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Note On Reflection and Transmission Coefficients for Converging-Diverging Ducts

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NOTE ON REFLECTION AND TRANSMISSION COEFFICIENTS FOR CONVERGING-DIVERGING DUCTS

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ABSTRACT

Simple formulas for calculating reflection and transmission coefficients for converging-diverging ducts are derived; they extend the method of Cho and Ingard to arbitrary, slowly varying ducts. These formulas involve two parameters: The first is a function of duct shape and the second is the ratio of the duct radius downstream of the throat to that upstream of the throat. An extension of the method to include mean flow is made for symmetric ducts.

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INTRODUCTION

There are three possible behaviors of acoustic waves propagating in converging-diverging ducts:

(1) They may propagate freely through the duct.

(2) They may propagate in the section upstream of the contraction and be cut off downstream, hence suffering total reflection at the contraction.

(3) They may propagate upstream and downstream of the contraction, but be cut off within the contraction, hence suffering partial reflection while being partially transmitted. It is the third case that is of concern in this report.

This case has been treated by Cho and Ingard (ref. 1) for waves in a circular-cosh duct. The present report extends their evaluations of reflection and transmission coefficients to nonuniform ducts that are not of circular-cosh shape. Our approach is to use the propagation equation of the circular-cosh duct as a comparison equation with one free parameter. An "eigenvalue" relation is derived for that parameter, and it, in turn, determines the reflection and transmission coefficients.

Our ultimate objective is to provide formulas for computing reflection and transmission coefficients for an arbitrary (slowly varying) duct. Such formulas are provided by equations (6) and (8). These equations require the duct shape as input, and after simple approximate or numerical integrations they yield the required coefficients. In an appendix the present method is extended to include a mean flow through the duct, although this can be done only if the duct is symmetric. The mean flow increases transmission.

Discussions with Dr. Y.-C. Cho of Lewis have been of great help in writing this report.

SYMBOLS

A	amplitude function, defined by eq. (3)
b	duct radius
B	defined by eq. (2a)
c	sound speed

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k	wave number
ℓ	defined below eq. (13)
M	Mach number
R	reflection coefficient
s	axial coordinate of circular-cosh comparison duct
T	transmission coefficient
U	mean flow
x	axial coordinate of duct
α	duct-mode eigenvalue
β	free parameter determining throat radius of circular-cosh duct
γ	defined by eq. (13)
δ	phase shift, defined by eq. (10)
ε	maximum wall slope
κ,λ	reduced wave numbers, defined by eq. (5a)
θ	defined below eq. (11)
τ	downstream radius of duct
ϕ	acoustic wave function for duct
χ	acoustic wave function for circular-cosh comparison duct

Subscripts:

−	upstream variable, $x \to -\infty$
0	variables at throat, $x = 0$
+	downstream variables, $x \to \infty$

The propagation equation for the n^{th}-radial, m^{th}-circumferential mode in a circular duct with walls of small maximum slope is

$$d^2\phi/dx^2 + [k^2 - \alpha^2/b^2(\varepsilon x)]\phi = 0 \tag{1}$$

(eq. (18) of ref. 1), where k is the wave number, b is the duct radius, α is the n^{th} zero of $J_m'(\alpha)$, x is the direction of propagation (axial direction), and $\varepsilon << 1$ is proportional to the maximum wall slope (fig. 1). Equation (1) applies in the absence of mean flow. In the appendix this analysis (for symmetric ducts) is extended to include a one-dimensional gas dynamics flow through the duct.

For the circular-cosh duct, equation (1) becomes

$$d^2\chi/ds^2 + [k^2 - \alpha^2/B^2(\varepsilon s)]\chi = 0 \tag{2}$$

where

$$B^{-2}(\varepsilon s) = \frac{1}{2}(1 + \tau^{-2})(1 + \beta^2 \operatorname{sech}^2(\varepsilon s)) + \frac{1}{2}(\tau^{-2} - 1)\tanh(\varepsilon s) \tag{2a}$$

For clarity, dependent and independent variables are denoted by $\chi(s)$ and s in equation (2), although equation (2) is just a special case of equation (1). Here the duct radius is unity upstream ($s \to -\infty$) and τ downstream ($s \to \infty$). The parameter β determines the shape of the circular-cosh comparison duct within its contraction section. (β is the free parameter alluded to in the Introduction.)

Equation (2) can be solved exactly by putting it into the form of a hypergeometric equation [ref. 1 and ref. 2, section 12.3]. However, for present purposes that solution is not required explicitly. It suffices to denote it by $\chi(s)$. Equation (2) is used as a comparison equation (ref. 3) for equation (1), and an approximate solution to the latter is sought in the form

$$\phi = A(\varepsilon x)\chi(s(\varepsilon x)/\varepsilon) \tag{3}$$

Substituting equation (3) into equation (1), using equation (2) to eliminate χ'', and equating to zero the coefficients of like powers of ε gives

$$(s')^2[k^2 - \alpha^2/B^2(s;\beta)] = [k^2 - \alpha^2/b^2(\varepsilon x)] \tag{4}$$

at $O(\varepsilon^0)$. Here primes indicate differentiation with respect to argument. The equation at $O(\varepsilon)$ determines $A(\varepsilon x)$, but it suffices here to note that $A(\pm\infty) = 1$. Thus the x-dependence of A does not affect reflection and transmission coefficients and so need not be evaluated.

Because we are dealing with case (3) of the first paragraph of this report, there are points x_a and x_b (resp. s_a and s_b) at which the right side (resp left side) of equation (4) vanishes. If equation (4) is integrated as

$$\int_{s_a}^{s} \kappa(s;\beta)ds = \int_{\varepsilon x_a}^{\varepsilon x} \lambda(x)dx \tag{5}$$

where

$$\kappa^2 = [k^2 - \alpha^2/B^2(s;\beta)]$$
$$\lambda^2 = [k^2 - \alpha^2/b^2(x)] \tag{5a}$$

an implicit equation for $s(\varepsilon x;\beta)$ is obtained with the parametric dependence on β noted. Now, for $x_a < x < x_b$ and $s_a < s < s_b$ both sides of equation (5) are imaginary because κ^2 and λ^2 are negative when x is in the interval x_a to x_b and s is in the interval s_a to s_b. For $x > x_b$ and $s > s_b$ both sides of equation (5) are complex and can only be equal if they have the same imaginary part. Equating these imaginary parts gives the eigenvalue relation

$$\int_{s_a(\beta)}^{s_b(\beta)} \kappa(s;\beta)ds = \int_{\varepsilon x_a}^{\varepsilon x_b} \lambda(x)dx \tag{6}$$

The free parameter must be adjusted to make the integrals in equation (6) equal. The functions $s_a(\beta)$ and $s_b(\beta)$ in equation (6) are the zeros of κ^2. Using equation (2a) and equating equation (5a) to zero gives

$$\tanh(s_{a,b}) = \frac{(1 - \tau^2)}{2\beta^2(1 + \tau^2)} \mp (2\beta^2)^{-1} \sqrt{\left(\frac{1 - \tau^2}{1 + \tau^2}\right)^2 + 4\beta^2 \left[1 + \beta^2 - \frac{2k^2}{\alpha^2(1 + \tau^{-2})}\right]} \tag{7}$$

Equation (6) with $s_a(\beta)$ and $s_b(\beta)$ given by equation (7) provides the required equation for β. The integral on the right side can be evaluated for any particular duct of known cross-sectional area. The expression on the left side then can be inverted (numerically, in general) to find β.

REFLECTION AND TRANSMISSION COEFFICIENTS

The reflection and transmission coefficients for the circular-cosh duct are given in references 1 and 2 as

$$\left.\begin{aligned} R &= \frac{\cosh[(k_+ - k_-)\pi/\varepsilon] + \cosh(\sigma\pi/\varepsilon)}{\cosh[(k_+ + k_-)\pi/\varepsilon] + \cosh(\sigma\pi/\varepsilon)} \\ \\ T &= \frac{2\sinh(k_+\pi/\varepsilon)\sinh(k_-\pi/\varepsilon)}{\cosh[(k_+ + k_-)\pi/\varepsilon] + \cosh(\sigma\pi/\varepsilon)} \end{aligned}\right\} \tag{8}$$

where

$$\left.\begin{aligned} k_+^2 &= k^2 - \alpha^2\tau^{-2} \\ \\ k_-^2 &= k^2 - \alpha^2 \\ \\ \sigma^2 &= 2(1 + \tau^{-2})\,\alpha^2\beta^2 - \varepsilon^2 = 4(k_0^2 + (k_+^2 + k_-^2)/2) - \varepsilon^2 \end{aligned}\right\} \tag{8a}$$

In the second form of σ^2, k_0 is defined by

$$k_0^2 = \frac{1}{2}(1 + \tau^{-2})\alpha^2\beta^2 - \frac{1}{2}(k_-^2 + k_+^2)$$

Clearly, k_+^2, k_-^2, and k_0^2 are $|\kappa^2|$ evaluated at ∞, $-\infty$, and 0. The value of β from equation (6) substituted into equations (8) and (8a) gives R and T.

The phase shifts that occur upon reflection and transmission of the wave can also be determined. Letting $x \to \pm\infty$ in equation (5), we find

$$s \to \begin{cases} \varepsilon x + s_a - \varepsilon x_a + \displaystyle\int_{\varepsilon x_a}^{-\infty}\left(\frac{\lambda}{k_-} - 1\right)dx - \int_{s_a}^{-\infty}\left(\frac{\kappa}{k_-} - 1\right)ds \equiv \varepsilon x - \varepsilon\delta_-; \; x \to -\infty \\ \\ \varepsilon x + s_b - \varepsilon x_b + \displaystyle\int_{\varepsilon x_b}^{-\infty}\left(\frac{\lambda}{k_+} - 1\right)dx - \int_{s_b}^{-\infty}\left(\frac{\kappa}{k_-} - 1\right)ds \equiv \varepsilon x - \varepsilon\delta_+; \; x \to \infty \end{cases} \tag{9}$$

The integrals appearing in the definitions of δ_+ and δ_- have been put into forms that converge at the upper limits. The upstream and downstream limits of the wave function (eq. (3)) are (ref. 2)

$$\phi \to \begin{cases} \dfrac{e^{ik_- s/\varepsilon}}{\sqrt{ik_-}} + \sqrt{\dfrac{R}{ik_-}}\, e^{-ik_-\{s/\varepsilon - \delta_R\}} & ; \ x \to -\infty \\ \\ \sqrt{T/ik_+}\, e^{ik_+\{s/\varepsilon + \delta_T\}} & ; \ x \to \infty \end{cases} \tag{10}$$

The first limit corresponds to the sum of the incident and reflected waves; the reflected waves are phase shifted with respect to the incident wave by $k_-\delta_R$. Similarly, the downstream limit corresponds to a transmitted wave with a phase shift of $k_+\delta_T$. Substituting equation (9) into equation (10), we find that in a non-circular-cosh duct the reflected wave suffers an additional phase shift of $2k_-\delta_-$, and the transmitted wave suffers a shift of $k_+(\delta_+ + \delta_-)$ - both measured with respect to the incident wave.

SYMMETRIC DUCTS

For the symmetric duct $\tau = 1$. With that value equation (6) becomes

$$\frac{k_0^2}{k_-} \int_0^1 \frac{(1 - y^2)^{1/2}\, dy}{\left[1 + (k_0^2/k_-^2)y^2\right]} = \Theta \tag{11}$$

where $\Theta \equiv \int_0^{x_b} |\lambda|\, dx$ is determined by the actual duct shape and hence is assumed to be known. The left side of equation (11) is obtained by substituting equation (2a), with $\tau = 1$, into equation (6).

Now, if k_- (which equals k_+ for symmetric ducts) is $O(1)$ and k_0 is also $O(1)$, then $R \simeq 1$ and $T \simeq 0$ for small ε; that is, the wave is totally reflected. However, if $k_- = O(1)$ and $k_0 = 0$, $T \simeq 1/2$ and $R \simeq 1/2$ (ref. 1). In general, with $k_- = O(1)$, both reflection and transmission occur only when $k_0/k_- \ll 1$. Present interest is in this nontrivial case.

These considerations suggest that equation (11) be expanded in powers of k_0/k_-:

$$\Theta = \frac{\sqrt{\pi}}{4} \frac{k_0^2}{k_-} \sum_0^{\infty} \left(\frac{k_0}{k_-}\right)^{2n} (-1)^n \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+2)} \tag{11a}$$

As asymptotic expansion for $k_0(\theta)$ results from inversion of equation (11a):

$$k_0^2 = \frac{4}{\pi} k_- \theta + \frac{4}{\pi^2} \theta^2 + O\left(\frac{k_0^5}{k_-^3}\right) \tag{12}$$

Because of equation (12) we have written σ as a function of k_0^2 below (eq. (8)): The β dependence of R and T appears implicity through k_0^2, and it is not necessary to evaluate β itself.

As an example, illustrating the present method, we might consider the symmetric duct

$$b^{-2}(x) = 1 + \gamma^2 - \gamma^2 x^2/(1 + |x|)^2 \tag{13}$$

for which

$$\theta = \frac{\ell_0^2}{k_-} \int_0^A \frac{(1 - y^2)^{1/2}\, dy}{\left[1 + y\ell_0/k_-(1 + \ell_0^2/k_-^2)^{-1/2}\right]}$$

$$= \frac{\ell_0^2}{k_-} \frac{\sqrt{\pi}}{4} \sum_0^{\infty} \frac{\ell_0^n}{k_-^n}\left(1 + \frac{\ell_0^2}{k_-^2}\right)^{-n/2} (-)^n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}$$

where $\ell_0^2 = \alpha^2\gamma^2 - k_-^2$. For $\ell_0/k_- << 1$, we find

$$k_0^2 \approx \ell_0^2 - \frac{4\ell_0^3}{3\pi k_-}\left(1 + \frac{\ell_0^2}{k_-^2}\right)^{-1/2} + \frac{\ell_0^4}{4k_-^2}\left[1 + \left(1 + \frac{\ell_0^2}{k_-^2}\right)^{-1}\right] \tag{14}$$

(To $O(\ell_0^2)$ this gives $\beta = \gamma$.) Equation (14) may be substituted into equations (8) and (8a) to obtain R and T for the duct (eq. ((13)). This example is considered further in the appendix.

DISCUSSION

It is noted above that, for symmetric ducts, when $k_0 = 0$, $T \approx 1/2$. The zeros of k occur at x_a and x_b; thus $k_0 = 0$ corresponds to $x_a = -x_b = 0$. In this case equation (6) is equivalent to $b(0) = B(0)$. Now, when $T \leq 1/2$, x_a is real and nonzero and equation (6) determines β. However, when $T > 1/2$, no real value of x_a exists and equation (6) is no longer valid. Of course, in this case there is no region in which the wave is cutoff; so we are considering case (1) rather than case (3) of the first paragraph. However, partial reflection still occurs. To continue to use the circular-cosh comparison equation when there is no cutoff, a criterion for determining β is required. The observation that the turning points, x_a and x_b, coalesce at a particular value of T ($T = 1/2$ for symmetric ducts), for which b and B become equal at the throat of the duct, suggests the appropriate criterion for determining β after the turning points have become imaginary is that B equal b at the duct throat.

APPENDIX - SYMMETRIC DUCTS WITH MEAN FLOW

When the duct is symmetric, the analysis of the text can be readily extended to include a one-dimensional gas dynamics flow through the duct. The density $\rho(\epsilon x)$, sound speed $c(\epsilon x)$, velocity $U(\epsilon x)$, and Mach number $M(\epsilon x) = U(\epsilon x)/c(\epsilon x)$ of that flow are expressed in terms of the duct cross-sectional area by well-known isentropic flow relations (sections 80 and 90 of ref. 4).

The propagation equation (cf. eq. (1)) is now

$$\frac{1}{\rho}\frac{d}{dx}\left(\rho\frac{d\phi}{dx}\right) - u\frac{d}{dx}\left(\frac{u}{c^2}\frac{d\phi}{dx}\right) - ikc_-u\left[\frac{d}{dx}\left(\frac{\phi}{c^2}\right) + \frac{1}{c^2}\frac{d\phi}{dx}\right] + \left(\frac{c_-^2k^2}{c^2} - \frac{\alpha^2}{b^2}\right)\phi = 0 \tag{A1}$$

where c_- is the upstream sound speed (so c_-k is the frequency of the acoustic wave). In analogy to equation (3) we seek an approximate solution to equation (A1) of the form

$$\phi = A(\epsilon x)\exp\left[ik\int_{x_a}^{x}\left(\frac{c_-}{c}\right)\frac{M}{(1 - M^2)}\,dx\right]\chi_M[s(\epsilon x)/\epsilon] \tag{A2}$$

and, analogously to equation (2), $\ddot{\chi}_M = -\kappa_M^2\chi_M/(1 - M_-^2)$ with

$$\kappa_M^2 = [k^2/(1 - M_-^2) - \alpha^2/B^2(s;\beta)]$$

Substituting equation (A2) into equation (A1) gives

$$\kappa_M^2(s)(s')^2/(1 - M_-^2) = \lambda_M^2(\epsilon x)/[1 - M^2(\epsilon x)] \tag{A3}$$

to $O(\epsilon^0)$. Here

$$\lambda_M^2(\epsilon x) = \left\{\frac{c_-^2k^2}{c^2(\epsilon x)[1 - M^2(\epsilon x)]} - \frac{\alpha^2}{b^2(\epsilon x)}\right\}$$

Integrating equation (A3) gives

$$\int_{s_a(\beta)}^{s_b(\beta)}\frac{\kappa_M(s)}{\sqrt{1 - M_-^2}}\,ds = \int_{\epsilon x_a}^{\epsilon x_b}\frac{\lambda_M(x)}{\sqrt{1 - M^2(x)}}\,dx \tag{A4}$$

in place of equation (6). Note that k^2 must be replaced by $k^2/1 - M_-^2$ in expression (7) for s_a and s_b, and x_a and x_b are now the roots of $\lambda_M^2(\epsilon x) = 0$. To present order of approximation, equations (8) for R and T remain valid (only for symmetric ducts) if $k_-^2 = k_+^2 = \kappa_M^2(\infty)/(1 - M_-^2)$ and $k_0^2 = |\kappa_M^2(0)|/(1 - M_-^2)$.

Equation (A4) shows that the effect of mean flow on reflection and transmission is to cause a Lorentz contraction of streamwise wavelengths. A Doppler shift also occurs because of the exponential in equation (A2), but this does not affect power transmission. Aside from the Doppler shift, there is no difference between upstream and downstream propagation.

In the cases of present interest, $k^2 - \alpha^2/b^2$ is small, so the effect of Mach number enters primarily through the factor $k^2/(1 - M^2)$ in κ_M and λ_M. Because M increases toward unity within the contraction, λ_M (which is a function of $M(x)$) is more affected by flow than is κ_M (which is a function of M_-). In brief, the Mach number effect will be to decrease Θ in equation (12) and hence increase transmission through the duct. Physically, this occurs because short waves are more easily transmitted than are long waves; thus the Lorentz contraction of wavelength enables acoustic waves to pass through the duct throat.

This phenomenon is illustrated in figure 2, where the transmission coefficient for the duct (eq. (13)) has been plotted as a function of Mach number M_-. For this calculation $k = 2.0$, $\alpha = 1.84118$ (corresponding to the (1,1) mode), and $\epsilon = 0.3$. When $\gamma^2 = 0.15$, there is no cutoff region, so T is greater than 1/2 at all Mach numbers: T increases with Mach number from its value of 0.85 in the absence of flow. At the other values of γ^2 shown in figure 1, T is zero in the absence of flow. Mean flow through the duct causes it to become acoustically transparent as M_- tends toward unity.

The analyses of this appendix is restricted to symmetric ducts because in κ_M^2 a constant Mach number of M_-, has been used. Extension of this method to asymmetric ducts is not trivial, for a constant Mach number then cannot be used in the comparison equation.

REFERENCES

1. Y. C. Cho and K. U. Ingard, 1980, American Institute of Aeronautics and Astronautics paper 80-1018, 6th AIAA Aeroacoustics Conference, Hartford, Conn. USA, Higher Order Mode Propagation in Nonuniform Circular Ducts.

2. P. M. Morse and H. Feshback, 1953, Methods of Theoretical Physics, New York: McGraw-Hill.

3. E. Zauderer, 1970, J. Math. Annal. Appl. 30 157-171, Uniform Asymptotic Solutions of the Reduced Wave Equation.

4. L. D. Landau and E. M. Lifshitz, 1959, Fluid Dynamics, London, Pergamon Press.

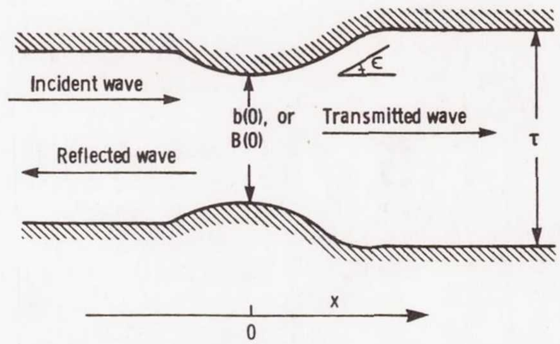

Figure 1. - Defining sketch for acoustic transmission through a converging-diverging duct.

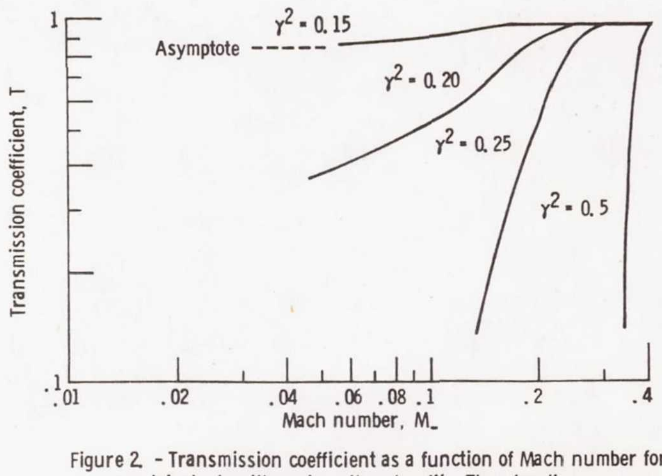

Figure 2. - Transmission coefficient as a function of Mach number for asymmetric ducts with various throat radii. Throat radius, $(1+\gamma^2)^{-1/2}$; wave number $k = 2.0$; duct-mode eigenvalue $\alpha = 1.84118$; maximum wall slope $\epsilon = 0.3$.

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15. Supplementary Notes

16. Abstract

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