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# Minimum Fuel Horizontal Flightpaths in the Terminal Area 

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# PART 1: TEE CASE OF RORSTMEULAR THREST 

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## I. Introduction

The literature on flightpath optimisation is extenaive. It can be classified according to pathe in the vertical plane, the horizontal plane, and the three-dimensional space; it can be further classified eccording to the type of aircraft and mission, and the performance index. We consider minimum-fuel, constant-altitude flightpaths of a transport airplane in the terminal area.

Most of the papers on flightpath optimization in the horizontal plane consider minimum time. ${ }^{1-7}$ In Refs. 1-3, the velocity is constant; in Refs. 4-7 the velocity is a state-variable as in our case but the assumptions, constraints, and numerical results correspond to a supersonic fighter ai:craft. Thus, although there is contact with our results, there is no overlap. References 8 and 9 consider minimum-fuel, horizontal rocket turns, but since the mass is variable, the results are quite different. An overview on flightpath optimization is given in Ref. 5.

Our objectives in investigating a minimum-fuel landing problem were to gain insight into the characteristics of minimum-fuel flightpaths by analysis and computation, and to use these results to improve the on-line, fuel-efficient capture algorithm of Ref. 10. Details of the refined algorithm (in the horizontal plane only) are reported in Ref. 11.

Following statements of the problem and of the necessary conditions in Secs. II and III, respectively, the optimality of straight-line and circular flightpath elements (which the algorithms of Refs. 10 and 11 use) is investigated in Sec. IV. It is shown that a straight-line segment can occur only at the beginning of a minimum-fuel flightpath;
this also revealed how atin pathe can be computed. Clrcular pathe are, fa senernl, aot fuel-optinel. Computation of reprecentative extromala is discuased in sec. V. The rasulting extremal flightpache can be grouped in three categories: centing extreand, deceleratiag mith zero thrust throughout; ahort-range turning extremale, where the initial and final positions, but not the headings, are relatively near, say, 1-3 n. mi.; and long-range extremals, characterized by a possible initial turn, followed by a long (say, 5-15 n. mi.) almost straight arc and ending with a possible final turn. Since the global optimality of some of these extremals was suspect, they were checked against near-optimal flightpaths produced by the algorithm of Ref. 11. This comparison, discussed in Sec. VI, established the existence and approximate location of Darboux points (beyond which the extremal ceases to be globally optimal; see Ref. 12).

## II. Problem Statement

The point-mass equations of motion in the horizontal plane are

$$
\begin{align*}
& \dot{x}=v \cos \psi  \tag{1}\\
& \dot{y}=v \sin \psi  \tag{2}\\
& \dot{\psi}=-g u / v  \tag{3}\\
& \dot{v}=(T-D) / m \tag{4}
\end{align*}
$$

Here, $x$ and $y$ are the coordinates of the horizontal plane, $\psi$ the heading angle measured counterclockwise from the $x$-axis, $v$ the velocity, $g$ the gravitational constant, and $m$ the mass; the control variables are the thrust $T$ and $u$, where $u=t a n ~$ and $\phi$ is the bank-angle, positive with right wing down. The drag $D$ is given by

$$
\begin{equation*}
D=k_{2} v^{2}+k_{2}\left(1+u^{2}\right) / v^{2} \tag{5}
\end{equation*}
$$

where $k_{1}$ and $k_{2}$ are constants (1.e., they are asoumed to be independent of $v$ for the low velocities in the terminal area). These equations were used in Refs. 4-7 and are derived by assuming zero wind velocity. constant mass, coordinated turns, and a small angle of attack which is automatically adjusted to maintain horizontal flight (see Appendix A). Constraints and final states are

$$
\begin{gather*}
|u(t)| \leq u_{\text {mu }}  \tag{6}\\
T_{\min } \leq T(t) \leq T_{\max }  \tag{7}\\
v(t) \leq v_{\max }  \tag{8}\\
x\left(t_{f}\right)=y\left(t_{f}\right)=0, \quad \psi\left(i_{f}\right)=k 2 \pi, \quad k=0, \pm 1, \ldots, v\left(t_{f}\right)=v_{f} \tag{9}
\end{gather*}
$$

The cost integral to be minimized is the fuel consumption

$$
\begin{equation*}
J=\int_{0}^{t_{f}}\left[c_{0}+c_{1} T(t)+c_{2} T^{2}(t)\right] d t, \quad c_{i}=\text { const }>0, \quad i=0,1,2 \tag{10}
\end{equation*}
$$

where $t_{f}$ is free. The terms $c_{0}$ and $c_{2} T^{2}$ in the fuel-flow-rate are often neglected in the literature. For the case considered here, $c_{0}$, the fuel-flow-rate at zero thrust, is not negligible. The term $c_{2} r^{2}$ is small but significant: when $c_{2}=0$, the optimal thrust is discontinuous and its intermediate values are singular. By changing the units of the $c_{i}$, the cost integral $J$ can be interpreted as a combination of the cost of time and fuel (the operating cost). The time-optimal problem, $c_{0}=1, c_{1}=c_{2}=0$, has been treated in Refs. 1-7 and is not considered here.

In sumary, the problem is to decerente the comtrols $u(t)$ and $Y(t)$ and the correaponding atate trajectory from an axblerary initial state at the time $t=0$ to the final state of Eq. (9) at a free tim $t-t_{f}$, subject to Eqs. (1)-(8), so at to minimise the fual consumtion [8q. (10)].

For our general analyeie, we make the following ewo aseumptions. which are easily satisfied for our (and indeed for most) numerical values.

Assumption 1. For the applicable range of velocities, the thrust that equals drag is intermediate. This implies that the two velocities for which $T_{\max }=\mathrm{D}$ are outside the applicable range. It also implies that $T_{\text {min }}$ is less than the minimal drag with respect to velocity, $D_{\text {min }}$. In the general case, $T_{\text {min }}$ can be negative, and we make Assumption 2.

Assumption 2. $T_{\text {min }}$ is such that the fuel flow rate at $T=T_{\text {min }}$ is positive and $T_{\text {min }}>-c_{2} /\left(2 c_{2}\right)$. The first part eliminates consideration of gliding flightpaths with shut-off engines; all such paths are fuel-optimal and trivially satisfy the necessary conditions embodied in the minimum principle. The significance of the second part is discussed in the next section.

Numerical values used in this study correspond to a $150,000-1 \mathrm{l}$ jet transport at sea level at velocities not below 150 knots.

$$
\begin{gathered}
v_{\max }=250 \text { knots }, \quad v_{f}=180 \text { knots } \\
T_{\max }=30,000 \mathrm{lb}, \quad T_{\min }=0 \\
\Phi_{m}=30^{\circ}, \tan \phi_{m}=u_{m}=0.577
\end{gathered}
$$

$$
k_{1}=0.08 \mathrm{lb} / \text { knot }^{2}, \quad k_{2}=2.127 \times 10^{\circ} \mathrm{ib} \text { knot }^{2}
$$

$c_{0}=0.808 \mathrm{lb} \mathrm{sec}^{-1}, \quad c_{1}=1.507 \times 10^{-4} \mathrm{sec}^{-1}, \quad c_{2}=5.4 \times 10^{-10} \mathrm{lb}^{-1} \mathrm{sec}^{-1}$ To get an idea on the percentage contributione of the teras $c_{0}, c_{1} T_{\text {, }}$ and $c_{2} T^{2}$ in Eq. (10) to the total fuel-flow-rate, assume a flight along a straig!t line at a constant velocity of 250 knots (at $T=D=8,403 \mathrm{lb}$ ). The contributions are then $38 \%$ for $c_{0}, 60 \%$ for $c_{1} T$, and $2 \%$ for $c_{2} T^{2}$. Assumption 1 is easily satisfied: we find that $T_{m a x}=D$ occurs at 83 knots and 607 knots, and that $D_{\text {min }}$ is $8,250 \mathrm{lb}$ at 227 knots. Evidently, Assumption 2 is also satisfied.

## III. The Necessary Conditions

We employ the minimum principle. The Hamiltonian is

$$
\begin{align*}
H= & \lambda_{0}\left(c_{0}+c_{1} T+c_{2} T^{2}\right)+\lambda_{x} v \cos \psi+\lambda_{y} v \sin \psi-\lambda_{\psi} g u / v \\
& +\lambda_{v}\left[T-k_{1} v^{2}-k_{2}\left(1+u^{2}\right) / v^{2}\right] / m+\eta\left(v-v_{\max }\right) \tag{11}
\end{align*}
$$

where $n \geq 0, n\left(v-v_{\max }\right) \equiv 0$ (Ref. 13). The costate variables are given by

$$
\begin{align*}
& \lambda_{0} \geq 0, \quad \lambda_{0}=\text { constant }  \tag{12}\\
& \dot{\lambda}_{x}=-H_{x}=0-\lambda_{x}=\text { constant }  \tag{13}\\
& \dot{\lambda}_{y}=-H_{y}=0 \Rightarrow \lambda_{y}=\text { constant }  \tag{14}\\
& \dot{\lambda}_{\psi}=-H_{\psi}=v\left(\lambda_{x} \sin \psi-\lambda_{y} \cos \psi\right)  \tag{15}\\
& \dot{\lambda}_{v}=-H_{v}=-\lambda_{x} \cos \psi-\lambda_{y} \sin \psi-\lambda_{\psi} g u / v^{2} \\
&  \tag{16}\\
&
\end{align*}
$$

where $u_{x}=2: / 8 x$, etc. since $t_{f}$ is not preseribed and $u_{t}-0_{\text {, we }}$ inve the mepasacy condition

$$
\begin{equation*}
\mathrm{H} \equiv 0, \quad \text { all } \in \in\left[0, t_{f}\right] \tag{17}
\end{equation*}
$$

If for some $t$

$$
\begin{equation*}
\lambda_{\psi}(t)=\lambda_{\psi}(t)=0-\lambda_{x}=\lambda_{y} \tag{18}
\end{equation*}
$$

then Eq. (17) and Assumption 2 require the vanishing of $\lambda_{0}$. Since having all costates zero is not optimal, an extremal where Eq. (18) occurs is not fuel optimal. Hencaforth, we consider only extremals with $\lambda_{0}>0$, and we normalize the costates by setting $\lambda_{0}=1$.

Minimization of $H$ with respect to $T$ yields the extremal thrust,

$$
T^{*}=\left\{\begin{array}{ccc}
T_{\max } & \text { if } & \tau \geq T_{\max }  \tag{19}\\
\tau & \text { if } & T_{\min }<\tau<T_{\max } \\
T_{\min } & \text { if } & \tau \leq T_{\min }
\end{array}\right.
$$

where

$$
\begin{equation*}
\tau=-\left(c_{1}+\lambda_{v} / n\right) /\left(2 c_{2}\right) \tag{20}
\end{equation*}
$$

We note that by Assumption 2. Eqs. (19) and (20) show at once that

$$
\begin{equation*}
T *=T_{\min } \text { if } \lambda_{v} \geq 0 \tag{21}
\end{equation*}
$$

Since the minimization of $H$ yields $T^{*}$ uniquely, it can be shown (see Ref. 13) that $T^{*}$ and $\lambda_{v}$ are continuous at junction times between the velocity-constrained and the unconstrained arcs. Thus, thrust is seen to be a continuous function of $\lambda_{v}$ and $t$. When $c_{2}$ is small, the range of $\lambda_{v}$ for intermediate thrust is narrow; when $c_{2}=0$, the intermediate thrust 1 :" singular. (In some cases, not considered in this paper, $c_{2}$ is negative; then, intermediate thrust is not fuel-optimal.)

We observe that Eqs. (3) and (15) for 1 and $\lambda_{\psi}$, respectively, and the fact that $\lambda_{x}$ and $\lambda_{y}$ are constant, imply

$$
\begin{equation*}
u \equiv 0 \text { if } \lambda_{\psi} \equiv 0 \text { on an interval } \tag{22}
\end{equation*}
$$

This is true irrespective of the minimization of $A$ with respect to $u$. From the latter, we obtain, by uaing $u_{u}=0$ and $H_{u u}>O_{\text {, }}$

$$
u^{*}=\left\{\begin{array}{cc}
\mu & \text { if }|\mu|<u_{m} \text { and } \lambda_{v}<0  \tag{23a}\\
u_{m} \operatorname{sgn} \mu=u_{m} \operatorname{sgn} \lambda_{\psi} & \text { if }|\mu| \geq u_{m} \text { and } \lambda_{v}<0
\end{array}\right.
$$

where

$$
\begin{equation*}
\mu=-g \operatorname{mv} \lambda_{\psi} /\left(2 k_{2} \lambda_{v}\right) \tag{23b}
\end{equation*}
$$

If $\lambda_{v} \geq 0$ and $\lambda_{\psi}$ does not vanish on an iscerval (denoted by $\lambda_{\psi} \neq 0$ ), the minimization of $H$ gives at once

$$
\begin{equation*}
u^{*}=-u_{m} \operatorname{sgn} \lambda_{\psi} \text { if } \lambda_{v} \geq 0 \text { and } \lambda_{\psi} \neq 0 \tag{23c}
\end{equation*}
$$

We note that $\lambda_{v}$ cannot be positive while $\lambda_{\psi}$ vaniahes on an interval because then minimization of $H$ implies $u^{*}= \pm u_{m}$ which is incompatible with Eq. (22). However, if $\lambda_{\psi}$ vanishes on an interval and $\lambda_{v}$ crosses zero from negative to positive values, say at $t=t_{2}$. then $u$ switches from $u(t)=0, t \leq t_{2}$ to $u\left(t_{2}^{+}\right)= \pm u_{m}$; this is a transition from a straight-line flightpath to a curved one.

We also note that the simultaneous vanishing on an interval of $\lambda_{\psi}$ and $\lambda_{v}$ is not fuel-optimal because it implies the vanishing of $i_{\psi}$ and $i_{v}$ which in turn implies $\lambda_{x}=\lambda_{y}=0$, that is, the nonoptimal case of Eq. (18).

In sumary, we see that $u$, and hence the bank-angle, are continuous in time when $\lambda_{v}$ is negative; $u$ is discontinuous when $\lambda_{v}$ is positive,
or, at the moment $\lambda_{0}$ crosene sero to bacome posctelve and $\lambda_{1}$ had bean sero on the previous interval. In view of Eq. (21). the disoontinulty of the bank-angle occure at mindinu thruat.

Lactiy, we evaluate $\eta_{\text {. Whan }}^{\nabla(t)}$ ㄹ veax, the thruat 10 intermediate under Aesumption 1. Than, by Eqe. (5) and (20), I = D gives

$$
-\left(c_{1}+\lambda_{v} / m\right) / 2 c_{2}-k_{2} v_{\max }^{2}+k_{2} / v_{\max }^{2}+k_{2} u^{2} / v_{\max }^{2}
$$

This equation gives an expression for $\lambda_{v}$ ead, upon differentiation, for $i_{v}$, which we substitute into Eq. (16) to obtain

$$
\begin{align*}
n= & -\lambda_{x} \cos \psi-\lambda_{y} \sin \psi-\lambda_{\psi} g u / v_{\max }^{2}+u_{m} c_{2} k_{2} u u^{2} v_{\max }^{2} \\
& -2\left\{c_{2}+2 c_{2}\left[k_{2} v_{\max }^{2}+k_{2}\left(1+u^{2}\right) / v_{\max }^{2}\right]\right\}\left[k_{2} v_{\max }-k_{2}\left(1+u^{2}\right) / v_{\max }^{3}\right] \tag{24}
\end{align*}
$$

For a velocity-constrained arc to be optimal, $n(t)$ given by Eq. (24) must be nonnegative.
IV. Optimality of Straight-Line and Circular Pata Elements

Since the suboptimal algorithms of Refs. 10 and 11 are based on piecing together circular arcs and at most one straight-line segment, we are interested in the optimality of these path elements.

We first show that there can be at most one straight-line segment In a fuel optimal path and, if it occurs, it must do 80 at the beginning of the path. A subsequent curved path, if any, starts by a suitch to . maximum bank angle $|\phi|=\phi_{m}$ at (continuous) minimum thrust $T=T_{m i n}$; the bank angle's mignitude and the thrust remain at $\phi_{m}$ and $T_{m i n}$. respectively, as long as $\lambda_{v}$ remains positive.

To prove thic proposition, we note that $111 g h t$ alone a atraight-line segment on a subinterval $\left\{t_{1}, t_{2}\right]$ ie characterised by

$$
\begin{equation*}
\psi(t)=\psi_{s}, \quad u(t)=0-\lambda_{\phi}(t), \quad \lambda_{v}(t)<0 \text { on }\left[t_{2}, t_{2}\right] \tag{25}
\end{equation*}
$$

We observe that for atraight-11me path, the point $\left(\psi=\psi_{s}, \lambda_{\psi}=0\right)$ is an equilibrium point" for Eqs. (3) and (15) for $\dot{i}$ and $i_{\phi}$, with $\because(t)$ as a continuous parameter. Hence the atraight-line segment can be entered at $t_{1}$ and exited at $t_{2}$ by an optimal control oniy if the control is discontinuous at $t_{1}$ and $t_{2}$. Therefore, as noted in the discussion following Eq. (24), it is necessary that $\lambda_{V}(t)$ cross zero at $t_{1}$ and $t_{2}$ according to

$$
\begin{equation*}
\lambda_{v}\left(t_{2}\right)=0, \quad i_{v}\left(t_{2}\right)<0 \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{v}\left(t_{2}\right)=0, \quad i_{v}\left(t_{2}\right)>0 \tag{27}
\end{equation*}
$$

(This causes $u(t)$ to switch from $u\left(t_{1}^{-}\right)= \pm u_{m}$ to $u(t)=0$ on $\left[t_{1}, t_{2}\right]$ and back tc $\left.u\left(t_{2}^{+}\right)= \pm u_{m}\right)$ Thus, $\lambda_{v}\left(t_{1}\right)=0,1=1,2 ;$ by Eq. (21) this implies $T\left(t_{i}\right)=T_{m i n}$. Since $i_{\psi}(t)$ is continuous, we have $i_{\psi}\left(t_{i}\right)=0$. Using $i_{\psi}\left(t_{i}\right)=0$ in Eq. (15), and $\lambda_{\psi}\left(t_{i}\right)=\lambda_{v}\left(t_{i}\right)=0$ and $T\left(t_{i}\right)=T_{\text {min }}$ in Eq. (17), gives two equations in $\lambda_{x}$ and $\lambda_{y}$ which yield
*An equilibrium point of differential equation $\dot{x}=f(x, t)$ is a point $x_{e}(t)$ that satisfies $0=f(x(t), t]$. If $f(x, t)$ is continuous in $x$ and $t$, then a solution $x(t)$ of the differential equation started ourside (at) the equilibrium point cannot reach it (depart from it) in finite time (see, e.g., Ref. 14).

$$
\begin{equation*}
\lambda_{x}=-\left(t_{\text {nata }} \cos \psi_{2}\right) / v\left(t_{1}\right), \quad 1=1.2 \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{y}=-\left(i_{\text {min }} \operatorname{ein} \omega_{y}\right) / v\left(\varepsilon_{2}\right) . \quad 1=1,2 \tag{29}
\end{equation*}
$$

where ${ }^{\frac{1}{a n}}$ in is fuel flow rate at gin

$$
\begin{equation*}
\xi_{\min }=c_{0}+c_{2} I_{\min }+c_{2} \operatorname{In}^{2} \tag{30}
\end{equation*}
$$

Uaing EqE. (28)-(30) in Eq. (16) gives

$$
\begin{equation*}
i_{v}\left(t_{1}\right)=f_{m i u} / v\left(t_{1}\right) . \quad 1=1,2 \tag{31}
\end{equation*}
$$

Since by Asoumption $2 \dot{f}_{\text {min }}$ is positive, Bq. (31) shows that $i_{v}\left(t_{1}\right)>0,1=1,2$. This contradicts Eq. (26) but confirme Eq. (27). Hence, a curved flightpath cannot precede, but can follow, a straightline segment. On the curved path, at least initially, $\lambda_{v}(t)>0, t>t_{2}$. Hence, by Eq. (23c), $|u(t)|=u_{m}$, and by Eq. (21), $T=T_{\text {min }}$. This completes the proof.

Next, we ask whether cruise at a constant velocity $v(t) \equiv \mathbf{v}_{\mathbf{c}}<\mathbf{v}_{\text {max }}$ on a straight-line segment is fuel optimal. Por the purpose of the inquiry, we temporarily lift the constraint $v(t) \leq V_{\text {max }}$. We find that a flight at constant velocity on a straight-line sagment of a flightpath can be fuel oprimal only if the entire flightpath is straight. flown at a constant optimal cruise velocity $v_{c} *$ given by the solution of the equation

$$
\begin{equation*}
3 c_{2} k_{1}^{2} v_{c}+c_{3} k_{1} v_{c}^{6}-\left(c_{0}+2 c_{2} k_{2} k_{2}\right) v_{c}-3 c_{2} k_{2} v_{c}^{2}-5 c_{2} k_{2}=0 ; \tag{32}
\end{equation*}
$$

Eq. (32) has one and only one real and positive solution.
Yroof. For flight on a straight path at a constant velocity $v_{c}$, the thrust is constant, $T=T_{c}=D\left(v_{c}\right)$, and, by Assumption 1, is intermediate; hence, $\lambda_{v}$ must be cunstant, $\lambda_{v}=\lambda_{v_{c}}$. according to Eq. (20). The point
$\left(\psi-\psi_{B}, v=v_{c}, \lambda_{申}=0, \lambda_{v}=\lambda_{v_{c}}\right.$ ) is an equilibrium point for the respentive set of differential equations [Bqe. (3), (9), (15), and (16)], whence follows the first part of the proposition. To derive Eq. (32), we eliminate $\lambda_{x}$ and $\lambda_{y}$ in Eq. (16) by using Eq. (17), and we subatitute the value of $\lambda_{v}=\lambda_{v_{c}}$ from Eq. (20). This yields

$$
\begin{align*}
i_{v} v_{c}= & \left\{c_{0}+c_{2}\left(k_{2} v_{c}^{2}+k_{2} / v_{c}^{2}\right)+c_{2}\left(k_{2} v_{c}^{2}+k_{2} / v_{c}^{2}\right)^{2}\right. \\
& \left.-2\left[2 c_{2}\left(k_{2} v_{c}^{2}+k_{2} / v_{c}^{2}\right)+c_{2}\right]\left(k_{2} v_{c}^{2}-k_{2} / v_{c}^{2}\right)\right\} \tag{33}
\end{align*}
$$

Since $\lambda_{v}$ is a constant, the right side of Eq. (33) vanishes, yielding Eq. (32). Let the right side of Eq. (33) be denoted by $f\left(v_{c}\right)$. Then

$$
\begin{equation*}
d f\left(v_{c}\right) / d v_{c}=-2\left(c_{2} k_{2} v_{c}+3 c_{2} k_{2} / v_{c}^{3}+6 c_{2} k_{3}^{2} v_{c}^{3}+10 c_{2} k_{2}^{2} / v_{c}^{8}\right) \tag{34}
\end{equation*}
$$

For sufficiently small $\mathbf{v}_{\mathrm{c}}, \mathrm{f}\left(\mathbf{v}_{\mathrm{c}}\right)>0$, and from Eq. (34). $\mathrm{df} / \mathrm{d} \mathbf{v}_{\mathrm{c}}<0$ for all positive $v_{c}$. Hence, $f\left(v_{c}\right)$ is monotonically decreasing and has one and only one real and positive zero $v_{c}=v_{c} *$. This completes the proof.

The significance of the velocity $v_{c} *$ is that when the entire flightpath is straight, $v_{c}$ * is an upper bound on $v(t), t c\left[0, t_{f}\right]$ in the sense

$$
\begin{equation*}
\text { if } v(0)<v_{c}^{*} \text { and } v_{f}<v_{c} * \text { then } v(t)<v_{c} * \tag{35}
\end{equation*}
$$

and a lower bound in the sense

$$
\begin{equation*}
\text { if } v(0)>v_{c} * \text { and } v_{f}>v_{c} * \text {, then } v(t)>v_{c} * \tag{36}
\end{equation*}
$$

Proof. Consider Eq. (i5), and suppose that the velecity rizes to a maximum $v\left(t^{0}\right) \geq v *$ there $\dot{v}\left(t^{0}\right)=0$, and $\ddot{v}\left(t^{0}\right)<0$ which we want to contradict. at $t=t^{\prime}$ we have

$$
\begin{equation*}
\ddot{v}\left(t^{0}\right)=\left[\dot{T}-(d D / d v) \dot{v}\left(t^{0}\right)\right] / a=\dot{T} / a=-2 c_{2} \dot{i} v\left(t^{0}\right) / m^{2} \tag{37}
\end{equation*}
$$

Since $\dot{v}\left(t^{\prime}\right)=0$, Eq. (33) applies with $v\left(t^{\prime}\right)$ replaciag $v_{c}$. For $v\left(t^{i}\right)=v_{c}{ }^{*}$, the left side of Eq. (33) vanishes. Since by Eq. (34) the right aide of Eq. (33) is decreasing, then, if $v\left(t^{\prime}\right) \geq v_{c}{ }^{*}$, we have $i_{v}\left(t^{\prime}\right) \leq 0$. It follows from Eq. (37) that $\ddot{\boldsymbol{v}}\left(\mathrm{t}^{\mathrm{l}}\right) \geq 0$, which is a contradiction. This proves Eq. (35); Eq. (36) is proved analogously. We note in passing that by linearizing the equations for $v$ and $\lambda_{v}$ around the equilibrium point ( $v_{c}{ }^{*}, \lambda_{v_{c}}$ ), this point can be shown to be a saddle point, which conforms with Eqs. (35) and (36).

The case of Eq. (36) is of little interest for landing because usually $\mathbf{v}_{\mathbf{f}} \leqslant \mathbf{v}_{\mathbf{c}} \mathbf{*}^{\text {. If }} \mathbf{v ( 0 )}$ and $\mathbf{v}_{\mathrm{f}}$ are on opposite sides of $\mathbf{v}_{\mathbf{c}}{ }^{*}$, then $v(t)$ can cross $v_{c} *$. Our numerical experience, described in the next section, shows that $v_{c}$ * acts as an upper bound on $v(t)$ also for relatively long-range flightpaths which are not strictly straight. Considering now the constraint $v(t) \leq v_{\text {max }}$ for the case of Rq. (35), it is clear that it will be inactive if $\mathbf{v a x}_{\max } \geq \mathbf{v}_{\mathrm{c}}$ * and is likely to be active if $\mathbf{v}_{\max }$ is much below $\mathbf{v}_{\mathbf{c}}$.

The optimal cruise velocity $v_{c} *$ given by Eq. (32) is the velocity that provides minimum fuel consumption per unit distance along a straight flightpath. One expects that $\mathbf{v}_{\mathrm{c}}{ }^{*}$ will be higher if $\mathrm{c}_{2}=0$ in the fuel consumption model of Eq. (10), and it will be lower if $c_{0}=0$. For our numerical values

$$
v_{c}^{*}= \begin{cases}349.5 \text { knots } & \text { if } c_{0} \neq 0, c_{2} \neq 0 \\ 359.0 \text { knots } & \text { if } c_{0} \neq 0, c_{2}=0 \\ 298.8 \text { knots } & \text { if } c_{0}=0, c_{2}=0\end{cases}
$$

For comparison, the minimum-drag velocity, which is the minimum fuel per unit time velocity, is

$$
v_{D_{\min }}=\left(k_{2} / k_{1}\right)^{1 / 4}=227 \text { knots }
$$

Examination of the naceseary conditions shows that a circular flightpath can be fuel optimal only if both the bank-angie and the velocity are at their reapective conatraint bounds. The proof is straightforward and is omitted.

## V. Computation of Extremals

The extremals are computed by numerical integration of the state and costate equations [Bqs. (1)-(5) and (12)-(16)], with the controls given by Eqs. (19) and (23), from a given state and with chosen initial values of the costate variables as parameters. One can thus obtain families of extremals but one cannot meet specified end-conditions.

In the computations, the state constraint $v \leq v_{\text {max }}=250$ knuts was imposed only on some of the long-range flightpaths; however, no extremal was extended beyond 350 knots. A state constraint $\mathbf{v} \geq \mathbf{v}_{\text {min }}$ was not explicitly incorporated in the necessary conditions; however, extremals where $v(t)$ dropped below 150 knots were rejected because the drag model of Eq. (5) would not be valid below that velocity.

In the following figures, distances are in nautical miles, the velocities $v$ are in knots, the thrust $T$ is in thousands of pounds, the fuel consumption $f$ to the final point is in pounds, and the time in seconds is from the integration's starting time $t_{s}=0$. The portions of the flightpath with $T>0$ are in bold curves. The maximal and minimal values of thrust and velocity are among the $T$ ' $s$ and $v$ 's shown. The arrows show the airplane's direction of flight. The starting values of the costates at $t_{s}=0$ are shown in the figures by

$$
\lambda_{y}^{\prime}=\lambda_{y} / c_{2}, \quad \lambda_{i}^{\prime}=\lambda_{\psi} / c_{2}, \quad \lambda_{y}^{\prime}=\lambda_{v} / c_{2} m
$$

(in our cempatations we divided the limeitondan by $c_{3}$ ).
It is convenient to atart integration beckwarde froe the known and fixed final state of Eq. (9). Computing $\boldsymbol{\lambda}_{\mathrm{x}}$ from the condition $\mathrm{E}=0$ at the integration's starting time $t_{8}$, the extremals are determined by the parameters $\lambda_{y}, \lambda_{\psi}\left(t_{g}\right)$, and $\lambda_{V}\left(t_{g}\right)$. Since the problem is timeinvariant, we set $t_{s}=0$. The sign and relative magnitude of both $\lambda_{y}$ and $\lambda_{\psi}\left(t_{f}\right)$ determine the direction of turn; it can be shown that by changing the sign of both, the flightpath is reflected about the $x$-axis (true if integration is started at $y=\psi=0$ ). Such backward integration produced coasting and short-range turning extremals, as those shown in Figs. 1 and 2.

Figure 1 shows coasting flightpaths, namely, deceierating paths with zero thrust throughout. For extremals 1 and 2 the bank angle switches between its bounds of $\pm 30^{\circ}$ because $\lambda_{v}(t)$ is positive throughout. This is typical of most, but not all, coasting extremals: extremal 3 ends with a smooth transition to a shallow bank angle, and extremal 4 starts with a straight-line segment. Coasting extremals may be significant for emergency landing.

Figure 2 shows (by solid curves) partly thrusting extremal paths whose backward computation was arbitrarily terminated at 200 knots. These paths represent short-range turning approaches starting at 200 knots, such as after an aborted landing. Typically, for a turn through a large angle, as in paths 1 and 2, the velocity first drops to achieve a tighter turn. This was noted in Ref. 6 for mirimum-time turns, but it is less intuitively obvious in the minimum-fuel case because fuel is later expended to


Fig. 1 Coasting (zero-thrust) flightpaths.


Fig. 2 Short-range turning flightpaths.
accelerate. The broken curves, here and in Fig. 4, will be commented on In the next section.

Backward integration from the final state did not produce longrange flightpaths. Such paths require a sustained intermediate thrust which is dictated by a narrow range of $\lambda_{v}$ [from Eq. (20) it follows at once that $T \geq T_{\min }=0$ if $\lambda_{v} / c_{2} m \geq-1.0$, and by computation, $T \leq T_{\max }$ if $\left.\lambda_{v} / c_{2} m \leq-1.215\right]$, resulting in extreme sensitivity to the choice of $\lambda_{v}\left(t_{s}\right)$. Therefore, long-range extremals, and other types of extremals with special conditions at intermediate points, were produced by forward and backward integration from an appropriate intermediate state. For convenience, integration was started at $x\left(t_{g}\right)=y\left(t_{g}\right)=\psi\left(t_{g}\right)=0$, $t_{s}=0$, with an appropriate $v\left(t_{s}\right)$. The resulting flightpath can then be shifted and rotated to satisfy the final condition of Bq . (9).

Integration of long-range extremals without the velocity constraint
[Eq. (8)] was started at a velocity below the optimal cruise velocity $v_{c} *=349.5$, with $\lambda_{v}$ such that $T=D$. The parameters are then $v\left(t_{s}\right)$, $\lambda_{y}$ and $\lambda_{\psi}\left(t_{s}\right), t_{s}=0$. The velocity profiles of such paths are shown in Fig. 3. Since $\dot{v}\left(t_{s}\right)=0, v\left(t_{s}\right)$ is the maximal velocity. Curves 1-3 correspond to straight-line flightpaths. Curves 4 and 5 correspond to paths with initial and final turns as indicated. For $t<0$, curves 3, 4, and 5 are indistinguishable. On curve 4, we note the dip in velocity at the large final turn to $v\left(t_{f}\right)=250$ knots. On curve 5 the velocity decreases faster than on curve 3 because of the added drag due to maximum bank-angle.

Extremals with a velocity-constrained arc were computed by starting on the arc at $v\left(t_{s}\right)=v_{\max }=250$ knots. The costate $\lambda_{v}\left(t_{s}\right)$ is


Fig. 3 Velocity profiles for long-range flightpaths without a velocity constraint.
determined by the requirement that thrust equal drag. The remaining parameters are $\lambda_{y}, \lambda_{\psi}\left(t_{8}\right)$, and two parameters to control the departures from the velocity-constrained arc according to preset times or according to the multiplier $\eta(t) \geq 0$ computed by Eq. (24). We considered extremals with only one constrained arc, of the long-range type.

Figure 4 shows long-range flightpaths, which typically consist of a possible initial turn, a long, almost straight arc, and a possible final turn. Path 1 is a velocity-constrained extremal, path 2 is unconstrained, and both were computed from the point indicated by $t=0$. For convenience of presentation, neither path is particularly long. In both cases the almost straight arc can be made as long as desired: in the constrained case by selecting $\lambda_{y}$ and $\lambda_{\psi}(0)$ sufficiently small, and in the unconstrained case by selecting, in addition, $v(0)$ sufficiently close to the optimal cruise velocity $\mathbf{v}_{c}$ *.

Integration from an intermediate state, rather than frcm an endstate, is mandatory also for the type of extremal discussed in Sec. IV: a straight-line segment followed by a switch to a maximal bank-angle turn. This extremal requires the satisfaction of Eq. (31) at the switch-time $\mathbf{t}_{\mathbf{2}}$. Since $\psi(t), u(t), \lambda_{\psi}(t)$, and $\lambda_{y}$ are zero on the straight-line segment, the remaining parameters are $v\left(t_{s}\right)$ and $\lambda_{v}\left(t_{s}\right)<0$. For our numerical values and range of velocity, these extremals (such as flightpath 4 in Fig. 1 whose integration started at the point indicated by $v=300$ knots) are all coasting extremals with maximal bank-angle throughout the turn.

The accuracy of the computed extremals was checked by decreasing the integration step and by observing the accuracy of the condition $\mathbf{H}=0$.


Fig. 4 Long-range flightpaths.

## VI. Optimal and Mear-Optimal Solutione

We compare now the fuel consumption of flightpathe produced by extremals with that of near-optimal pathe. We thus test the quality of the near-optimal paths as well as the global optimality of the extremals. The near-optimal algorithm of Ref. 10 generates a flyable statetrajectory between any two end states; it is sufficiently fast to be operated on-board and in real time. The algorithm, based on the results in Ref. 1 , generates the shortest flyable path consisting of circular paths joined by at most one straight-line segment. In Ref. 11, the algorithm is further refined in the light of the results of this study; in particular, the curved path is created by a succession of $30^{\circ}$ circular arcs of varying radii. In view of the results of Sec. III, it is clear that such a synthesized flightpath cannot be fuel-optimal. However, it is evident from Fig. 4 that such paths can closely approximate the optimal ones for the most common and important type of path, the longrange path. The details of the algorithm as well as numerous comparisons are reported in Ref. 11. For example, for 28 long-range paths of about 20 miles, without a velocity constraint, the worst approximation was 2.95\% off the minimum fuel consumption, the best $0.5 \%$, and the average 1.52\% (Ref. 11).

The extremals exhibited in Sec. V satisfy only necessary conditions for optimality. Are they optimal? We have no proof, but fuel-optimality can be argued for at least the coasting extremals, which are also minimumtime coasting extremals. Consider flightpaths 1 and 2 in Fig. 1. Along these the magnitude of the bank-angle, and hence the drag, are at all
times maxiral. Therefore, any other coanting filghtpath with $x\left(c_{f}^{f}\right)=y\left(t_{f}^{j}\right)=\phi\left(t_{f}\right)=0_{f} t_{f}<\varepsilon_{f}$, must have $v\left(t_{f}\right)>v_{f}=180$ knots. The fuel-optimality of pathe 3 and 4 can be aupported by aimilar, albeit somewhat weaker argumente.

The situation is far less clear for the thruating extremale. These, as the integration continued, often started twisting and looping in a manner which appears increasingly nonoptimal, as shown by the broken curves of Figs. 2 and 4. Such behavior of extrema_ 3 is likely to have been observed priviously; there is an allusion to loss of global optimality in Ref. 7. In general, as an extremal is extended by integration from some starting point, a time $t_{D}$ may be reached beyond which the extremal ceases to be globally optimal; $t_{D}$ is called a Darboux point with respect to the starting point (Ref. 12).

The existence of Darboux points on the extremals of Figs. 2 and 4 is demonstrated in Fig. 5 by comparison of fuel consumption with nearoptimal paths. Figure 5a shows path 1 of Fig. 4 which, as the forward integration is extended, has in the final turn three points with $v=180$ knots. The comparison confirms tice nonoptimality of end-points 2 and 3; the Darhoux point appears to be about midway between points 1 and 2 . We note that also path 2 of Fig. 4 has two possible end-points with $v=180$ knots. In this case, however, the extremal path to the second end-point is deemed to be optimal. Figure 5b shows the fuel consumption to the final point $x=y=0$ of a turning extremal path and of nearoptimal paths, for several points along the extremal; three such nearoptimal paths are shown. Evidently, for points beyond $v=320$ knots or so, the near-optimal paths use less fuel. Although we do not have the

(a) long-range pathe

Fig. 5 Comparison of extremal and near-optimal fiightpaths.


Fig. 5 Concluded.
exact location if the Darboux point, we are confident that if the extremal 1s ternduated at, exy, 250 knots, it is optimal.

Thus, although one cannot use the near-optimal paths to prove optimality, one can gat a rough idea of the location of the Darboux point, particularly for the long-range pathe, where the near-optimal approximation is very good. Of course, portions of nonoptimal extremals may be optimal; for example, the coasting portions of the extremale in Fig. 2 are likely to be optimal.

## VII. Sumary and Concluding Remarke

The characteristics of minimum fuel horizontal flightpathe in the terminal area were investigated analytically and computationally. Anslysis of the necessary conditions showed the following.

1) Thrust is continuous, but the bank-angle may be switching for certain values of the costates $\lambda_{v}$ and $\lambda_{\psi}$ [see Eq. (23c)].
2) A straight-line sagment may be fuel-optimal only if it is at the beginning of the flightpath; a subsequent curved path, if any, must start by a switch to maximum bank angle while the thrust is at minimum value.
3) The optimal cruise velocity, given by the solution of Eq. (32). acts as an upper bound on the velocity for straight (or almost straigist) minimum-fuel ilightpaths.
4) A circular filghtpath may be fuel optimel only if both the bank angle and the velocity are at their respective constraiat bounds.

The computation of extremals produced many representative minimumfuel flightpaths that can be categorized as long-range paths, short-range turning pathe, and coasting (zero-thrust) pathe. We found that:

1) Extreme sensitivity to choice of $\lambda_{v}$ for long-range paths could be overcome by starting integration at an appropriate intermediate state, rather than an end-state.
2) Long-range paths with a large initial turn (over $100^{\circ}$ ) start by deceleration followed by an acceleration in the remainder of the turn.
3) Long-range paths with a final turn up to $140^{\circ}$ end with zero thrust and turn with maximum bank-angle magnitude. However, if the final turn is large, and in particular if in addition the final velocity $\boldsymbol{v}\left(\mathrm{t}_{\mathrm{f}}\right)$ is higher than 180 knots, the turn is executed by decelerating below $v\left(t_{f}\right)$ and a final acceleration at maximum bank-angle magnitude. It is shown in Ref. 11 how these findings made possible the refinement of an existing on-line algorithm of Ref. 10 to the point where the fuel consumption of long-range near-optimal paths is well within 1-3\% of that of optimal paths.

The near-optimal algorithm was very helpful also in alleviating the problem of finding the Darboux points. We found that:

1) Turning extremals that require thrust toward the end of the path, produced by backward integration from $v\left(t_{f}\right)=180$ knots, inevitably became nonoptimal at some point beyond $\mathbf{v}=250$ knots.
2) Turning but coasting extremals, on the other hand, appear to be optimal no matter how long they were extended (backwards from $v\left(t_{f}\right)=180$ knots).
3) Optimality of long-range extremals may be lost if integration is extended so that the initial or final turns are much larger than $180^{\circ}$.

Evaluation of the optimality of extremals by a near-optimal algorithm proved to be a practical solution to the Darboux point question for flightpath problems. The general problem, however, is unresolved, for
there is no test for Darboux points. Could a test (e.g., Ref. 15) for conjugate points (beyond which an extremal ceases to be locally optimal) be helpful? The answer appears to be negative; certainly so for a special two-dimensional case of the present problem, in which $x\left(t_{f}\right)$ is free and $v(t)=$ constant, which we examined in detail in Ref. 16. The question of global optimality, highlighted here by computation of extremals, is inherent (though perhaps less visible) in other optimization techniques. The problem of Darboux points remains a challenge for future research.

The work reported here is of course but one element in the development of a practical, fuel-efficient, and safe system for terminal operation. In particular, an extension to include the third dimension, altitude, is to be studied.

## Appendix A: Equations of Motion

The lateral, longitudinal, and vertical force equations are, respectively,

$$
\begin{align*}
\operatorname{mv\dot {\psi }} & =-(L+T \sin \alpha) \sin \phi  \tag{Al}\\
m \dot{v} & =T \cos \alpha-D  \tag{A2}\\
m g & =(L+T \sin \alpha) \cos \phi \tag{A3}
\end{align*}
$$

For small angle of attack $\alpha$, Eqs. (3) and (4) result, where $u=t a n \phi$.
Lift and drag are given by $L=C_{L_{\alpha}} \alpha q S, D=C_{D_{0}} q S+\varepsilon L \alpha$, where $\varepsilon$ is the efficiency factor and $q=(1 / 2) \rho v^{2}$ is the dynamic pressure, $S$ the wing area, and the coefficients $C_{L_{\alpha}}$ and $C_{D_{0}}$ are assumed to be independent of the velocity. Now,

$$
\varepsilon L \alpha=\varepsilon L^{2} /\left(C_{L_{\alpha}} q S\right)=\varepsilon[m g / \cos \phi-T \sin \alpha]^{2} /\left(C_{L_{\alpha}} q S\right)
$$

Neglecting $T$ sin $a, D$ is of the form of $E q$. (5), with $k_{1}=(1 / 2) C_{D_{0}} p s$, $k_{2}=2 \mathrm{Em}^{2} \mathrm{~g}^{2} /\left(C_{L_{\alpha}} \rho S\right)$. Using the values $C_{D_{0}}=0.015, C_{L_{\alpha}}=0.08 / \mathrm{deg}$. $\varepsilon=0.004 / \mathrm{deg},(1 / 2) \rho=295^{-1} \mathrm{lb} / \mathrm{ft}^{2} / \mathrm{knnt}^{2}, \mathrm{~S}=1560 \mathrm{ft}^{2}, \mathrm{mg}=150,000 \mathrm{ib}$, gives the values of $k_{1}$ and $k_{2}$ in Sec. II.

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## Eliezer Kreindler

## Introduction

In Ref. 1, the fuel flow rate model includes a small term quadratic in thrust. When this term is neglected, as is usual, the intermediate thrust is singular. In this note the singular thrust is derived, and the Generalized Legendre-Clebsch condition is examined for various ranges of bank-angle and velocity. The results are summarized in the last section. The literature on flightpath optimization in the horizontal plane is reviewed in Ref. 1.

## Problem Formulation

The point-mass equations of motion in the horizontal plane are

$$
\begin{align*}
& \dot{x}=v \cos \psi  \tag{la}\\
& \dot{y}=v \sin \psi  \tag{lb}\\
& \dot{\psi}=-g u / v  \tag{1c}\\
& \dot{v}=(T-D) / m \tag{1d}
\end{align*}
$$

Here $x$ and $y$ are the coordinates of the horizontal plane, $\psi$ the heading angle measured counterclockwise from the $x$-axis, $v$ the velocity, $g$ the gravitational constant, and $m$ the mass. The control variables are the thrust $T$ and $u$, where $u=\tan \phi$ and $\phi$ is the bank-angle, positive with right wing down. The drag $D$ is given by

$$
\begin{equation*}
D=k_{1} v^{2}+k_{2}\left(1+u^{2}\right) / v^{2} \tag{2}
\end{equation*}
$$

where $k_{1}$ and $k_{2}$ are constants. These equations are derived by assuming zero wind velocity, constant mass, coordinated turns, and a small angle of attack which is automatically adjusted to maintain horizontal flight (Ref. 1).

The controls are constrained by

$$
\begin{gather*}
|u(t)| \leq u_{m}  \tag{3}\\
T_{m i n} \leq T(t) \leq T_{\max } \tag{4}
\end{gather*}
$$

and the fuel consumption to be minimized is given by

$$
\begin{equation*}
J=\int_{0}^{t} f\left[c_{0}+c_{1} T(t)\right] d t \tag{5}
\end{equation*}
$$

where $c_{0}$ and $c_{1}$ are constants and $t_{f}$ is free. In Ref. 1 , the fuel flow rate model includes a quadratic term $c_{2} T^{2}(t)$ with a small $c_{2}$. Application of the Minimum Principle

The Hamiltonian is given by
$H=c_{0}+c_{1} T+\lambda_{x} v \cos \psi+\lambda_{y} v \sin \psi-\lambda_{\psi} g u / v+\lambda_{v}\left(T-k_{1} v^{2}-k_{2}\left(1+u^{2}\right) / v^{2}\right) / m$

The costate variables are given by

$$
\begin{align*}
\dot{\lambda}_{x}=-H_{x} & =0 \Rightarrow \lambda_{x}=\text { constant }  \tag{7a}\\
i_{y}=-H_{y} & =0 \Rightarrow \lambda_{y}=\text { constant }  \tag{7b}\\
i_{\psi}=-H_{\psi} & =v\left(\lambda_{x} \sin \psi-\lambda_{y} \cos \psi\right)  \tag{7c}\\
\dot{\lambda}_{v}=-H_{v}= & -\lambda_{x} \cos \psi-\lambda_{y} \sin \psi-\lambda_{\psi} g u / v \\
& +2 \lambda_{v}\left[k_{z} v-k_{2}\left(1+u^{2}\right) / v^{3}\right] / m \tag{7d}
\end{align*}
$$

where $H_{x}=\partial H / \partial x$, etc. Since $t_{f}$ is not prescribed and $H_{t}=0$, we have the necessary condition

$$
\begin{equation*}
u=0, \quad \text { all } t \in\left[0, t_{f}\right] \tag{8}
\end{equation*}
$$

The Hamiltonian is linear in $T$, with the term multiplying $T$ given by

$$
\begin{equation*}
H_{T}=c_{2}+\lambda_{v} / m \tag{9}
\end{equation*}
$$

Minimization of $H$ with respect to $T$ gives

$$
T^{*}= \begin{cases}T_{\max } & \text { if } H_{T}<0 \\ T_{\min } & \text { if } H_{T}>0 ;\end{cases}
$$

the thrust can be intermediate only in the singular case

$$
\begin{equation*}
H_{T}=0 \quad \text { on a subinterval of }\left[0, t_{f}\right] \tag{10}
\end{equation*}
$$

In this note, we are concerned only with the case (10); therefore,

$$
\lambda_{v}=-c_{1} m
$$

Since in this note $\lambda_{v}$ is negative, minimization of $H$ with respect to $u$ yields

$$
u^{*}= \begin{cases}g v \lambda_{\psi} /\left(2 c_{2} k_{2}\right) & \text { if }\left|g v \lambda_{\psi} /\left(2 c_{1} k_{2}\right)\right|<u_{m}  \tag{1la}\\ u_{m} \operatorname{sgn} \lambda_{\psi} & \text { if }\left|g v \lambda_{\psi} /\left(2 c_{1} k_{2}\right)\right| \geq u_{m}\end{cases}
$$

The singular case (10) implies the vanishing of all time derivatives of $H_{T}$. Let the first time derivative of $H_{T}$ in which $T$ appears explicitly be the $2 q$ th (it is always even); $q$ is the order of the singular arc. The Generalized Legendre-Clebsch necessary condition requires (Ref. 2) that

$$
\begin{equation*}
(-1)^{q}\left[H_{T}^{(2 q)}\right]_{T} \geq 0 \tag{12}
\end{equation*}
$$

Consider first the case $|u|<u_{m}$. Using Eqs. (16) and (18a) in $\dot{\mathbf{H}}_{\mathrm{T}}=0=\dot{\lambda}_{\mathrm{V}}$, we compute $\ddot{\mathrm{H}}_{\mathrm{T}}$ to be

$$
\ddot{u}_{T}=-g\left(\lambda_{x} \sin \psi-\lambda_{y} \cos \psi\right) u / m v-2 c_{1}\left(k_{2} v^{4}+3 r_{2}\right)(T-D) / m^{2} v^{4}
$$

First, we note that

$$
\left(\ddot{H}_{T}\right)_{T}=-2 c_{1}\left(k_{1} v^{4}+3 k_{2}\right) / m^{2} v^{4}<0
$$

so that the Generalized Legendre-Clebsch necessary condition of Eq. (12) is satisfied. Second, setting $\ddot{H}_{T}=0$ yields the intermediate, singular thrust:

$$
\begin{equation*}
T=D-\frac{\operatorname{mgv}\left(\lambda_{x} \sin \psi-\lambda_{y} \cos \psi\right) u}{2 c_{1}\left(k_{1} v^{2}+3 k_{2} / v^{2}\right)} \tag{13}
\end{equation*}
$$

Adding the condition $H=0$ to $H_{T}=\dot{H}_{T}=0$, we obtain

$$
\lambda_{\psi}= \pm 2 c_{1} \sqrt{k_{2}\left(3 k_{2}-k_{2} v^{4}+c_{0} v^{2} / c_{2}\right)} / g v
$$

and hence, using Eq. (11a),

$$
\begin{equation*}
u= \pm \sqrt{\left(3 k_{2}-k_{1} v^{4}+c_{0} v^{2} / c_{2}\right) / k_{2}} \tag{14}
\end{equation*}
$$

Using Eq. (14), we find that the velocities for which $|u|<u_{m}$ are outside the range $\left[v_{1}, v_{2}\right]$, where $v_{1}$ and $v_{2}$ are the real and positive roots of

$$
\begin{equation*}
k_{1} v^{4}-c_{0} v^{2} / c_{1}+k_{2}\left(u_{m}^{2}-3\right)=0 \tag{15}
\end{equation*}
$$

(Of course, real and positive $v_{1}$ or $v_{2}$, or both $v_{1}$ and $v_{2}$, may not exist; in particular, if $u_{m}{ }^{2}>3$ and $\left(c_{0} / c_{2}\right)^{2}<4 k_{2} k_{2}\left(u_{m}{ }^{2}-3\right)$, then $|u|<u_{m}$ for all v.)

We observe that for a straight-line flight, satting $u=0$ in Eq. (13) gives $T=D$, which implies $v(t)=v_{c}=$ constant. Setting $u=0$ in Eq. (14), we obtain the optimal cruise velocity $v_{c}$ *

$$
\begin{equation*}
\left(v_{c}^{*}\right)^{2}=c_{0} /\left(2 k_{1} c_{1}\right)+\left[\left(c_{0}^{2} /\left(2 k_{2} c_{1}\right)^{2}+3 k_{2} / k_{2}\right]^{1 / 2}\right. \tag{16}
\end{equation*}
$$

We now consider the case $|u|=u_{\text {n }}$ which occurs when the velocity enters the range $\left[v_{1}, v_{2}\right]$. Proceeding as in the previous case for $|u|<u_{m}$, we obtain

$$
\begin{align*}
\left(H_{T}\right)_{T} & =-3\left(c_{1} D-c_{0}\right) /(m v)^{2},  \tag{17}\\
T & =D-\frac{2 m g v\left(\lambda_{x} \sin \psi-\lambda_{y} \cos \psi\right) u_{m} \operatorname{sgn} \lambda_{\psi}}{3\left(c_{2} D-c_{0}\right)} . \tag{18}
\end{align*}
$$

and

$$
\left|\lambda_{\psi}\right|=c_{1}\left[3 k_{2}\left(1+u_{m}^{2}\right)-k_{1} v^{4}+c_{0} v^{2} / c_{1}\right] /\left(2 g v u_{m}\right)
$$

We observe from Eq. (17) that the Generalized Legendre-Clebsch condition of Eq. (12) is satisfied as long as

$$
D=k_{1} v^{2}+k_{2}\left(1+u_{n}{ }^{2}\right) / v^{2}>c_{0} / c_{1}
$$

or

$$
\begin{equation*}
k_{1} v^{4}-c_{0} v^{2} / c_{2}+k_{1}\left(1+u_{m}^{2}\right)>0 \tag{19}
\end{equation*}
$$

The left side of Eq. (19) has no real zeros, and hence Eq. (19) holds if

$$
\begin{equation*}
c_{0}^{2} /\left(2 c_{1}\right)^{2}<k_{1} k_{2}\left(1+u_{m}^{2}\right) \tag{20}
\end{equation*}
$$

If Eq. (20) is violated (as when $c_{0}$ is increased to create a combined time and fuel cost functional), then there exists a range of velocties [ $v_{1}^{\prime}, v_{2}^{\prime}$ ], where $v_{1}^{\prime}$ and $v_{2}^{\prime}$ are the real and positive zeros of the left side of Eq. (19), for which the singular thrust of Eq. (18) is nonoptimal.

We observe that $\left[v_{1}^{\prime}, v_{2}^{\prime}\right] \subset\left[v_{1}, v_{2}\right]$, and therefore the Generalized Legendre-Clebsch condition, is satisfied whenever $v$ is such that $|u| \leqslant u_{m}$.

We note that letting $u=u_{m}, T$ in Eq. (13) is different from that In Eq. (18) (the author is indebted to Dr. H. Erzberger for this observation). This shows that the intermediate thrust is discontinuous at a time $t_{1}$, say, when the bank angle saturates. This is because $\ddot{H}_{T}$ contains $\dot{u}$ which is discontinuous at $t_{1}$ and so must be $T\left(t_{1}\right)$ in order to satisfy $\mathrm{H}_{\mathrm{T}} \equiv 0$.

Sumary of Results

For intermediate bank-angle, the intermediate singular thrust is given by Eq. (13) and it satisfies the Generalized Legendre-Clebsch condition. The bank-angle is given (except for sign) by Eq. (14) and the velocity is outside the range $\left[v_{2}, v_{2}\right]$, with $v_{1}$ and $v_{2}$ being the real and positive roots of Eq. (15). On a straight-line flight, the singular thrust is constant and the constant velocity is the optimal cruise velocity $v_{c}$ * given by Eq. (16).

When the velocity enters the range $\left[v_{1}, v_{2}\right]$ the bank-angle saturates, $|u|=u_{m}$, and the intermediate singular thrust, now given by Eq. (18), undergoes a jump. The singular thrust satisfies the Generalized Legendre-Clebsch condition if Eq. (20) holds; if Eq. (20) does not hold, there exists a range of velocities $\left[v_{1}^{\prime}, v_{2}^{\prime}\right] \subset\left[v_{1}, v_{2}\right]$, where $v_{1}^{\prime}$ and $v_{2}^{\prime}$ are the zeros of the left side of Eq. (19), such that intermediate thrust is nonoptimal.

## Referencee

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