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Calculations of the Room-Temperature Shapes of Unsymmetric Laminates

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Preface

The enclosed report is a copy of a paper submitted for publication in the _Journal of Composite Materials_. The issuing of this report is not intended to duplicate the Journal's publication effort. This report is intended to accelerate the dissemenation of information resulting from investigations of advanced composite materials.

Abstract

The cured shape of unsymmetric laminates do not always conform to the predictions of classical lamination theory. Classical lamination theory predicts the room-temperature shapes of all unsymmetric laminates to be a saddle. Experimental observations, however, indicate some unsymmetric laminates have cylindrical room-temperature shapes. In addition, some unsymmetric laminates, exhibit two stable room-temperature configurations, both cylindrical. This paper presents a theory which explains these characteristics. The theory is based on an extension of classical lamination theory which accounts for geometric nonlinearities. A Rayleigh-Ritz approach to minimizing the total potential energy is used to obtain quantitative information regarding the room-temperature shapes of square T300/5208 $[0_2/90_2]_T$ and $[0_4/90_4]_T$ graphite-epoxy laminates. It is shown that, depending on the thickness of the laminate and the length of the side of the square, the saddle shape configuration is actually unstable. For values of length and thickness that render the saddle shape unstable, it is shown that two stable cylindrical shapes exist. The predictions of the theory are compared with existing experimental data.

Introduction

Most calculations used to predict the response of laminates to static, dynamic, or thermal loadings are based on what has become to be known as classical lamination theory [1], [2], [3]. This is a linear theory and is based on the following major assumptions:

(1) the displacements are continuous throughout the laminate,

(2) the Kirchhoff hypothesis regarding undeformed normals is assumed to be valid,

(3) the strain-displacement relationship is linear,

(4) the material is linearly elastic, and

(5) the through-the-thickness stresses are small in comparison to the in-plane stresses.

The theory smears the individual lamina properties by integrating the constitutive equations through the thickness of the laminate. As a result of this integration, force and moment resultants are defined. In addition, the well-known A, B, and D matricies are defined. While this theory is quite capable of accurately predicting static deflections, natural vibration frequencies and mode shapes, buckling loads and mode shapes, and thermal expansion coefficients of laminates, there are physical situations for which the theory fails to predict the correct answer. Two situations of note are: the inability of the theory to predict the response of thicker laminates, and; its inability to explain the behavior of laminates near edges. The former problem has been studied by several investigators [4], [5] and satisfactory corrections to the theory have been obtained. The latter problem is now a classic and has been studied by many individuals, at least for the case of the straight free-edge. A survey of the edge problem has been put forth in [6].

There appears to be another situation for which classical lamination theory fails to give the correct answer. Specifically, it appears the theory is unable to correctly predict the room-temperature shapes of thin unsymmetric laminates. Hyer [7] has documented the room-temperature shapes of several families of unsymmetric laminates and found that the room-temperature shapes of some thin unsymmetric laminates are closely approximated by right circular cylinders. In addition, some thin laminates have two room-temperature cylindrical shapes. These results are in contrast to the predictions of the classical theory. The classical theory predicts the room-temperature shapes of all unsymmetric laminates to be a saddle with unique (single-valued) curvature characteristics. Specifically, Hyer found that 100 x 100 mm and 150 x 150 mm $[0_2/90_2]_T$ T300/5208 graphite-epoxy laminates cured to become cylindrical at room temperature. In addition, they exhibited a snap-through or oil-canning phenomenon. The cylindrical shape could be snapped into another cylindrical shape which had the same characteristics as the first shape. However, the second cylinder was oriented perpendicular to the first cylinder and the curvature of the second cylinder was of opposite sign. Hyer showed that thicker (say, 8-layer) 100 x 100 mm unsymmetric laminates conformed to the predictions of the theory.

To explain the behavior of unsymmetric laminates, it was assumed that it would be necessary to incorporate a nonlinear effect into classical lamination theory. The existence of <u>two</u> room-temperature shapes (i.e. the two cylinders) essentially ruled out a linear extension to the theory since a linear extension would lead to the prediction of a unique shape, albiet perhaps not a saddle shape. Furthermore, since the out-of-plane deflections of the unsymmetric laminates were on the order of many laminate thicknesses, geometric nonlinearities were felt to be an important effect. Thus, in an effort to explain the behavior of unsymmetric laminates, classical lamination theory was extended to include geometric nonlinearities through the strain-displacement relationship. This extension was applied to the analysis of the $[0_n/90_n]_T$, $n=1,2,\ldots$ family of laminates. This family was chosen for study because this class of laminates always exhibits a snap-through phenomenon, there is data available for different thickness $[0_n/90_n]_T$ laminates, and due to some of the A_{ij}, B_{ij}, and D_{ij} terms being zero with this family, the algebra associated with this family of laminates is simplier than the algebra associated with other families. The analysis of the room-temperature shapes of this family of laminates is the subject of this paper. The paper traces the development of the analysis which successfully predicts the existence of two room-temperature cylindrical shapes, and compares the predictions with the available data.

Problem Formulation

Since a problem formulation which includes geometric nonlinearities would result in nonlinear governing equations, it was assumed from the beginning that obtaining a closed-form exact solution for the unsymmetric laminate problem would be difficult and not really necessary. The occurence of the cylindrical shape is so prevalant with thin laminates that it was hypothesised that one is dealing with a fundamental phenomenon rather than some higher order effect. Thus any good approximate theory would reveal the mechanics of the problem. The problem is idealized as follows. A cured laminate (and uncured prepreg) is flat at the elevated curing temperature, fig. 1a. As the laminate cools, it is assumed it is free from any external mechanical forces which produce net work. It is assumed the out-of-plane deflections develop only because of the differences in thermal expansion properties of the individual lamina. This idealization ignores the effects of any mechanical constraints the autoclaving and vacuum bagging process may exert on the laminate. Upon cooling, the laminate deforms into one of the shapes given by figs. 1b, c and d. Of these shapes, the one that actually occurs is the one associated with a minimum of the total potential energy. The shape scenario given by figs. 1b, c and d includes the saddle shape observed for the thicker laminates and predicted by the classical theory, and the two possible cylindrical shapes observed for the thinner laminates. It is assumed that in attaining these shapes, geometric nonlinearities are important.

Since it is assumed external tractions are not important during the cooling process, the total potential energy, including the effects of thermal expansion, is given by [8],

$$W = \int_{Vol} \mathcal{W} dVol, \tag{1}$$

$$\mathcal{W} = \frac{1}{2} C_{ijkl} e_{ij} e_{kl} - \beta_{ij} e_{ij} \Delta T, \tag{2}$$

where $\mathcal{W}$ equals the strain energy density. The C_{ijkl} are the elastic constants of the material and the β_{ij} are coefficients related to the elastic constants and the coefficients of thermal expansion of the material. Both the elastic properties and the thermal expansion coefficients are assumed to be temperature-independent. The e_{ij} are the strains in the material and ΔT is the temperature change in the material due to cooling from curing. In eq. (2), since the problem is a plane-stress formulation, i and j assume the values 1 and 2. These values are not directly related to the principal material directions of the lamina, but rather, 1 and 2 are associated with the x and y directions of the laminates (see fig. 1). Thus the following relations apply:

$$e_{11} = \varepsilon_x^\circ - z \frac{\partial^2 w}{\partial x^2} , \tag{3}$$

$$e_{22} = \varepsilon_y^\circ - z \frac{\partial^2 w}{\partial y^2} , \tag{4}$$

$$e_{12} = \varepsilon_{xy}^\circ - z \frac{\partial^2 w}{\partial y \partial x} , \tag{5}$$

with

$$\varepsilon_x^\circ = \frac{\partial u^\circ}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 \tag{6}$$

$$\varepsilon_y^\circ = \frac{\partial v^\circ}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2 , \text{ and} \tag{7}$$

$$\varepsilon_{xy}^\circ = \frac{1}{2} \left(\frac{\partial u^\circ}{\partial y} + \frac{\partial v^\circ}{\partial x} + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right). \tag{8}$$

As usual, z=0 is the midplane of the laminate. The quantity u° is the laminate midplane displacement in the x-direction, v° is the laminate midplane displacement in the y-direction, and w is the out-of-plane displacement of the midplane.

Equations (6) - (8) represent the principal departure from classical lamination theory and include the usual approximations associated with thin-plate theory when employing the nonlinear geometric effects in the strain-displacement relations. These approximations assume the elongation and shearing strains and the squares of the rotations are the same order of magnitude and this order is small compared to unity [9], [10].

For a laminate the C_{ijkl}'s can be related to the $\bar{Q}_{ij}$'s, the reduced stiffnesses, and the β_{ij}'s can be related to the $\bar{Q}_{ij}$'s and α_x, α_y, and α_{xy}, the laminate thermal expansion coefficients in the x-y coordinate system. Expanding eq. 2 yields

$$W = \frac{1}{2}\bar{Q}_{11}e_{11}^2 + \bar{Q}_{12}e_{11}e_{22} + 2\bar{Q}_{11}e_{12}^2 + \frac{1}{2}\bar{Q}_{22}e_{22}^2 - (\bar{Q}_{11}\alpha_x + \bar{Q}_{12}\alpha_y)e_{11}\Delta T - (\bar{Q}_{12}\alpha_x + \bar{Q}_{22}\alpha_y)e_{22}\Delta T \quad . \tag{9}$$

($\alpha_{xy}=0$ for this family of laminates.) The problem has now been reduced to one of finding the deformation u°, v°, and w as functions of x and y which, through eqs. (2) - (9), minimize eq. (1). As previously mentioned, approximate solutions to u°, v°, and w are sought. In seeking realistic approximate solutions, two basic assumptions are made. First, it is assumed that even in attaining the cylindrical shape, the midplane elongation strains, ε_x° and ε_y°, do not vary much from the linear prediction (i.e. ε_x° and ε_y° independent of x and y). Second, it is assumed that to the order of the nonlinearity considered here, the midplane shear strains are negligible i.e. $\varepsilon_{xy}^\circ = 0$. Since for $[0_n/90_n]_T$ laminates classical lamination theory predicts ε_{xy}° to be zero and ε_x° and ε_y° to be constant, these two assumptions could be lumped into one by saying it is assumed that even in attaining the cylindrical shapes, magnitude of the midplane strains do not vary much from the predictions of the classical theory. However, for rationalizing the choice of the functional form of the approximate solutions, the two issues are separated.

It is assumed w(x,y) is of the form

$$w(x,y) = \frac{1}{2}(ax^2 + by^2) \;, \tag{10}$$

a and b being constants. With this functional form for w, both the classical lamination solution, $a = -b$, and either of the two cylindrical shapes, fig. 1c and d, can be approximated. For fig. 1c the solution can be $a \neq 0$, $b = 0$ while for fig. 1d the solution can be $a = 0$, $b \neq 0$. Using the kinematic assumptions regarding the midplane strains, ε_x°, ε_y°, and ε_{xy}°, the approximate solutions for u° and v° are given by

$$u^\circ(x,y) = cx - \frac{a^2x^3}{6} - \frac{abxy^2}{4} \tag{11}$$

$$v^\circ(x,y) = dy - \frac{b^2y^3}{6} - \frac{abx^2y}{4} \;, \tag{12}$$

c and d being constants. Using eqs. (11) and (12) in eqns. (6) - (8) yields

$$\varepsilon_x^\circ = c - \frac{aby^2}{4} \tag{13}$$

$$\varepsilon_y^\circ = d - \frac{abx^2}{4} \tag{14}$$

$$\varepsilon_{xy}^\circ = 0. \tag{15}$$

Note that if it were not required to have ε_{xy}° be zero, the third term in each of eqs. (11) and (12) would not need to be included and then the second term in eqs. (13) and (14) would not appear. However, with the $[0_n/90_n]_T$ family, it is felt in-plane shear strains are impossible and this is the prime factor in choosing the functional form for u° and v°.

At this point a, b, c and d are considered as generalized coordinates and are to be determined. The problem of finding a minimum for the total potential energy, W, becomes a problem of finding solutions to the values of a, b, c, and d so that the first variation of W is zero, i.e.

$$\delta W = (\frac{\partial W}{\partial a})\delta a + (\frac{\partial W}{\partial b})\delta b + (\frac{\partial W}{\partial c})\delta c + (\frac{\partial W}{\partial d})\delta d \equiv 0 \ . \tag{16}$$

This variation is done with all the assumptions in the Introduction with the exception of (3).

Calculations Associated with the Solution

The x-y-z coordinate system in fig. 1 is assumed to be situated such that at the elevated curing temperature the laminate is defined by the region

$$\begin{aligned} -L_x/2 &\leq x \leq L_x/2 \\ -L_y/2 &\leq y \leq L_y/2 \\ -h/2 &\leq z \leq h/2 \ . \end{aligned} \tag{17}$$

With these limits on the spatial variables and with the various material properties involved, eq. (1) takes the form

$$W = \int_{x=-L_x/2}^{L_x/2} \int_{y=-L_y/2}^{L_y/2} \int_{z=-h/2}^{h/2} \mathcal{W}(a,b,c,d,\bar{Q}_{ij},\alpha_x,\alpha_y,\Delta T,x,y,z)dxdydz. \tag{18}$$

The involved, but straight-forward, process of substituting eqs. (10) and (13) - (15) into eqs. (3) - (5), substituting these results into eq. (9), performing the spatial integrations in eq. (18), and finally taking the first variation, eq. (16), leads to an equation of the form:

$$\begin{aligned} \delta W = f_1(a,b,c,d)\delta a + f_2(a,b,c,d)\delta b \ + \\ f_3(a,b,c,d)\delta c + f_4(a,b,c,d)\delta d = 0 \ . \end{aligned} \tag{19}$$

Equation (19) immediately leads to four equations:

$$f_1(a,b,c,d) = -C_1cb + C_2ab^2 + 2C_3ab - B_{11}c + D_{11}a - C_4cb + 2C_5ab^2 - C_6db + D_{12}b - C_7db + C_8ab^2 + C_9b^2 + (L_y^2/48)N_x^Tb + M_x^T + (L_x^2/48)N_y^Tb = 0 \ , \tag{20}$$

$$f_2(a,b,c,d) = -C_1ac + C_2a^2b + 2C_3a^2 - C_4ac + 2C_5a^2b + D_{12}a - C_6da - C_7da + C_8a^2b + 2C_9ab - B_{22}d - D_{22}b + (L_y^2/48)N_x^Ta + (L_x^2/48)N_y^Ta + M_y^T = 0 \ , \tag{21}$$

$$f_3(a,b,c,d) = A_{11}c - C_1ab - B_{11}a + A_{12}d - C_4ab - N_x^T = 0 \ , \tag{22}$$

$$f_4(a,b,c,d) = A_{12}c - C_6ab - B_{22}b + A_{22}d - C_7ab - N_y^T = 0 \ . \tag{23}$$

The constants $C_1 - C_9$ are defined in the Appendix and A_{ij}, B_{ij} and D_{ij} have the familiar definitions associated with laminates. The other definitions used in eqs. (20) - (23) are,

$$N_x^T = \Delta T \int_{-h/2}^{h/2} (\bar{Q}_{11}\alpha_x + \bar{Q}_{12}\alpha_y)dz \ , \tag{24}$$

$$N_y^T = \Delta T \int_{-h/2}^{h/2} (\bar{Q}_{12}\alpha_x + \bar{Q}_{22}\alpha_y)dz \ , \tag{25}$$

$$M_x^T = \Delta T \int_{-h/2}^{h/2} (\bar{Q}_{11}\alpha_x + \bar{Q}_{12}\alpha_y)zdz \ , \tag{26}$$

$$M_y^T = \Delta T \int_{-h/2}^{h/2} (\bar{Q}_{12}\alpha_x + \bar{Q}_{22}\alpha_y) z dz \ . \tag{27}$$

These quantities are immediately recognizable as the effective in-plane thermal loads, N_x^T and N_y^T, and the effective thermal moments, M_x^T and M_y^T. It should be noted that when $L_x = L_y = 0$, the coefficients C_1 through C_9 are all zero and eqs. (20) - (23) reduce to the equations of classical lamination theory.

Solution of Equations, Numerical Results

Solutions to eqs. (20) - (23) were obtained by solving eqs. (22) and (23) for c and d in terms of a and b and substituting these relations into eqs. (20) and (21). Thus eqs. (20) and (21) become coupled cubic equations for the quantitites a and b. These two resulting equations have the characteristics of being able to be reduced to a single cubic equation for either a or b. Such an equation would have either one or three real roots. This reduction approach was not used, however, and eqs. (20) and (21), in terms of a and b, were solved numerically. Solutions were obtained for several laminates using elastic and thermal expansion properties of T300/5208 graphite-epoxy. It was assumed the curing temperature of T300/5208 is 177° C (350° F) and that the laminates are cooled to a room temperature of 21° C (70° F). The material properties used in the calculations were:

E_1 = 181 GPa (26.2 x 10^6 psi)

E_2 = 10.3 GPa (1.49 x 10^6 psi)

ν_{12} = 0.28

G_{12} = 7.17 GPa (1.04 x 10^6 psi)

$\alpha_1 = -0.106 \times 10^{-6}/° C\ (-0.059 \times 10^{-6}/° F)$

$\alpha_2 = 25.6 \times 10^{-6}/° C\ (14.2 \times 10^{-6}/° F)$

The elastic properties were taken from [2] while the thermal expansion coefficients were taken from [11]. Solutions were obtained for square laminates ($L_x=L_y=L$) ranging from 0 to 150 mm in length on a side. Two thicknesses were considered, $[0_2/90_2]_T$ and $[0_4/90_4]_T$. Experimental data were available for some checking of these two thickness cases. Figure 2 shows the characteristics of the predicted room-temperature shapes of the $[0_2/90_2]_T$ laminate and fig. 3 shows the characteristics for the thicker laminate. Each figure illustrates the effect of the size of the laminate, L, on the room temperature shapes.

Immediately obvious from the figures is the existence of three possible room-temperature shapes of the laminate if the lengths of the sides are greater than some critical value. For both laminates, at L=0 the room-temperature shape is the saddle predicted by classical lamination theory, a=-b. This solution is denoted by point A on figs. 2 and 3. As the sides of the laminate increase in length, say to L=25 mm for the $[0_2/90_2]_T$ laminate of fig. 2, the shape is still predicted to be a saddle but one which is shallower than the one predicted by the classical theory. As the lengths of the sides increase, the saddle shape is still predicted to exist but it gets shallower and shallower. At some critical length, the solution bifurcates. For the thinner laminate the critical length is 35 mm while for the thicker laminate the critical length is 71 mm. The bifurcation point is denoted as B on the figures. For lengths greater than the critical length, three room temperature shapes, each represented by a different branch on the figures, can possibly exist. These branches are denoted as BC, BD, and BE on the figures. Branch BD represents a continuation of the saddle shape ($a=-b$) but the other two branches represent a radical departure from a saddle shape. Branch BC represents a shape which has a large curvature in the x-direction and practically no curvature in the y-direction, fig. 1c. On the other hand, branch BE represents a shape which has a large curvature in the y-direction and very little curvature in the x-direction, fig. 1d. The shapes associated with these latter branches can be considered cylindrical because as the laminate gets larger, i.e. L increases, the one curvature asymptotically approaches zero while the other curvature asymptotically approaches a non-zero constant value. As seen from the figures, the latter two branches have certain symmetry characteristics. These symmetry characteristics are such that for a given length, the values of a and b associated with branch BC are equal, respectively, to the values of -b and -a associated with branch BE.

Figures 2 and 3 show both laminates exibit similar behavior. There are differences however. The two main differences are that the curvatures for the thicker laminate are less than the curvatures for the thinner laminate, and, the critical length for the thicker laminate is greater. Thus, compared to a $[0_2/90_2]_T$ laminate, a $[0_4/90_4]_T$ can be made larger before the triple-shape phenomenon occurs.

Stability of the Predicted Shapes

With multiple solutions to a nonlinear problem, the question arises as to the stability of the various solutions. If any of the solutions do not represent a stable solution, those solutions will not be physically realizable. Equating the first variation of the total potential energy to zero yields equilibrium positions of the laminate which either maximize or minimize the total energy. For stable equilibrium, the total potential energy must be mimimized. Thus, for stable equilibrium the second variation of the total potential energy, $\delta^2 W$, must be positive definite. From stability theory [12] for this discretized system, stability of the equilibrium positions for the laminate is possible if and only if the following matrix of coefficients is positive definite:

$$\begin{vmatrix} \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} & \frac{\partial f_1}{\partial d} \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c} & \frac{\partial f_2}{\partial d} \\ \frac{\partial f_3}{\partial a} & \frac{\partial f_3}{\partial b} & \frac{\partial f_3}{\partial c} & \frac{\partial f_3}{\partial d} \\ \frac{\partial f_4}{\partial a} & \frac{\partial f_4}{\partial b} & \frac{\partial f_4}{\partial c} & \frac{\partial f_4}{\partial d} \end{vmatrix} . \tag{28}$$

For a given equilibrium solution at a given length, e.g. the saddle solution of the triple-valued solution at L=100 mm, each element of the matrix is evaluated numerically by substituting in the values of a, b, c, and d corresponding to that solution. If each of the principal minors of the matrix are positive definite, the matrix is positive definite and the solution corresponds to a stable equilibrium solution. Otherwise the solution corresponds to an unstable equilibrium solution. Using this scheme for the solutions shown in figs. 2 and 3, it was found that the saddle solutions corresponding to the single-valued solutions, segments AB in the figures, were <u>stable</u>. On the other hand, the saddle solutions corresponding to the triple-valued solutions, segments BC in the figures, were <u>unstable</u>. The other two branches of the triple-valued solutions, segments BD and BE, represent stable solutions. Physically this means that for square laminates of the $[0_2/90_2]_T$ family, if the length of the sides of the laminate exceed 35 mm, the saddle shape does not exist. Instead, two cylindrically shaped equilibrium configurations exist. If the laminate is from the thicker $[0_4/90_4]_T$ family, the length of a side must exceed 71 mm before the saddle-shape equilibruim configuration disappears and dual cylindrical shapes appear. The fact that two stable cylindrical equilibrium solutions are predicted to exist is felt to be significant since it correlates well with the the reported snap-through phenomena associated with these types of laminates.

Experimental Results

Shown on fig. 2 are two data points. These points correspond to the curvatures of cylindrical $[0_2/90_2]_T$ laminates as measured by Hyer. Figure 3 shows one data point. This point corresponds to the curvature of a $[+45_4/-45_4]_T$ saddle-shaped T300/5208 graphite-epoxy laminate as measured by Pagano and Hahn [13]. The comparison between magnitudes of the predicted and experimentally measured curvatures is fair. More importantly, however, the character of the measured shapes, i.e. cylindrical or saddle, compares well with the predictions. For the 150 x 150 mm laminate shown in fig. 2, only the major curvature of 11 laminates were measured in the original work and the average curvature and the range are shown here arbitrarily as a y-direction curvature, b. These curvatures could have just as easily been called an x-direction curvature. In this case the experimental data would have been associated with the variable a. Also the major curvature from the one 100 x 100 mm specimen is also arbitrarily associated with the y-direction curvature. For the data point in fig. 3, the curvature of the specimen was never measured directly. The out-of-plane deflection across the diagonals of a 63.5 mm (2.50 in.) square laminate was measured and the curvature was computed from this measure. Again this single curvature measurement was arbitrarily associated with the variable b.

It should be noted in fig. 2 that the x-direction curvature for the 100 x 100 mm laminate was measured to be slightly negative. The theory predicts this curvature to be slightly positive. The reason for the descrepancy is not clear. However, it is not felt to be due to measuring error in the experimental determination of curvature. This points needs further investigation.

Discussion

Despite the lack of large amounts of quantitative experimental data to compare with the complete range of the numerical predictions, the results of the work reported here are quite encouraging. First, the theoretical calculations predict the disappearance of the saddle shape, a phenomenon observed by many investigators. Second, the snap-through or appearance of two stable equilbrium states is predicted, another phenomenon observed by investigators. Finally, the transition from stable single-valued saddle solutions to stable cylindrical solutions is linked with a size effect. The figures show that both the thickness of the laminate and the length of the side determine whether the cylindrical shape exists or whether the saddle shape exists. This investigator, as well as others, has felt a size effect exists in unsymmetric laminates and the model put forth here lends some credence to that notion.

While the predictions presented here exhibit all the important features associated with unsymmetric laminates, several comments are in order before closure. First, the solution presented here is a one-term Galerkin, or Rayleigh-Ritz, solution. Thus the solution is, as with all one-term Galerkin solutions, over-constrained. This deficiency can be remedied by using more terms, and hence more generalized coordinates, in the assumed functional forms for u°, v°, and w. Modifying the current approach this way would probably change the numerical values associated with each solution branch of figs. 2 and 3 but not the main features of bifurcation and triple-solution. Second, the effects of moisture absorption, viscoelastic relaxation, or any other mechanism that alters the internal stress state of the laminate is felt to be important for laminates with lengths near the critical length. Alteration of the internal stress state most likely influences the numerical value of the critical length. Thus a laminate sized just above the critical length could, with time, actually be sized just below the critical length due to any of the above mentioned time-dependent effects. Related to this is the fact that unsymmetric laminates sized near their critical length could exhibit "strange" behavior, requiring practically no force to snap them from one shape to another. More than likely, these multiple shapes would be some barely stable combination of shallow cylinders and shallow saddles.

Finally, the analysis presented here is based on symmetric curing (symmetric about the z=0 plane) of the laminate and, as noted above, the lack of any effects to alter the internal stress state over and above that due to temperature change during the cool-down from curing. The result is that for lengths greater than the critical length, two simiiliar shapes are possible. Each of these shapes has the same possibility of actually occurring. However, any external perturbation which is unsymmetric with respect to the midplane will cause one or the other of the two possible cylindrical shapes to be favored. Such a perturbation could be unsymmetric curing (due to unsymmetric cooling), unsymmetric moisture absorbtion, or any other process over which our control is limited. Thus in reality it may require, for example, more force to snap the cylinder of fig. 1c to the cylinder of fig. 1d than it does to make the reverse snap.

Appendix

$$C_1 = A_{11}L_y^2/48 \; ; \qquad C_2 = A_{11}L_y^4/1280$$

$$C_3 = B_{11}L_y^2/48 \; ; \qquad C_4 = A_{12}L_x^2/48$$

$$C_5 = A_{12}L_x^2L_y^2/2304 \; ; \qquad C_6 = A_{12}L_y^2/48$$

$$C_7 = A_{22}L_x^2/48 \; ; \qquad C_8 = A_{22}L_x^4/1280$$

$$C_9 = B_{22}L_x^2/48$$

The coefficients A_{11}, A_{12}, A_{22}, B_{11} and B_{22} have the familiar definitions associated with classical lamination theory.

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Fig. 1 Laminate Shapes: (a) at the elevated curing temperature, and at room-temperature, (b) a saddle shape, (c) a cylindrical shape, (d) another cylindrical shape.

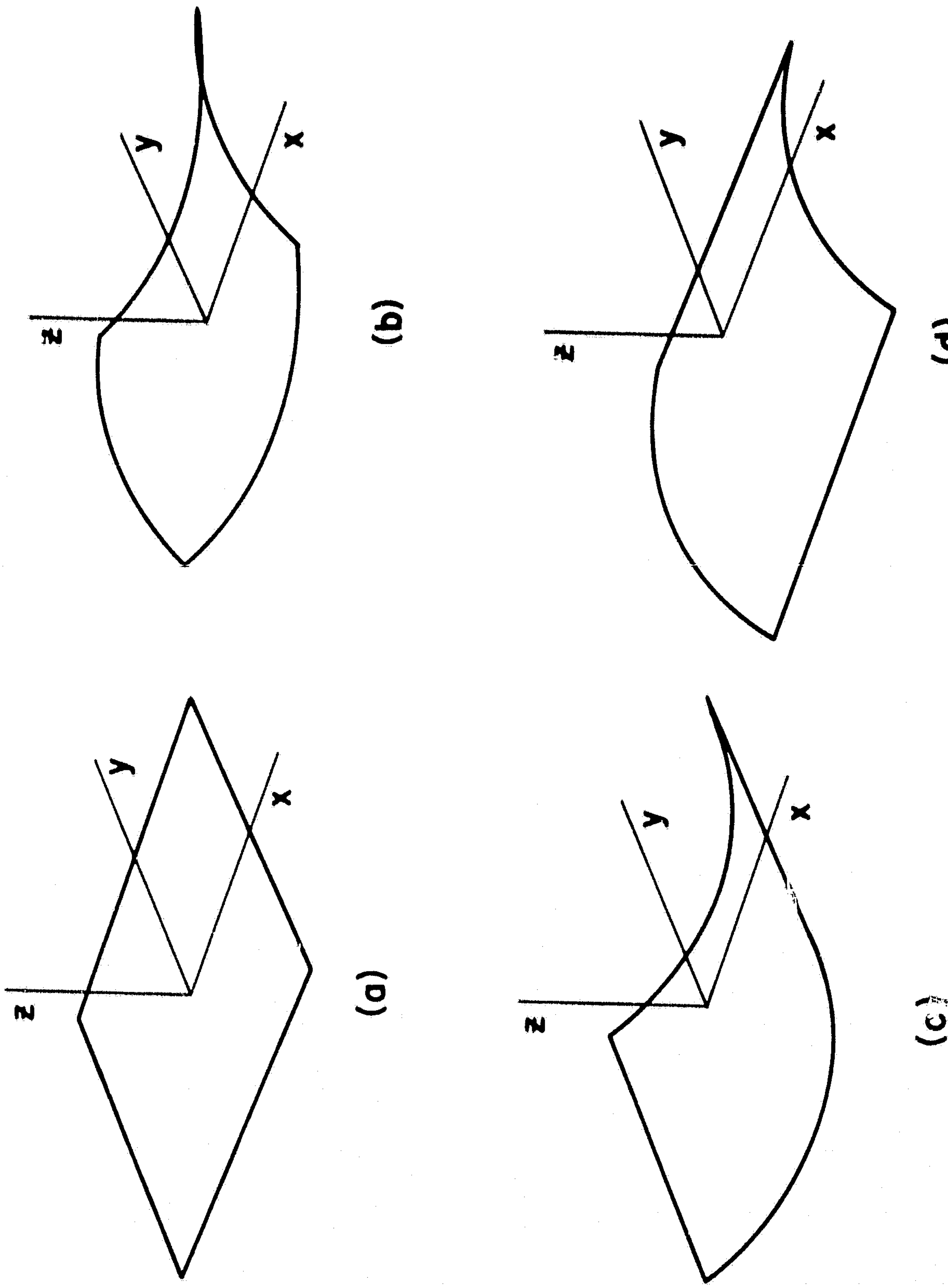

z
y
x
(a)
z
y
x
(b)
z
y
x
(c)
z
y
x
(d)

Fig. 2 Room-temperature shapes of square $[0_2/90_2]_T$ T300/5208 graphite-epoxy laminates

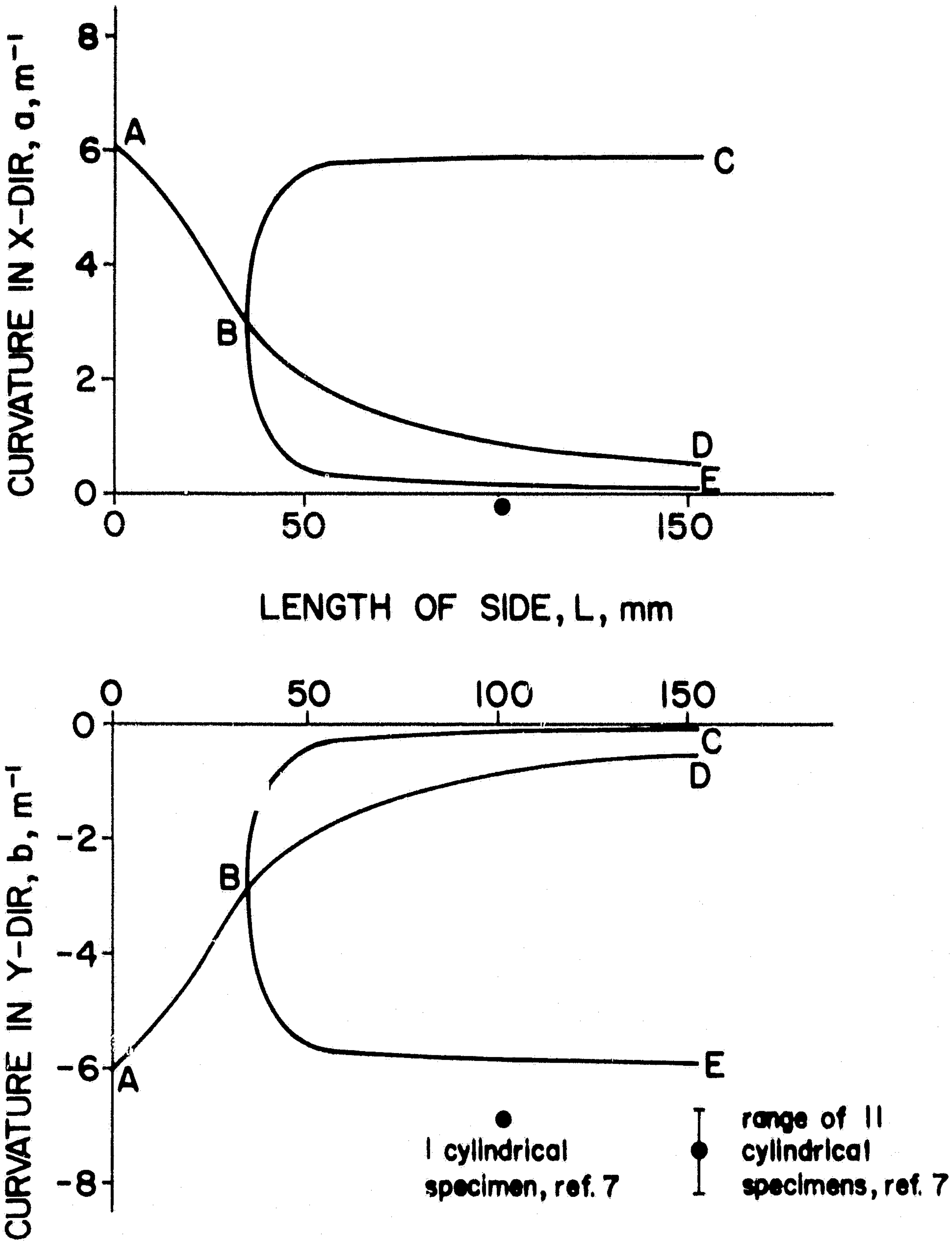
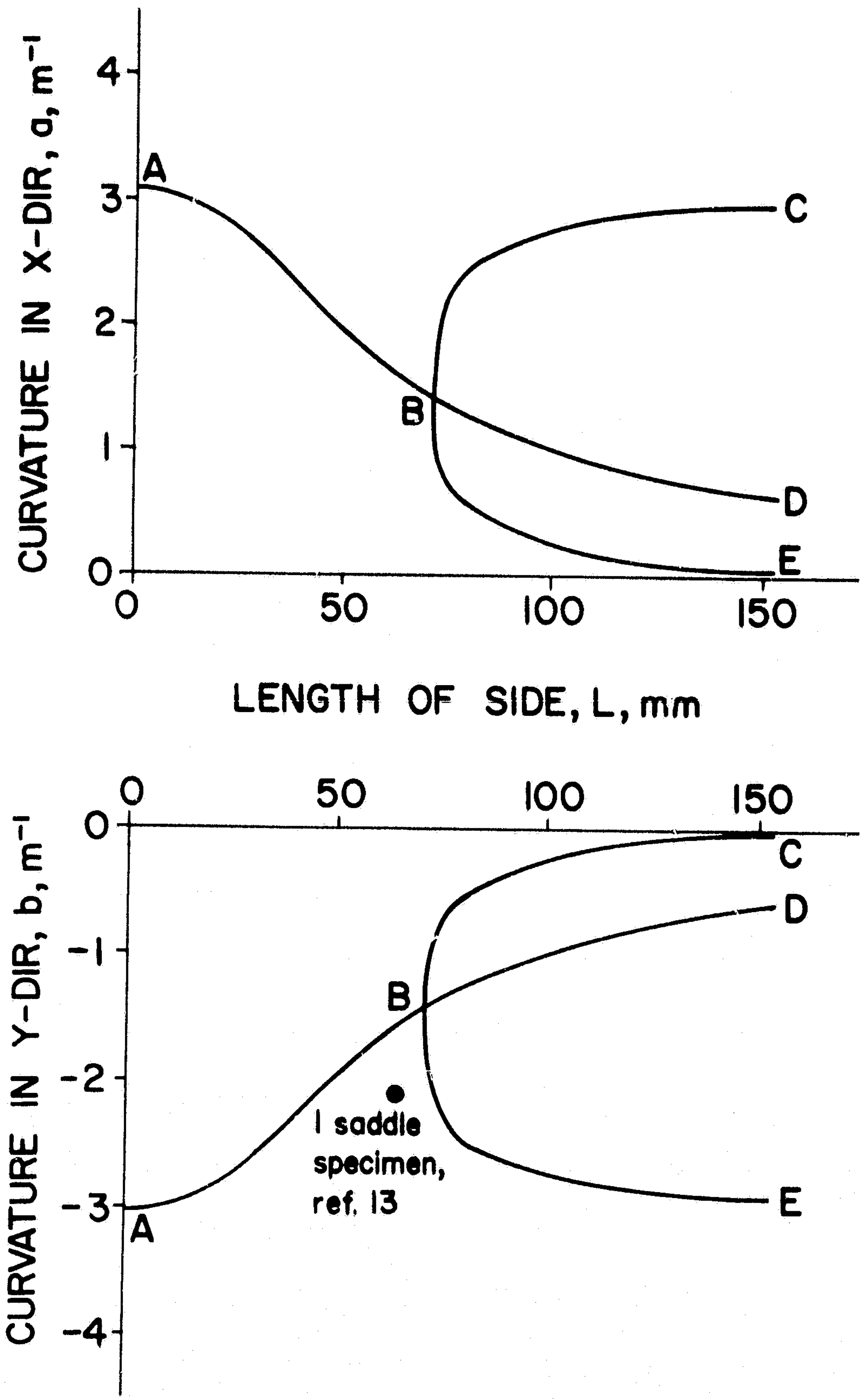

CURVATURE IN X-DIR, a, m⁻¹
8
6
4
2
0
A
B
C
D
E
0
50
150
LENGTH OF SIDE, L, mm
0
50
100
150
0
-2
-4
-6
-8
CURVATURE IN Y-DIR, b, m⁻¹
A
B
C
D
E
I cylindrical specimen, ref. 7
range of 11 cylindrical specimens, ref. 7

Fig. 3 Room-Temperature shapes of square $[0_4/90_4]_T$ T300/5208 graphite-epoxy laminates


CURVATURE IN X-DIR, a, m^{-1}
4
3
2
1
0
A
B
C
D
E
0
50
100
150
LENGTH OF SIDE, L, mm
0
50
100
150
0
-1
-2
-3
-4
CURVATURE IN Y-DIR, b, m^{-1}
A
B
C
D
E
I saddle
specimen,
ref. 13

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