N O T I C E

(NASA-CR-160940) AN ALGORITHM FOR THE RAPID LOCATION OF AN EXTREME OF A FUNCTION SUBJECT ONLY TO GEOMETRIC RESTRICTIONS (Lockheed Engineering and Management) 7 p HC A02/MF A01 CSCL 09B G3/61 N81-23812 Unclas 23969

AN ALGORITHM FOR THE RAPID LOCATION OF AN EXTREMUM OF A FUNCTION SUBJECT ONLY TO GEOMETRIC RESTRICTIONS

George R. Terrell

ABSTRACT

If a function of a single variable is convex and symmetric in a neighborhood of an extremum, the extremum may be approximated to a precision that increases by at least a power of two per functional evaluation. This procedure may be used to drive a complex optimization procedure (such as the Davidon-Fletcher-Powell) in the kind of multivariate area estimation problem encountered in remote sensing.

Key words: Optimization, convex functions

LOCKHEED

ENGINEERING AND MANAGEMENT SERVICES COMPANY, INC.

JSC-17061

Ref: 643-81-084
Contract NAS 9-15800
Job Order 73-306

TECHNICAL MEMORANDUM

AN ALGORITHM FOR THE RAPID LOCATION OF AN EXTREMUM OF A FUNCTION SUBJECT ONLY TO GEOMETRIC RESTRICTIONS

NASA CR-
160940

by

George R. Terrell

Approved By: T. C. Minter
T. C. Minter, Supervisor
Techniques Development Section

January 1981

LEMSCO-15991

1830 NASA ROAD 1, Houston, Texas 77058 • Tel (713) 333-5411

A Subsidiary of Lockheed Corporation

1. INTRODUCTION

Let $f(X)$ have a minimum on an interval $[X_0, X_2]$ and assume further that f is convex upward there and symmetric around its minimum. Then we know the following fact about the minimum: (Let $X_1 = \frac{X_0 + X_2}{2}$).

Theorem: Assume without loss of generality that $f(X_0) \leq f(X_2)$. Then f assumes its minimum at a point between

$$\frac{1}{2}(X_0 + X_1) + \frac{1}{2}\ (X_2 - X_1)\frac{f(X_0) - f(X_1)}{f(X_2) - f(X_1)}$$

and whichever of X_0 and X_1 that has smaller functional value $f(X)$.

2. PROOF OF THEOREM

Case I: $f(X_1) \leq f(X_0) \leq f(X_2)$.

Let X^* be such that $f(X^*) = f(X_0)$ and $X_1 \leq X^* \leq X_2$. It exists by symmetry about the minimum and convexity upward. Similarly, by upward convexity $\{X^*, f(X^*)\}$ is below a segment joining the points $\{X_1, f(X_1)\}$ and $\{X_2, f(X_2)\}$ in the graph of f, so

$$f(X_0) = f(X^*) \leq f(X_1) + \frac{(X^* - X_1)}{(X_2 - X_1)}[f(X_2) - f(X_1)]$$

So $X^* \geq X_1 + \dfrac{f(X_0) - f(X_1)}{f(X_2) - f(X_1)} \ (X_2 - X_1)$

By symmetry of f around its minimum

$$X_{min} = \frac{X_0 + X^*}{2}$$

So

$$X_{min} \geq \frac{1}{2} (X_0 + X_1) + \frac{1}{2}(X_2 - X_1) \frac{f(X_0) - f(X_1)}{f(X_2) - f(X_1)}$$

Now $X^* \leq X_2$ implies $X_{min} = \dfrac{X_0 + X^*}{2} \leq \dfrac{X_0 + X_2}{2} = X_1$

So $X_1 \geq X_{min}$ and we have case I.

Case II: $f(X_0) \leq f(X_1) \leq f(X_2)$

Again, let X^* be such that $f(X^*) = f(X_0)$ but $X_0 \neq X^*$. $X^* \leq X_1$ by convexity. Also by upward convexity, $\{X_1, F(X_1)\}$ is below the segment connecting $\{X^*, f(X^*)\}$ to $\{X_2, f(X_2)\}$, so

$$f(X_1) \leq f(X^*) + \frac{(X_1 - X^*)}{(X_2 - X^*)} \ (F(X_2) - f(X^*))$$

and this may be manipulated to

$$X^* \leq X_1 + (X_2 - X_1)\,\frac{f(X_0) - f(X_1)}{f(X_2) - f(X_1)}$$

Again by symmetry

$$X_{min} = \frac{X_0 + X^*}{2}, \text{ so}$$

$$X_{min} \leq \frac{1}{2}\left[X_0 + X_1\right] + \frac{1}{2}(X_2 - X_1)\,\frac{f(X_0) - f(X_1)}{f(X_2) - f(X_1)}$$

But $X_0 \leq X_{min}$ by assumption, so we have case II.

The case $f(X_0) \leq f(X_2) < f(X_1)$ violates convexity upward, so

Q.E.D.

<u>Corollary</u>: The new sub-interval containing the minimum of f is at most one fourth the length of $\left[X_0, X_2\right]$.

<u>Proof</u>: The computed boundary in the formula is clearly from its formula nearer the other boundary than is

$$\frac{X_0 + X_1}{2} = 3/4X_0 + 1/4X_2.$$ Q.E.D.

3. APPLICATION

These results become an algorithm for the minimization or maximization of a function meeting or nearly meeting the requirements of symmetry and convexity. This method involves replacing $[X_0, X_2]$ by the new interval at successive iterations. Convergence is at least by powers of one fourth at a cost of two functional evaluations per iteration.