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EXACT FINITE ELEMENTS FOR CONDUCTION AND CONVECTION

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EXACT FINITE ELEMENTS FOR CONDUCTION AND CONVECTION

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SUMMARY

An approach for developing exact one-dimensional conduction-convection finite elements is presented. Exact interpolation functions are derived based on solutions to the governing differential equations by employing a nodeless parameter. Exact interpolation functions are presented for combined heat transfer in several solids of different shapes, and for combined heat transfer in a flow passage. Numerical results demonstrate the exact one dimensional elements offer significant advantages over elements based on approximate interpolation functions.

1. INTRODUCTION

A significant development in finite element methodology for forced convection analysis is the concept of upwind weighting functions [1]. For the one-dimensional convective-diffusion equation upwind weighting functions are expressed in terms of an upwind parameter which can be varied to adjust the accuracy of the finite element solution. Optimal values of the upwind parameter lead to exact values of the temperature at nodal points. Other elements in one-dimensional thermal and structural analysis also predict exact nodal values. For example, one dimensional solutions [2] for heat conduction with internal heat generation by linear interpolation functions yield exact temperatures at the nodes. The ability of finite elements to predict exact nodal temperatures has previously been regarded as a property of a few particular equations and to be of limited generality. There is, however,

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a broad class of one-dimensional problems governed by linear ordinary differential equations for which finite elements can be developed to yield exact solutions. The one-dimensional convective-diffusion equation and the one-dimensional conduction equation are special cases of this class of problems. In a previous paper [3], the authors have presented two examples of one-dimensional thermal-structural finite elements which yield exact solutions for linear problems.

The purpose of the paper is to present an approach for developing one-dimensional conduction-convection finite elements which yield exact nodal temperatures and an exact variation of temperature within an element for steady-state linear analysis. The one-dimensional conduction-convection problems considered are first discussed. A nodeless variable approach for deriving exact interpolation functions is then presented and illustrated for several cases. Next, the general form of the element equations is presented and discussed. Finally, the benefits of the exact finite element formulation are illustrated for two problems by comparing numerical solutions from finite elements with exact and linear interpolation functions.

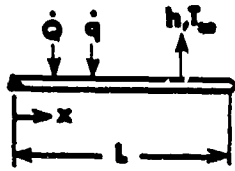
2. EXACT ELEMENT FORMULATION

2.1 Governing Equations and Boundary Conditions

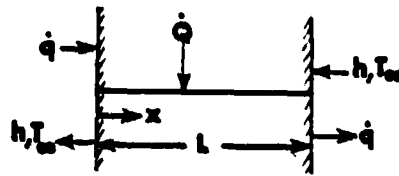
The geometry and terminology for eight one-dimensional conduction-convection cases are shown in Fig. 1. Cases 1-7 are solids of various shapes where heat transfer consists of conduction which may be combined with: (a) surface convection, (b) internal heat generation, and (c) surface heating. Case (8) is a one-dimensional flow where the heat transfer consists of fluid conduction and mass-transport convection which may be combined with: (a) surface convection, and (c) surface heating. In the figure, \dot{Q} is the volumetric heat generation rate, \dot{q} is the surface heating rate, h is the convective heat transfer coefficient and T_∞ is the environmental temperature for the convection heat exchange. Heat transfer in Cases 1-8 is described by differential equations of the form

$$a_0(x) \frac{d^2 T}{dx^2} + a_1(x) \frac{dT}{dx} + a_2(x) T = a_3(x) \quad (1)$$

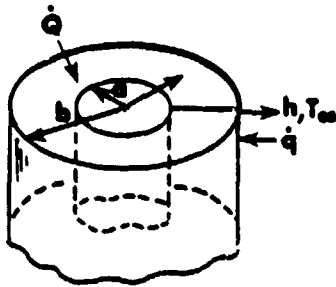
where a_i , $i = 0, 1, 2, 3$, are functions which depend on the geometry and thermal parameters of each case. Finite element equations for eq. (1) may be formulated by the method of weighted residuals, but due to the presence of the odd-order derivative an unsymmetrical coefficient matrix will result. However, if eq. (1) is first cast into self-adjoint form, the differential operator will be symmetric. Then, finite element matrices can be derived by the method of weighted residuals or a variational method, and coefficient matrices will be



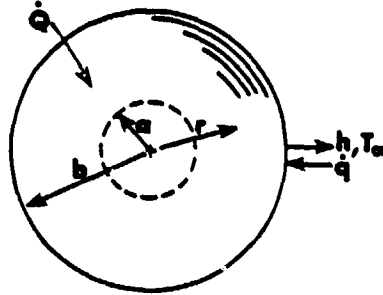
Case 1. ROD



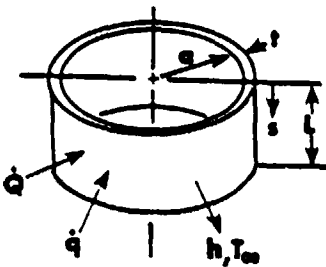
Case 2. SLAB



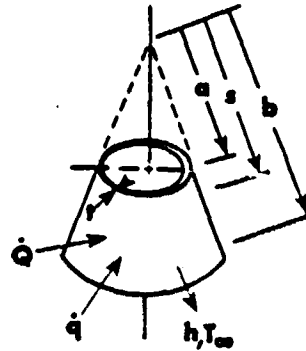
Case 3. HOLLOW CYLINDER



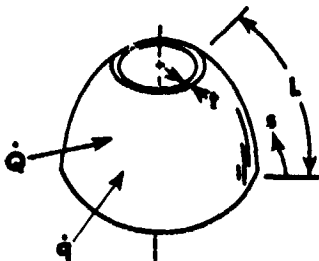
Case 4. HOLLOW SPHERE



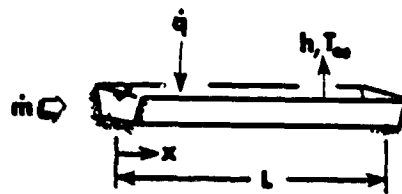
Case 5. CYLINDRICAL SHELL



Case 6. CONICAL SHELL



Case 7. SPHERICAL SHELL



Case 8. FLOW PASSAGE

Figure 1
ONE DIMENSIONAL CONDUCTION AND CONVECTION CASES

symmetrical, [2]. Eq. (1) is written in self-adjoint form by multiplying it by the factor

$$P = \exp[\int (a_1/a_0) dx]$$

such that

$$\frac{d}{dx} [P(x) \frac{dT}{dx}] + Q(x)T = R(x) \quad (2)$$

where $Q(x) = a_2 P/a_0$ and $R(x) = a_3 P/a_0$.

The governing self-adjoint differential equations for Cases 1-8 are shown in Table 1. Additional terms not previously defined are k the thermal conductivity, p the perimeter of the rod or flow passage, A the cross-sectional area of the rod or flow passage, m the mass flow rate and c the specific heat. Table 1 shows the coefficients of eq. (2) for each case. Note that for each case a different differential equation may

TABLE 1
GOVERNING SELF-ADJOINT DIFFERENTIAL EQUATIONS

$$\frac{d}{dx} [P(x) \frac{dT}{dx}] + Q(x)T = R(x)$$

Case	Conduction	Heat Loads			Surface Flux (c)
		Convection (a)	Convection (a)	Source (b)	
1	$-\frac{d}{dx}[\frac{dT}{dx}]$	$\frac{hp}{kA} T$	$\frac{hp}{kA} T_\infty$	$\frac{\dot{Q}}{k}$	$\frac{\dot{q}p}{kA}$
2	$-\frac{d}{dx}[\frac{dT}{dx}]$	--	--	$\frac{\dot{Q}}{k}$	--
3	$-\frac{d}{dr}[r \frac{dT}{dr}]$	--	--	$\frac{\dot{Q}}{kr}$	--
4	$-\frac{d}{dr}[r^2 \frac{dT}{dr}]$	--	--	$\frac{\dot{Q}}{kr^2}$	--
5	$-\frac{d}{ds}[\frac{dT}{ds}]$	$\frac{h}{kt} T$	$\frac{h}{kt} T_\infty$	$\frac{\dot{Q}}{k}$	$\frac{\dot{q}}{kt}$
6	$-\frac{d}{ds}[s \frac{dT}{ds}]$	$\frac{h}{kt} sT$	$\frac{h}{kt} sT_\infty$	$\frac{\dot{Q}}{k} s$	$\frac{\dot{q}}{kt} s$
7	$-\frac{d}{ds}[\cos \frac{s}{a} \frac{dT}{ds}]$	--	--	$\frac{\dot{Q}}{k} \cos \frac{s}{a}$	$\frac{\dot{q}}{kt} \cos \frac{s}{a}$
8	$-\frac{d}{dx}[P \frac{dT}{dx}]$	$\frac{hp}{kA} P T$	$\frac{hp}{kA} P T_\infty$	--	$\frac{\dot{q}p}{kA} P$

where $P = \exp(-\dot{m}cx/kA)$

result by combining conduction with each of the three heat loads or combinations of these heat loads.

The boundary conditions for eq. (2) consist of specifying the temperature or the temperature gradient at the endpoints of the solution domain, $a < x < b$. For cases 1-8, the boundary conditions considered herein are:

$$\begin{aligned} T &= \text{constant, or} \\ -k \frac{dT}{dx} &= q, \text{ or} \\ -k \frac{dT}{dx} &= h(T - T_\infty) \end{aligned} \quad (3)$$

2.2 Interpolation Functions

Exact interpolation functions for finite element formulations of eq. (2) can be derived from the general solution to the differential equation. The general solution has the form,

$$T(x) = c_1 f_1(x) + c_2 f_2(x) + G(x) \quad (4)$$

where $f_1(x)$ and $f_2(x)$ are linearly independent solutions of the homogeneous equation, c_1 and c_2 are constants of integration, and $G(x)$ is a particular solution. A finite element with two nodes is formulated by imposing the conditions

$$T(x_1) = T_1 \quad T(x_2) = T_2 \quad (5)$$

where x_i , $i = 1, 2$ are nodal coordinates, and T_i , $i = 1, 2$ are the nodal temperature values. To accommodate the particular solution $G(x)$, the element temperature interpolation is written as

$$T(x) = N_0(x) T_0 + N_1(x) T_1 + N_2(x) T_2 \quad (6)$$

where T_0 is a nodeless parameter. The nodeless parameter T_0 is similar to a nodeless variable [2] except the nodeless parameter is uniquely determined for each differential equation. Note that $N_0(x)$ is zero at the nodes to satisfy (5). The interpolation functions N_i , $i = 0, 1, 2$ are determined by imposing the boundary conditions, eq. (5), on eq. (4) and then writing the result in the form of eq. (6). The nodeless parameter is identified by writing the particular solution $G(x) = T_0 g(x)$. Thus, in general

$$\begin{aligned} N_0(x) = g(x) + & \left[\frac{f_2(x_1)g(x_2) - f_2(x_2)g(x_1)}{D} \right] f_1(x) + \\ & \left[\frac{f_1(x_2)g(x_1) - f_1(x_1)g(x_2)}{D} \right] f_2(x) \end{aligned} \quad (7a)$$

$$N_1(x) = \frac{f_2(x_2)f_1(x) - f_1(x_2)f_2(x)}{D} \quad (7b)$$

$$N_2(x) = \frac{f_1(x_1)f_2(x) - f_2(x_1)f_1(x)}{D} \quad (7c)$$

where $D = f_1(x_1)f_2(x_2) - f_1(x_2)f_2(x_1)$.

Element interpolation functions derived in the form of eq. (2) are not generally the same as used in conventional finite elements since they result from the differential equation solution. Interpolation functions, in fact, may range from simple polynomials to higher transcendental functions depending on the equation. The exact interpolation functions, eqs. (7), are derived for specified temperature boundary conditions, but they also yield exact solutions for the gradient boundary conditions, eq. (3), when these boundary conditions are consistently incorporated in the finite element equations. Since the interpolation functions are exact, temperature gradients and fluxes computed from these functions are also exact.

Nodeless parameters and element interpolation functions have been derived for Cases 1-8 with heat loads (a) - (c), but for brevity only typical results are presented herein. Typical nodeless parameters appear in Table 2 and typical interpolation functions appear in Table 3.

TABLE 2
NODELESS PARAMETERS

<u>Case</u>	<u>T_o</u>	<u>Case</u>	<u>T_o</u>
1(a)	T _∞	7(b)	Q̇a ² /k
1(b)	Q̇L ² /2k	8(b)	q̇pL/ṁc
3(b)	Q̇b ² /4kw	8(a)	T _∞
4(b)	Q̇/6k		

where $w = \ln(b/a)$

TABLE 3
ELEMENT INTERPOLATION FUNCTIONS

Case	N_0	N_1	N_2
1(a)	$1-N_1-N_2$	$\frac{\sinh m(L-x)}{\sinh mL}$	$\frac{\sinh mx}{\sinh mL}$
1(b)	$N_1 N_2$	$1-\frac{x}{L}$	$\frac{x}{L}$
3(b)	$\frac{a^2}{b^2} \frac{N_1}{w} + \frac{N_2}{w} - \frac{r^2}{b^2} w$	$\frac{1}{w} \ln \left(\frac{b}{r}\right)$	$\frac{1}{w} \ln \left(\frac{r}{a}\right)$
4(b)	$\frac{(r-a)(b-r)(r+a+b)}{r}$	$\frac{a(b-r)}{r(b-a)}$	$\frac{b(r-a)}{r(b-a)}$
7(b)	$\log(\cos \frac{s}{a}) - N_2 \log(\cos \frac{L}{a})$	$1-N_2$	$\frac{\log \left[\frac{1+\sin \frac{s}{a}}{1-\sin \frac{s}{a}} \right]}{\log \left[\frac{1+\sin \frac{L}{a}}{1-\sin \frac{L}{a}} \right]}$
8(b)	$\frac{x}{L} - N_2$	$1-N_2$	$\frac{1 - e^{2\alpha x}}{1 - e^{2\alpha L}}$
8(a)	$1-N_1-N_2$	$e^{\alpha x} \frac{\sinh \beta(L-x)}{\sinh \beta L}$	$e^{\alpha(x-L)} \frac{\sinh \beta x}{\sinh \beta L}$

where

$$m = (hp/kA)^{\frac{1}{2}}$$

$$w = \ln(b/a)$$

$$\alpha = mc/2kA$$

$$\beta = (m^2 + \alpha^2)^{\frac{1}{2}}$$

2.3 Element Matrices

For general interpolation functions, element matrices for eq. (2) are

$$K_{ij} = \int_{x_1}^{x_2} P \frac{dN_i}{dx} \frac{dN_j}{dx} dx + \int_{x_1}^{x_2} Q N_i N_j dx \quad (8a)$$

$$F_i = \int_{x_1}^{x_2} R N_i dx \quad (8b)$$

where K_{ij} is a typical term in the symmetric conductance matrix, and F_i is a typical term in the heat load vector. For a node on the boundary additional terms are required to represent surface heating or a convective heat exchange. For the exact interpolation functions, element matrices have the general form

$$\begin{bmatrix} K_{00} & K_{01} & K_{02} \\ K_{10} & K_{11} & K_{12} \\ K_{20} & K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} T_0 \\ T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ F_1 \\ F_2 \end{Bmatrix}$$

since $i = 0, 1, 2$. Because T_0 is known the first equation may be uncoupled from the nodal unknowns in the second and third equations. Thus, the exact element matrices have the same size as a conventional linear element and can be written as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} - T_0 \begin{Bmatrix} K_{10} \\ K_{20} \end{Bmatrix} \quad (9)$$

Eq. (9) can be simplified further by noting that $K_{10} = K_{20} = 0$ for self-adjoint equations. The last result can be shown by observing that N_i , $i=1,2$ are solutions of the ordinary differential equation, (2). Multiplying this equation by N_0 and integrating by parts shows that $K_{10} = K_{20} = 0$. Thus, for self-adjoint equations, element matrices need to be evaluated only for $i=1,2$.

As emphasized previously, the development of the element matrices, eqs. (8), and element equations, eq. (9), are based upon the differential equation being written in the self-adjoint form, eq. (2). As an alternate approach, exact finite element matrices may also be derived from the standard form of the differential equation, eq. (1). However, element matrices

will not be symmetrical due to the odd-order derivative, and the exact element equations will have $K_{0i} \neq 0$, $i=1,2$.

The principal advantage of the exact one-dimensional finite elements is the superior accuracy in comparison to elements based on approximate interpolation functions. The principal disadvantage of the exact elements is the additional effort required to form the more complex interpolation functions and to evaluate the element matrices. This disadvantage has been overcome, in part, by using a computer-based symbolic manipulation language MACSYMA to perform the algebra and calculus required for these derivations.

3. NUMERICAL EXAMPLES

Two numerical examples are presented to illustrate the benefits of the exact elements; other examples for Cases 1 and 3 appear in [4]. The solution of problems with exact elements typically requires only a few elements because mesh refinement is not required in regions of large temperature gradients. The number of elements used is determined by changes in geometry, material properties or heating variations. After nodal temperatures are computed, temperatures are computed at several points within each element by using the element interpolation functions, eq. (6).

3.1 Coffee Spoon with Conduction and Convection

Rod elements with conduction and convection (Table 1, Case 1(a)) are used to model one-dimensional heat transfer in a coffee spoon (Fig. 2). The lower one-half of the spoon is convectively heated by the coffee at a specified temperature of 150°F, and the upper one-half of the spoon is convectively cooled by the atmosphere at a specified temperature of 50°F. The ends of the one-dimensional spoon model are assumed to have negligible heat transfer. Element matrices for the exact finite elements appear in reference [4]. Temperatures for the spoon model are computed from: (1) two exact finite elements, (2) two elements with linear temperature interpolation, (3) ten elements with linear temperature interpolation.

The temperature distributions (Fig. 2) show the exact temperature distribution computed by the two exact elements with three nodal unknown temperatures compared with predictions from the elements with linear temperature interpolation. Two linear elements predict the temperatures at the three nodes with fair accuracy, but the linear interpolation functions are not capable of representing the zero temperature gradient boundary conditions. Temperatures computed from ten linear elements, however, show excellent agreement with the exact finite element solution. Solutions for similar problems have also shown this trend; typically about five to ten elements with linear interpolation per exact element are required for

acceptable accuracy.

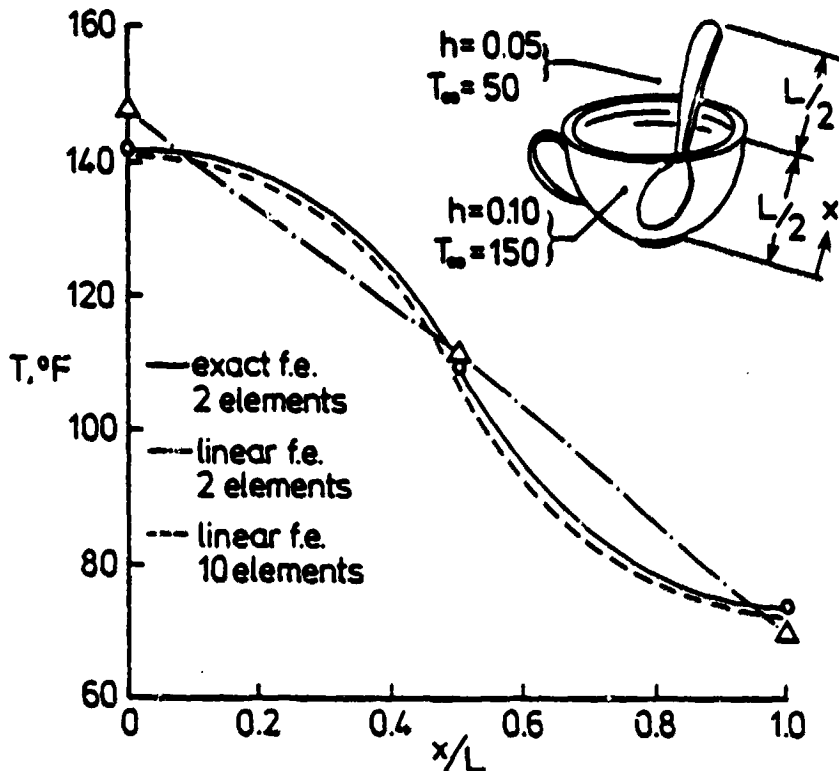


Figure 2
ROD ELEMENT ANALYSIS OF COFFEE SPOON
WITH CONDUCTION AND CONVECTION

3.2 Heat Transfer in Merging Flows

In merging flows, significant temperature gradients may occur at the flow confluence; therefore, merging flows provide a good test for finite element evaluations. In reference [4], the authors compared conventional (Bubnov-Galerkin) finite element solutions with upwind (Petrov-Galerkin) solutions for steady-state and transient merging flows. Fig. 3 presents the geometry and terminology of a merging flow where conduction is combined with mass-transport and surface convection.

Temperatures for the merging flow (Fig. 3) were computed from: (1) three exact finite elements with four nodes, and (2) 75 conventional elements with 76 nodes. There is good agreement between the two solutions except in the vicinity of the flow confluence where the conventional element shows oscillations indicating a need for mesh refinement in the region of large temperature gradients. The clear superiority of the exact elements is demonstrated. Exact nodal temperatures are also predicted in problems of this type (without surface convection) by optimum upwind elements, but a larger

number of nodes are required since upwind elements do not predict exact temperatures within an element.

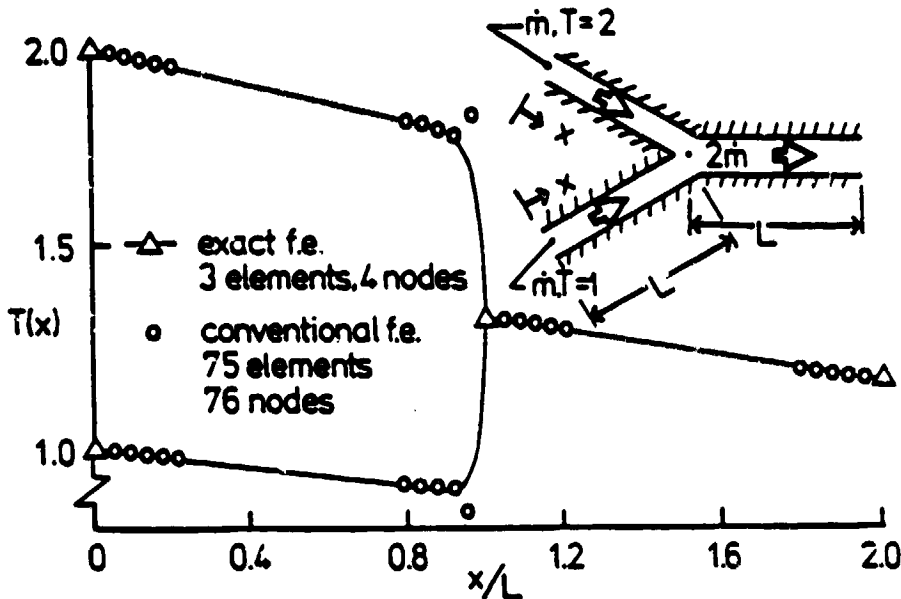


Figure 3

EXACT AND CONVENTIONAL FINITE ELEMENT ANALYSIS OF MERGING FLOW WITH CONDUCTION, MASS TRANSPORT CONVECTION AND SURFACE CONVECTION.

4. CONCLUDING REMARKS

An approach for developing one-dimensional, exact finite elements for linear, steady conduction-convection analysis is presented. The element interpolation functions employ the solution from the governing second-order differential equation by utilizing a nodeless parameter approach. The elements require two nodes and yield exact nodal temperatures and an exact variation of the temperature within an element. Exact nodal interpolation functions are presented for several cases of conduction and convection. Numerical results are presented for conduction and surface convection in a rod element model, and for conduction, mass-transport convection and surface convection in a merging, one-dimensional flow model. The examples demonstrate that for one-dimensional analysis, the exact elements offer significant advantages over conventional elements based on approximate interpolation functions.

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