TECHNIQUES FOR INCREASING THE EFFICIENCY OF EARTH GRAVITY CALCULATIONS FOR PRECISION ORBIT DETERMINATION*

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ABSTRACT

Two techniques for increasing the efficiency of Earth gravity calculations are analyzed. The first is a representation using Chebyshev expansions in three-dimensional cells. Mathematical formulas are given for converting the standard spherical harmonic representation (e.g., GEM10B 36 x 36) to the Chebyshev representation. The error in the truncated Chebyshev representation was measured as a function of cell size and degree of truncation. For example, with a sixth degree Chebyshev expansion, the maximum gravity error is about 10^{-10}g for a 36 x 36 parent representation in a cell extending 5 degrees in both latitude and longitude and having a thickness of 600 kilometers. Computer storage requirements and relative CPU time requirements are presented. The Chebyshev gravity representation can provide a significant reduction in CPU time in precision orbit calculations, but at the cost of a large amount of direct-access storage space, which is required for a global model.

The second technique employs a temporary file for storing the components of the nonspherical gravity force. In

*This work was supported by the Operations Analysis Section, Operational Orbit Support Branch, Goddard Space Flight Center, National Aeronautics and Space Administration, under Contract NAS 5-24300.

differential correction orbit solutions it is often unnecessary to repeat computations for most of the gravity terms during subsequent iterations for which the satellite's position changes only slightly. By saving a direct-access file of gravitational forces and partial derivatives it is possible to reduce CPU time without significantly affecting orbit accuracy. The gravity file is updated whenever the position tolerance is exceeded. The Goddard Trajectory Determination System was temporarily modified to test this technique, and the results of the test are presented.

1. INTRODUCTION

As the orbit determination accuracy for Earth-orbiting spacecraft is improved through the use of increasingly more accurate Earth gravity models, the computer time requirements increase rapidly. Using the customary global spherical harmonic expansion, the amount of computation time increases approximately as the square of the maximum degree and order of the expansion. For currently available gravity models, for example, the Goddard Earth Model 10B (GEM10B), most of the computation for an orbit solution is devoted to evaluations of the gravity force. Clearly, less time-consuming methods of gravity evaluation are required, particularly if precise gravity models are needed for future operational orbit determination. The need for faster methods is enhanced by the fact that the utilization of more precise gravity models requires the use of correspondingly smaller step sizes for numerical integration of the spacecraft equations of motion.

Table 1 shows the amounts of computer time (GSFC IBM S-360/75) currently required for orbit solutions calculated using the Goddard Trajectory Determination System (GTDS). In order to isolate the dependence of the computer time on the specified value of the maximum degree and order in the

Table 1. GTDS Computer Time Usage for Various Sizes of the Spherical Harmonic Gravity Expansion

SPACECRAFT: SEASAT-1
NUMERICAL INTEGRATOR: COWELL FIXED STEP, 12TH ORDER
FORCE MODEL:

- GRAVITY: SOLAR, LUNAR, GEM9
- DRAG, WITH HARRIS-PRIESTER ATMOSPHERE
- SOLAR RADIATION FORCE
- MEAN OF 1950.0 SYSTEM FOR INTEGRATION

EPOCH: 18^h ON JULY 10, 1978 ARC LENGTH: 30 HOURS
EPOCH – ARC LENGTH: 18^h ON JULY 10, 1978 – 30 HOURS
OBSERVATIONS: 391 DOPPLER USB, 100 LASER RANGE

SIZE OF EARTH GRAVITY MODEL	IBM S-360/75 COMPUTER TIME USAGE (MIN)			
	90-SECOND STEP SIZE		45-SECOND STEP SIZE	
	CPU	I/O	CPU	I/O
		EPHEM PROGRAM		
4 x 4	0.888	0.241	1.544	0.239
8 x 8	1.007	0.241	1.613	0.239
21 x 21	1.280	0.252	2.306	0.249
36 x 36 (GEM10B)	3.210	0.329	5.058	0.330
		DC PROGRAM[1]		
4 x 4	7.448	1.804	11.015	1.725
8 x 8	8.322	1.805	12.051	1.727
21 x 21	10.419	1.817	15.482	1.739
36 x 36 (GEM10B)	20.577	1.938	35.952	1.855

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[1]SIX ITERATIONS AND CONVERGENCE

spherical harmonic expansion, all other input parameters for these solutions were identical. Computer times for both GTDS Ephemeris Generation (EPHEM) and GTDS Differential Correction (DC) Program runs are shown in this table.

Two methods for efficiency improvement are examined in this paper. Section 2 outlines a gravity representation using Chebyshev polynomials rather than spherical harmonics. Section 3 considers a procedure for making use of previously computed values of the gravity force during the later iterations of differential correction orbit solutions. This procedure, unlike the Chebyshev representation, is not generally applicable to orbit prediction. Section 4 assesses the merit of these two methods and indicates directions for future work.

2. REPRESENTATION OF THE EARTH'S GRAVITY FIELD USING CHEBYSHEV POLYNOMIALS

2.1 OUTLINE OF THE METHOD

In order to accurately represent the Earth's gravity using Chebyshev polynomials, the region of interest is partitioned into cells, and for each cell the gravity force components are expressed as a series of Chebyshev polynomials. The numerical values of the expansion coefficients for a given cell are, in general, different from those of any other cell. With a suitable selection of the cell dimensions, the convergence of the Chebyshev series is sufficiently fast that the computational effort for its evaluation is significantly less than the effort required to evaluate the standard spherical harmonic expansion. In exchange for the reduction in computational effort, however, the Chebyshev representation requires a large data set containing the expansion coefficients for all of the cells.

The evaluation of the gravity force in GTDS is accomplished with the following standard spherical harmonic expansion:

$$F_r = -g \sum_{n=0}^{n_{max}} (n + 1) \left(\frac{1}{r}\right)^n \sum_{m=0}^{n} P_n^m (\sin \phi) \cdot \left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) \tag{1}$$

$$F_\phi = g \sum_{n=0}^{n_{max}} \left(\frac{1}{r}\right)^n \sum_{m=0}^{n} \left[P_n^{m+1} (\sin \phi) - m \tan \phi \, P_n^m (\sin \phi)\right] \cdot \left(C_n^m \cos m\lambda + S_n^m \sin m\lambda\right) \tag{2}$$

$$F_\lambda = \frac{g}{\cos \phi} \sum_{n=0}^{n_{max}} \left(\frac{1}{r}\right)^n \sum_{m=0}^{n} m \, P_n^m (\sin \phi) \left(S_n^m \cos m\lambda - C_n^m \sin m\lambda\right) \tag{3}$$

where
- r = radial distance in Earth radii (a)
- ϕ = geocentric latitude
- λ = geocentric longitude
- P_n^m = Legendre function of degree n and order m
- n_{max} = maximum degree of the spherical harmonic expansion for the Earth's gravity field
- g = $GM/(ar)^2$, where G is the universal constant of gravitation, M is the Earth's mass, a is the Earth's radius, and r is defined above
- C_n^m S_n^m = nonnormalized spherical harmonic expansion coefficients for the geopotential field model considered

The Chebyshev expansions used in this paper also yield the radial, latitudinal, and longitudinal gravity components, F_r, F_ϕ, F_λ. The Chebyshev expansions are applied only to that part of the gravity force described by spherical harmonic terms of degree greater than 4. Terms of degree less than or equal to 4 are still evaluated using spherical harmonics.

In each cell, independent position variables, x, y, and z, are designated. These variables are related to r, ϕ, and λ by means of the following equations:

$$\frac{1}{r} = \frac{1}{r_o} + Ax \qquad (|x| \leq 1) \tag{4}$$

$$\sin \phi = \sin \phi_o + Cy \qquad (|\phi| \leq 45^\circ,\ |y| \leq 1) \tag{5}$$

$$\cos \phi = \cos \phi_o + Cy \qquad (|\phi| > 45^\circ,\ |y| \leq 1) \tag{6}$$

$$\cos \lambda = \cos \lambda_o + Dz \qquad (|\lambda - 90^\circ| \leq 45^\circ,\ |z| \leq 1) \tag{7}$$

The cell origin is (r_o, ϕ_o, λ_o) and the physical size of a cell is controlled by the three parameters A, C, and D. The position variables x, y, and z describe displacements, relative to the cell origin, in the radial, latitudinal, and longitudinal directions, respectively. The locus of points such that x = +1 or x = -1 describes spherical surfaces bounding the top and bottom of a cell. The locus of points such that y = $\pm$1 defines cones of constant latitude bounding the north and south sides, and the locus of points such that x = $\pm$1 describes longitudinal planes bounding the cell on the east and west sides. This choice of independent variables leads to cell crowding near the poles, but allows a fast and efficient procedure for calculation of the Chebyshev expansion coefficients.

As indicated by Equations (5) and (6), the latitude-like variable, y, is defined differently for the polar and equatorial regions. This difference is necessary to avoid slow convergence of the Chebyshev expansions close to the poles and close to the equator. This slow convergence problem also exists for $\lambda = 0$ or $\lambda = \pi$ using the definition given for z by Equation (7). However, it is only necessary to apply a longitude shift when the problem occurs (by suitably adjusting the C_n^m's and S_n^m's) and thus avoid a double definition.

The expansion of each factor of a typical spherical harmonic

$$\frac{1}{r^{n+1}} P_n^m (\sin \phi) \begin{matrix} \cos \\ \sin \end{matrix} m\lambda$$

into a series of Chebyshev polynomials follows the equations (for each cell)

$$\frac{1}{r^n} = \sum_{i=0}^{\infty} \left[\frac{2 - \delta_{io}}{\pi}\right] X_{ni} \; T_i(x) \tag{8}$$

$$P_n^m (\sin \phi) = \sum_{j=0}^{\infty} \left[\frac{2 - \delta_{jo}}{\pi}\right] Y_{nj}^m \; T_j(y) \tag{9}$$

$$\cos m\lambda = \sum_{k=0}^{\infty} \left[\frac{2 - \delta_{ko}}{\pi}\right] Z_{mk}^{(1)} \; T_k(z) \tag{10}$$

$$\sin m\lambda = \sum_{k=0}^{\infty} \left[\frac{2 - \delta_{ko}}{\pi}\right] Z_{mk}^{(2)} \; T_k(z) \tag{11}$$

The Chebyshev polynomials, T_i, are functions of x, y, or z and satisfy the recurrence relation

$$T_{i+1}(x) = 2x \; T_i(x) - T_{i-1}(x) \tag{12}$$

where the subscript indicates the degree of the polynomial. In several cases, the Chebyshev expansions indicated by Equations (8) through (11) are finite, not infinite, as a result of the definitions of x, y, and z. The X's, Y's and Z's are the Chebyshev expansion coefficients and their values depend on the cell parameters r_o, ϕ_o, λ_o, A, C, and D, in addition to the order and degree of the spherical harmonic.

The X's, Y's, and Z's are combined in the following way, according to Equations (1) through (3), to form the three subscripted Chebyshev expansion coefficients, e.g., $C_{ijk}^{(1)}$, used for the calculation of the force components:

$$C_{ijk}^{(1)} = Q \sum_{n=4}^{n_{max}} (n + 1) \; X_{ni} \sum_{m=0}^{n} Y_{nj}^{m} \left(C_n^m Z_{mk}^{(1)} + S_n^m Z_{mk}^{(2)}\right) \tag{13}$$

$$C_{ijk}^{(2)} = Q \sum_{n=4}^{n_{max}} X_{ni} \sum_{m=0}^{n} Y_{nj}^{m+1} \left(C_n^m Z_{mk}^{(1)} + S_n^m Z_{mk}^{(2)}\right) \tag{14}$$

$$C_{ijk}^{(3)} = Q \sum_{n=4}^{n_{max}} X_{ni} \sum_{m=0}^{n} mY_{nj}^{m} \left(C_n^m Z_{mk}^{(1)} + S_n^m Z_{mk}^{(2)}\right) \tag{15}$$

$$C^{(4)}_{ijk} = Q \sum_{n=4}^{n_{max}} X_{ni} \sum_{m=0}^{n} mY^{m}_{nj} \left(S^{m}_{n} Z^{(1)}_{mk} - C^{m}_{n} Z^{(2)}_{mk}\right) \tag{16}$$

$$Q \equiv \frac{(2 - \delta_{0i})\ (2 - \delta_{0j})\ (2 - \delta_{0k})}{\pi^3} \tag{17}$$

The three gravity force components are then calculated in the following way:

$$F_r = -g \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} C^{(1)}_{ijk}\ T_i(x) T_j(y) T_k(z) \tag{18}$$

$$F_\phi = g \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} \left(C^{(2)}_{ijk} - \tan\phi\ C^{(3)}_{ijk}\right) T_i(x) T_j(y) T_k(z) \tag{19}$$

$$F_\lambda = \frac{g}{\cos\phi} \sum_{i=0}^{I} \sum_{j=0}^{J} \sum_{k=0}^{K} C^{(4)}_{ijk}\ T_i(x) T_j(y) T_k(z) \tag{20}$$

These three equations represent the calculation of gravity as it might be performed in an orbit determination program, using precalculated coefficients.

The formulation used in this paper required four types of three-subscripted Chebyshev expansion coefficients. With additional work, it should be possible to also expand the function

$$\tan\phi\ P^{m}_{n}(\sin\phi)$$

in a Chebyshev series, leading to a formulation using only three types of coefficients. This additional complication was omitted for the present for simplicity.

As indicated by Equations (8) through (16) the three-subscripted coefficients depend on the gravity model coefficients, C_n^m and S_n^m, the cell location, and the cell dimensions. The combined set of three-subscripted coefficients for all cells constitutes a Chebyshev representation for the given gravity model.

The calculation of the Chebyshev coefficients for the spherical harmonic factors, that is, the calculation of the X's, Y's, and Z's, can be easily accomplished using recurrence relations. These recurrence relations are as follows:

Recurrence relations for the radial Chebyshev coefficients:

$$X_{n+1,i} = \frac{A}{2} \left(X_{n,i+1} + X_{n,i-1}\right) + \frac{1}{r_0} X_{n,i} \qquad (n \geq 0,\ \text{all } i) \quad (21)$$

$$X_{n+1,0} = \frac{1}{n+1}\left[(2n+1)\frac{1}{r_0} X_{n,0} + n\left(A^2 - \frac{1}{r_0^2}\right) X_{n-1,0}\right] \qquad (n > 0) \quad (22)$$

Recurrence relations for the longitudinal Chebyshev coefficients:

$$Z^{(1)}_{m+1,k} = D\left(Z^{(1)}_{m,k-1} + Z^{(1)}_{m,k+1}\right) + 2\cos\lambda_0\, Z^{(1)}_{m,k} - Z^{(1)}_{m-1,k} \qquad (\text{all } m,\ \text{all } k) \quad (23)$$

$$Z^{(2)}_{m+1,k} = D\left(Z^{(2)}_{m,k-1} + Z^{(2)}_{m,k+1}\right) + 2\cos\lambda_0\, Z^{(2)}_{m,k} - Z^{(2)}_{m-1,k} \qquad \text{(all m, all k)} \tag{24}$$

Recurrence relations for the latitudinal Chebyshev coefficients ($|\phi| \leq 45^{\circ}$):

$$Y^{m}_{n+1,j} = \frac{2n+1}{n-m+1}\,\frac{C}{2}\left(Y^{m}_{n,j-1} + Y^{m}_{n,j+1}\right) + \frac{2n+1}{n-m+1}\sin\phi_0\, Y^{m}_{n,j} - \frac{n+m}{n-m+1}\, Y^{m}_{n-1,j} \qquad (\text{all } j,\ n \geq m \geq 0) \tag{25}$$

$$Y^{n}_{n,j} = (2n-1)(2n-3)\left[-\frac{C^2}{4}\left(Y^{n-2}_{n-2,j-2} + Y^{n-2}_{n-2,j+2}\right) - C\sin\phi_0\left(Y^{n-2}_{n-2,j-1} + Y^{n-2}_{n-2,j+1}\right) + \left(1 - \sin^2\phi_0 - \frac{C^2}{2}\right) Y^{n-2}_{n-2,j}\right] \qquad (\text{all } j,\ n \geq 2) \tag{26}$$

Recurrence relations for the latitudinal Chebyshev coefficients ($|\phi| > 45^{\circ}$):

$$Y^m_{n+2,i} = \frac{(2n+3)}{(n+1-m)(n+2-m)} \Bigg\{ \left[- \frac{(n+m)(n+m-1)}{(2n-1)} \right] Y^m_{n-2,i}$$
$$+ \left[- (2n+1)(C \cos \phi_o) \right] \left(Y^m_{n,i+1} + Y^m_{n,i-1} \right)$$
$$+ \left[- (2n+1) \frac{C^2}{4} \right] \left(Y^m_{n,i+2} + Y^m_{n,i-2} \right) \tag{27}$$
$$+ \left[- \frac{(n+1-m)(n+1+m)}{(2n+3)} \right.$$
$$+ (2n+1) \sin^2 \phi_0 - (2n+1) \frac{C^2}{2}$$
$$\left. - \frac{(n+m)(n-m)}{(2n-1)} \right] Y^m_{n,i} \Bigg\} \quad (\text{all } i,\ n \geq m \geq 0)$$

$$Y^{n+1}_{n+1,i} = (2n+1) \left[\cos \phi_0 \, Y^n_{n,i} + \frac{C}{2} \left(Y^n_{n,i+1} + Y^n_{n,i-1} \right) \right] \quad (\text{all } i,\ n \geq 0) \tag{28}$$

The derivation of these recurrence relations is omitted here; some detail is given in Reference 1. It should be noted that, although the same symbol is used in each case, the Y's of Equations (25) and (26) are defined differently than the Y's of Equations (27) and (28). There should be no confusion since Equations (25) and (26) are intended only for the equatorial region, while Equations (27) and (28) apply to the polar regions.

2.2 ERROR MEASUREMENTS FOR THE CHEBYSHEV REPRESENTATION

This section addresses the question of how closely a Chebyshev gravity representation matches the gravity field defined by the parent spherical harmonic representation. In order to study the Chebyshev expansion error, a computer program was written to numerically evaluate the error for any selected cell. The program first constructs the Chebyshev expansion coefficients for the given spherical harmonic expansion, using the recurrence relations given in Section 2.1. These Chebyshev expansion coefficients are functions of the C_n^m's and S_n^m's; the cell parameters r_o, ϕ_o, and λ_o; and A, C, and D. Then, for a selected maximum degree, the three gravity force components, F_r, F_ϕ and F_λ generated by the Chebyshev expansions (Equations (18) through (20)) are numerically compared with the corresponding force components calculated from the spherical harmonic expansion (Equations (1) through (3)), using a minimum degree of 4. This comparison is made at many points uniformly distributed throughout the given cell, and the maximum difference between the two representations provides a measure of the Chebyshev expansion error. All of the error measurements in this paper apply to Chebyshev representations based upon the GEM10B 36 x 36 gravity model.

Figures 1 and 2 show the numerically computed error as a function of the cell size parameter A. For simplicity, the latitude size parameter C, and the longitude size parameter D, remained equal to A as A was varied. Figures 1 and 2 show the error for cells at reference heights of 967 kilometers and 255 kilometers, respectively. On each figure, a reference error level at 10^{-10}g is indicated. Order of magnitude estimates place the resultant orbit error at less than 0.1 meters for a 5-day orbit propogation subject to a high-frequency gravity error having this amplitude. The maximum degrees for each of the Chebyshev components were equal to one another and are indicated for each group of curves in the figure. For example, in Figure 1, the upper group of curves represents the error in the three-force components as a function of A for a 3 x 3 x 3 Chebyshev expansion.

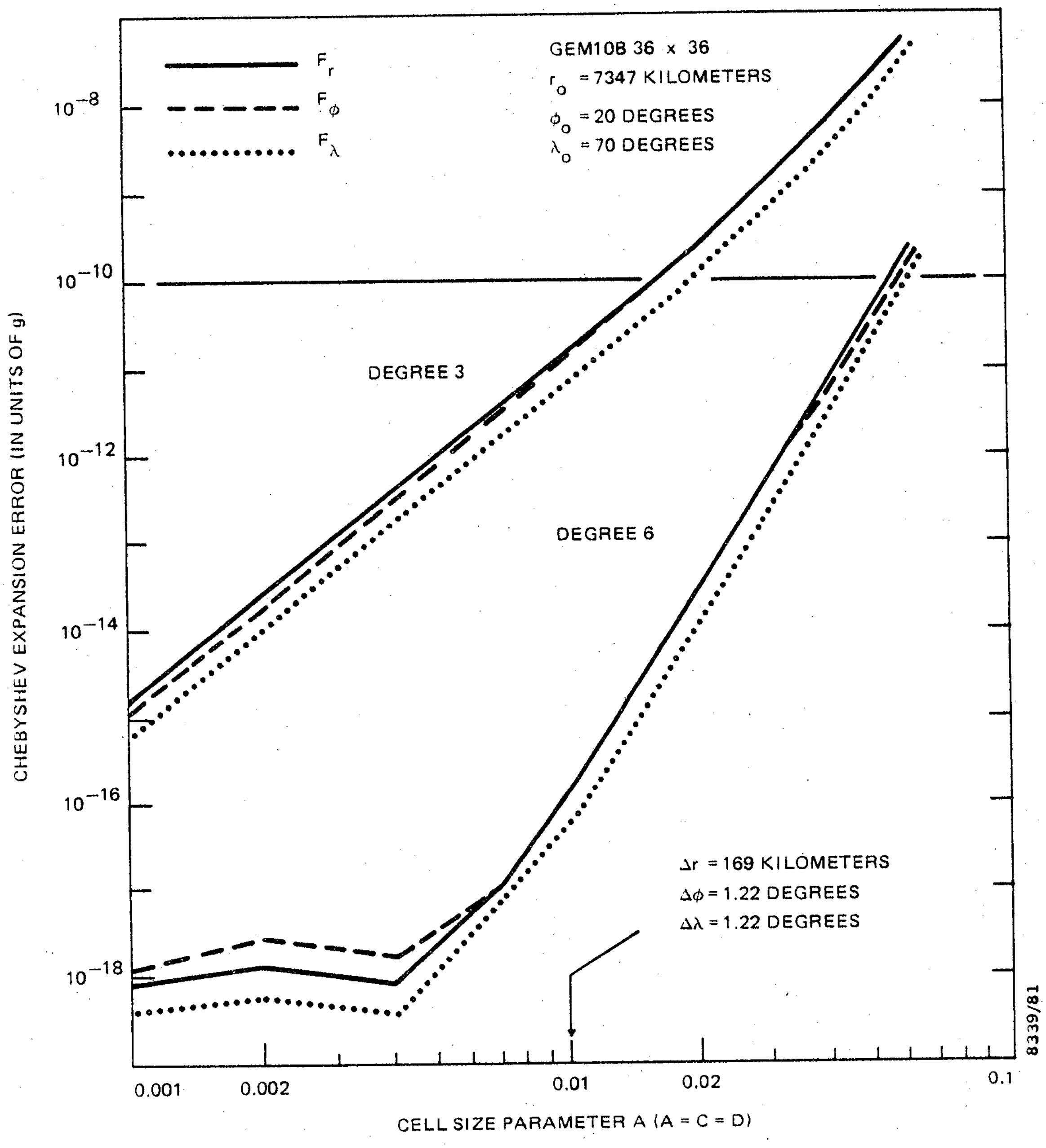

Figure 1. Numerical Measurement of Chebyshev Gravity Representation Error as a Function of Cell Size and Expansion Degrees (Height of Cell Center = 967 Kilometers)

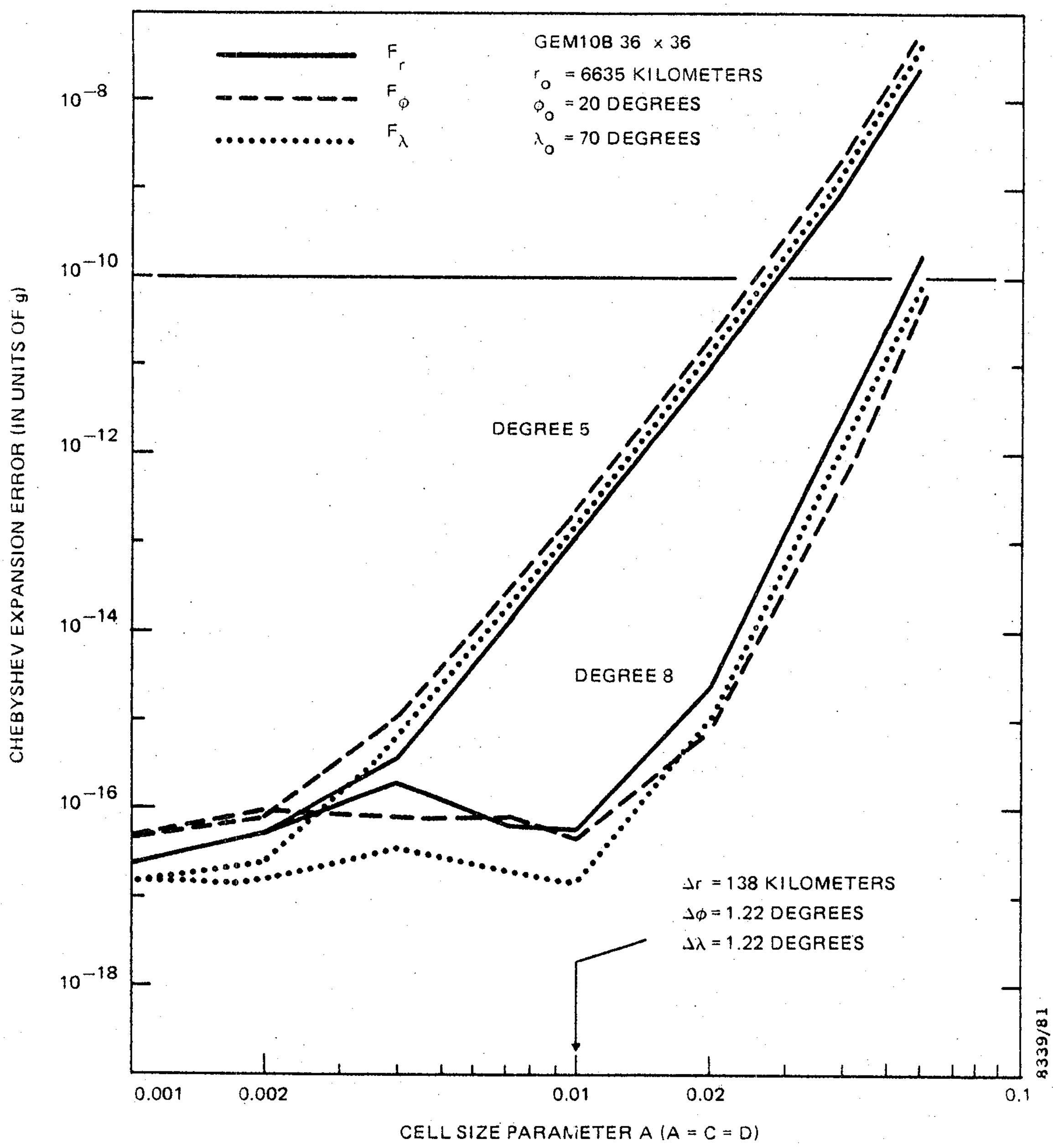

Figure 2. Numerical Measurement of Chebyshev Gravity Representation Error as a Function of Cell Size and Expansion Degrees (Height of Cell Center = 255 Kilometers)

Each of the error curves in Figures 1 and 2 has a range, for intermediate values of A, where the curve is nearly a straight line. In this range, the slope of this straight line, on a log-log scale, is one greater than the maximum degree of the Chebyshev expansion; i.e., the error varies as the cell size to the $K_{max}+1$ power, where K_{max} is the maximum Chebyshev degree. (This rule does not seem to be accurate for the larger values of K_{max}.) For larger values of A, the curves bend away from the straight line. For very small values of A, a numerical noise level is reached and the error reaches a lower limit--about 10^{-18}g for Figure 1 and 3×10^{17}g for Figure 2.

Figures 3 and 4 show the numerical error as a function of latitude for a $5^{\circ} \times 5^{\circ}$ cell, using a 6 x 6 x 6 polynomial degree expansion. The cell thickness was chosen to be small, at a value of 12.8 kilometers, to eliminate the effects of radial variation on the error. The results in Figure 3 were obtained using the equatorial zone formulation (Equations (5), (25), and (26)) and those in Figure 4 were obtained using the polar zone formulation (Equations (6), (27), and (28)). The former diverges near the poles and the latter diverges near the equator, so that a global Chebyshev gravity model must be based upon a combination of these two formulations. In Figures 3 and 4, the maximum error in each cell is plotted at the cell center, so that cells centered at 2.5 degrees latitude extend to the equator and cells centered at 87.5 degrees extend to within 0.001 degrees of the pole.

The slight rise in error near the pole in Figure 4 occurs at error sampling points that are 0.75 degrees from the pole. This slight rise is presumably due to factors of $\cos^{-1}\phi$ and an associated loss of precision in the calculation of F_{ϕ} and F_{λ} (Equations (2) and (3)).

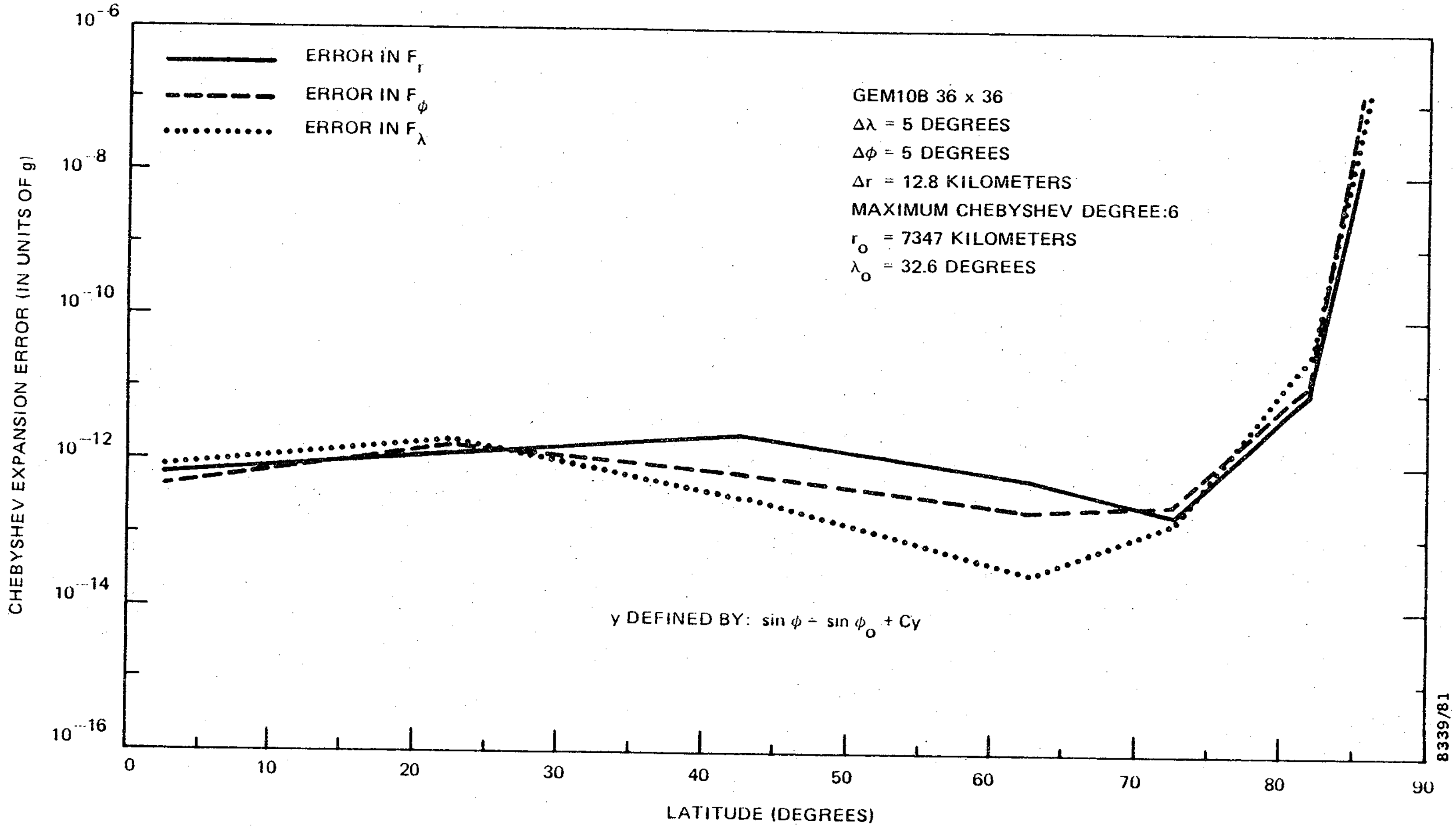

Figure 3. Numerical Measurement of Chebyshev Gravity Representation Error as a Function of Latitude (Equatorial Zone Expansion Used)

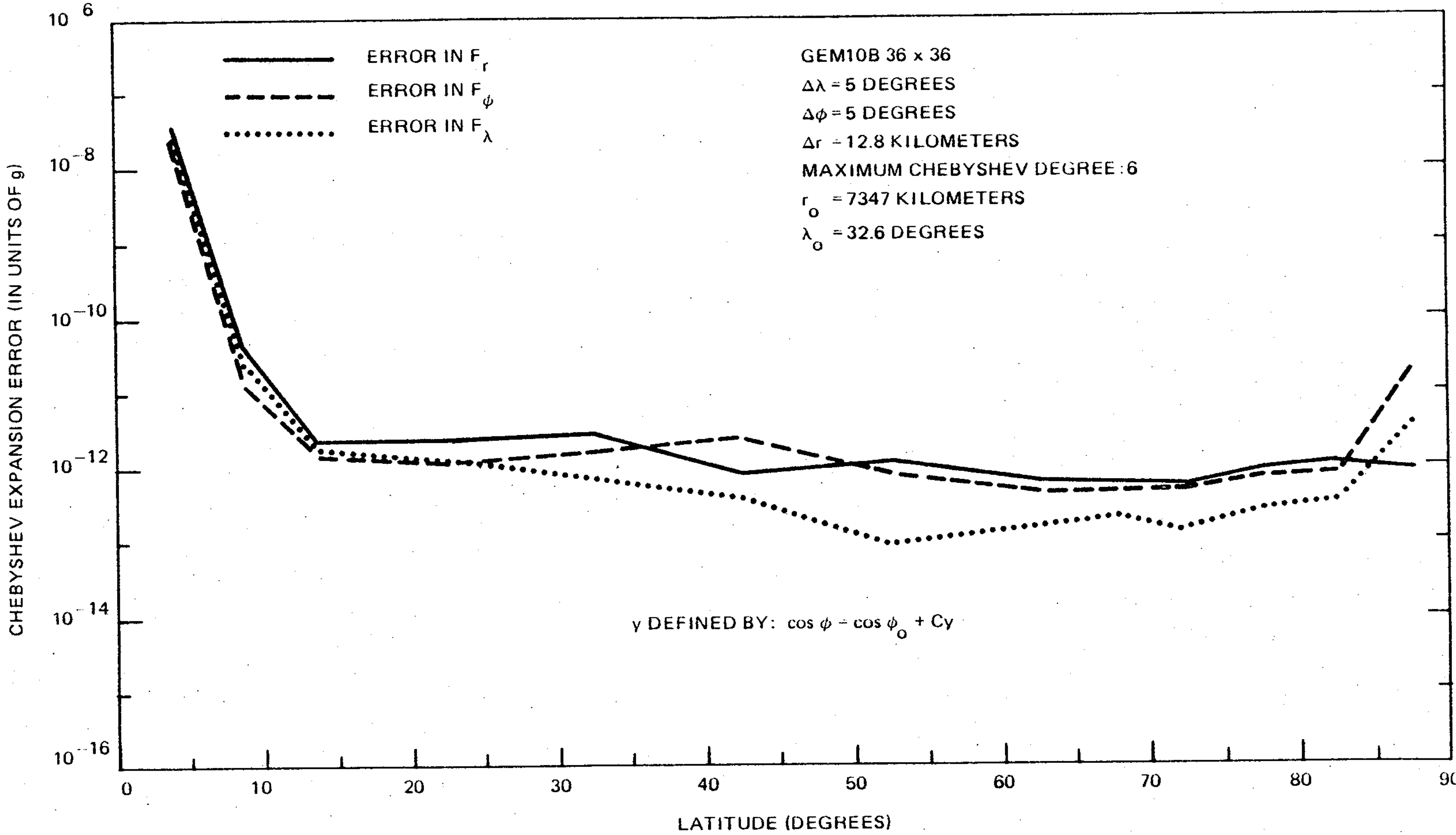

Figure 4. Numerical Measurement of Chebyshev Gravity Representation Error As A Function of Latitude (Polar Zone Expansion Used)

Outside of the latitude regions in which divergence of the Chebyshev expansions is approached, it is clear from Figures 3 and 4 that a uniform level of error is obtained using cells of constant latitudinal and longitudinal dimensions. The solid angle of these cells is much smaller near the poles than near the equator; leading to an unpleasant crowding of cells near the poles in a global Chebyshev model.

2.3 ESTIMATED CHARACTERISTICS OF A GLOBAL CHEBYSHEV GRAVITY REPRESENTATION

The use of the Chebyshev representation for precise satellite orbit determination requires a large, direct-access data set that contains the three-subscripted Chebyshev coefficients for a distribution of cells covering the entire spatial region of interest. The orbit determination program would retain in main memory the coefficients for a small number of cells and would update this working storage as necessary, drawing from the large, direct-access data set. In this section the general characteristics of a sample global Chebyshev representation are estimated.

Table 2 provides data for estimating the speed of the Chebyshev representation, relative to the spherical harmonic representation. For each representation, the table shows the number of machine multiplication or division operations required to evaluate the three force components at a single spatial point. The numbers given assume efficient coding. The maximum degree used in the Chebyshev representation, K_{max}, is assumed to be chosen to be the same for all three indices in the expansions. Comparing the 36 x 36 spherical harmonic representation with the 6 x 6 x 6 Chebyshev representation, the latter requires about 75 percent less time for force evaluation (1,736 operations versus 6,933 operations).

Table 2. Number of Computer Multiplication or Division Operations Needed for Gravity Force Evaluation in the Chebyshev and Spherical Harmonic Gravity Force Representations

CHEBYSHEV REPRESENTATION	
MAXIMUM DEGREE (K_{max})	NUMBER (N_1) OF MULTIPLICATIONS OR DIVISIONS*
3	332
4	640
5	1,098
6	1,736
8	3,669
10	6,685

*$N_1 = 5(K_{max} + 1)^3 + 3K_{max}$

SPHERICAL HARMONIC REPRESENTATION	
MAXIMUM DEGREE (n_{max})	NUMBER (N_2) OF MULTIPLICATIONS OR DIVISIONS**
4	116
8	409
16	1,473
21	2,463
30	4,875
36	6,933
48	12,129

**$N_2 = 5n_{max}^2 + 13n_{max} - 15$

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Since the number of operations in the Chebyshev representation increases as the third power of K_{max}, while the number of operations in the spherical harmonic representation increases as only the square of the maximum degree, it is desirable to choose as small a value as possible for K_{max} in order to achieve a computation time advantage. In order to simultaneously meet accuracy requirements, it is then necessary to properly adjust the cell dimensions.

The characteristics of the Chebyshev model presented in Figure 5 were based upon Table 2 and the results of Section 2.2. This sample model covers the range of many NASA low-altitude spacecraft; an additional layer could be added to extend the model to higher altitudes. The estimate of the total number of three-subscripted Chebyshev coefficients assumes that only three types were necessary. Although the formulation presented in Section 2.1 employed four types of these coefficients, it is expected that there would be no difficulty in modifying the formulation to require only three types.

From Figure 5, it is clear that the computation time advantage of the Chebyshev representation is accompanied by the need for a large, but not unreasonable, amount of direct-access storage.

3. FILE RETRIEVAL FOR GRAVITY FORCE EVALUATION

3.1 FILE RETRIEVAL METHOD

In standard GTDS Differential Correction orbit solutions, the full force model is reevaluated during every iteration. Except for the first and second iterations, corrections to the orbital position are generally so small that the change in position has a negligible effect on the numerical values of most of the spherical harmonic terms in the gravity model.

- ACCURACY: 10^{-10}g FOR GEM10B 36 x 36
- MAXIMUM DEGREE OF EXPANSION: 6 x 6 x 6
- NUMBER OF CHEBYSHEV COEFFICIENTS FOR EACH CELL: 3 x (7 x 7 x 7) = 1029
- CELL SIZE: Δh = 607 KILOMETERS (A = 0.04)
  $\Delta\phi$ = 5 DEGREES
  $\Delta\lambda$ = 5 DEGREES
- CELL DISTRIBUTION: SINGLE LAYER (r_o = 6954 KILOMETERS)
  h_{MIN} = 284 KILOMETERS
  h_{MAX} = 891 KILOMETERS
- NUMBER OF CELLS: 36 x 72 = 2592
- NUMBER OF CHEBYSHEV COEFFICIENTS IN STORAGE: 2592 x 1029 = 2.7 MILLION
- CPU TIME FOR GRAVITY EVALUATION (RELATIVE TO SPHERICAL HARMONICS): 0.25

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Figure 5. Characteristics of a Sample Chebyshev Gravity Model

Rough estimates have indicated that, for a 1-day orbit, a 10-meter error in the argument of the portion of the gravity force that does not include the monopole and quadrupole terms leads to orbital position errors that are well below 0.01 meter. These estimates suggest that considerable computation time could be saved, particularly for a 36 x 36 gravity model, if a file of gravity values was saved for use during the later iterations.

The method of gravity evaluation tested is shown in Figure 6. This figure is a flowchart representing the GTDS subroutine that evaluates the gravity force, F(N x N), for a given input position. A test is first made to determine whether a gravity file value exists for the given integration point. (This method is valid only for fixed-step numerical integration.) If the file value exists, then the position associated with the file is compared with the input position. If the difference is less than a prescribed tolerance, ε, then the file value is accepted. The file value describes that part of the gravity force represented by spherical harmonic terms of degree greater than four. This value is added to the 4 x 4 force calculated for the input position, F(4 x 4), to produce the total gravity force F(N x N).

If the file gravity value does not exist, or if the position deviation $|\vec{\Delta r}|$ is greater than the specified tolerance, ε, then the file is not used. Instead F(N x N), F(4 x 4), and F(FILE) are calculated, F(FILE) is stored for later use, and F(N x N) is returned by the subroutine. The resultant orbit precision of this method is controlled by the specified value of ε.

Not shown in Figure 6 is the treatment for partial derivatives of the gravity force with respect to position. These are stored, retrieved, and calculated in a manner parallel to that of the force components themselves.

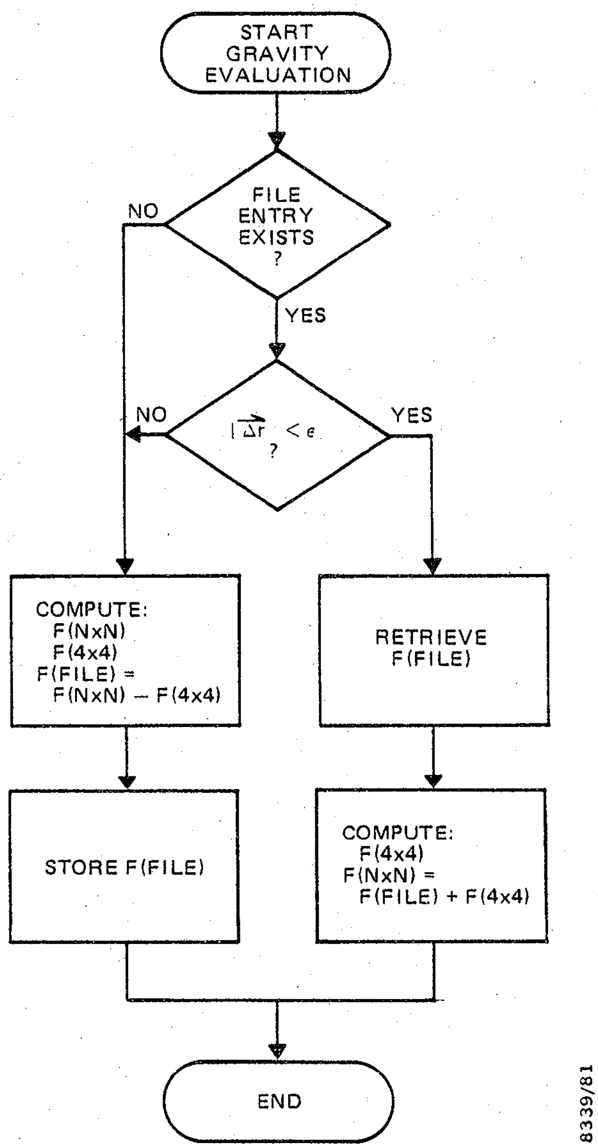

Figure 6. Method for Gravity Force Evaluation Using File Retrieval

3.2 FILE RETRIEVAL RESULTS

In order to test the file retrieval method, two GTDS differential correction orbit solutions, 12 hours in length, were calculated using a 36 x 36 Earth gravity model and using Unified S-Band and laser tracking data. One solution was calculated in the standard way, and the other used the file retrieval method. For the latter solution, the position tolerance, ε, was specified to be 500 meters. Each solution required four iterations to converge, and each differential correction solution was followed by 12-hour ephemeris generation, using the converged orbital elements. The a priori elements for the two solutions were identical, differing from the converged elements by about 80 meters.

A direct comparison between the ephemerides of the two solutions is shown in Figure 7. The position difference between the two solutions is plotted over the solution time interval. Examination of the intermediate results showed that for the first hour, the gravity file was built, but never subsequently updated since the 500-meter tolerance was never exceeded. On the other hand, for the following 11 hours, the gravity file was built during the first iteration, and since the 500-meter tolerance was exceeded during the second iteration (because the first-iteration orbit error progressively worsened with time, and this first-iteration orbit was the basis for the first-iteration file) the file was automatically updated, using positions generally accurate to 5 meters. The last two iterations were calculated with no further updates to the file. This file update history explains the sharp drop in orbit error over the first half hour in Figure 12--from 42 millimeters to the 5-millimeter level.

It is clear from this file update history that the file retrieval method reduces the number of standard gravity force

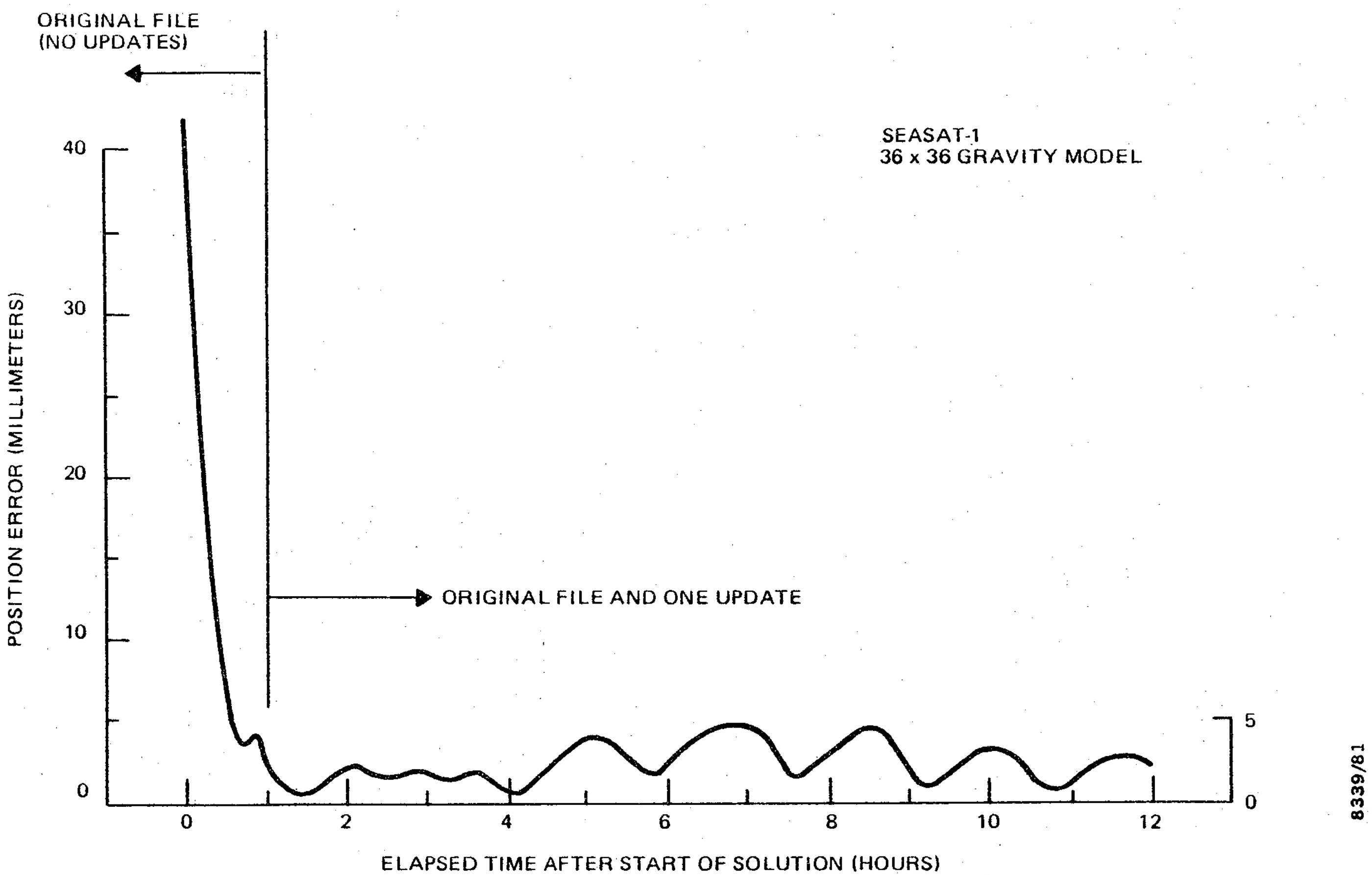

Figure 7. Orbit Error Resulting From Use of Gravity File With Position Tolerance Specified at 500 Meters

evaluations by more than a factor of two without substantial orbit precision loss. The CPU times for the two solutions were 1.23 minutes and 0.69 minutes (IBM S-360/95) for the standard and file retrieval solutions, respectively. These CPU times do not accurately show the full potential computation time reduction of the file retrieval method because, for simplicity, these test calculations did not incorporate file usage into the numerical integration starting algorithms.

4. CONCLUSIONS

The results presented in this paper show that the Chebyshev representation should provide substantial computation time savings for orbit determination using precise Earth gravity models, although its disadvantage is the requirement for a large file of pre-calculated Chebyshev coefficients. Tests of this representation in actual orbit calculations need yet to be performed.

Two areas for possible improvement for the Chebyshev representation are evident. First, truncation of terms in the three-dimensional expansion should be explored. Rather than summing over terms such that i, j, and k range from 0 to K_{max}, it may be possible to sum over terms such that $i + j + k$ ranges from 0 to K_{max}. This type of summation reduction could save a factor of approximately three in both execution time and in direct-access storage. The second improvement would be to extend the formulation so that Cartesian components of the gravity force are directly calculated, rather than spherical components. This would require the derivation of additional recurrence relations for evaluation of the Chebyshev coefficients.

The file retrieval method for gravity evaluation has been shown to be an effective method for reducing computation time without sacrificing orbit accuracy. Combined with the Chebyshev representation, it could almost eliminate computation time problems in orbit determination using currently available, precise gravity models.

REFERENCES

1. Computer Sciences Corporation, CSC/TR-81/6008, A Chebyshev Representation of the Earth's Gravity Field for Precision Satellite Orbit Calculations, R. L. Smith, June 1981