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"FINITE PART" ELECTRIC AND MAGNETIC STORED ENERGIES FOR PLANAR ANTENNAS

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SUMMARY

A pair of formulas representing the time-average "finite part" electric and magnetic stored energies for planar antennas are derived. It is also shown that the asymptotic reciprocal relationship between quality factor and relative bandwidth exists for planar antennas.

INTRODUCTION

Several years ago, Rhodes published a pair of formulas which represented the time-average "physically observable" electricl and magnetic stored energies for planar antennas [1]. Since his formulas arise through the use of the complex Poynting Theorem in which the volume integrals of the electric and magnetic fields appear only as a difference, their uniqueness were questioned by Borgiotti [2] and Collin [3]. Rhodes defended his formulas by offering a physical interpretation which supported his mathematical conclusions given in his earlier paper [reply following refs. 2,3].

Rhodes derived his formulas by adding and substracting terms (which were finite) to the magnetic and electric volume integral representations (which were infinite). These infinities come from the volume integrals of the field components that do not vanish outside the aperture of a planar antenna [1]. The added and subtracted finite terms were identical to the terms produced by the remaining field components. The subtracted terms were then grouped with the volume integrals whose contributions were infinite. When the electric and magnetic volume integrals were differenced, the grouped terms cancelled identically, leaving what Rhodes defined as time-average "physically observable" stored energies.

The purpose of this paper is to derive expressions which represent time-average electric and magnetic stored energies for planar antennas by using the concepts of Hadamard's "finite part" of divergent integrals and Schwartz's "distribution" functions [4,5]. The time-average stored-energy formulas, based on these concepts, come directly from the time-average electric and magnetic volume integrals; the complex Poynting Theorem is not used in their derivation. These time-average "finite part" stored energies are shown to give exactly the same reactive power expressions as the ones given in references [1-3 and 6] (with the proper notation change). But, more importantly, these "finite part" stored energies are shown to establish the asymptotic reciprocal relationship between quality factor and relative bandwidth for planar antennas.

APPROACH TO THE PROBLEM

For definiteness the problem of an aperture in an infinite perfectly conducting plane, lying in the x-y plane, is considered (see fig. 1). The tangential electric field components (E_x,E_y) and the normal magnetic field component (H_z), therefore, vanish outside the aperture on the ground plane. It is the nonvanishing components (E_z,H_x,H_y) on this ground plane that causes the contributions to the volume integrals to become infinite. These divergent integrals are given explicitly as [1]

$$\frac{\varepsilon}{4}\int_{z=0}^{\infty}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}|E_z|^2\,dx\,dy\,dz = \frac{1}{(2\pi)^2}\frac{\varepsilon}{4}$$

$$\times\iint_{k_x^2+k_y^2\geq k^2}\frac{|k_xF_x+k_yF_y|^2}{2(k_x^2+k_y^2-k^2)^{3/2}}\,dk_x\,dk_y \qquad (1)$$

$$\frac{\mu}{4} \int_{z=0} \int_{x=-\infty} \int_{y=-\infty} |H_x|^2 \, dx \, dy \, dz = \frac{1}{(2\pi)^2} \frac{1}{\omega^2 \mu} \frac{1}{4}$$

$$\times \iint_{k_x^2 + k_y^2 \geq k^2} \frac{|k_x k_y F_x + (k^2 - k_x^2) F_y|^2}{2(k_x^2 + k_y^2 - k^2)^{3/2}} \, dk_x \, dk_y \tag{2}$$

$$\frac{\mu}{4} \iiint |H_y|^2 \, dx \, dy \, dz = \frac{1}{(2\pi)^2} \frac{1}{\omega^2 k} \frac{1}{4}$$

$$\times \iint_{k_x^2 + k_y^2 \geq k^2} \frac{|(k^2 - k_y^2) F_x + k_x k_y F_y|^2}{2(k_x^2 + k_y^2 - k^2)^{3/2}} \, dk_x \, dk_y \tag{3}$$

where

$$F_{\substack{x \\ y}} = \iint_{\text{aperture}} E_{\substack{xa \\ ya}}(x,y,k) e^{jk_x x} e^{jk_y y} \, dk_x \, dk_y \tag{4}$$

and $E_{\substack{xa \\ ya}}$ are electric fields in the aperture. Equations (1) to (3) do not exist in the ordinary sense; however, treating the integrands as distributions will enable one to attach some meaning to these divergent integrals [4,5]. More will be said later about the treatment of the integrands as distributions in this section.

The terms of the integrands multiplying the F functions are rewritten as

$$
\left.
\begin{aligned}
\frac{k_x^2}{(k_x^2 + k_y^2 - k^2)^{3/2}} &= \frac{1}{\sqrt{k_x^2 + k_y^2 - k^2}} + \frac{k^2 - k_y^2}{(k_x^2 + k_y^2 - k^2)^{3/2}} \\
&= \frac{1}{\sqrt{k_x^2 + k_y^2 - k^2}} + \frac{\partial^2}{\partial k_x^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \\
\frac{k_x k_y}{(k_x^2 + k_y^2 - k^2)^{3/2}} &= \frac{\partial^2}{\partial k_x \, \partial k_y} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \\
\frac{k_y^2}{(k_x^2 + k_y^2 - k^2)^{3/2}} &= \frac{1}{\sqrt{k_x^2 + k_y^2 - k^2}} + \frac{k^2 - k_x^2}{(k_x^2 + k_y^2 - k^2)^{3/2}} \\
&= \frac{1}{\sqrt{k_x^2 + k_y^2 - k^2}} + \frac{\partial^2}{\partial k_y^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)
\end{aligned}
\right\} \quad (5)
$$

where they are to interpreted as distribution functions because interpretations as ordinary functions do not exist. Substitution of these equations into equations (1) to (3) combined with volume integrals (which do exist in the ordinary sense)

$$
\left.
\begin{aligned}
&\frac{\varepsilon}{4} \iiint |E_x|^2 \, dx\, dy\, dz = \frac{1}{(2\pi)^2} \frac{\varepsilon}{4} \iint \frac{|F_x|^2 \, dk_x \, dk_y}{2\sqrt{k_x^2 + k_y^2 - k^2}} \\
&\frac{\varepsilon}{4} \iiint |E_y|^2 \, dx\, dy\, dz = \frac{1}{(2\pi)^2} \frac{\varepsilon}{4} \iint \frac{|F_y|^2 \, dk_x \, dk_y}{2\sqrt{k_x^2 + k_y^2 - k^2}} \\
&\frac{\mu}{4} \iiint |H_z|^2 \, dx\, dy\, dz = \frac{1}{(2\pi)^2} \frac{1}{\omega^2 \mu} \frac{1}{4} \\
&\qquad \times \iint \frac{|k_y F_x - k_y F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \, dk_x \, dk_y
\end{aligned}
\right\} \quad (6)
$$

the total electric and magnetic volume integrals are written, respectively, as

$$\frac{\varepsilon}{4} \iiint |E|^2 \, dx \, dy \, dz = \frac{1}{(2\pi)^2} \frac{\varepsilon}{4} \left\{ \iint \frac{|F_x|^2 + |F_y|^2}{\sqrt{k_x{}^2 + k_y{}^2 - k^2}} \, dk_x \, dk_y \right.$$

$$+ \frac{1}{2} \iint \left[|F_x|^2 \frac{\partial^2}{\partial k_x{}^2} + (F_x F_y{}^* + F_x{}^* F_y) \frac{\partial^2}{\partial k_x \, \partial k_y} + |F_y|^2 \frac{\partial^2}{\partial k_y{}^2} \right]$$

$$\left. \left(-\sqrt{k_x{}^2 + k_y{}^2 - k^2}\right) dk_x \, dk_y \right\} \qquad (7)$$

$$\frac{\mu}{4} \iiint |H|^2 \, dx \, dy \, dz = \frac{1}{(2\pi)^2} \; \frac{1}{\omega\mu} \; \frac{1}{4} \left\{ \iint \frac{|k_y F_x - k_y F_y|^2}{\sqrt{k_x{}^2 + k_y{}^2 - k^2}} \, dk_x \, dk_y \right.$$

$$+ \frac{k^2}{2} \iint \left[|F_x|^2 \frac{\partial^2}{\partial k_x{}^2} + (F_x F_y{}^* + F_x{}^* F_y) \frac{\partial^2}{\partial k_x \, \partial k_y} + |F_y|^2 \frac{\partial^2}{\partial k_y{}^2} \right]$$

$$\left. \left(-\sqrt{k_x{}^2 + k_y{}^2 - k^2}\right) dk_x \, dk_y \right\} \qquad (8)$$

The integration on the right hand side of equations (7) and (8) is performed over the region $k_x{}^2 + k_y{}^2 \geq k^2$. The first term in each of these equations is the exact "physically observable" stored energy relationship given by Rhodes [1], (eqs. (16), (15)). The last term in these equations diverge when the integrands are interpreted as ordinary functions but converges when the concepts of "finite part" and distribution "distribution" are introduced into the interpretation [4,5]. Interpretation of last integrands in equations (7) and (8) as ordinary functions, therefore, lead to infinite stored energies in both electric and magnetic fields. Rhodes' concept of "physically observable" stored energies is based on the complex Poynting Theorem in which the volume integrals given in equations (7) and (8) appear only as a difference. It was through this difference that he was able to arrive at his stored energy pair (first terms in equations (7) and (8) since the last terms cancelled identically) [1], (eqs. (16), (15)). In essence, therefore, the last terms in equation (7) and (8) have been neglected in the definition of "physically observable" stored energies. In the next section, the last terms in equations (7) and (8) are not neglected but redefined in terms of the concept of "finite parts" of divergent integrals [4,5].

The validity of introducing the concept "finite parts" in divergent integrals may be justified by recalling the mathematical steps leading up to the divergent integrals. In determining the volume integral representations for the E_z, H_x, and H_y components, an interchange in the order of integration was assumed valid. This interchange is guaranteed to be valid only if the integrand is continuous over the range of integrations and if the integral is uniformly convergent. Even though E_z, H_x, and H_y have integrable singularities at $k_x^2 + k_y^2 = k^2$, they produce discontinuous integrands in the volume integrals. It is at the points on this circle that causes the volume integrals of these components to diverge. To include these points in the existing stored energy integral representations of these components necessitates special consideration in order to produce meaningful results. The use of the concept of "finite parts" of divergent integrals allows one to take only the finite parts of the integrals and disregard the infinities associated with ends points of the integrals. Therefore, these points are included in the evaluation of these divergent integrals but the infinities are moved by the concept of "finite parts" of divergent integrals. The inclusion of this circle of values in the other field components contributes nothing to their energy integrals, and hence, can be included without special attention.

"FINITE PARTS" OF THE DIVERGENT INTEGRALS OF THE ELECTRIC AND MAGNETIC VOLUME INTEGRALS

In this section the divergent integrals in equations (7) and (8) (last term in each) are examined in detail from the standpoint of interpreting the integrands as "distributions" and not as ordinary functions; the latter interpretation, as noted earlier, leads to divergent integrals.

Let

$$I_1 = \iint_{k_x^2 + k_y^2 \geq k^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \frac{\partial^2}{\partial k_x^2} |F_x|^2 \, dk_x \, dk_y$$

which converges due to the properties of F_x [1]. Since the integration is outside the circle $k_x^2 + k_y^2 = k^2$ and the integrand is an even function of both k_x and k_y,

$$I_1 = 4 \left\{ \int_0^k \left[\int_{\sqrt{k^2-k_y^2}}^{\infty} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \frac{\partial^2}{\partial k_x^2} |F_x|^2 \, dk_x \right] dk_y + \int_k^{\infty} \left[\int_0^{\infty} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \frac{\partial^2}{\partial k_x^2} |F_x|^2 \, dk_x \right] dk_y \right\} \qquad (9)$$

Integration by parts twice yields

$$I_1 = \iint\limits_{k_x{}^2 + k_y{}^2 \geq k^2} \frac{\partial^2}{\partial k_x{}^2} \left(-\sqrt{k_x{}^2 + k_y{}^2 - k^2}\right) |F_x|^2 \, dk_x \, dk_y$$

$$+ 4\int_0^k \frac{\partial}{\partial k_x} \left(-\sqrt{k_x{}^2 + k_y{}^2 - k^2}\right) |F_x|^2 \Bigg]_{k_x = \sqrt{k^2 - k_y{}^2}} dk_y$$

$$+ 4\int_k^\infty \sqrt{k_y{}^2 - k^2} \; \frac{\partial}{\partial k_x} |F_x|^2 \Bigg]_{k = 0} dk_y \qquad (10)$$

Since the left side of equation (10) converges the right side must converge as a whole although the individual terms may produce divergent parts (first and second terms); the existence of the first two terms on the right side has been called by Hadamard the "finite part" of the divergent integral [4,5]. With the notation FP denoting the "finite part,"

$$FP \iint\limits_{k_x{}^2 + k_y{}^2 \geq k^2} \frac{\partial^2}{\partial k_x{}^2} \left(-\sqrt{k_x{}^2 + k_y{}^2 - k^2}\right) |F_x|^2 \, dk_x \, dk_y$$

$$\triangleq \iint \frac{\partial^2}{\partial k_x{}^2} \left(-\sqrt{k_x{}^2 + k_y{}^2 - k^2}\right) |F_x|^2 \, dk_x \, dk_y$$

$$+ 4\int_0^k \frac{\partial}{\partial k_x} \left(-\sqrt{k_x{}^2 + k_y{}^2 - k^2}\right) |F_x|^2 \Bigg]_{k_x = \sqrt{k^2 = k_y{}^2}} dk_y \qquad (11)$$

equation (10) becomes

$$I_1 = FP \iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial^2}{\partial k_x^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_x|^2 \, dk_x \, dk_y$$

$$+ 4\int_k^\infty \sqrt{k_y^2 - k^2} \left. \frac{\partial}{\partial k_x} |F_x|^2 \right]_{k_x = 0} dk_y \tag{12}$$

where $\partial^2/\partial k_x^2 \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)$ exist not as an ordinary function but as a distribution. Therefore,

$$FP \iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial^2}{\partial k_x^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_x|^2 \, dk_x \, dk_y$$

$$= \iint_{k_x^2 + k_y^2 \geq k^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \frac{\partial^2}{\partial k_x^2} |F_x|^2 \, dk_x \, dk_y$$

$$-4\int_k^\infty \sqrt{k_y^2 - k^2} \left. \frac{\partial}{\partial k_x} |F_x|^2 \right]_{k_x = 0} dk_y \tag{13}$$

or

$$FP \iint_{k_x^2 + k_y \geq k^2} \frac{\partial^2}{\partial k_x^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_x|^2 dk_x \, dk_y$$

$$= \iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial}{\partial k_x} \sqrt{k_x^2 + k_y^2 - k^2} \; \frac{\partial}{\partial k_x} \; |F_x|^2 \, dk_x \, dk_y \tag{14}$$

A similar argument shows that

$$I_2 = \iint_{k_x^2 + k_y^2 \geq k^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \frac{\partial^2}{\partial k_y^2} |F_y|^2 \, dk_x \, dk_y \tag{15}$$

and

$$FP \iint \frac{\partial^2}{\partial k_y^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_y|^2 dk_x \, dk_y$$

$$\triangleq \iint \frac{\partial}{\partial k_y^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_y|^2 \, dk_x \, dk_y$$

$$+ 4\int_0^\infty \frac{\partial}{\partial k_y} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_y|^2 \Big]_{k_y = \sqrt{k^2 - k_x^2}} \, dk_x \tag{16}$$

Therefore, I_2 becomes

$$I_2 = FP \iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial^2}{\partial k_y^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_y|^2 \, dk_x \, dk_y + 4\int_k^\infty \sqrt{k_x^2 - k^2} \left. \frac{\partial}{\partial k_y} |F_y|^2 \right]_{k_y = 0} dk_x \tag{17}$$

where $\partial^2/\partial k_y^2 \left(\sqrt{k_x^2 + k_y^2 - k^2}\right)$ is interpreted as a distribution. Therefore,

$$FP \iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial^2}{\partial k_y^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_y|^2 \, dk_x \, dk_y$$

$$= \iint_{k_x^2 + k_y^2 \geq k^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \frac{\partial^2}{\partial k_y^2} |F_y|^2 \, dk_x \, dk_y$$

$$- 4\int_k^\infty \sqrt{k_x^2 - k^2} \left. \frac{\partial}{\partial k_y} |F_y|^2 \right]_{k_y = 0} \tag{18}$$

or

$$\text{FP} \iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial^2}{\partial k_y^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) |F_y|^2 \, dk_x \, dk_y$$

$$= \iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial}{\partial k_y} \sqrt{k_x^2 + k_y^2 - k^2} \; \frac{\partial}{\partial k_y} |F_y|^2 \, dk_x \, dk_y \tag{19}$$

The second terms in the divergent parts of equations (7) and (8) are now examined in terms of their finite parts; let,

$$I_3 = \iint_{k_x^2 + k_y^2 \geq k^2} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \frac{\partial^2}{\partial k_x \, \partial k_y} (F_x F_y^* + F_x^* F_y) \, dk_x \, dk_y \tag{20}$$

which converges due to the properties of F_x and F_y [1].

Integrating by parts first on k_y and then k_x, I_3 becomes

$$I_3 = \Bigg[\iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial^2}{\partial k_y \, \partial k_x} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)(F_x F_y^* + F_x^* F_y) \, dk_x \, dk_y$$

$$+ 4\int_0^k \frac{\partial}{\partial k_y} \left(-\sqrt{k_x^2 + k_y^2 - k^2} \; (F_x F_y^* + F_x^* F_y)\right]_{k_x = \sqrt{k^2 - k_y^2}} dk_y \Bigg]$$

$$+ 4\int_k^\infty \sqrt{k_x^2 - k^2}\ \frac{\partial}{\partial k_x} (F_xF_y^* + F_x^*F_y)\Big]_{k_y = 0} dk_x$$

$$+ 4\int_k^\infty \sqrt{k_y^2 - k^2}\ \frac{\partial}{\partial k_y} (F_xF_y^* + F_x^*F_y)\Big]_{k_x = 0} dk_y \tag{21}$$

The finite part of the double integral is defined as the terms in the square brackets:

$$FP \iint_{k_x^2 + k_y^2 \geq k^2} \frac{\partial^2}{\partial k_x\, \partial k_y} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)(F_xF_y^* + F_x^*F_y)\, dk_x\, dk_y$$

$$\stackrel{\Delta}{=} \iint_{k_x^2+k_y^2\geq k^2} \frac{\partial^2}{\partial k_x\, \partial k_y} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)(F_xF_y^* + F_x^*F_y)\, dk_x\, dk_y$$

$$+ 4\int_0^k \frac{\partial}{\partial k_y} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)(F_xF_y^* + F_xF_y)\Big]_{k_x = \sqrt{k^2 - k_y^2}} dk_y \tag{22}$$

so that I_3 becomes

$$I_3 = FP\iint \frac{\partial^2}{\partial k_y\, \partial k_x} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)(F_x F_y^* + F_x^* F_y)\, dk_x\, dk_y$$

$$+ 4\int_k^\infty \sqrt{k_x^2 - k^2}\, \frac{\partial}{\partial k_x}(F_x F_y^* + F_x^* F_y)\Big]_{k_y = 0}\, dk_x$$

$$+ 4\int_k^\infty \sqrt{k_y^2 - k^2}\, \frac{\partial}{\partial k_y}(F_x F_y^* + F_x^* F_y)\Big]_{k_x = 0}\, dk_y \qquad (23)$$

Therefore,

$$FP\iint \frac{\partial^2}{\partial k_y\, \partial k_x} \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)(F_x F_y^* + F_x^* F_y)\, dk_x\, dk_y$$

$$= \iint \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \frac{\partial^2}{\partial k_x\, \partial k_y}(F_x F_y^* + F_x^* F_y)\, dk_x\, dk_y$$

$$- 4\int_k^\infty \sqrt{k_x^2 - k^2}\, \frac{\partial}{\partial k_x}(F_x F_y^* + F_x^* F_y)\Big]_{k_y = 0}\, dk_x$$

$$-4\int_k^\infty \sqrt{k_y^2 - k^2}\, \frac{\partial}{\partial k_y}(F_x F_y^* + F_x^* F_y)\Big]_{k_x = 0}\, dk_y \qquad (24)$$

or

$$FP\iint \frac{\partial^2}{\partial k_y\,\partial k_x}\left(-\sqrt{k_x^2 + k_y^2 - k^2}\right)(F_x F_y^* + F_x^* F_y)\,dk_x\,dk_y$$

$$= \iint \frac{\partial}{\partial k_x}\left(\sqrt{k_x^2 + k_y^2 - k^2}\right)\frac{\partial}{\partial k_y}(F_x F_y^* + F_x^* F_y)\,dk_x\,dk_y$$

$$- 4\int_k^\infty \sqrt{k_x^2 - k^2}\,\frac{\partial}{\partial k_x}\left(F_x F_y^* + F_x^* F_y\right)\Big]_{k_y = 0}\,dk_x \qquad (25)$$

The time-average electric and magnetic stored energies are now defined in terms of finite parts as

$$\langle W_e \rangle_{FP} = \frac{\varepsilon}{4}\,FP \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty |E|^2\,dx\,dy\,dz \qquad (26)$$

or

$$\langle W_e \rangle_{FP} = \frac{\varepsilon}{4}\,\frac{1}{(2\pi)^2}\left\{\iint_{k_x^2 + k_y^2 \geq k^2} \frac{|F_x|^2 + |F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}}\,dk_x\,dk_y\right.$$

$$+ \frac{1}{2}\,FP \iint_{k_x^2 + k_y^2 \geq k^2}\left[|F_x|^2 \frac{\partial^2}{\partial k_x^2} + (F_x F_y^* + F_x^* F_y)\frac{\partial^2}{\partial k_x\,\partial k_y}\right.$$

$$+ |F_y|^2 \frac{\partial^2}{\partial k_y^2}\Bigg] \left(\sqrt{k_x^2 + k_y^2 - k^2}\right) \; dk_x \; dk_y \Bigg\} \qquad (27)$$

and

$$<W_m>_{FP} = \frac{\mu}{4} \text{ FP } \int_o^\infty \int_\infty^\infty \int_\infty^\infty |H|^2 \, dx \, dy \, dz \qquad (28)$$

or

$$<W_m>_{FP} = \frac{1}{4} \frac{1}{(2\pi)^2} \frac{1}{\omega^2 \mu} \iint_{k_x^2 + k_y^2 \geq k^2} \frac{|k_y F_x - k_x F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \, dk_x \, dk_y$$

$$+ \frac{k^2}{2} \text{ FP } \iint_{k_x^2 + k_y^2 \geq k^2} \Bigg[|F_x|^2 \frac{\partial^2}{\partial k_x^2} + (F_x F_y^* + F_x^* F_y) \frac{\partial^2}{\partial k_x \, \partial k_y}$$

$$+ |F_y|^2 \frac{\partial^2}{\partial k_y^2}\Bigg] \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \; dk_x \; dk_y \qquad (29)$$

where the individual finite parts are given by equations (13) or (14), (18), or (19) and (24) or (25).

The reactive power, which will be needed in determining the relative bandwidth, is given as*

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*Note that no new definition is needed here because the difference in the usual volume integrals is the same as $<W_e>_{FP} - <W_m>_{FP}$

$$2\omega(\langle W_e \rangle_{FP} - \langle W_m \rangle_{FP}) = \frac{1}{(2\pi)^2} \frac{1}{2\omega\mu}$$

$$\left\{ \iint_{k_x^2 + k_y^2 \geq k^2} \frac{(k^2(|F_x|^2 + |F_y|^2) - |k_y F_x - k_x F_y|^2)}{\sqrt{k_x^2 + k_y^2 - k^2}} dk_x\, dk_y \right\} \tag{30}$$

which agrees with the representations given in references [1] to [3], and [6].

Q AND RELATIVE BANDWIDTH

The quality factor Q is defined in terms of the finite part stored as

$$Q \stackrel{\Delta}{=} \left. \frac{\omega[\langle W_e \rangle_{FP} + \langle W_m \rangle_{FP}]}{Pr} \right|_{\omega \,=\, \text{angular resonant frequency}} \tag{31}$$

where $\langle W_e \rangle_{FP}$ and $\langle W_e \rangle_{FP}$ are given, respectively, by equations (27) and (29) and Pr is the radiated power given by [1]

$$Pr = \frac{1}{(2\pi)^2} \frac{1}{2kZ_o} \iint_{k_x^2 + k_y^2 < k^2}$$

$$\frac{(k^2 - k_x^2)|F_y|^2 + (k^2 - k_y^2)|F_x|^2 + 2k_x k_y \mathrm{Re} F_x F_y^*}{\sqrt{k^2 - k_x^2 - k_y^2}} dk_x\, dky \tag{32}$$

where Z_o is the free space impedance and Re denotes the real part.

Substituting equations (27) and (29) into equation (31) Q becomes

$$Q \triangleq \left\{ \frac{1}{Pr} \frac{1}{(2\pi)^2} \frac{1}{\omega\mu} \left[\iint_{k_x^2 + k_y^2 \geq k^2} \frac{[|k_y F_x - k_x F_y|^2 + k^2(|F_x|^2 + |F_y|^2]}{\sqrt{k_x^2 + k_y^2 - k^2}} dk_x\, dk_y \right.\right.$$

$$+ k^2\, FP \iint_{k_x^2 + k_y^2 \geq k^2} \left[|F_x|^2 \frac{\partial^2}{\partial k_x^2} + (F_x F_y^* + F_x^* F_y) \frac{\partial^2}{\partial k_x\, \partial k_y} \right.$$

$$\left.\left.\left. + |F_y|^2 \frac{\partial^2}{\partial k_y^2} \right] \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) dk_x\, dk_y \right] \right\} \tag{33}$$

ω = angular resonant frequency

The relative bandwidth is defined as*

$$B.W. \triangleq \left[\frac{Pr}{\omega \frac{d}{d\omega}\left[\omega(\langle W_e \rangle_{FP} - \langle W_m \rangle_{FP})\right]} \right] \tag{34}$$

ω = angular resonant frequency

*Note that no new definition is needed here because the difference in the usual volume integrals is the same as $\langle W_e \rangle_{FP} - \langle W_m \rangle_{FP}$

The purpose of this section is to show that at resonance the asymptotic relationship between Q and relative bandwidth is given by

$$Q = \frac{1}{B.W.} \tag{35}$$

This will be accomplished by explicitly taking the frequency derivative shown in equation (34):

$$\frac{1}{B.W.} = \left\{ \frac{1}{Pr} \frac{d}{d\omega} [\omega(\langle W_e \rangle_{FP} - \langle W_m \rangle_{FP})] \right\}_{\text{at resonance}} \tag{36}$$

$$\frac{1}{B.W.} = \left\{ \frac{k}{Pr} \frac{d}{dk} \left[\frac{1}{(2\pi)^2} \frac{\sqrt{\mu\varepsilon}}{k\mu} \iint_{k_x^2 + k_y^2 \geq k^2} \frac{k^2(|F_x|^2 + |F_y|^2) - |k_y F_x - k_x F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \, dk_x \, dk_y \right] \right\}_{\text{at resonance}} \tag{37}$$

$$\frac{1}{B.W.} = \left\{ \frac{1}{Pr} \frac{k}{(2\pi)^2 Z_0} \left[\iint_{k_x^2 + k_y^2 \geq k^2} \left[\frac{k^2(|F_x|^2 + |F_y|^2) - |k_y F_x - k_x F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \right] dk_x \, dk_y \left(- \frac{1}{k^2} \right) \right. \right.$$

$$\left. \left. + \frac{1}{k} \frac{d}{dk} \iint_{k_x^2 + k_y^2 \geq k^2} \left[\frac{k^2(|F_x|^2 + |F_y|^2) - |k_y F_x - k_x F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \right] dk_x \, dk_y \right] \right\} \qquad (38)$$

$$\frac{1}{\text{B.W.}} = \left\{ \frac{1}{Pr} \frac{k}{(2\pi)^2 Z_0} \left(-\frac{1}{k^2} \iint_{k_x^2 + k_y^2 \geq k^2} \left[\frac{k^2(|F_x|^2 + |F_y|^2 - |k_y F_x - k_x F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \right] dk_x \, dk_y \right.\right.$$

$$+ \frac{1}{k} \frac{d}{dk} \left[4 \int_{k_y=0}^{k} \int_{k_x=\sqrt{k^2-k_y^2}}^{\infty} \frac{(k^2 - k_y^2)|F_x|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \, dk_x \, dk_y \right.$$

$$+ 4 \int_{k_y=k}^{\infty} \int_{k_x=0}^{\infty} \frac{(k^2 - k_y^2)|F_x|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \, dk_x \, dk_y + 4 \int_{k_x=0}^{k} \int_{k_y=\sqrt{k^2-k_x^2}}^{\infty} \frac{(k^2 - k_x^2)|F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \, dk_y dk_x$$

$$+ 4 \int_{k_x=0}^{\infty} \int_{k_y=0}^{\infty} \frac{(k^2 - k_x^2)|F_y|^2}{\sqrt{k_x^2 + k_y^2 - k^2}} \, dk_y \, dk_x + 4 \int_{k_y=0}^{k} \int_{k_x=\sqrt{k^2-k_x^2}}^{\infty} \frac{k_x k_y (F_x F_y^* + F_x^* F_y)}{\sqrt{k_x^2 + k_y^2 - k^2}} dk_x dk_y$$

$$\left.\left.\left. + 4 \int_{k_y=k}^{\infty} \int_{k_x=0}^{\infty} \frac{k_x k_y (F_x F_y^* + F_x^* F_y)}{\sqrt{k_x^2 + k_y^2 - k^2}} \right] \right) \right\} \qquad (39)$$

at resonance

The derivative with respect to k of the double integrals in equation (39) is determined via the generalized Leibnitz formula [7], with the modifications that the integrands and their derivatives with respect to k exist as distributions. Performing the differentiations, therefore, 1/B.W. becomes

$$\frac{1}{B.W.} = \Bigg\{ \frac{1}{Pr} \frac{k}{(2\pi)^2 Z_o} \Bigg(\iint_{k_x{}^2+k_y{}^2 \geq k^2} \frac{|F_x|^2 + |F_y|^2}{\sqrt{k_x{}^2 + k_y{}^2 - k^2}} \, dk_x \, dk_y$$

$$+ \frac{1}{k^2} \iint_{k_x{}^2+k_y{}^2 \geq k^2} \frac{|k_y F_x - k_x F_y|^2}{\sqrt{k_x{}^2 + k_y{}^2 - k^2}} \, dk_x \, dk_y$$

$$+ \left[\iint_{k_x{}^2+k_y{}^2 \geq k^2} \frac{(k^2 - k_y{}^2)|F_x|^2}{(k_x{}^2 + k_y{}^2 - k^2)^{3/2}} \, dk_x \, dk_y - 4\int_o^k \left[\frac{k^2 - k_y{}^2 |F_x|^2}{\sqrt{k_x{}^2 + k_y{}^2 - k^2}} \right]_{k_x = \sqrt{k^2 - k_y{}^2}} dk_y \right]$$

$$+ \left[\iint_{k_x{}^2+k_y{}^2 \geq k^2} \frac{(k^2 - k_x{}^2)|F_y|^2}{(k_x{}^2 + k_y{}^2 - k^2)^{3/2}} \, dk_x \, dk_y - 4\int_o^k \left[\frac{\overline{k^2 - k_x{}^2} |F_y|^2}{\sqrt{k_x{}^2 + k_y{}^2 - k^2}} \right]_{k_y = \sqrt{k^2 - k_x{}^2}} dk_x \right]$$

$$+ \left[\iint_{k_x{}^2+k_y{}^2 \geq k^2} \frac{k_x k_y (F_x F_y{}^* + F_x{}^* F_y)}{(k_x{}^2 + k_y{}^2 - k^2)^{3/2}} \, dk_x \, dk_y - 4\int_o^k \left[\frac{(k_y (F_x F_y{}^* + F_x{}^* F_y)}{\sqrt{k_x{}^2 + k_y{}^2 - k^2}} \right]_{k_x = \sqrt{k^2 - k_y{}^2}} dk_y \right]$$

$$+ \frac{1}{k} \left\{ \iint\limits_{k_x^2+k_y^2 \geq k^2} \frac{(k^2 - k_y^2)}{\sqrt{k_x^2 + k_y^2 - k^2}} \frac{d|F_x|^2}{dk} dk_x\, dk_y \right.$$

$$+ \iint\limits_{k_x^2+k_y^2 \geq k^2} \frac{(k^2 - k_x^2)}{\sqrt{k_x^2 + k_y^2 - k^2}} \frac{d|F_y|^2}{dk} dk_x\, dk_y$$

$$\left. \left. \left. + \iint\limits_{k_x^2+k_y^2 \geq k^2} \frac{k_x k_y}{\sqrt{k_x^2 + k_y^2 - k^2}} \frac{d}{dk} (F_x F_y^* + F_x^* F_y)\, dk_x\, dk_y \right\} \right) \right\}_{\text{at resonance}} \qquad (40)$$

At resonance the derivatives of the F's* are zero and the terms in the square brackets are precisely the finite parts of integrals defined, respectively, in equations (11), (16) and (22); therefore,

---

*For a parallel circuit $I = V(k)\,Y(k)$ and at resonance Y is a minimum; therefore, the voltage $V(k)$ is a maximum, hence, $dV/dk\big|_{\text{resonance}} = 0$ since $V(k) \propto E_x$, then $\partial E_x/\partial k\big|_{\text{resonance}} = 0$. Therefore, $\frac{\partial F_y}{\partial k}\Big|_{\text{resonance}} \iint \frac{\partial E_x}{\partial k}(x,y,k)\Big|_{\text{resonance}} e^{jk_x x} e^{jk_y y}\, dx\, dy = 0.$

$$\frac{1}{B.W.} = \left\{ \frac{1}{Pr} \frac{1}{(2\pi)^2 \omega\mu} \left[\iint_{k_x^2+k_y^2 \geq k^2} \left[\frac{|k_y F_x - k_y F|^2 + k^2(|F_x|^2 + |F_y|^2)}{\sqrt{k_x^2 + k_y^2 - k^2}} \right] dk_x \, dk_y \right. \right.$$

$$+ k^2 \, FP \iint_{k_x^2+k_y^2 \geq k^2} \left[|F_x|^2 \frac{\partial^2}{\partial k_x^2} + (F_x F_y^* + F_x^* F_y) \frac{\partial^2}{\partial k_x \, \partial k_y} \right.$$

$$\left. \left. + |F_y|^2 \frac{\partial^2}{\partial k_y^2} \right] \left(-\sqrt{k_x^2 + k_y^2 - k^2}\right) \; dk_x \, dk_y \right] \right\} \qquad (41)$$

ω = angular resonant frequency

This equation is identical to equation (33), thus, proving the asymptotic relationship between them as given by equation (35).

CONCLUSION

The concepts of Hadamard's "finite part" of divergent integrals and Schwartz's "distribution" functions are used to derive a pair of formulas representing the time-average "finite part" electric and magnetic stored energies for planar antennas. The asymptotic reciprocal relationship between quality factor and relative bandwidth known to exist in circuits is shown to be valid for planar antennas.

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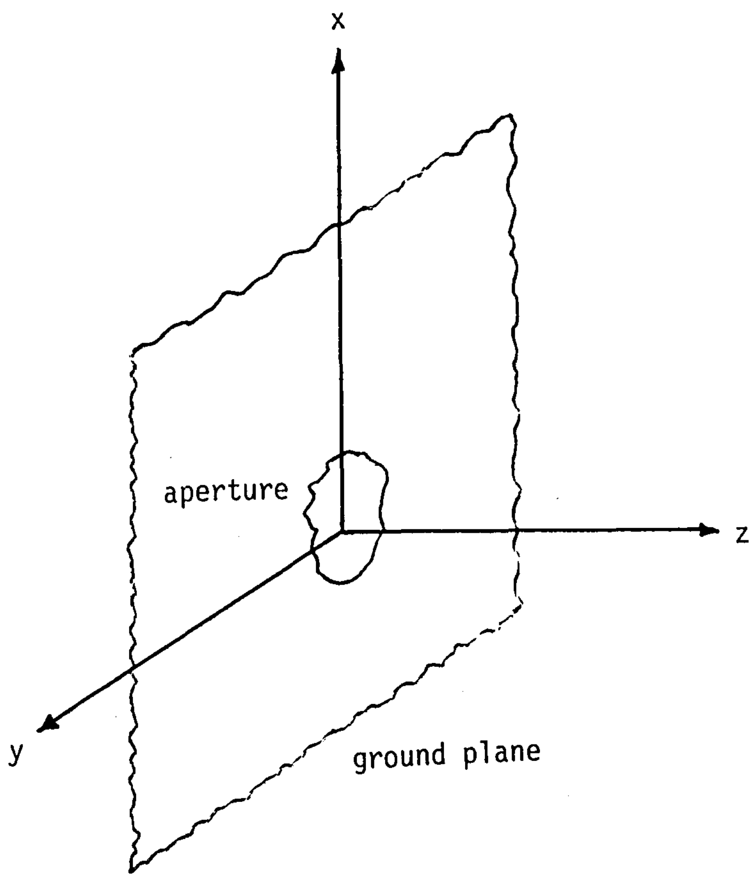

Figure 1.- Geometry of the problem.

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15. Supplementary Notes

16. Abstract

A pair of formulas representing the time-average "finite part" electric and magnetic stored energies for planar antennas are derived. It is also shown that the asymptotic reciprocal relationship between quality factor and relative bandwidth exists for planar antennas.

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