$$\|u(x)\| = \left[\int_{-\infty}^{\infty} u*(x)u(x)dx \right]^{1/2}$$

where $u*$ is the transpose and complex conjugate of u.

Consider the Cauchy problem

$$\left. \begin{array}{l} u_t + Au_x + Bu = 0 \qquad -\infty < x < \infty, \ t \geq 0 \\ \\ u(x,0) = f(x) \end{array} \right\} \qquad (A1)$$

where A and B are $N \times N$ constant matrices, and u and f are vectors with dimension N.

Definition A.2: For all initial values $f(x)$ with $\|f(x)\| < \infty$, the Cauchy problem (A1) is well-posed if there are constants k, α (independent of $f(x)$) such that for all solutions and all t, there exists an estimate.

$$\|u(x,t)\| \leq K \ e^{\alpha t} \|u(x,0)\| \qquad (A2)$$

where

$$\|u(x,t)\| = \left[\int_{-\infty}^{\infty} u*(x,t)u(x,t)dx \right]^{1/2}$$

Example: Consider the first-order scalar equation

$$\left. \begin{array}{l} u_t(x,t) - \alpha u(x,t) = 0 \qquad -\infty < x < \infty, \ t \geq 0 \\ \\ u(x,0) = f(x) \end{array} \right\} \qquad (A3)$$

with $\|u(x,0)\| < \infty$ and known real constant α. The solution of (A3) is

$$u(x,t) = f(x)e^{\alpha t}$$

Since

$$\|u(x,t)\| = \left|e^{\alpha t}\right| \|u(x,0)\|$$

the solution of (A3) satisfies (A2). Therefore (A3) is well-posed.

The definition of hyperbolic and strictly hyperbolic systems is as follows: The system (A1) is <u>hyperbolic</u> if A is diagonalizable and with real eigenvalues. It is <u>strictly hyperbolic</u> if all the eigenvalues are real and distinct.

There is a simple equivalent algebraic condition for definition (A.2) to hold. This condition (ref. 2) is found by Fourier transforming equation (A1) in x and studying the norm of the Fourier transformed variable. Through this method, it can be shown that a hyperbolic system (A1) with all $\|u(x,0)\| < \infty$ and B = 0 are well posed.

Let us define $P(i\omega) = -i\omega A$ with ω real. Then the algebraic condition is: The Cauchy problem for (A1) is well-posed if and only if there are constants K and α such that

$$\max_{\omega} \left|e^{P(i\omega)t}\right| \le K\, e^{\alpha t}$$

Example: For the scalar hyperbolic equation

$$u_t + c u_x = 0$$

$$u(x,0) = f(x)$$

with c real and $\|u(x,0)\| < \infty$. We have

$$P(i\omega) = -i\omega c$$

thus $\max_{\omega}\left|e^{-i\omega c}\right| \le 1$. If we take K = 1, $\alpha = 0$, then the algebraic condition is satisfied. That is $\|u(x,t)\| = \|u(x,0)\|$. For the hyperbolic system (A1), the well-posed algebraic condition is immediately satisfied since there is a unitary matrix T s.t.

$$TAT^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & \lambda_N \end{pmatrix}, \quad \lambda_j \text{ real and } \left|T\right| = \left|T^{-1}\right| = 1$$

Well-Posedness of Initial Boundary Value Problem (IBVP)

Consider the IBVP of the strictly hyperbolic system in the quarter plane $(0 \le x < \infty)$

$$u_t + Au_x = F(x,t) \qquad\qquad 0 \le x < \infty,\ t \ge 0 \qquad\qquad (A4a)$$

$$u(x,0) = f(x) \qquad\qquad (A4b)$$

$$u^I(0,t) = Su^{II}(0,t) + g(t) \qquad\qquad (A4c)$$

where A is an $N \times N$ diagonal constant coefficient matrix with

$$A = \begin{pmatrix} A^I & 0 \\ 0 & -A^{II} \end{pmatrix}$$

$$A^I = \begin{pmatrix} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \lambda_\ell \end{pmatrix} > 0$$

$$A^{II} = \begin{pmatrix} \lambda_{\ell+1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \lambda_N \end{pmatrix} > 0$$

and

$$u^I = (u_1, \ldots, u_\ell)^T, \qquad u^{II} = (u_{\ell+1}, \ldots, u_N)^T$$

where λ_j, $j = 1, \ldots, N$ are real and distinct, S is an $\ell \times (N - \ell)$ matrix, and $f(x)$ is smooth. (It is no restriction to assume that A is in diagonal form because the system is strictly hyperbolic and can always be written in this form after a suitable transformation.) For simplicity, we will consider the homogeneous initial data $u(x,0) = f(x) = 0$. The assumption of homogeneous initial data is no restriction since we can always subtract that solution of the nonhomogeneous Cauchy (initial value) problem and obtain exactly this situation.

Definition A.3: We will say that the quarter-plane problem (A4) with homogeneous initial data is well-posed if the estimate

$$\int_0^\tau \|u(x,t)\|^2 \, dt \le K_\tau \left(\int_0^\tau |g(t)|^2 \, dt + \int_0^\tau \|F(x,t)\|^2 \, dt \right) \qquad (A5)$$

holds with a constant K_τ independent of g and F, but perhaps depending on τ. In here, the L_2 norm of $u(x,t)$ is defined as

$$\|u(x,t)\| = \int_0^\infty u^*(x,t)u(x,t)dx$$

We can extend this definition for higher dimension systems of equations with a slight modification. Consider the IBVP of a two-dimensional strictly hyperbolic system (see sec. 2 for definition) in the quarter plane $(0 \leq x < \infty, -\infty < y < \infty)$

$$\left.\begin{array}{ll} u_t + Au_x + Bu_y = F(x,y,t) & 0 \leq x < \infty \\[2mm] u(x,y,0) = 0 & -\infty < y < \infty \\[2mm] u^I(0,y,t) = Su^{II}(0,y,t) + g(y,t) & t \geq 0 \end{array}\right\} \qquad \text{(A6)}$$

where A, S, u^I and u^{II} are defined as before (with A replaced by $\omega_1 A + \omega_2 B$, where ω_1 and ω_2 are real and $\omega_1^2 + \omega_2^2 = 1$).

Definition A.4: We will say that the quarter-plane problem (A6) is well-posed if the estimate

$$\int_0^\tau \|u(0,y,t)\|_y^2 \, dt + \int_0^\tau \|u(x,y,t)\|_{x,y}^2 \, dt$$

$$\leq K_\tau \left(\int_0^\tau \|g(y,t)\|_y^2 \, dt + \int_0^\tau \|F(x,y,t)\|_{x,y} \, dt \right)$$

holds. Here K_τ depends on τ but not on F and g. Where the L_2 norm are defined as

$$\|u(0,y,t)\|_y^2 = \int_{-\infty}^\infty u^*(0,y,t)u(0,y,t)dy$$

and

$$\|u(x,y,t)\|_{x,y}^2 = \int_{-\infty}^\infty \int_0^\infty u^*(x,y,t)u(x,y,t)dx \, dy$$

with similar definitions for $\|g(y,t)\|_y^2$ and $\|F(x,y,t)\|_{x,y}$.

For the one-dimensional systems, we can get the same conditions as in definition (A3) by using the method of characteristics. This is not the case for higher dimensional systems (refs. 2, 6). The application of the method of characteristics is discussed in detail in section 2. Here we will state the necessary and sufficient algebraic conditions for definition (A3). This is a

simplified version of the main theorem of Kreiss (ref. 1). The theory of
Kreiss (ref. 1) treats problems in any number of space dimensions. Interested
readers should refer to reference 1 for extension to more space dimensions.
For the two-dimensional and three-dimensional quasilinear systems of inviscid
gas dynamics equations, please refer to Oliger and Sunström (ref. 6) and Oliger
(ref. 7).

The so-called "normal-mode analysis" algebraic conditions for defini-
tion (A3) will be stated after the following brief preliminary background. Let
us Laplace transform (A4a) and (A4c) with respect to t and denote $s = \eta + i\xi$
as the variable dual to t. We obtain

$$\left.\begin{array}{ll} s\hat{u} + A\hat{u}_x = \hat{F} & \text{for} \quad x \geq 0 \\ \hat{u}^I = S\hat{u}^{II} + \hat{g} & \text{for} \quad x = 0 \end{array}\right\} \tag{A7}$$

The symbol ($\hat{\ }$) is the Laplace transformation of the variable ( ).

Associated with (A7) is the following eigenvalue problem. A square-
integrable function $\phi(x)$ for $0 \leq x < \infty$ is an eigenfunction of (A7) corre-
sponding to an eigenvalue s if ϕ is a solution of the problem

$$s\phi + A\phi_x = 0 \qquad \text{for} \quad 0 \leq x < \infty \tag{A8}$$

$$\phi^I = S\phi^{II} \qquad \text{for} \quad x = 0 \tag{A9}$$

We do not want s with $\eta = \text{Re}(s) > 0$ to be an eigenvalue of (A8) and (A9).
If this happens, ϕ is not in L_2 (ϕ is not in L_2 means ϕ is not square
integrable). Therefore, we have to decide whether s with $\text{Re}(s) > 0$ is an
eigenvalue or not. Equation (A8) is an ordinary differential equation whose
general solution in L_2 for $\text{Re}(s) > 0$ can be expressed as a linear combina-
tion of ℓ linearly independent normalized eigensolutions (see Kreiss, ref. 1,
for details). That is, the general solutions in L_2 depend on ℓ free
parameters $\sigma = (\sigma_1, \ldots, \sigma_\ell)^T$. Introducing the solution into (A9), we get
a linear system of equations:

$$R(s)\sigma = 0 , \qquad R \text{ a matrix function of } s \tag{A10}$$

and s is an eigenvalue if and only if

$$\text{Det}[R(s)] = 0 \tag{A11}$$

Kreiss has shown that $\text{Det}[R(s)]$ is a continuous function of s for
$\text{Re}(s) \geq 0$ and he defines $s = i\xi$ to be a generalized eigenvalue if
$\text{Det}[R(i\xi)] = 0$.

Now we can state the necessary and sufficient conditions for the estimate
of type (A5) to hold.

Theorem A.1: The IBVP for (A4) is well-posed in L_2 if and only if the
eigenvalue problem (A8) and (A9) has no eigenvalue or generalized eigenvalue
for $\text{Re}(s) \geq 0$.

Next, we want to give an example to show that for a one-dimensional system, using the method of characteristics is equivalent to Theorem A.1. Here we assume that readers are either familiar with the subject of the method of characteristics or will consult references 2-4 (or sec. 2) for details. Consider the following quarter-plane problem for the wave equation

$$w_{tt} = w_{xx} \qquad 0 \le x < \infty, \ t \ge 0 \tag{A12}$$

with initial conditions

$$w(x,0) = f(x)$$

$$w_t(x,0) = g(x)$$

and boundary conditions

$$w(0,t) = w_t(0,t) = 0$$

We can recast the problem into a system of first-order hyperbolic form by letting $v = w_t$, $u = w_x$, $z_1 = v - u$, and $z_2 = v + u$. Then (A12) becomes

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_x = 0 \tag{A13}$$

with initial conditions

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} g - f_x \\ g + f_x \end{pmatrix} \qquad \text{at} \quad t = 0 \tag{A14}$$

and boundary conditions

$$z_2 = -z_1 \qquad \text{at} \quad x = 0 \tag{A15}$$

From the method of characteristics, we can see that the initial condition (A14) together with the boundary condition (A15) determine the solution of (A13) uniquely.

Now we turn to Theorem A.1. The general solution of the associated eigenvalue problem for (A13) in L_2 with $\text{Re}(s) \ge 0$ is

$$\phi = \sigma_1 e^{-sx} e_1 \tag{A16}$$

where e_1 is the normalized eigenvector of

$$-A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

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corresponding to $Re(s) \geq 0$. The normalized eigenvector e_1 is found to be

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Introducing (A16) into (A15), we get

$$\sigma_1 e^{-sx}\Big|_{x=0} = \sigma_1 = 0 \quad \Rightarrow \quad \phi \equiv 0$$

Therefore, there are no non-trivial solutions in L_2 for $Re(s) \geq 0$ and the problem is well-posed.

CONDITIONS ON WELL-POSEDNESS OF THE INVISCID EQUATIONS
OF GAS DYNAMICS

Freeze the coefficients of the Jacobian matrix A and rewrite equations (10) and (11):

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0 \tag{B1}$$

$$\frac{\partial \tilde{U}}{\partial t} + \tilde{A} \frac{\partial \tilde{U}}{\partial x} = 0 \ , \quad \tilde{U}_t = M^{-1}U_t \tag{B2}$$

and the characteristic equation

$$\frac{\partial W}{\partial t} + \Lambda \frac{\partial W}{\partial x} = 0 \tag{B3}$$

with

$$W = T^{-1}\tilde{U} \tag{B4}$$

or

$$W = T^{-1}M^{-1}U \tag{B5}$$

where U, $\tilde{U}$, W are the conservative, primitive and the characteristic variables of (B1), (B2), and (B3), respectively. The matrices M^{-1} and T^{-1} are defined as

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{-u_o}{\rho_o} & \dfrac{1}{\rho_o} & 0 \\ \dfrac{(\gamma - 1)u_o^2}{2} & (1 - \gamma)u_o & (\gamma - 1) \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 1 & 0 & \dfrac{-1}{c_o^2} \\ 0 & \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}\rho_o c_o} \\ 0 & \dfrac{-1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}\rho_o c_o} \end{pmatrix}$$

and u_o, c_o, ρ_o are the values of u, c, ρ due to freezing of coeffic s.
We want to discuss two sets of well-posed boundary conditions: (1) imp   i
boundary conditions (case 1) that are in the form of the individual na   l
physical variables alone, that is, the primitive or the conservative '   lf;
and (2) imposed boundary conditions (case 2) that are in the form of   .near
combination of the physical variables, that is, $a_1\rho + a_2 m + a_3 e = g(t$ or
$b_1\rho + b_2 u + b_3 p = \breve{g}(t)$ where the a_i's, b_i's, $g(t)$ and $\breve{g}(t)$ are the   .ven
known quantities, not ρ, m, e, u, or p.

Case 1

Assume that we want to impose the analytical boundary condition in terms
of the conservative variables. (We use the term "analytical boundary condi-
tions" as the boundary conditions that are required for the partial differen-
tial equation.) The boundary is assumed to be at the left of domain and the
flow direction is from left to right. Thus the number of positive eigenvalues
(equal to the number of analytical boundary conditions) and negative eigen-
values is known. The procedure to check for well-posedness consists of two
steps. First we reorder (B5) as

$$\begin{pmatrix} w^I \\ w^{II} \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} \begin{pmatrix} U^I \\ U^{II} \end{pmatrix}$$

where W^I and W^{II} are the characteristic variables corresponding to positive
and negative eigenvalues of A, respectively. And u^I corresponds to the
proposed analytical boundary condition variables and u^{II} represents the rest
of the variables. Second, we have to check whether Q_4^{-1} exists or Q_4 is
empty. Thus the necessary and sufficient condition for well-posedness is
Q_4^{-1} exists or Q_4 is empty.

Similarly, if we want to impose the primitive variables as analytical
boundaries, we can reorder (B4) as

$$\begin{pmatrix} w^I \\ w^{II} \end{pmatrix} = \begin{pmatrix} \tilde{Q}_1 & \tilde{Q}_2 \\ \tilde{Q}_3 & \tilde{Q}_4 \end{pmatrix} \begin{pmatrix} \tilde{U}^I \\ \tilde{U}^{II} \end{pmatrix}$$

where well-posedness here means $\tilde{Q}_4^{-1}$ exists or $\tilde{Q}_4$ is empty.

Therefore, under a type of inflow-outflow condition, once we have decided
on a set of analytical boundary conditions, the way to check for well-posedness
of (B1) or (B2) is to see if the determinant of (Q_4) or ($\tilde{Q}_4$) is equal to zero
or not. The following are the determinants of Q_4 and $\tilde{Q}_4$ (if it is not empty)
for various choices of inflow, outflow conditions. Again, we want to empha-
size that the boundary is assumed to be at the left of the domain. Therefore,
we only need to investigate the determinant of Q_4 (or $\tilde{Q}_4$). The form of a
Q_4 (or $\tilde{Q}_4$) depends on how we order the variables in w^{II} and u^{II} (or $\tilde{u}^{II}$),
which differ by a change of rows and columns or both; but the absolute value
of the determinants are the same.

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Pure supersonic: ($u > c$ for inflow and $-c < u < 0$ for outflow):

Primitive variables: inflow — There are three positive eigenvalues. We require three analytical boundary conditions. Thus $\tilde{Q}_4$ is empty.

Primitive variables: outflow — There are no positive eigenvalues. We do not have to impose any analytical boundary condition. Thus $\tilde{Q}_4 = T^{-1}$ and $\text{Det}(\tilde{Q}_4) = \text{Det}(T^{-1})$. Note that $\text{Det}(\tilde{Q}_4)$ means determinant of $\tilde{Q}_4$.

Conservative variables: The situation is the same as in the case of primitive variables. Therefore, the well-posedness conditions are to impose all three variables for the supersonic inflow case and none for the supersonic outflow case.

Subsonic outflow: ($-c < u < 0$): There is one positive eigenvalue. We require one analytical boundary condition. Therefore, we can propose the following three choices.

Primitive variables: analytical boundary condition — p

$$\tilde{Q}_4 = \begin{pmatrix} 1 & 0 \\ & \\ 0 & -\dfrac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{Det}(\tilde{Q}_4) = -\dfrac{1}{\sqrt{2}}$$

Primitive variables: analytical boundary condition — u

$$\tilde{Q}_4 = \begin{pmatrix} 1 & -\dfrac{1}{c_o^2} \\ & \\ 0 & \dfrac{1}{\sqrt{2}\rho_o c_o} \end{pmatrix} \qquad \text{Det}(\tilde{Q}_4) = \dfrac{1}{\sqrt{2}\rho_o c_o}$$

Primitive variables: analytical boundary condition — ρ

$$\tilde{Q}_4 = \begin{pmatrix} 0 & -\dfrac{1}{c_o^2} \\ & \\ \dfrac{-1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}\rho_o c_o} \end{pmatrix} \qquad \text{Det}(\tilde{Q}_4) = \dfrac{-1}{\sqrt{2}c_o^2}$$

Thus, imposing any one of the variables ρ, u, or p will result in a well-posed condition.

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Conservative variables: analytical boundary condition — e

$$Q_4 = \begin{pmatrix} 1 - \dfrac{(\gamma - 1)u_o^2}{2c_o^2} & \dfrac{(\gamma - 1)u_o}{c_o^2} \\[3mm] \dfrac{u_o}{\sqrt{2}\rho_o} + \dfrac{(\gamma - 1)u_o^2}{2\sqrt{2}\rho_o c_o} & \dfrac{-1}{\sqrt{2}\rho_o} + \dfrac{(1 - \gamma)u_o}{\sqrt{2}\rho_o c_o} \end{pmatrix}$$

$$\text{Det}(Q_4') = \dfrac{1}{\sqrt{2}\rho_o} + \dfrac{(\gamma - 1)u_o^2}{2\sqrt{2}\rho_o c_o} - \dfrac{(\gamma - 1)u_o^2}{2\sqrt{2}\rho_o c_o^2}$$

Conservative variables: analytical boundary condition — m

$$Q_4 = \begin{pmatrix} 1 - \dfrac{(\gamma - 1)u_o^2}{2c_o^2} & - \dfrac{(\gamma - 1)}{c_o^2} \\[3mm] \dfrac{u_o}{\sqrt{2}\rho_o} + \dfrac{(\gamma - 1)u_o^2}{2\sqrt{2}\rho_o c_o} & \dfrac{(\gamma - 1)}{\sqrt{2}\rho_o c_o} \end{pmatrix}$$

$$\text{Det}(Q_4) = \dfrac{(\gamma - 1)(c_o + u_o)}{\sqrt{2}\rho_o c_o^2}$$

Conservative variables: analytical boundary condition — ρ

$$Q_4 = \begin{pmatrix} \dfrac{(\gamma - 1)u_o}{c_o^2} & \dfrac{-(\gamma - 1)}{c_o^2} \\[3mm] \dfrac{-1}{\sqrt{2}\rho_o} + \dfrac{(1 - \gamma)u_o}{\sqrt{2}\rho_o c_o} & \dfrac{(\gamma - 1)}{\sqrt{2}\rho_o c_o} \end{pmatrix} \qquad \text{Det}(Q_4) = \dfrac{-(\gamma - 1)}{\sqrt{2}\rho_o c_o^2}$$

Again, for well-posedness, imposing any one of the variables ρ, m, or e will result in a well-posed condition.

Subsonic inflow: (0 < u < c):

Primitive variables: analytical boundary conditions — u, p

$$\tilde{Q}_4 = 0 = \text{Det}(\tilde{Q}_4) \ . \ . \ . \ \underline{\text{not well-posed}}$$

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Primitive variables: analytical boundary conditions — ρ, p

$$\tilde{Q}_4 = \frac{-1}{\sqrt{2}} = \text{Det}\,(\tilde{Q}_4)$$

Primitive variables: analytical boundary conditions — ρ, u

$$\tilde{Q}_4 = \frac{1}{\sqrt{2}\rho_o c_o} = \text{Det}\,(\tilde{Q}_4)$$

In this case <u>imposing u and p will produce an ill-posed problem</u>.

Conservative variables: analytical boundary conditions — m, e

$$Q_4 = \frac{u_o}{\sqrt{2}\rho_o} + \frac{(\gamma - 1)u_o^{\,2}}{2\sqrt{2}\rho_o c_o} = \text{Det}\,(Q_4)$$

Conservative variables: analytical boundary conditions — ρ, e

$$Q_4 = \frac{-1}{\sqrt{2}\rho_o} + \frac{(1 - \gamma)u_o}{\sqrt{2}\rho_o c_o} = \text{Det}\,(Q_4)$$

Conservative variables: analytical boundary conditions — ρ, m

$$Q_4 = \frac{\gamma - 1}{\sqrt{2}\rho_o c_o} = \text{Det}\,(Q_4)$$

In this case, imposing any pair (ρ,m), (ρ,e) or (m,e) will result in a well-posed condition. From the above examination, the only analytical boundary condition set that produces an ill-posed problem is (u,p) for the subsonic inflow case.

Case 2

In this case, we only can impose the characteristic variables corresponding to the positive eigenvalues of A (or $\tilde{A}$) in order to obtain a well-posed condition.

For supersonic inflow, we can specify all three characteristic variables w_1, w_2, and w_3, that is,

$$\rho - \frac{1}{c_o^{\,2}}\,p = \tilde{g}_1(t)$$

$$\frac{1}{\sqrt{2}} u + \frac{1}{\sqrt{2}\rho_o c_o} p = \tilde{g}_2(t)$$

$$-\frac{1}{\sqrt{2}} u + \frac{1}{\sqrt{2}\rho_o c_o} p = \tilde{g}_3(t)$$

for the primitive-nonconservative form (11) or conservative form (9), and

$$\left[1 - \frac{(\gamma - 1)u_o^2}{2c_o^2} \right] \rho + \left[\frac{(\gamma - 1)u_o}{c_o^2} \right] m - \left[\frac{\gamma - 1}{c_o^2} \right] e = g_1(t)$$

$$\left[-\frac{u_o}{\sqrt{2}\rho_o} + \frac{(\gamma - 1)u_o^2}{2\sqrt{2}\rho_o c_o} \right] \rho + \left[\frac{1}{\sqrt{2}\rho_o} + \frac{(1 - \gamma)u_o}{\sqrt{2}\rho_o c_o} \right] m + \frac{(\gamma - 1)}{\sqrt{2}\rho_o c_o} e = g_2(t)$$

$$\left[\frac{u_o}{\sqrt{2}\rho_o} + \frac{(\gamma - 1)u_o^2}{2\sqrt{2}\rho_o c_o} \right] \rho + \left[\frac{-1}{\sqrt{2}\rho_o} + \frac{(1 - \gamma)u_o}{\sqrt{2}\rho_o c_o} \right] m + \left[\frac{(\gamma - 1)}{\sqrt{2}\rho_o c_o} \right] e = g_3(t)$$

for the conservative form (eq. (9)) where $\tilde{g}_i$'s and g_i's are the values which are supposed to be specified.

For subsonic inflow, we only can specify w_1 and w_2, that is,

$$\rho - \frac{1}{c_o^2} p = \tilde{g}_1(t)$$

$$\frac{1}{\sqrt{2}} u + \frac{1}{\sqrt{2}\rho_o c_o} p = \tilde{g}_2(t)$$

for system (11) or (9) and

$$\left[1 - \frac{(\gamma - 1)u_o^2}{2c_o^2} \right] \rho + \left[\frac{(\gamma - 1)u_o}{c_o^2} \right] m - \left[\frac{(\gamma - 1)}{c_o^2} \right] e = g_1(t)$$

$$\left[-\frac{u_o}{\sqrt{2}\rho_o} + \frac{(\gamma - 1)u_o^2}{2\sqrt{2}\rho_o c_o} \right] \rho + \left[\frac{1}{\sqrt{2}\rho_o} + \frac{(1 - \gamma)u_o}{\sqrt{2}\rho_o c_o} \right] m + \left[\frac{(\gamma - 1)}{\sqrt{2}\rho_o c_o} \right] e = g_2(t)$$

for system (9). Again the $\tilde{g}_i$'s and g_i's are the values which are supposed to be specified.

APPENDIX C

EXAMPLES OF THE THEORY OF GUSTAFSSON, KREISS, AND SUNDSTRÖM
(NORMAL MODE ANALYSIS)

Here we briefly review the stability theory of Gustafsson, Kreiss, and Sundström for the initial boundary value problem of the hyperbolic type for the leapfrog method. Please refer to their original paper (ref. 9) for more details.

Consider the following equation

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} + c\,\dfrac{\partial u}{\partial x} = 0 \qquad 0 \le x \le 1,\ t \ge 0 \\[2em] u(x,0) = f(x) \end{array}\right\} \qquad (C1)$$

In addition to equation (C1), we specify boundary conditions

$$u(0,t) = g_0(t) \qquad \text{if} \quad c > 0$$

$$u(1,t) = g_1(t) \qquad \text{if} \quad c < 0$$

But, numerically, one needs boundary conditions at both $x = 0$ and $x = 1$. Therefore, a separate procedure is used to determine the numerical boundary conditions.

Let us solve (C1) by the leapfrog scheme with Δt as the time-step, and Δx as the mesh spacing. We will use the notation

$$v_j^{\,n} = v(j\Delta x,t) \approx u(j\Delta x,t)$$

$$t = n\Delta t$$

Assume for the moment $c = -1$, $0 \le x < \infty$, and approximate (C1) by

$$v_j^{\,n+1} = v_j^{\,n-1} + \frac{\Delta t}{\Delta x}\,(v_{j+1}^{\,n} - v_{j-1}^{\,n}) \qquad\qquad (C2a)$$

$$v_j^{\,0} = f(j\Delta x) \qquad\qquad (C2b)$$

and the numerical boundary condition at $x = 0$ by

$$v_0^{\,n} = v_1^{\,n} \qquad\qquad (C3)$$

The Gustafsson et al. stability theory for this case seeks a general solution of (C2a) and (C3) of the form

$$v_j{}^n = z^n \kappa^j$$

for appropriate complex scalars z and κ. This substitution is made in both the difference scheme in the interior and on the boundary. The basic scheme (C2a) is assumed stable for the Cauchy problem, that is, $(\Delta t / \Delta x) = \lambda < 1$ for the interior points.

By substituting $v_j{}^n = z^n \hat{v}_j(x)$ into (C2a) and (C3), we obtain

$$\hat{v}_{j+1} - \frac{z^2 - 1}{\lambda z} \hat{v}_j - \hat{v}_{j-1} = 0 \qquad j = 1, 2, \ldots \qquad \text{(C4a)}$$

$$\hat{v}_0 = \hat{v}_1 \qquad \text{(C4b)}$$

Equation (C4) is defined as the <u>resolvent equation</u>.

Letting the solution of (C2) be $z^n \kappa^j$, we obtain the <u>characteristic equation</u> for (C2) as

$$\kappa^2 - \frac{z^2 - 1}{\lambda z} \kappa - 1 = 0 \qquad \text{(C5)}$$

A necessary and sufficient condition for stability of the IBVP is that (C4) have no nontrivial bounded solutions

$$\sum_{j=0}^{\infty} |\hat{v}_j|^2 < \infty$$

with $|z| \geq 1$. An eigenvalue to the associated equation (C4) is defined as a nontrivial solution to (C4) with bounded $\hat{v}_j = \kappa^j (|\kappa| < 1)$ and $|z| > 1$. A generalized eigenvalue to the associated (C4) is defined as a nontrivial solution to (C4) with $\hat{v}_j = \kappa^j$, and $|\kappa| = 1$ and $|z| = 1$, such that all solutions $\bar{z}, \bar{\kappa}$ of (C5) with $|\bar{z}| > 1$, and sufficiently close to z and κ, have $|\bar{\kappa}| < 1$. The equivalent necessary and sufficient condition for stability is that the associated (C4) have no nontrivial eigenvalues or generalized eigenvalues.

The stability analysis consists of the following four stages:

1. The order d ($d = 2$ in this case) of the resolvent (difference) equation (C4a) determines the general solution of $\hat{v}_j(x)$ — a linear combination of d solutions

$$\hat{v}_j = c_1 \kappa_1{}^j + c_2 \kappa_2{}^j + \ldots + c_d \kappa_d{}^j$$

where the κ_i's are the roots of (C5).

2. The root structure of (C5) determines the type of solution for $\hat{v}_j(x)$. In this case the roots of (C5) have the following properties (see ref. 9 for detail). If $|z| > 1$, then $|\kappa_1| < 1$ and $|\kappa_2| > 1$. This is an immediate consequence of the Cauchy stable scheme of (C2a). If $z = e^{i\theta}$, then

$$|\kappa_1| > 1, \ |\kappa_2| > 1 \quad \text{for} \quad |\sin\theta| > \lambda$$

$$|\kappa_1| = |\kappa_2| = 1 \quad \text{for} \quad |\sin\theta| \leq \lambda$$

$$\kappa_1 = -1, \ \kappa_2 = 1 \quad \text{for} \quad \theta = 0$$

$$\kappa_1 = 1, \ \kappa_2 = -1 \quad \text{for} \quad \theta = \pi$$

$$\kappa_1 = \kappa_2 = \pm i \quad \text{for} \quad \sin\theta = \pm\lambda$$

3. The assumption that the interior scheme (C2a) is Cauchy stable helps delete the unbounded solutions of $\hat{v}_j$ — all solutions with $|\kappa_1| > 1$. The theory says that the general bounded solution of (C4a) is then

$$\hat{v}_j = \sum_{|\kappa_i| < 1} c_i \kappa_i^{\,j}$$

From lemma (5.1) of reference 9, only one root of the quadratic (C5) has modulus less than one. When $|z| = 1$, one or both of the roots of (C5) may have modulus one. If this is the case, the κ_1 for the general bounded solution of (C4a) is defined by continuity to be that root which is the limit of the root $\kappa(\tilde{z})$, $|\kappa(\tilde{z})| < 1$ for $|\tilde{z}| > 1$, as $|\tilde{z}| \to 1$. Thus

$$\hat{v}_j = c_1 \kappa_1^{\,j}$$

4. After substitution of $\hat{v}_j = \kappa^j$ in (C4b), if there exists a nontrivial bounded solution for $|z| \geq 1$, the difference schemes (C2a) and (C3) will be unstable. In this case

$$(\kappa - 1) = 0 \tag{C6}$$

Therefore, when $\kappa = 1$, (C5) and (C6) have a nontrivial solution with $z = -1$. From item (2) above we know that this is a generalized eigenvalue and thus stability is violated.

In many instances, the root structure of the characteristic equation (C5) is difficult to analyze. Another way of testing for generalized eigenvalues is as follows:

With $z = -1$, we want to find out whether $\kappa = 1$ is κ_1 or κ_2. We therefore make a perturbation calculation, and study (C5) in the neighborhood of $z = -1$. Let $z = -1 - \delta$, $\delta > 0$ and $\kappa = 1 + \epsilon$ with δ, ϵ small. From (C5) we get

$$\kappa = \frac{\dfrac{z^2 - 1}{\lambda z} \pm \sqrt{\left(\dfrac{z^2 - 1}{\lambda z}\right)^2 + 4}}{2}$$

$$\epsilon_{1,2} = \frac{\dfrac{(1 + \delta)^2 - 1}{-\lambda(1 + \delta)} \pm \sqrt{\left[\dfrac{(1 + \delta)^2 - 1}{-\lambda(1 + \delta)}\right]^2 + 4}}{2} - 1$$

Since $\delta > 0$, at least one of the ϵ_i, $i = 1,2$ is negative, and $\kappa = 1 + \epsilon$ is κ_1, not κ_2. Therefore $z = -1$ is a generalized eigenvalue and thus stability is violated.

Now, consider using

$$v_0^{n+1} = v_1^n \tag{C7}$$

instead of (C3). The equivalent of equation (C6) becomes

$$(z - \kappa_1)c_1 = 0$$

For $|\kappa_1| \leq 1$ and $|z| \geq 1$

$$|z - \kappa_1| > 0$$

Thus, (C2a) and (C7) constitute a stable difference method for the right half-plane problem.

Stability of some other explicit and implicit schemes, using the above approach, can be found in Oliger (ref. 15), Gottlieb and Turkel (ref. 24), Sloan (ref. 23), and Sköllermo (ref. 21). For multistep schemes, the stability criteria of this method are often very difficult to verify. Here, we are going to discuss an unconditionally stable scheme in which we use it for the quasi-1-D nozzle problem. Let us solve (C1) by backward Euler in time and central difference in space. The numerical boundary condition at $x = 0$ is by linear extrapolation

$$v_j^{n+1} - v_j^n = \lambda\left(v_{j+1}^{n+1} - v_{j-1}^{n+1}\right) , \qquad \lambda = \frac{\Delta t}{2\Delta x} \tag{C8a}$$

$$v_0^{n+1} = 2v_1^{n+1} - v_2^{n+1} \tag{C8b}$$

The characteristic equation of (C8a)

$$\kappa(z - 1) = \lambda z(\kappa^2 - 1) \tag{C9}$$

and the boundary scheme (C8b) satisfies

$$(\kappa - 1)^2 = 0 \quad \Rightarrow \quad \kappa = 1 \tag{C10}$$

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The Cauchy stability of (C8a) implies that the roots $\kappa_{1,2}$ of (C9) satisfy

$$|\kappa_1| < 1, \quad |\kappa_2| > 1 \quad \text{for} \quad |z| > 1$$

The only problem is that

$$z = 1 \quad \text{when} \quad \kappa = 1$$

Therefore, we have to prove whether there is any generalized eigenvalue (J. Oliger and B. Gustafsson, private communication) for (C9) and (C10). For stability, we do not want $|\kappa_1| \to 1$ from below as $|z| \to 1$ from above. Therefore we want to find out if $\kappa_1 = 1$. Let $z = 1 + \delta$, $\delta > 0$, and $\kappa = 1 + \varepsilon$ with δ, ε small, we get

$$(1 + \varepsilon)(1 + \delta - 1) = \lambda(1 + \delta)[(1 + \varepsilon)^2 - 1]$$

$$\delta = \frac{\lambda\varepsilon(2 + \varepsilon)}{(1 + \varepsilon) - \lambda\varepsilon(2 + \varepsilon)}$$

Since $\delta > 0$, this implies $\varepsilon > 0$; thus, $\kappa = 1 + \varepsilon$ is κ_2, not κ_1. Therefore $z = 1$ is not a generalized eigenvalue, and the entire scheme is <u>unconditionally stable</u>.

By applying the same procedure, it can be shown that the boundary approximation (C8b), that is, spatial linear extrapolation, together with the interior schemes (a) central difference in space and (b) two-step backward Euler in time, form an unconditionally stable scheme for the model equation (C1).

APPENDIX D

MATRIX METHOD AND POSITIVE REAL FUNCTION METHOD

Consider a scalar hyperbolic equation:

$$\left.\begin{array}{ll} \dfrac{\partial u}{\partial t} + c \dfrac{\partial u}{\partial x} = 0 & 0 \le x \le 1, \; t \ge 0 \\[2mm] u(x,0) = f(x) & c > 0 \\[2mm] u(0,t) = g(t) & \end{array}\right\} \tag{D1}$$

The above equation constitutes a well-posed problem. Let $v_j = v(j\Delta x, t)$ be the difference solution of (D1) at $x = j\Delta x$, where Δx is the step size. Let us discuss the method of lines approach by using central difference in space. We will examine the stability of this difference scheme by the matrix method (ref. 29) and by the positive real-function method (ref. 30) for fixed Δx.

A word of caution: these methods only show the stability of the ODE for a fixed Δx. In order to show that the original PDE is stable, the related ODE has to be stable as $\Delta x \to 0$. That is, additional analysis is required. The additional requirement involves the analysis of infinite dimension matrices. Here, we only show the method for fixed Δx, and want to point out that stability of the ODE for fixed Δx does not rule out the possibility that the ODE might become unstable as $\Delta x \to 0$.

By central differencing in space, (D1) becomes

$$\frac{dv_j}{dt} + c\left(\frac{v_{j+1} - v_{j-1}}{2\Delta x}\right) = 0 \; , \quad j = 2, \ldots, J - 1$$

$$\frac{dv_1}{dt} + c\left(\frac{v_2}{2\Delta x}\right) - c\left(\frac{g(t)}{2\Delta x}\right) = 0 \; , \quad j = 1$$

At the right boundary (the numerical boundary) we use the backward difference scheme,

$$\frac{\partial v_J}{\partial x} \approx \frac{v_J - v_{J-1}}{\Delta x}$$

and, therefore, we have

$$\frac{dv_J}{dt} + c\left(\frac{v_J - v_{J-1}}{\Delta x}\right) = 0$$

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In matrix notation

$$\frac{dv}{dt} = \frac{c}{2\Delta x} Av + \tilde{g}(t) \qquad (D2)$$

with

$$A = \begin{pmatrix} 1 & -1 & & & & \\ 1 & 0 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & -1 \\ & & & & 2 & -2 \end{pmatrix}$$

$$\tilde{g} = \begin{pmatrix} c \dfrac{g(t)}{2\Delta x} \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

If the real part of all the eigenvalues of A are negative, we can apply a stable ODE solver to integrate (D2). The particular type of ODE solver depends heavily on the spectrum of the eigenvalues of A, that is, on the stiffness of the system. The matrix A cannot be transformed to a diagonally dominant matrix with all its diagonal elements positive. We cannot get an explicit bound for Δx. We have to actually compute the eigenvalues of A. Gary (ref. 29) has shown that A is a stable matrix for various mesh spacings.

We now turn to the use of positive real functions in an investigation of numerical stability of (D2) with fixed Δx. For details of the theory, please refer to Dahlquist's original paper (ref. 30) on this subject.

Let $z = (2\Delta x/c)\lambda$, with λ the eigenvalues, N the dimension of A, and

$$D_N(z) \equiv \det(zI - A) = 0$$

Then $D_N(z)$ is of the form

$$D_N(z) = \begin{vmatrix} z & 1 & 0 & & & & \\ -1 & z & 1 & & & & \\ & -1 & z & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & z & 1 \\ & & & & & -2 & z+2 \end{vmatrix}$$

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It is easy to see that

$$D_{n+1}(z) = zD_n(z) + D_{n-1}(z) \qquad n \geq 3$$

If it turns out that $D_n(z) \neq 0$ for $\text{Re}(z) > 0$ and that imaginary zeros are simple, then for each n, all solutions of the ODE's are bounded, and any A-stable method can be used for the integration in time.

Let

$$\phi_n = \frac{D_n}{D_{n-1}}$$

then

$$\phi_{n+1} = z + \frac{1}{\phi_n} \qquad n \geq 3$$

$$\phi_3 = \frac{D_3(z)}{D_2(z)} = \frac{\begin{vmatrix} z & 1 & 0 \\ -1 & z & 1 \\ 0 & -2 & z+2 \end{vmatrix}}{\begin{vmatrix} z & 1 \\ -2 & z+2 \end{vmatrix}}$$

or

$$\phi_3 = z + \frac{z+2}{z(z+2)+2} \tag{D3}$$

Since

$$D_1(z) = z + 2 = 0 \qquad\qquad z = -2$$

$$D_2(z) = z(z+2) + 2 = 0 \qquad z = -1 \pm i$$

have their only zeroes in the left half-plane, it is sufficient (though not necessary) to show that $\phi_n(z)$ are positive functions for $n \geq 3$. Let us look at the second part of (D3). Recall that for an arbitrary complex number W that if $\text{Re}(W) > 0$ then $\text{Re}(W^{-1}) > 0$. Since

$$f(z) = \frac{z(z+2)+2}{z+2} = z + \frac{2}{z+2}$$

is a positive function, it follows that ϕ_3 is a positive function. By applying the proof by induction, we can easily show that ϕ_{n+1}, for $n \geq 3$, are

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positive functions. Thus, the central scheme is stable for $c > 0$ for fixed Δx.

APPENDIX E

SUFFICIENT STABILITY CONDITIONS

From the discussion of well-posedness of hyperbolic initial boundary value problems, no new difficulties arise if we have smooth variable coefficients and quasi-linear equations with smooth solutions. We will concentrate on a general strictly hyperbolic constant coefficient system with well-posed coupled boundary conditions. With this system in mind, we will give a detailed description in the following order: (1) the basic idea, (2) dissection of the problem, (3) reduction of the system to scalar equations, and (4) sufficient stability conditions.

Basic Idea

The sufficient stability conditions only involve properties of methods for related Cauchy problems. We want stable schemes for the related Cauchy problem applied at the interior, and stable and uncentered dissipative schemes for the related Cauchy problem applied at the boundary. The stabilities of the related Cauchy problems are usually known or can be verified by standard techniques. The main theories behind these are based on the Cauchy stability of the composite method, and the matching of stable schemes, which has been examined by Ciment (ref. 13) and Oliger (ref. 15). The usefulness of these results is fourfold: (1) stability can be easily verified by standard techniques; (2) the result can be used to guide us in the construction of stable methods for the entire problem; (3) the Cauchy stability of the composite method is especially useful and efficient for higher order schemes; and (4) the result can help to simplify the verification of the necessary and sufficient conditions tremendously if the use of higher order schemes is desired.

Dissection of the Problem

We will discuss the approximation of the well-posed strictly hyperbolic system

$$\left.\begin{array}{ll} w_t + Aw_x = 0 & 0 \le x \le 1, \ t \ge 0 \\[2mm] w(x,0) = f(x) \\[2mm] L_1 w(0,t) = f_1(t) \\[2mm] L_2 w(1,t) = f_2(t) \end{array}\right\} \tag{E1}$$

where A is a $N \times N$ constant matrix, and L_1 and L_2 are rectangular matrices. After an appropriate nonsingular transformation T, we can transform (E1) into

$$\left.\begin{array}{l} u_t + \Lambda u_x = 0 \\[2mm] u(x,0) = f(x) \end{array}\right\} \qquad \text{(E2a)}$$

where

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & -\Lambda_2 \end{pmatrix}$$

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_\ell \end{pmatrix} > 0 \quad , \qquad \Lambda_2 = \begin{pmatrix} \lambda_{\ell+1} & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} > 0$$

and

$$u^I = (u^1, \ldots, u^\ell)^T$$

$$u^{II} = (u^{\ell+1}, \ldots, u^N)^T$$

$$u = Tw$$

with boundary conditions

$$\left.\begin{array}{l} u^I(0,t) = S_I u^{II}(0,t) + \tilde{g}_I(t) \\[3mm] u^{II}(1,t) = S_{II} u^I(1,t) + \tilde{g}_{II}(t) \end{array}\right\} \qquad \text{(E2b)}$$

where S_I is an $\ell x(N - \ell)$ matrix, S_{II} is a $(N - \ell) \times (\ell)$ matrix, and $\tilde{g}_I = Tf_1$, $\tilde{g}_{II} = Tf_2$.

From the well-known theorem of Gustafsson et al. (ref. 9), the stability of the approximation for an initial boundary value problem on $0 \le x \le 1$ is equivalent to the stability of two related quarter-plane problems. Therefore we can split (E2) into the related left and right quarter-plane problems. The related right quarter-plane problem on $0 \le x < \infty$, $t > 0$ is obtained by simply removing the boundary at $x = 1$ and extending the definition of our initial data and interior approximation to $x = \infty$, that is,

$$\left.\begin{array}{ll} u_t + \Lambda u_x = 0 & 0 \le x < \infty, \ t \le 0 \\[3mm] u(x,0) = f(x) & \\[3mm] u^I(0,t) = S_I u^{II}(0,t) + \tilde{g}_I(t) & \end{array}\right\} \qquad \text{(E3)}$$

The related left quarter-plane problem on $-\infty < x \leq 1$, $t > 0$ is defined

$$
\left.
\begin{aligned}
u_t + \Lambda u_x &= 0 & -\infty < x \leq 1, \; t \geq 0 \\
u(x,0) &= f(x) \\
u^{II}(1,t) &= S_{II} u^I(1,t) + \check{g}_{II}(t)
\end{aligned}
\right\} \tag{E4}
$$

Therefore, the discussion of the right quarter-plane problem (E3) is sufficient for our purpose. The analysis of (E4) is similar.

Reduction of the System to Scalar Equations

We want to solve system (E3) using finite difference schemes. Divide the x-axis into subintervals of length Δx and the t-axis into subintervals of length Δt. Denote the grid points by $x_j = j\Delta x$ and grid functions by $v_j(t) = v(x_j,t)$, $t = n\Delta t$ and approximate (using one step in time for illustration; theory holds for multistep) (E3a) in the interior of the domain by

$$
v_j(t + \Delta t) = \sum_{i=-p}^{p} A_i v_{j+i}(t) \qquad j = r, \; r + 1, \; r + 2, \; \ldots \tag{E5}
$$

where p is the order of the spatial differencing for the interior scheme, and the approximation grid points are defined as in figure 12 (without the right boundary present) and A_i are fixed $N \times N$ diagonal matrices.
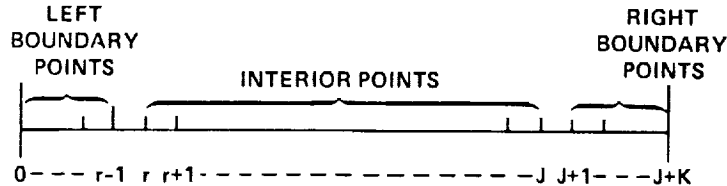

Figure 12.- Grid point definition

For the outflow unknowns (variables with negative eigenvalues), we approximate the boundary conditions by the following uncentered scheme

$$
\sum_{i=-m}^{s} c_{ji}^{(0)} v_{j+i}^{II}(t + \Delta t) = \sum_{i=-m}^{s} c_{ji}^{(1)} v_{j+i}^{II}(t) \qquad j,m = 0, \; \ldots, \; r - 1, \; m < j \tag{E6}
$$

where $c_{ji}^{(k)}$ are fixed diagonal $(N - \ell) \times (N - \ell)$ matrices and s is the order of the spatial differencing for the boundary scheme. Note that for $m = 0$ the scheme is one-sided.

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The following are a few of the spatially one-sided and uncentered schemes:

$$\frac{\partial v_j(t)}{\partial x} \approx \frac{v_{j+1}(t) - v_j(t)}{\Delta x}$$

$$\frac{\partial v_j(t)}{\partial x} \approx \frac{-\frac{1}{2} v_{j+2}(t) + 2v_{j+1}(t) - \frac{3}{2} v_j(t)}{\Delta x}$$

$$\frac{\partial v_j(t)}{\partial x} \approx \frac{-v_{j+2}(t) + 6v_{j+1}(t) - 3v_j(t) - 2v_{j-1}(t)}{6\Delta x}$$

The first two are one-sided and are of order of accuracy Δx and Δx^2, respectively. The last one is uncentered and is of order Δx^3.

For the inflow part (variables with positive eigenvalues), we have the analytical boundary condition

$$v_0^I(t) = S_I v^{II}(t) + g_I(t) \tag{E7}$$

together with $r - 1$ additional approximations of the form

$$v_j^I(t) = \sum_{i=0}^{q} D_{ji} v_i^{II}(t) + g_j(t) \tag{E8}$$

where D_{ji} are fixed $\ell \times (N - \ell)$ matrices, q is a positive integer, and the $g_j(t)$ are vectors depending on Δx and $g_I(t)$. See Goldberg and Tadmor (ref. 14) for derivation of (E8).

Since the A_j are diagonal, we can split the scheme (E5) into its inflow and outflow parts (ref. 14):

$$v_j^I(t + \Delta t) = \sum_{i=-p}^{p} A_i^I v_{j+i}^I(t) \qquad j = r, r + 1, \ldots$$

$$\tag{E9}$$

$$v_j^{II}(t + \Delta t) = \sum_{i=-p}^{p} A_i^{II} v_{j+i}^{II}(t) \qquad j = r, r + 1, \ldots$$

where

$$A_i = \begin{pmatrix} A_i^I & 0 \\ 0 & A_i^{II} \end{pmatrix}$$

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Now we can see that equations (E5) and (E6) can be partitioned into the fo- lowing problems

$$v_j(t + \Delta t) = \sum_{i=-p}^{p} A_i^I v_{j+i}^I(t) \qquad j = r, r + 1, \ldots$$

(E10a)

$$v_o^I(t) = S_I v_o^{II}(t) + \check{g}_I(t)$$

(E10b)

$$v_j^I(t) = \sum_{i=0}^{q} D_{ji} v_i^{II}(t) + g_j(t) \qquad j = 1, \ldots, r - 1$$

(E10c)

$$\left. \begin{array}{l} v_j^{II}(t + \Delta t) = \sum_{i=-p}^{p} A_i^{II} v_{j+i}^{II}(t) \qquad j = r, r + 1, \ldots \\[2em] \sum_{i=-m}^{s} c_{ji}^{(0)} v_{j+i}^{II}(t + \Delta t) = \sum_{i=-m}^{s} c_{ji}^{(1)} v_{j+i}^{II}(t) \qquad m,j = 0, \ldots, r - 1, m < j \end{array} \right\}$$

(E11)

The outflow problem (E11) is self contained, while the inflow problem (E10) depends on the outflow part to the extent that the outflow computations provide the inhomogeneous boundary values in (E10b) and (E10c). Therefore the stability of the right quarter plane under the above approximation is equiva- lent to the following two separate parts,

1. Stability of the inflow problem (E10) with inhomogeneous boundary values

2. Stability of the outflow problem (E11)

Since all the A_i and $c_{ji}^{(0)}$, $c_{ji}^{(1)}$ are diagonal matrices, the inflow prob- lem splits into ℓ independent approximations and the outflow problem splits into $(N - \ell)$ independent approximations. Similarly, we can split our left quarter-plane problem into the equivalent form. Therefore, the stability study of a system of the form (E1) reduces to a study of a single scalar equa- tion with two related quarter-plane problems as follows

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$$u_t + cu_x = 0 \qquad 0 \leq x < \infty$$

$$u(x,0) = f(x) \qquad t \geq 0, \; c > 0$$

$$u(0,t) = \breve{g}_I(t)$$

$$u_x + cu_x = 0 \qquad -\infty < x \leq 1$$

$$u(x,0) = f(x) \qquad t \geq 0, \; c > 0$$

(E12)

For $c > 0$, or

$$u_t + cu_x = 0 \qquad 0 \leq x < \infty$$

$$u(x,0) = f(x) \qquad t > 0, \; c < 0$$

$$u_t + cu_x = 0 \qquad -\infty < x \leq 1$$

$$u(x,0) = f(x) \qquad t \geq 0, \; c < 0$$

$$u(1,t) = \breve{g}_{II}(t)$$

(E13)

For $c < 0$.

Sufficient Stability Conditions

Let us assume $c > 0$, and discuss stability analysis of difference approximations to equations (E12). Using the same difference approximation as before

$$v_j(t + \Delta t) = \sum_{i=-p}^{p} A_i v_{j+i}(t) \qquad j = r, \; r+1, \; \ldots$$

$$v_o(t) = \breve{g}_I(t)$$

$$v_j(t) = g_j(t) \quad \text{if} \quad r > 1 \quad j = 1, \ldots, r-1$$

(E14)

$$v_j(t + \Delta t) = \sum_{i=-p}^{p} A_i v_{j+i}(t) \qquad j \leq J \qquad \text{(E15a)}$$

$$\sum_{i=-s}^{m} C_{ji}^{(0)} v_{j+i}(t + \Delta t) = \sum_{i=-s}^{m} C_{ji}^{(1)} v_{j+i}(t) \qquad j = J+1, \ldots, J+K \quad \text{(E15b)}$$

$$j < m + J \leq J + K$$

where the approximation grid points are defined as in figure 12 (without the left boundary present), and the A_i, C_{ji} are now scalar constants and $g_j(t)$ are obtained by Taylor series expansions of the solution in the neighborhood of the boundary in terms of the physical boundary data $\tilde{g}_I(t)$. This has been shown to acquire the desired accuracy of order d if the data is sufficiently smooth. The form of $g_j(t)$ is as follows:

$$g_j(t) = \sum_{i=0}^{d} \frac{(j\Delta x)^i}{i!} \frac{\partial^i}{\partial x^i} u(0,t) + 0(\Delta x^{d+1})$$

$$= \sum_{i=0}^{d} \frac{(j\Delta x)^i}{i!} (-c)^{-i} \frac{d^i}{dt^i} \tilde{g}_I(t) + 0(\Delta x^{d+1})$$

In (E14) $v_j(t)$, $j = 1, \ldots, r - 1$ can be approximated or extrapolated by other uncentered methods (see Oliger, ref. 15). But a stability proof will be more complicated. Now, the stability of the inflow, right quarter-plane problem (E14) is an immediate consequence of the stability of the interior approximation, so the stability discussion will only deal with the outflow left quarter-plane problem (E15). We need the following definition and assumptions:

Definition: An approximation is said to be <u>Cauchy stable</u> if it is stable for the related Cauchy problem.

Assumptions: (1) We assume that our interior approximations and boundary approximations are stable for the related Cauchy problems; (2) we assume our boundary approximations are dissipative (or at least one of the scheme is dissipative).

The sufficient conditions rest on the following three results:

1. The theory of matching of stable schemes (Ciment, ref. 13; Oliger, ref. 15).

2. The theory of successively constructing Cauchy stable methods — composite method (Oliger, ref. 10).

3. The theory of Gustafsson et al. (ref. 9) — if the method is Cauchy stable, then it is stable for the left quarter-plane problem.

<u>Matching of stable schemes</u>— If a Cauchy stable scheme of the form (E15a) is used for all $j \leq n_0$ and a Cauchy stable dissipative approximation of the form (E15b) is used for all $j > n_0$, the resulting approximation is Cauchy stable. This is based on the result of Ciment and Oliger's theorem on the matching of stable schemes. The result depends solely on the Cauchy stability of both methods and the dissipativity of at least one method. The result is best illustrated by figure 13.
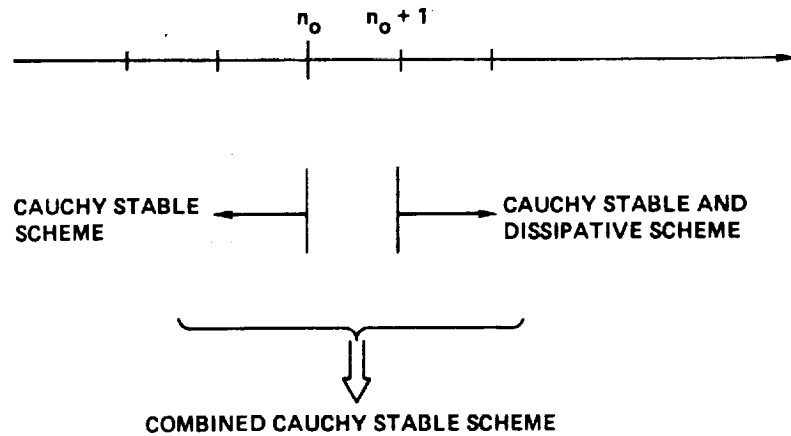
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Figure 13.- Sufficient condition.

Successively constructing Cauchy stable methods- By applying the previous
method of "matching of stable schemes" on $n_0 = J$ (see fig. 12), with scheme
(E15a) for $j \leq J$ and scheme (E15b) defined for $j = J + 1$ for all
$j \geq J + 1$, the combined scheme is Cauchy stable. We can construct a second
composite method using the one we have just constructed with scheme (E15a) for
$j < J + 1$ and the scheme (E15b) defined for $j = J + 2$ for all $j \geq J + 2$.
This in turn again is Cauchy stable by the method of matching of stable
schemes. We proceed in this way until we get to $j = J + K$. This is illus-
trated by the diagram in figure 14.

Theory of Gustafsson et al.- By successive construction of a Cauchy stable
scheme using the composite method, and the assumption we made for (E15), the
result of Gustafsson et al. (ref. 9) says that the left quarter-plane (outflow)
problem is stable.

Therefore, the key to constructing stable schemes for the initial bound-
ary value problem for the hyperbolic equations is to have Cauchy stable schemes
for the interior points and the boundary points, and at least one of the
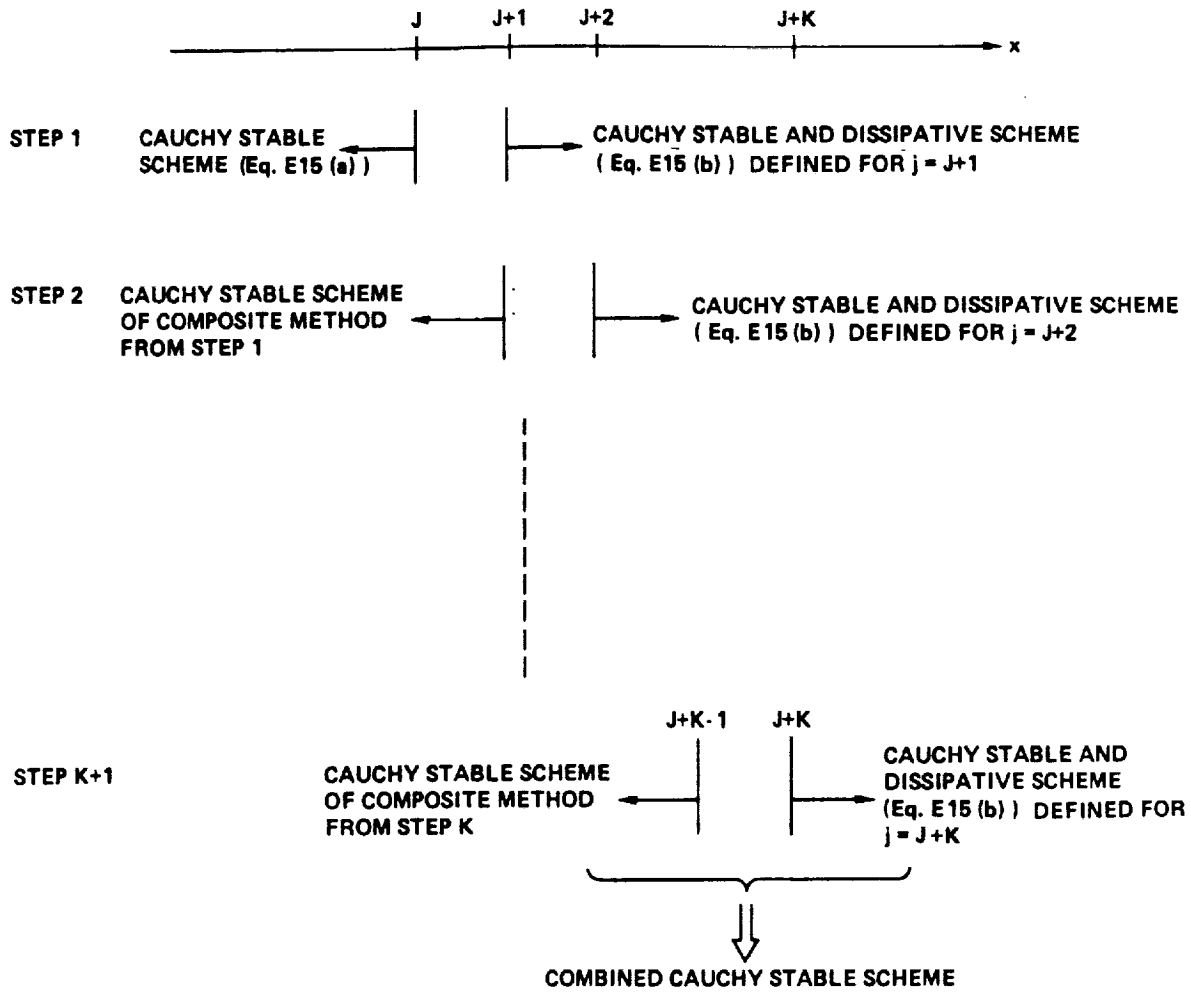schemes is dissipative.

Figure 14.- Successively constructing Cauchy stable methods.

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