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Shape Determination and Control for Large Space Structures

Connie J. Weeks

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Preface

The work contained in this report was performed while Dr. Weeks was employed at the Jet Propulsion Laboratory. Dr. Weeks is currently an Assistant Professor in the Mechanical and Aerospace Engineering Department at Princeton University, Princeton, New Jersey.

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Abstract

An integral operator approach is used to derive solutions to static shape determination and control problems associated with large space structures. Problem assumptions include a linear self-adjoint system model, observations and control forces at discrete points, and quadratic performance criteria for the comparison of estimates or control forces.

Results are illustrated by simulations, in the one dimensional case with a flexible beam model, and in the multidimensional case with a finite element model of a large space antenna.

Modal expansions for terms in the solution algorithms are presented, using modes from the static or associated dynamic model. These expansions provide approximate solutions in the event that a closed form analytical solution to the system boundary value problem is not available.

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Abbreviations

BVP	Boundary Value Problem
FEM	Finite Element Model
LSS	Large Space Structures

Greek Letters

Ω	The domain in R^{ℓ}
Γ	The boundary of Ω
ψ	The desired shape
Λ	A diagonal matrix with diagonal elements λ_i (section 6.4)
Φ_i	Eigenfunctions (modes) of the associated BVP (multidimensional)
γ_i	Linear combinations of Green's functions (section 2.4)
γ	Free space solution (Chapter 4, Appendix A)
δ	The dirac delta function
ζ_i	The noise in the observation y_i
θ	Angular coordinate of the point P
λ_j	Lagrange multiplier (section 2.6)
λ_j	Eigenvalue of the BVP
μ_i	Eigenvalue of the integral operator K. $\mu_i = \frac{1}{\lambda_i}$ (for $\lambda_i \neq 0$)
ξ,η	x,y coordinates of the point Q in R^2 (Chapter 4)
ρ,ϕ	Polar coordinates of Q (Chapter 4)
ρ_i,ϕ_i	Polar coordinates of P_i (Chapter 4)
ϕ	A test function on Ω (Appendix A)
ϕ_i	Normalized eigenfunction (mode) of the associated boundary value problem
ω_i	System frequencies: $\omega_i^2 = \lambda_i$
∇^2	The Laplacian Operator (Chapter 4)

Upper Case English Letters

A	Matrix in the shape control/determination algorithms
B	Vector in the one dimensional shape control law
B_i	$1 \leq i \leq k_o$, the boundary operators for the system $LU = F$
C_i	Constant matrices operating on the control vectors F_i or the state vectors $U(P_i)$ in the observations
D	A vector in the shape control solution (Chapters 5-6)
D^i	A partial differential operator (Appendix A)
F	The forcing function in the multidimensional BVP
F	The vector of forces $(f_1 \ldots f_m)^T$ or $(F_1^T \ldots F_m^T)^T$
F_i	The control forces applied at positions P_i in the multidimensional problem
$G(P\|Q)$	The n x n matrix Green's function
$G_j(P\|Q)$	The jth column of G
H	Variation in the vector of disturbances F in the estimation problem (section 5.4)
I	An identity matrix of appropriate dimension
J	The performance criterion in the shape determination or shape control problem
K	The integral operator representing the inverse of L
K	The stiffness matrix in a FEM (Chapter 6)
L	The linear differential operator, or n x n matrix of differential operators, which acts on the shape function U
M(P)	The mass matrix in the dynamic BVP
M	The mass matrix in the FEM
N	The sum $\sum_{i=1}^{m} n(i)$.
P,Q,R	Points in Ω
P_i	$1 \leq i \leq m$, the points in Ω where control forces are to be applied, or observations taken
R^ℓ	Euclidean ℓ space

R_i, R_i^{-1}	n(i) by n(i) dimensional weighting matrices in the performance criteria of the control, estimation problems
R, R^{-1}	Block diagonal matrices with diagonal blocks R_i, R_i^{-1} (Chapters 5-6)
R, R^{-1}	Diagonal matrices with diagonal elements r_i, r_i^{-1} (Chapters 2-4)
T	(superscript) denotes transpose
U	The multidimensional (vector) shape function
$\bar{U}$	The vector formed by stacking the vectors $C_j\ U(P_j)$ (Chapters 5-6)
V	Another vector function defined on Ω
V_i	$1 \leq i \leq s$, solutions of the multidimensional homogeneous boundary value problem
W, W^{-1}	Piecewise-continuous weighting matrices in the multidimensional performance criteria (Chapters 5-6)
X	The vector of optimal pointwise shape estimates $(u(P_1)\ldots u(P_m))^T$
X	The state vector in the FEM (Chapter 6)
Y	The vector of observations $(y_1\ldots y_m)^T$ or $(Y_1^T\ldots Y_m^T)^T$
Y_i	The n(i) dimensional observation of $C_i\ U(P_i)$
Z	The vector of observation noises $(\zeta_1\ldots\zeta_m)^T$
Z_i	The noise in the observation Y_i

Lower Case Letters

a_{ij}	Coefficients of the matrix A in the control/estimation laws (Chapters 2-4)
b_i	Coefficients of the vector B in the control law (Chapters 2-4)
c_i, c_{ij}	Constant coefficients
e_j	The jth column vector of the identity matrix
f	Scalar function representing non-conservative forces acting on the system
f_i	Constant scalar force at point P_i (or x_i)

$g(P|Q), g(x|y)$ The scalar Green's function

h A variation in the unknown disturbance function f (section 2)

i, j, k, ℓ Indices of sequences

k_o The number of boundary conditions

ℓ The dimension of the domain

ℓ The length of the flexible beam (Chapter 3)

m The number of observations, or control forces

n The dimension of the state

$n(i)$ The dimension of the observation vector or control force at the point P_i

n_m The number of modes (eigenfunctions) used in approximations

p^i, q^i The ith coordinate of P, Q

r The radial coordinate of P (Chapter 4)

r_i, r_i^{-1} Scalar weights in the performance criteria (Chapters 2-4)

s The number of solutions of the homogeneous BVP

u The scalar shape function (Chapters 2-4)

v_i $1 \leq i \leq s$, the solutions of the homogeneous BVP

x_i x coordinate of the point P_i

y_i Observation of $u(P_i)$

Norms and Products

$\langle U,V \rangle$ $\int_\Omega U^T(P)\, V(P)\, dP$ where U and V are vector functions on Ω

$\langle X,Y \rangle$ where X and Y are vectors, is X^TY

$\langle X,X \rangle_R$ is the weighted inner product X^TRX

The norms are those induced by the inner products:

$$||U||^2 = \langle U,U \rangle \qquad ||X||_R^2 = X^TRX \qquad ||X||^2 = X^TX$$

Other notation is as defined locally.

Static Shape Determination and Control for Large Space Structures

Chapter 1. Introduction and Summary

This report presents the results of the development and simulations of algorithms for the static shape determination and shape control of large space structures (LSS). Observations of positions on the structure, and actuators for subsequent shape control, are assumed located at a relatively few discrete points along its surface.

Quadratic performance criteria are defined to provide a means of determining "best" shape estimates and control forces. The resulting constrained optimization problems are solved using an integral operator approach, which proves ideal for the mixture of continuous and discrete problem elements.

Results are illustrated in the one dimensional case with a flexible beam, and in the multidimensional case for a large space antenna.

1.1 Background

The development of the space shuttle has made it possible to design space structures larger than ever before, which may be carried into space and deployed or assembled there. Examples of such structures include the space platform, which would support experiments, laboratories, observation instruments and even habitation modules, and the solar power satellite, which would collect and transmit solar energy.

Large space antennae, ranging in diameter from 50 meters to one kilometer, are also being planned. They will assist in earth communications, radio and high energy astronomy, the deep space network as orbital relay antennae, and the remote sensing of soil moisture, salinity concentration and climatic conditions on the earth. The latter information would assist agricultural productivity around the world.

Satisfactory performance of these large space structures depends upon the competence of their control systems. Three kinds of control systems must be developed: shape, attitude, and orbit transfer and stationkeeping.

In the past, the major deleterious influence on shape was the interaction between the control system, or systems, and the structural dynamics of the spacecraft. Such interactions were minimized at the design stage, by guaranteeing a large separation between the modal frequencies of the structure and the control system bandwidth. This is accomplished either by stiffening the structure, which increases its natural frequency (and often its weight), or by reducing the control system bandwidth, which usually reduces the control system performance.

However, in the case of the space structures now being designed, the enormous size, coupled with shuttle payload considerations, requires the use of lightweight, flexible materials. On the other hand, the performance criteria are extremely stringent. Furthermore, other influences, in particular gravity and temperature gradients, will exert significant torques on the structure. Thus design considerations are no longer adequate for the maintenance of appropriate shape.

The shape control problem is actually the dual problem of shape determination followed by shape control. Shape determination must be accomplished by the processing of possibly inaccurate observations of a number of predetermined positions along the structure. After the shape is estimated, shape control must be accomplished by means of actuators (control devices) placed at a finite number of discrete (isolated) points, which produce forces or torques in one or more directions at these points. Since the sensing devices and actuators are likely to be both expensive and heavy, in comparison with other structural elements, they will be limited in number and in the choice of their positions.

Thus we require methods for determining and controlling the shape of continuous structures by means of discrete or pointwise observations and control devices. This is referred to as the continuous-discrete nature of the problem.

Within shape control four categories have been identified: dynamic shape control (control of active vibrations), static shape control, model verification, and engineering verification. This report deals with the problem of static shape control for large space structures.

1.2 The Model

In formulating the general system model it is helpful to consider the shape of the dish of a large space antenna. Its ideal or rest shape is a parabolic shape embedded in three dimensional space. If P is a point on the rest shape, the shape of a distorted antenna may be described by a three or six dimensional shape function U(P), which represents the translational and/or rotational displacements in R^3 of the distorted shape from the ideal shape.

Thus we consider an n dimensional state function U(P), defined on a simply connected domain $\Omega \in R^{\ell}$. We assume the state is governed by linear dynamics

$$L\,U(P) = F(P) \qquad \text{for } P \in \Omega, \tag{1}$$

where L is an n x n matrix of differential operators.

Associated with the dynamics (1) is a set of linear homogeneous boundary conditions

$$B_i(U) = 0 \ , \qquad 1 \le i \le k_o, \tag{2}$$

on Γ, the boundary of Ω, which will determine the number of degrees of freedom of the antenna as a whole. The conditions (2) may represent portions of the boundary which are pinned, simply supported, or free.

We will assume the system (1-2) is self-adjoint.

The n dimensional vector function F(P) in (1) represents forces or torques acting on the system. In the shape estimation problem, F represents the unknown forces producing the shape distortion. F is to be determined, along with the shape itself, by means of a set of, possibly inaccurate, observations

$$Y_i = C_i \ U(P_i) + Z_i, \qquad 1 \le i \le m \ , \tag{3}$$

of the shape at the m positions P_i.

In the shape control problem the vector F has the form

$$F(P) = \sum_{i=1}^{m} C_i F_i \ \delta(P-P_i) \ . \tag{4}$$

The representation (4) for F corresponds to the assumption that the forces F_i are to be applied in one or more dimensions at the positions P_i. A force applied to a rotational coordinate is a torque.

To provide a measure of the optimal estimates of the shape and disturbance functions, or alternatively the optimal set of control forces, we will define quadratic performance criteria.

Thus the shape determination and shape control problems become constrained optimization problems, consisting of the following problem elements: A continuous state which satisfies a self-adjoint linear boundary value problem, together with a set of m observations or forces applied at discrete points on the structure, and a quadratic performance criterion, which includes both continuous and discrete components, and serves as a means of comparison of estimates or control forces.

1.3 Approach and Procedure

We will apply an integral operator approach to the solution of both the static shape determination and shape control problems in the following manner: for a given forcing function F, the solution U of the boundary problem (1-2) may be expressed in terms of an integral operator K:

$$U(P) = KF = \int_{\Omega} G(P|Q)\ F(Q)\ dQ \tag{5}$$

where the function $G(P|Q)$ is the Green's function, or influence coefficient, corresponding to the system (1-2). The integral operator K in (5) represents the inverse of the operator L on an appropriate space of functions. The use of the integral expression (5) in place of the differential boundary value problem (1-2) eliminates some or all of the constraints in the optimization problem, and proves particularly advantageous in the case of a continuous-discrete problem mix.

Procedure

We will begin by solving the static shape control and estimation problems for a one-dimensional shape function u, in Chapter 2. The results will be illustrated in Chapter 3 by simulations of a flexible beam, for both simply supported and pinned-free boundary conditions.

Consideration of the one dimensional case has several advantages: It is easier to use intuition about the results, and it is possible to be specific about the identity of the operator L and its inverse K. Thus exact solutions may be computed, and compared with solutions from modal approximations of the type which must be used in the multidimensional case.

In Chapter 4 the results derived in Chapter 2 are applied to the case that L is a partial differential operator. The static shape distortion of a circular membrane and a rectangular plate are considered as examples. The analytical results are similar to those for an ordinary differential operator, but it is clear that even when the operator L is known, the specific Green's function for a system governed by a partial differential equation may be difficult or impossible to compute. Approximate algorithms using the system modes (eigenfunctions), which can still be computed analytically, are also presented.

In Chapters 5 and 6 multidimensional shapes, corresponding to most LSS models, are considered. In Chapter 5 the theory is developed. It parallels the theory for the one dimensional case, with some exceptions. The differential operator and the Green's function are matrix operators. Observations and control forces may be applied to only some of the components of the state at each point. Furthermore, in most cases the differential operator L and the system modes are not explicitly known. Thus the modes must be computed experimentally, or by a modeling method such as the finite element method. Approximate solutions based on eigenfunction expansions corresponding to the static model are presented.

In Chapter 6, in order to apply results to a finite element model of a large space antenna, the methods of Chapter 5 are adapted to the use of eigenfunctions supplied by a dynamic (time-varying) model. A description of the finite element method is presented. The control problem is used to demonstrate the exact correspondence between solutions of the continuous static problem and the finite dimensional static model of the finite element method. Finally, results are illustrated by simulations, using data from a finite element model of a large space antenna.

Conclusions and future work are stated in Chapter 7.

The appendices include program listings and outputs for the simulations of the flexible beam (Appendix B) and the LSS antenna (Appendix C).

Appendix A contains a simplified sketch of distribution theory, the mathematical theory within which the use of the delta "function" may be considered legitimate. It also contains a proof of the identity of the free space solution of $\nabla^4\gamma = -\delta(P-Q)$, which is a part of the Green's function for the operator ∇^4.

1.4 A Comment on the Approach

The integral operator approach is ideally suited to the continuous-discrete problems of LSS shape control and determination. Physically the Green's function represents the response of the system to a unit impulsive force at one point. Thus, the shape control problem, for example, becomes merely the problem of determining the linear combination of Green's functions or responses at each point which produce the best approximation to the desired shape.

The analytical problem of handling a continuous-discrete mathematical mixture can prove messy or awkward. The integral equation approach reduces the elements of the shape control and determination problems either to purely discrete or purely continuous problems which are more easily handled.

In addition, no approximations, other than the initial assumptions of linearity and pointwise application of forces or observations, which are common to most engineering approaches, are applied until the final computation of the solution algorithms. This approach has value in both its simplicity and its generality. Intuition about the behavior of the system can be retained to the final computation stage.

For example, it is easy to determine the additional constraints which must be applied in the case that the system has rigid body modes (eigenfunctions corresponding to zero frequencies), and to understand their physical interpretation.

Furthermore, the shape control and estimation algorithms are not dependent on a particular model, since the only dynamical assumptions are that the system is linear and self-adjoint. A change in the model does not necessitate a change in the method, only a change in the eigenfunctions used to approximate elements in the algorithms. The eigenfunctions can be provided by lumped mass finite element models, which are themselves linear and self-adjoint.

Finally, the use of integral operators rather than differential ones possesses these general advantages:

(1) The expression of a solution as an integral equation automatically incorporates the boundary conditions, which must be stipulated separately if the problem is stated as a differential equation.

(2) The integral operator is usually bounded and often completely continuous, whereas differential operators are unbounded. Thus results concerning eigenfunction expansions, solutions of nonhomogeneous equations etc. are more easily obtained.

(3) Numerical approximations and variational techniques which include several other methods of solving problems with constraints are more easily applied to integral rather than differential equations.

Chapter 2. Static Shape Control in One Dimension

2.1 Introduction

In this chapter we present the general theory for a one dimensional shape, which will be illustrated by a flexible beam model in Chapter 3. While the shape of a large space structure is usually modeled as multidimensional, consideration of the one dimensional case possesses several advantages:

1) It is possible to be explicit about the identities of the differential operator L and its inverse, the integral operator K. Thus exact solutions to the shape determination and control problems may be computed.

2) Intuition about the physical meaning of results may be applied more easily to the one dimensional case.

Procedure

In section 2.2 we define the general linear boundary value problem (BVP) satisfied by a one dimensional shape function u, and discuss the existence of solutions. In section 2.3 we define the corresponding Green's function, and demonstrate its role in the solution of the BVP. We discover a mathematical distinction between the problem of shape control and those of attitude control and stationkeeping.

We will state general shape control and determination problems for a one dimensional state in section 2.4 and 2.5, and use the Green's function to derive algorithms for their solution.

In section 2.6 we will present eigenfunction expansions which may be truncated to provide approximations to elements of the shape control and estimation algorithms. Since in the multidimensional case approximations must be used, it is interesting to compare them to the exact solutions available in the one dimensional case.

Conclusions are stated in section 2.7.

2.2 The Boundary Value Problem

Consider a surface which occupies a simple connected region $\Omega \epsilon R^{\ell}$ and is bounded by the curve Γ.

Assume the surface is acted on at each point $P \epsilon \Omega$ by a force $f(P)$, and that the static deformation $u(P)$ of the surface satisfied the partial differential equation

$$Lu = f \tag{6}$$

where L is a linear ordinary or partial differential operator, related to the stiffness of the structure, which also satisfies linear boundary conditions

$$B_i(u) = 0 \;, \qquad 1 \leq i \leq k_o, \quad \text{for } P \epsilon \Gamma. \tag{7}$$

Assume the boundary conditions (7) are such that the operator L is self-adjoint. That is

$$\langle Lu,\, v\rangle = \langle u, Lv\rangle \tag{8}$$

for any pair of functions (u,v) in an appropriate class which satisfy the boundary conditions. (The term "appropriate class" is purposely vague. See Appendix A.) The inner product $\langle u,v\rangle$ is defined to be the integral

$$\langle u,v\rangle = \int_{\Omega} u(Q)\; v(Q)\; dQ \;. \tag{9}$$

Solutions of boundary value problems do not always exist. Before the Green's function can be defined and its role in the solution of (6-7) discussed, it is helpful to recall the following rule from linear differential equations, which gives sufficient reasons for the existence of a solution: Consider the self-adjoint boundary value problem (6-7) and its corresponding homogeneous problem

$$Lv = 0, \qquad B_i(v) = 0 \;, \qquad 1 \leq i \leq k_o. \tag{10}$$

Then (a) The system (6-7) has a unique solution for each f if and only if the homogeneous system (10) has only the trivial solution.

(b) If (10) has non-trivial solutions, the problem (6-7) has no solution unless the consistency condition

$$\langle f, v\rangle = \int_{\Omega} f(Q)\, v(Q)\, dQ = 0 \tag{11}$$

is satisfied for every v(P) which is a solution of (10). This rule is a simplification of Theorem 5.1 in Chapter 5.

<u>Remark 2.1</u>: If a solution u(P) of (6-7) exists, and $v_1, \ldots, v_s$ are independent non-trivial solutions of (10), then u is not a unique solution, since

$$u + \sum_{i=1}^{s} c_i v_i \tag{12}$$

is a solution of (6-7) for any set of constants c_i.

<u>Remark 2.2</u>: The consistency condition (11) becomes reasonable when we consider that seeking a solution to (6-7) for any function f in some space is equivalent to seeking the inverse of the operator L on that space. If the null space of L is zero (i.e. the solution of (10) is only the trivial solution) then L is one to one and its inverse may be defined. If (10) has non-trivial solutions, L is not one to one and L^{-1} may be defined, if at all, not uniquely on the range of L. The "consistency condition" guarantees that f has no component in the null space of L, hence (with a little more work) that it is in the range of L.

2.3 The Green's Function

We first consider case (a) of the rule in the previous section. The corresponding homogeneous problem (10) has only the trivial solution. Then the Green's function for the problem (6-7) satisfies

$$Lg(P|Q) = \delta(P-Q) \qquad \text{for } P,\ Q \in \Omega , \tag{13}$$

$$B_i(g) = 0 , \quad 1 \leq i \leq k_o, \quad \text{for } P \in \Gamma . \tag{14}$$

It represents the response of the system at the point P to a unit impulsive force at Q. $\delta(P-Q)$ is the dirac delta function.

Since L is self-adjoint, and both u and g satisfy the boundary conditions, we have

$$\langle u,Lg\rangle = \langle Lu,g\rangle \tag{15}$$

which implies that

$$u(P) = \int u(Q)\ \delta(P-Q)\ dQ = \int_\Omega g(P|Q)\ f(Q)\ dQ . \tag{16}$$

Remark 2.3: Because the BVP (6-7) is self-adjoint, $g(P|Q)$ is symmetric, that is $g(P|Q) = g(Q|P)$. [2] This is proved in the multidimensional case as Theorem 5.2 in Chapter 5.

Remark 2.4: The Green's function is the kernel of the compact integral operator K such that

$$Kf = \int_\Omega G(P|Q)\ f(Q)\ dQ . \tag{17}$$

K is clearly the inverse of the operator L, where defined on the range of L, since $KLu = Kf = u$ and $LKf = Lu = f$.

Remark 2.5: The solution of (13) is called a fundamental solution. The equation (13) is satisfied in a distributional rather than a pointwise sense. That is

$$\langle Lg,\phi\rangle = \langle G,L^*\phi\rangle = \phi(\xi) \tag{18}$$

for all test functions ϕ. (A test function is an infinitely differentiable function defined on R^ℓ which has compact support. See Appendix A.)

The Modified Green's Function

We now consider case (b). Suppose the problem (10) has s independent solutions $v_1,\ldots,v_s$, which we assume have been made orthonormal with respect to the inner product (9). We may not define the Green's function as in (13-14) because

$$\langle\delta(P-Q),\ v_i\rangle = \int_\Omega v_i(Q)\ \delta(P-Q)\ dQ = v_i(P) \neq 0\ . \tag{19}$$

Thus the consistency condition (11) is not satisfied. Therefore, we define the modified Green's function $g(P|Q)$ which satisfies

$$Lg(P|Q) = \delta(P-Q) - \sum_i v_i(P)\ v_i(Q) \tag{20}$$

$$B_i(g) = 0\ , \qquad 1 \leq i \leq k_o\ . \tag{21}$$

We have subtracted the offending components of $\delta(P-Q)$ which lie in the nullspace of L. A solution to this system does exist. It is not unique, however, since the addition of any linear combination of the solutions $v_1,\ldots,v_s$ is also a solution of (20-21). We therefore impose an additional constraint on g:

$$\langle g(P|Q),\ v_i\rangle = 0\ , \qquad 1 \leq i \leq s\ . \tag{22}$$

The function which satisfies (20-22) is the unique Green's function of minimum norm, that is, the Green's function which itself has no component in the nullspace of the operator L.

We apply the relation (15) to the modified Green's function. We note that

$$\langle g,\ Lu\rangle = \int_\Omega g(P|Q)\ f(Q)\ dQ$$

and

$$\langle u, Lg\rangle = \int_\Omega u(Q)\ (\delta(P-Q) - \sum_i v_i(P)\ v_i(Q))$$

$$= u(P) - \sum_{i=1}^{s} (\int_\Omega u(Q)\ v_i(Q)\ dQ)\ v_i(P)$$

$$= u(P) - \sum_{i=1}^{s} c_i\ v_i(P)\ .$$

Thus

$$u(P) = \int_\Omega g(P|Q)\ f(Q)\ dQ + \sum_{i=1}^{s} c_i\ v_i(P)\ . \tag{23}$$

The arbitrary constants c_i are an expected consequence of Remark 2.1. For reasons given in the next segment, we may neglect the last term of (23).

Rigid Body Modes

As will be seen in the examples, the solutions of the homogeneous BVP(10) are the rigid body modes, or degrees of freedom, of the system. They represent changes in position the structure may take as a rigid body. The pinned-free beam in section 3.3 has one rigid body mode: it may rotate about the pinned endpoint.

If a structure has free-free boundary conditions, which represent a structure floating freely in space, it may rotate or translate without a change in its shape. In three dimensions this implies up to six rigid body modes.

If the boundary is firmly fixed, the structure will have no rigid body modes. This is the case with the simply supported beam in Chapter 3, the distorted membrane and plate of Chapter 4, and the large space antenna with fixed hub in Chapter 6.

Since shape distortion is measured with respect to the structure itself, it is reasonable to define a structure-centered coordinate system: the origin and axes are defined to be along the structure. To such a coordinate system the rigid body modes are invisible, and the solution of (6-7) for case (b) becomes (16), as for case (a). Since the constants in (23) are arbitrary, no generality is lost in this assumption.

The consistency condition (11) will be seen to imply that no net forces or torques may be applied in the direction of any degree of freedom. Were this not so, an acceleration would result, contradicting the assumed boundary conditions.

This condition (11), coupled with condition (22) on g and the arbitrary constants in (23), imply that the rigid body modes are both invisible to the shape control system and beyond its powers of influence. Translational and rotational motions must be controlled by the other control systems. Attitude control, orbit transfer and stationkeeping. This is the mathematical distinction between the systems mentioned in section 2.1.

2.4 The Static Shape Control Problem

In this section we define a general shape control problem for one dimensional shape functions. We first solve the control problem assuming case (a) of the rule in section 2.3. We then discuss the solution for case (b), which is slightly more complicated, due to extra constraints imposed by the consistency condition.

We assume the control devices are located at the points P_i, $1 \leq i \leq m$, along the structure. The general model for the control problem is

$$Lu = \sum_{i=1}^{m} f_i \, \delta(P-P_i) \tag{24}$$

$$B_j(u) = 0 , \qquad 1 \leq j \leq k_o \tag{25}$$

where $u(P)$ is the shape, L is a linear differential operator as before, f_i is a force to be applied at the point P_i, and (25) denotes an appropriate set of boundary conditions.

Let ψ be the desired shape of the space structure. Define the criterion

$$J(F,u) = \frac{1}{2} \sum_{i=1}^{m} f_i^{\,2} \, r_i + \frac{1}{2} \int_{\Omega} (\psi(Q) - u(Q))^2 \, dQ \tag{26}$$

as a measure of performance. The constants r_i are arbitrary weights and $F = (f_1 \ldots f_m)^T$.

The control problem is to determine the vector of forces F* which together with the corresponding solution u* of (24-25) minimizes J overall admissible sets (F,u).

Solution of the Control Problem

There are two basic approaches to constrained optimization problems. One is to use Lagrange multiplier theory. We will use this method to solve part of the shape estimation problem.

The other, perhaps more direct method, is to solve the constraints for an expression for some of the variables in terms of the others. This expression is substituted into the function of fewer variables, which can be minimized without constraints.

We will use the second approach in the control problem. We first assume the system has no rigid body modes:

The solution of (24-25) is given by

$$u(P) = \int_\Omega g(P|Q) \, [\sum_{i=1}^{m} f_i \, \delta(P_i - Q)] \, dQ$$

$$= \sum_{i=1}^{m} f_i \, g(P|P_i) \qquad (27)$$

where $g(P|Q)$ satisfies (13-14). Substitution of (27) into the criterion (26) yields

$$J(F) = \frac{1}{2} \sum_{i=1}^{m} f_i^{\,2} \, r_i + \frac{1}{2} \int_\Omega (\psi(Q) - \sum_{i=1}^{m} f_i \, g(Q|P_i))^2 \, dQ \, . \qquad (28)$$

The constrained optimization problem (24-26) has become the simpler problem of minimizing a function of m unknown constants without constraints. Simultaneous solution of the equations

$$\frac{\partial J}{\partial f_i} = 0 \ , \qquad 1 \le i \le m \ , \tag{29}$$

leads to the following necessary condition for an optimal solution $F^* = (f_1{}^* \ldots f_m{}^*)^T$.

$$(R + A)\ F^* = B \tag{30}$$

The m x m matrices R and A have coefficients

$$R_{ij} = r_i\ \delta(i-j) \tag{31}$$

$$A_{ij} = \int_\Omega g(P_i|Q)\ g(P_j|Q)\ dQ \tag{32}$$

and the m dimensional vector B has coefficients

$$B_i = \int_\Omega g(P_i|Q)\ \psi(Q)\ dQ\ . \tag{33}$$

Once the optimal forces are determined, the optimal shape u* is given by (27).

Solution of the Control Problem: Case (b)

We assume that the homogeneous BVP corresponding to (24-25) has s independent solutions $v_1,\ldots,v_s$. This is, of course, equivalent to the assumption that the structure governed by (24-25) has s rigid body modes.

In order for a solution to (24-25) to exist, the consistency condition

$$0 = \langle v_j,\ \sum_{i=1}^m f_i\ \delta(P-P_i)\rangle = \sum_{i=1}^m f_i\ v_j\ (P_i) \tag{34}$$

must be satisfied for each function v_j. Thus the control problem is to determine the set of forces $\{f_i\}$ and shape function u which minimize the criterion (26) subject to the constraints (24-25) and (34).

We will assume the coordinate system is centered on the spacecraft (recall the segment "Rigid Body Modes"). The solution of (24-25) is given by

$$u(P) = \sum_{i=1}^{m} f_i \, g(P|P_i) \tag{35}$$

where g is the modified Green's function which satisfies (20-22).

We first solve the s constraints (34) for the forces $f_1,\ldots,f_s$ in terms of the remaining forces $f_{s+1},\ldots,f_m$.

$$f_i = \sum_{j=s+1}^{m} c_{ij} \, f_j \,, \qquad 1 \le i \le s \,. \tag{36}$$

It is clear that a necessary condition for any solution to exist is that the number m of forces applied must be at least as great as s, the number of rigid body modes. If we wish to obtain an optimal solution m must be greater than s, since for m = s the condition (34) determines the forces uniquely.

Substitution of (36) into (35) yields

$$u(P) = \sum_{i=s+1}^{m} \left(g(P|P_i) + \sum_{j=1}^{s} c_{ji} \, g(P|P_j) f_i\right) \tag{37}$$

Define

$$\gamma_i(P) = g(P|P_i) + \sum_{j=1}^{s} c_{ji} \, g(P|P_j) \tag{38}$$

then

$$u(P) = \sum_{i=s+1}^{m} \gamma_i(P) \, f_i \tag{39}$$

We substitute expressions (36) and (39) into the performance criterion, which results in

$$J = \frac{1}{2} \sum_{i=1}^{s} \left(\sum_{j=s+1}^{m} c_{ij} \, f_j\right)^2 r_i + \sum_{i=s+1}^{m} f_i^{\,2} \, r_i$$

$$+ \frac{1}{2} \int_{\Omega} \left(\psi(P) - \sum_{s+1}^{m} \gamma_i(P) \, f_i\right)^2 dP \,. \tag{40}$$

The criterion is now a function of the (m-s) constants $f_{s+1},\ldots,f_m$, without constraints. J is minimized by solving simultaneously the (m-s) conditions

$$\frac{\partial J}{\partial f_i} = 0 \,, \qquad i = s+1,\ldots,m \,. \tag{41}$$

Let $\hat{F}$ and $\hat{B}$ be (m-s) dimensional vectors with components

$$\hat{F}_i = f^*_{i+s} \tag{42}$$

$$\hat{B}_i = \int_\Omega \gamma_{i+s}(P)\ \psi(P)\ dP \tag{43}$$

and the (m-s) square matrices $\hat{R}$ and $\hat{A}$ with components

$$\hat{R}_{ij} = r_{i+s}\ \delta(i-j) \tag{44}$$

$$\hat{A}_{ij} = \sum r_k\ c_{ki}\ c_{kj} + \int_\Omega \gamma_i(P)\ \gamma_j(P)\ dP\ . \tag{45}$$

Then the optimal control law for the control problem (24-26)(34) is

$$(\hat{R} + \hat{A})\ \hat{F} = \hat{B}\ . \tag{46}$$

Once the optimal forces $f^*_{s+1},\ldots,f^*_m$ are determined from (46), the optimal forces $f^*_1,\ldots,f^*_s$ may be found from (36), and the resulting optimal shape is given by (35).

The non-constant terms in $\hat{A}$ and $\hat{B}$ are linear combinations of terms of the form (32) and (33) respectively.

2.5 The General Estimation Problem

For the estimation problem we assume the shape u(P) satisfies the boundary value problem

$$Lu = f,\quad B_i(u) = 0\ ,\qquad 1 \le i \le k_o\ , \tag{47}$$

where f(P) is an unknown function representing disturbances or inaccuracies in the model. Sensors placed at the positions P_i, $1 \le i \le m$, yield the observations

$$y_i = u(P_i) + \zeta_i \tag{48}$$

where ζ_i is an unknown constant representing inaccuracy in the observation at P_i. Let $Z = (\zeta_1 \ldots \zeta_m)$. We define the performance criterion

$$J(Z,f) = \frac{1}{2}\sum_{i=1}^{m} \zeta_i^{\,2}\, r_i^{\,-1} + \frac{1}{2}\int_\Omega f^2(Q)\, dQ \; .$$

$$= \frac{1}{2}\sum_{i=1}^{m} (y_i - u(P_i))^2\, r_i^{\,-1} + \frac{1}{2}\int_\Omega f^2(Q)\, dQ \; . \tag{49}$$

The estimation problem is to determine the pair (u^*,f^*) which jointly satisfy (47-48) and minimize the criterion (49) over all admissible pairs (u,f).

Solution of the Estimation Problem: Case (a)

We assume there are no rigid body modes. Then the solution to (47) is given by

$$u(P) = \int_\Omega g(P|Q)\, f(Q)\, dQ \tag{50}$$

where $g(P|Q)$ again satisfies (13-14). Thus

$$u(P_i) = \int_\Omega g(P_i|Q)\, f(Q)\, dQ \; . \tag{51}$$

We substitute (51) into the criterion (49), which produces the criterion

$$J(f) = \frac{1}{2}\sum_{i=1}^{m} (y_i - \int_\Omega g(P_i|Q) f(Q)\, dQ)^2\, r_i^{\,-1} + \frac{1}{2}\int_\Omega f^2(Q)\, dQ \; . \tag{52}$$

The problem is now to minimize the functional J without constraints. A necessary condition for a minimum of J at f^* is that the differential

$$\partial J(f^*,h) = 0 = \sum_{i=1}^{m} r_i^{-1}(y_i - \int_\Omega g(P_i|(Q) f^*(Q) dQ)(-\int_\Omega g(P_i|Q) h(Q) dQ)$$

$$+ \int_\Omega f^*(Q) h(Q) dQ \; . \tag{53}$$

for all admissible variation h. (The unknown noise function f and variation h may be assumed to be in $L_2(\Omega)$, for example.) Thus it may be concluded that

$$f^*(P) = \sum_{i=1}^{m} r_i^{\,-1}\, g(P|P_i)(y_i - u^*(P_i)) \; . \tag{54}$$

Substitution of this relation into (50) yields the optimal shape estimate

$$u^*(P) = \sum_{i=1}^{m} [r_i^{\,-1}(y_i - u^*(P_i)) \int_\Omega g(P|Q)\, g(P_i|Q)\, dQ] \; . \tag{55}$$

Note that $u^*(x)$ is expressed in terms of the unknown discrete shape estimates $u^*(P_i)$. Let

$$X = (u^*(P_1) \; . \; . \; . \; u^* \; (P_m))^T \tag{56}$$

and

$$Y = (y_1 \; . \; . \; . \; y_m)^T \; . \tag{57}$$

Evaluation of (55) at $x = x_j$, $j = 1,\ldots, m$ yields the following necessary condition for the vector X:

$$(I + AR^{-1})X = AR^{-1} \; Y \tag{58}$$

where A and R are the matrices of coefficients (31-32).

Once the vector X has been determined the optimal shape estimate is given by (55).

Solution of the Estimation Problem: Case (b)

We now assume the structure described by (47) has s rigid body modes $v_1,\ldots,v_s$, which are orthonormal with respect to the inner product (9). The estimation problem is to determine the pair (u^*,f^*) which minimizes the criterion (49) over all admissible pairs (u,f) which satisfy (47) and the set of consistency conditions

$$\langle f,v_j\rangle = \int_\Omega f(Q) \; v_j(Q) \; dQ = 0 \; , \qquad 1 \leq j \leq s \; . \tag{59}$$

We will show that the solution of the estimation problem for case (b) has the same form as that for case (a):

The solution to (47) is given by (60), where g is the modified Green's function (20-22).

$$u(P) = \int_\Omega g(P|Q) \; f(Q) \; dQ \tag{60}$$

We evaluate (60) at P_i, $1 \leq i \leq m$, and substitute into the criterion (49) producing the criterion (52). Thus we have eliminated part of the constraints, the boundary value problem (47). The estimation problem becomes the problem of minimizing the criterion (52) subject to the remaining constraints (59).

We will apply the Lagrange multiplier theorem [3]. We adjoin the constraints to the criterion by means of scalar multipliers $\{\lambda_j\}$:

$$\tilde{J} = \frac{1}{2} \sum_{i=1}^{m} (v_i - \int_\Omega g(P_i|Q)\ f(Q)\ dQ)^2\ r_i^{-1} + \frac{1}{2} \int_\Omega f^2(Q)\ dQ$$

$$+ \frac{1}{2} \sum_{j=1}^{s} \lambda_j \int_\Omega f(Q)\ v_j(Q)\ dQ\ . \tag{61}$$

A necessary condition for a minimum of $\tilde{J}$ at f^* is that the differentials of $\tilde{J}$ with respect to f and λ_j, $1 \le j \le s$, are 0. We have

$$\frac{\partial J}{\partial \lambda_j} = 0 = \int_\Omega f(Q)\ v_i(Q)\ dQ\ , \qquad 1 \le j \le s\ , \tag{62}$$

and

$$\partial J(f,h) = \sum_{i=1}^{m} (y_i - u(P_i))r_i^{-1}\ (-\int_\Omega g(P_i|Q)h(Q)dQ)$$

$$+ \int_\Omega f\ (Q)\ h(Q)\ dQ + \sum_{j=1}^{s} \lambda_j \int_\Omega h(Q)\ v_j(Q)\ dQ = 0\ . \tag{63}$$

We factor the variation h to one side:

$$\int_\Omega h(Q)\ dQ\ [-(y_i - u(P_i))r_i^{-1}\ g(P_i|Q) + f(Q) + \sum_{j=1}^{s} \lambda_j\ v_j(Q)] = 0$$

Since this must be true for all admissible variations h, the bracketed term must be zero. We have

$$f(Q) = (y_i - u(P_i))r_i^{-1}\ g(P_i|Q) - \sum_{j=1}^{s} \lambda_j\ v_j(Q)\ . \tag{64}$$

We apply the other necessary conditions (62).

$$0 = \langle f, v_k\rangle = \sum_{i=1}^{m} r_i^{-1}(y_i - u(P_i))(\int_\Omega g(P_i|Q)\ v_k(Q)\ dQ)$$

$$- \sum_{j=1}^{s} \lambda_j \langle v_j, v_k\rangle\ , \qquad k = 1,\dots, s\ . \tag{65}$$

The set v_j was chosen orthonormal, so $\langle v_j, v_k \rangle = \delta(j-k)$.

Furthermore, from condition (22) on the modified Green's function we know that

$$\int_\Omega g(P_i|Q)\, v_k(Q)\, dQ = 0 , \qquad k = 1,\ldots, s . \tag{66}$$

Thus we have $\lambda_k = 0$ for $k = 1,\ldots, s$.

We may conclude that

$$f^*(Q) = \sum_{i=1}^{m} g(P_i|Q)\, r_i^{-1}(y_i - c_i\, u(P_i)) \tag{67}$$

as in case (a), and therefore that the optimal shape estimate u^* is also given by (55).

Remark 2.6: Note that because of condition (22) on g, the optimal shape estimate has no component in the direction of the rigid body modes. There may be components in the actual shape, but a shape control system has no means of determining them.

2.6 Approximations

If the Green's function is known, the shape determination and shape control problems may be solved exactly by the methods of this chapter. However, it will be seen in Chapter 4 that when L is a partial differential operator it can be difficult to determine the Green's function. For large space structures, which are multidimensional, the determination of the matrix differential operator L, and consequently the Green's function, is usually impossible.

However, the Green's function, and the terms in the shape control and determination algorithms which involve the Green's function, may be expressed in terms of series expansions involving eigenvalues and eigenfunctions corresponding to the BVP (6-7). Truncations of those series can serve as approximations of the relevant terms. Even when L is not known the eigenfunctions and frequencies can be computed numerically, for example by the finite element method.

Let $\phi_1, \phi_2, \ldots$ be the normalized eigenfunctions of the boundary value problem (6-7), corresponding to the non-zero eigenvalues $\lambda_1, \lambda_2, \ldots$. Then $\{\phi_j\}$ and $\{\lambda_j\}$ satisfy

$$L\,\phi_j(P) = \lambda_j\,\phi_j(P) \qquad \text{for } P \in \Omega, \tag{68}$$

$$B_i(\phi_j) = 0\,, \qquad 1 \le i \le k_o \quad \text{for } P \in \Gamma. \tag{69}$$

Eigenfunctions corresponding to zero eigenvalues are rigid body modes. We have the following expansions:

$$g(P|Q) = \sum_j \frac{1}{\lambda_j}\,\phi_j(P)\,\phi_j(Q) \tag{70}$$

and

$$\int_\Omega g(P|Q)\,f(Q)\,dQ = \sum_j \frac{1}{\lambda_k}\,\phi_j(P)\,\langle\phi_j, f\rangle\ . \tag{71}$$

Substitution of (70) for f in (71) yields

$$\int_\Omega g(P|Q)\,g(Q|R)\,dQ = \sum_j \frac{1}{\lambda_j^2}\,\phi_j(P)\,\phi_j(R)\ . \tag{72}$$

The expressions (71) and (72) provide approximations for the terms B_j and A_{ij} defined by (33) and (32) in the control and estimation algorithms.

The series expansions (70-72) are standard results of linear operator theory [2]. They are based on the assumptions that the integral operator K defined by

$$Kf = \int_\Omega g(P|Q)\,f(Q)\,dQ$$

is a symmetric Hilbert-Schmidt operator. The symmetry follows from the self-adjointness of the boundary value problem. An operator K is Hilbert-Schmidt if

$$||K|| = (\int_\Omega \int_\Omega |g(P|Q)|^2 dPdQ)^{1/2} < \infty . \tag{73}$$

In the case that L is an ordinary linear differential operator, as in Chapter 3, the Green's function is continuous on the compact domain Ω, which implies (73). If g is not known precisely, the property (73) must be assumed.

2.7 Conclusions

An integral operator approach to the continuous-discrete optimization problems of static shape estimation and control proves ideal for these problems. Solutions reduce to the solution of linear equations of dimension less than or equal to the number of observations, or control forces.

A distinction must be drawn between the solutions for systems with rigid body modes and those without. The control law for a system with rigid body modes is more complicated, due to the imposition of extra constraints on the forces, which represent the requirement of zero net forces and/or torques in the directions of these modes.

The estimation procedure for a system with rigid body modes is the same as for a system without them, but the resulting estimate has no component in the direction of the rigid body modes, because they are invisible to the shape estimator. The rigid body modes represent changes in attitude and translational movement, which must be the concern of the attitude control, orbit transfer and stationkeeping systems.

In the event that the Green's function cannot be precisely known, approximations to the terms in the control and estimation algorithms may be computed from eigenfunction expansions available from linear operator theory. The eigenfunctions, often called modes or mode shapes, may be computed numerically even when the operator L is not known.

Chapter 3. Static Shape Control for the Flexible Beam

3.1 Introduction

A flexible beam provides a perfect illustration of static shape distortion and subsequent shape control. Consider a flexible beam which is supported at the end points, and is intended to serve as a bookshelf. The desired shape, or rest shape in the absence of outside forces, is strictly horizontal. However, the forces of gravity act continuously along the beam, causing it to sag in the center.

In order to achieve the desired horizontal shape, we apply a third support under the center of the beam. The natural stiffness of the beam together with the application of this additional force at the center approximately counteract the effects of the gravitational force. Thus we observe static shape control by means of one pointwise force in the ordinary brick and board bookcase.

In this chapter we will solve static shape control and estimation problems for a flexible beam of length ℓ, and boundary conditions representing simply supported, or pinned-free endpoints.

3.2 Shape Control for a Simply Supported Beam

Consider the problem of controlling the static deflection of an elastic beam of length ℓ. Define a coordinate system such that the x-axis passes through the endpoints of the beam, with one end at the origin and the other at $x = \ell$. Suppose control is to be implemented by means of transverse forces f_i at positions x_i, $1 \leq i \leq m$, where $0 < x_1 < x_2 \ldots < x_m < \ell$. See Fig. 3.1.

At each point $x \in [0,\ell]$ denote the deflection by $u(x)$. Assuming no net tensile force on a cross-section, the shape of the beam is governed by the differential equation

$$\frac{d^4u}{dx^4} = \sum_{i=1}^{m} f_i \; \delta(x-x_i) \tag{74}$$

The ends of the beam satisfy the boundary conditions

$$u(0) = u''(0) = 0 \qquad u(\ell) = u''(\ell) = 0 \; . \tag{75}$$

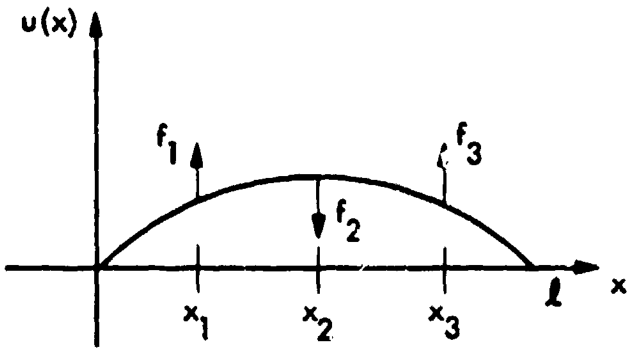


Figure 3.1 The Simply Supported Beam

Let $\psi(x)$ be the desired shape of the beam. As a measure of performance we define the criterion

$$J(u,F) = \frac{1}{2} \sum_{i=1}^{m} f_i^{\,2} \, r_i + \frac{1}{2} \int_0^{\ell} (u(x) - \psi(x))^2 \, dx \tag{76}$$

where F is the vector of forces $(f_1 \ldots f_m)^T$ and r_i are non-negative constant weights whose values are optional.

The object is to determine the set of forces f_i* which together with the solution u*(x) of (74) minimizes (76) over all possible pairs (u,F).

The existence and uniqueness of a solution to (74-75) follows from the fact that the associated homogeneous system

$$\frac{d^4v}{dx^4} = 0 \; , \quad v(0) = v''(0) = 0 \; , \quad v(\ell) = v''(\ell) = 0 \tag{77}$$

has only the trivial solution. Consequently the solution of (74-75) is given by

$$u(x) = \sum_{i=1}^{m} g(x|x_i)\, f_i \tag{78}$$

where $g(x|\xi)$ is the Green's function which satisfies

$$\frac{d^4 g(x|\xi)}{dx^4} = \delta(x-\xi) \tag{79}$$

$$g(0|\xi) = g''(0|\xi) = 0\ , \quad g(\ell|\xi) = g''(\ell|\xi) = 0\ . \tag{80}$$

The Green's function represents the response of the beam shape to a unit impulsive force at $x = \xi$.

The solution of (79-80) is

$$g(x|\xi) = \begin{cases} \dfrac{(\xi-\ell)x}{6\ell}\,(x^2 - 2\ell\xi + \xi^2) & 0 \le x \le \xi \\[2ex] \dfrac{(x-\ell)\xi}{6\ell}\,(x^2 - 2\ell x + \xi^2) & \xi \le x \le \ell. \end{cases} \tag{81}$$

Figure 3.2 displays the Green's functions which correspond to impulsive forces at positions $\xi = n\,(\frac{\ell}{8})$, $n = 1, \ldots, 7$.

The solution of the control problem: Substitution of the solution into the criterion (76) yields

$$J(F) = \frac{1}{2}\sum_{i=1}^{m} f_i^{\,2}\, r_i + \frac{1}{2}\int_0^{\ell} \Big(\sum_{i=1}^{m} g(x|x_i) f_i - \psi(x)\Big)^2\, dx \tag{82}$$

The problem of minimizing the criterion (76) subject to the constraints (74-75) has become the problem of minimizing a function of m unknown constants without constraints. A necessary condition for J to have a minimum at F* is

$$\frac{\partial J}{\partial f_i}\,(F^*) = 0 \qquad 1 \le i \le m \tag{83}$$

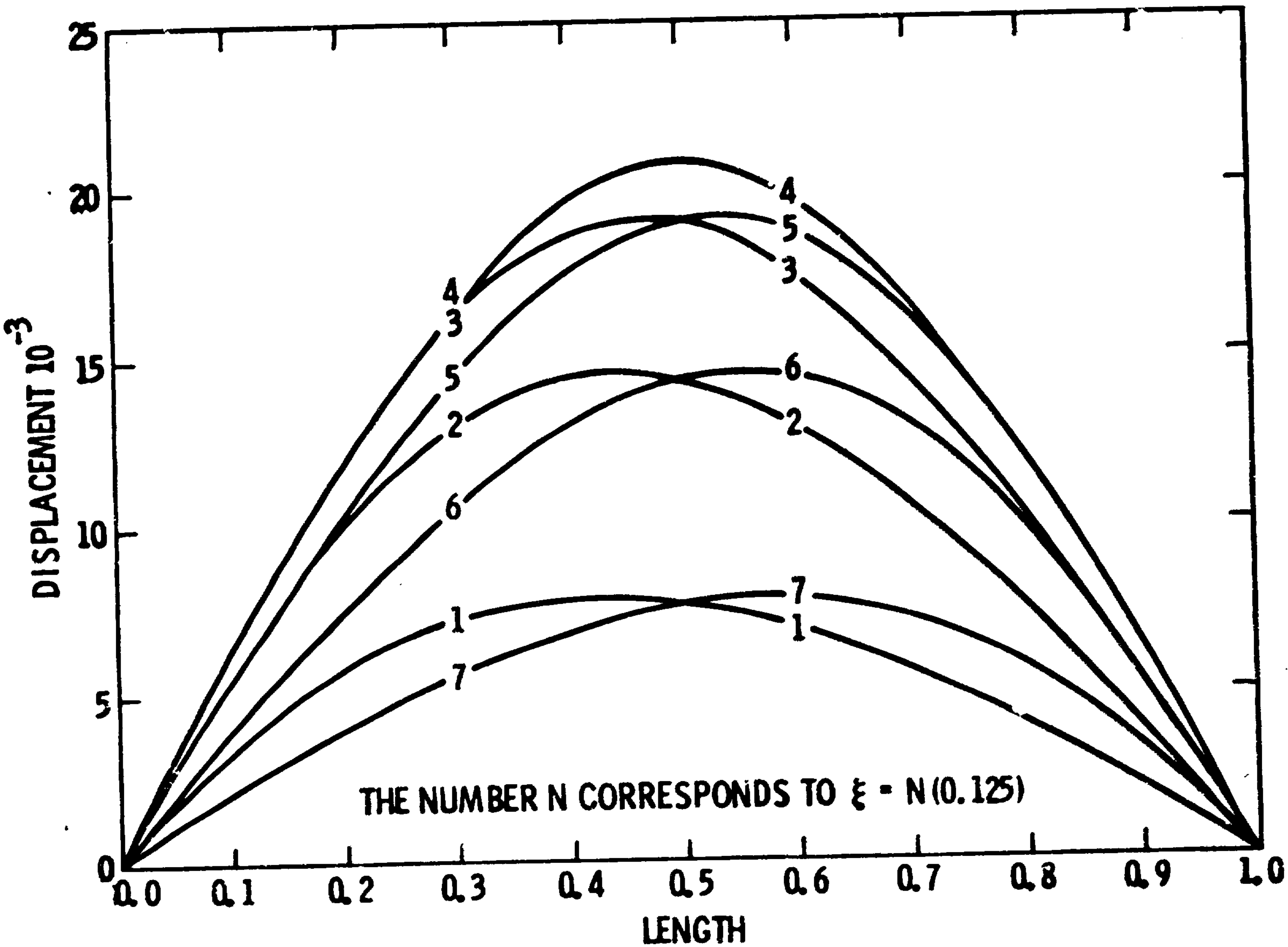


Figure 3.2 The Green's Function for the Simply Supported Beam

This condition becomes

$$f_i\, r_i + \sum_{j=1}^{m} f_j \left(\int_0^{\ell} g(x|x_i)\, g(x|x_j)\, dx \right) = \int_0^{\ell} \psi(x)\, g(x|x_i)\, dx \; . \qquad (84)$$

If we define

$$a_{ij} = \int_c g(x|x_i)\, g(x|x_j)\, dx \; , \qquad 1 \leq i, j \leq m \qquad (85)$$

and

$$b_i = \int_0^{\ell} \psi(x)\, g(x|x_i)\, dx \; , \qquad 1 \leq i \leq m \; , \qquad (86)$$

then the necessary condition for a minimum of J at F* is that F* satisfy

$$(R + A)\, F^* = B \qquad (87)$$

where R is the m x m diagonal matrix

$$R = \begin{pmatrix} r_1 & & \bigcirc \\ & \ddots & \\ \bigcirc & & r_m \end{pmatrix} \qquad (88)$$

A is the m x m matrix with coefficients (85), and B is the m dimensional vector with coefficients (86).

The Shape Control Algorithm for the Simply Supported Beam

1) Compute the constants a_{ij} and b_j defined by (85-86). Define R, A, B.

2) Solve (87) to obtain F*.

3) The optimal shape $u^*(x) = \sum_{i=1}^{m} f_i^* g(x|x_i)$.

Figure 3.3 displays the optimal shape vs. the desired shape $\psi(x) = \sin \frac{2\pi x}{\ell}$, the second mode of the system (74-75), for two actuators at $1/4\ell$ and $3/4\ell$.

$$\psi = \sin \frac{2\pi x}{\ell}$$

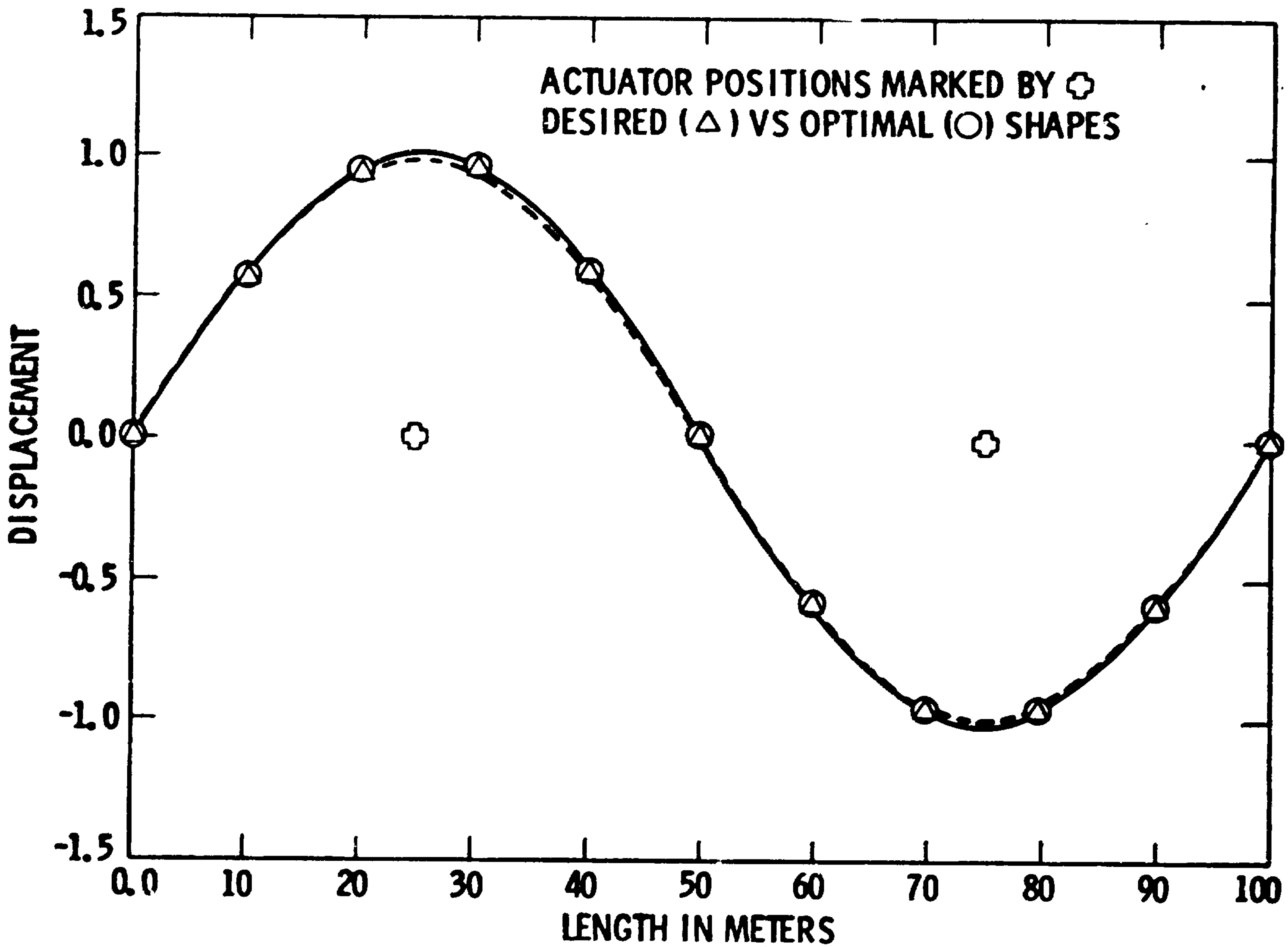


Figure 3.3 Optimal Shape vs. Desired Shape for the Simply Supported Beam with Two Actuators

3.3 The Control Problem for the Pinned-Free Beam

A modification of the control algorithm is necessary if the system has rigid body modes, as is the case with the pinned-free beam.

The beam with one pinned and one free end point satisfies the differential equation (74) with boundary conditions

$$u(0) = u''(0) = 0 \qquad u''(\ell) = u'''(\ell) = 0 \,. \tag{89}$$

We will again use the performance criterion (76). The object is to determine the set of forces $\{f_i\}$ which together with the solution $u(x)$ of (74) (89) minimizes (76) over all possible pairs $(\{f_i\}, u)$.

The system (74) (89) has the rigid body mode $v_1(x) = \sqrt{\frac{3}{\ell^3}}\; x$ (normalized). Physically this means the beam can have a non-zero slope or tilt as a rigid body. Mathematically it means that the corresponding homogeneous system

$$\frac{d^4 v}{dx^4} = 0 \qquad v(0) = v''(0) = 0 \quad v''(\ell) = v'''(\ell) = 0 \tag{90}$$

has the non-trivial solution $v_1(x)$. Thus the system (74)(89) has a solution only if the inner product

$$\Big(\sum_{i=1}^{m} f_i\ \delta(x-x_i),\ v_1\Big) = \sqrt{\frac{3}{\ell^3}}\ \sum_{i=1}^{m} f_i x_i = 0 \,. \tag{91}$$

The additional constraint (91) must be added to the problem of determining the optimal control forces.

A solution to (79) with pinned-free boundary conditions does not exist because the inner product $<\delta(x-\xi),\ v_1>$ is not zero. The "modified" Green's function which is appropriate to the system (74)(89) satisfies

$$\frac{d^4 g_m(x|\xi)}{dx^4} = \delta(x-\xi) - \frac{3}{\ell^3}\ x\xi \tag{92}$$

$$g_m(0|\xi) = g_m''(0|\xi) = 0 \qquad g_m''(\ell|\xi) = g_m'''(\ell|\xi) = 0 \tag{93}$$

We make the additional requirement that $g_m(x|\xi)$ have no component in the subspace spanned by the rigid body mode.

$$(g_m(x|\xi), v_1) = \sqrt{\frac{3}{\ell^3}} \int_0^\ell g_m(x|\xi)\ x\ dx = 0 \tag{94}$$

The modified Green's function which satisfies (92-94) is given by

$$g_m(x|\xi) = x\xi \left(\frac{33\ell}{140} + \frac{\xi^2+x^2}{4\ell} - \frac{\xi^4+x^4}{40\ell^3}\right) - \begin{cases} \dfrac{\xi^3 x}{2} + \dfrac{x^3}{6} & 0\le x\le\xi \\ \dfrac{x^2\xi}{2} + \dfrac{\xi^2}{6} & \xi\le x\le\ell \end{cases} \tag{95}$$

Condition (94) guarantees that $g_m(x|\xi)$ is symmetric and of minimum norm among all solutions of (92,93). Figure 3.4 displays $g_m(x|\xi)$ for impulsive forces at intervals of 1/8 ℓ.

The Green's function (95) represents the response of the pinned-free beam to one of a set of unit impulsive forces which satisfy (91). Figure 3.4 displays the Green's functions for impulsive forces at positions $n\ (\frac{\ell}{8})$, $n = 1,\ldots,7$.

The solution of (74)(89)(91) is given by

$$u(x) = \sum_{i=1}^{m} f_i\ g_m\ (x|x_i) \tag{96}$$

We solve (91) for f_1 in terms of the outer forces and substitute that expression together with (94) into the criterion (76), which results in

$$\begin{aligned} J(\hat{F}) = \frac{r_1}{2} \left(\sum_{i=2}^{m} \frac{-x_i}{x_1} f_i\right)^2 + \frac{1}{2} \sum_{i=2}^{m} f_i^{\ 2}\ r_i \\ + \frac{1}{2} \int_0^\ell \left(\sum_{i=2}^{m} f_i \left(g_m(x|x_i) - \frac{x_i}{x_1} g(x|x_i)\right) - \psi(x)\right)^2 dx \end{aligned} \tag{97}$$

where $\hat{F}$ is the vector $(f_2 \ldots f_m)^T$.

Again, the optimization problem is reduced to one of minimizing a function of unknown constants.

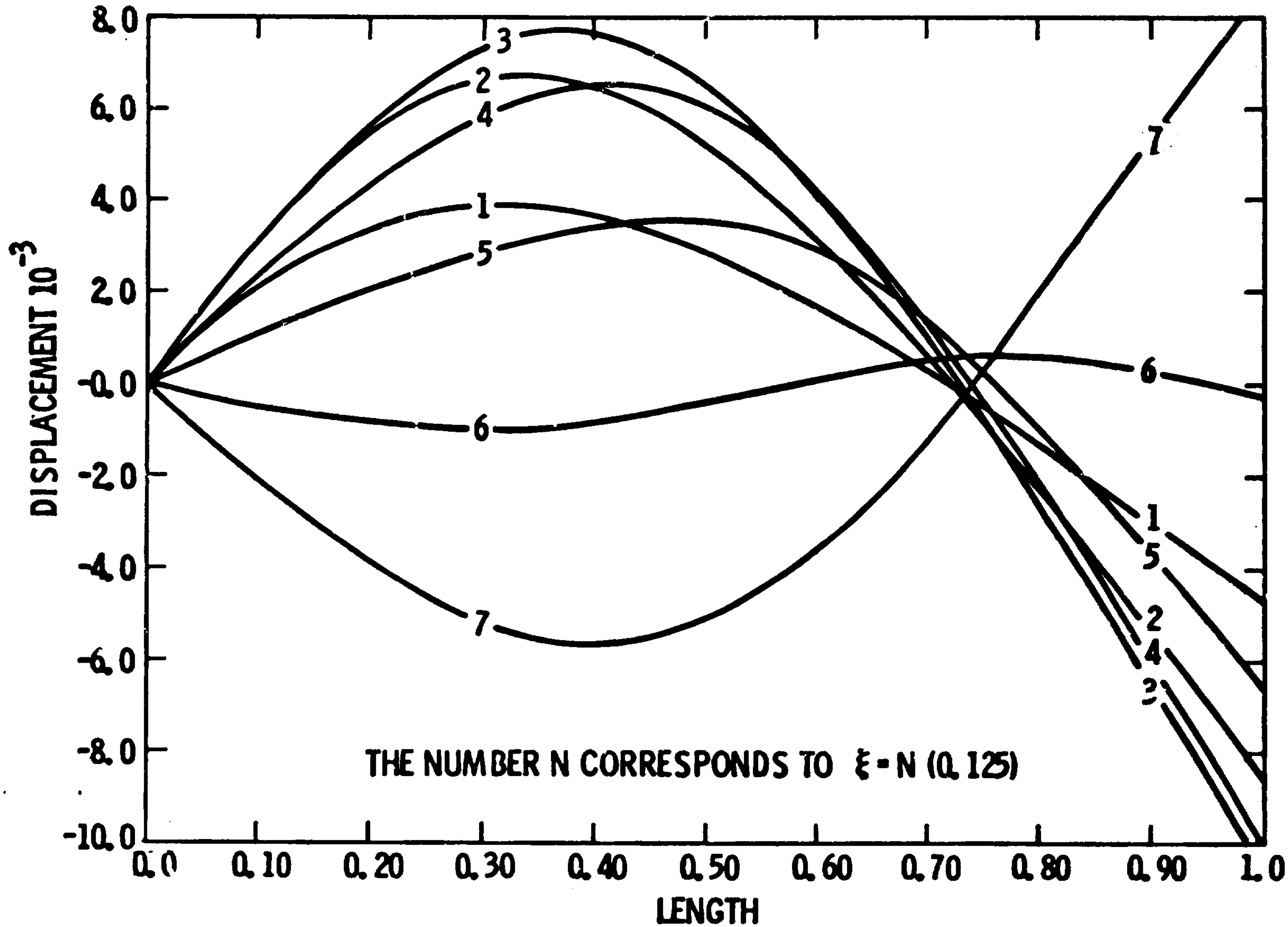


Figure 3.4 The Green's Function for the Pinned-Free Beam

The necessary condition for a minimum at F* is

$$\frac{\partial J}{\partial f_i} (\hat{F}^*) = 0 \qquad 2 \leq i \leq m . \tag{98}$$

These conditions result in the following algorithm.

(i) Compute the m dimensional vector B and m x m matrix A whose coordinates are

$$b_i = \int_0^{\ell} g_m(x|x_i)\, \psi(x)\, dx \tag{99}$$

$$a_{ij} = \int_0^{\ell} g_m(x|x_i)\, g_m(x|x_j)\, dx . \tag{100}$$

(ii) Compute the (m-1) dimensional vector $\hat{B}$ and (m-1) x (m-1) matrix $\hat{A}$ whose coordinates are

$$\hat{b}_i = b_{i+1} - \frac{x_{i+1}}{x_1} b_1 \tag{101}$$

$$\hat{a}_{ij} = r_1 + a_{11}) \frac{x_{i+1}x_{j+1}}{x_1^{\,2}} + a_{i+1,j+1} - a_{1,i+1} \frac{x_{j+1}}{x_1} - a_{1,j+1} \frac{x_{i+1}}{x_1} \tag{102}$$

Let $\hat{R}$ be the (m-1) x (m-1) diagonal matrix

$$\hat{R} = \begin{pmatrix} r_2 & & 0 \\ & \ddots & \\ 0 & & r_m \end{pmatrix} \tag{103}$$

(iii) The vector $\hat{F}^*$ of optimal forces satisfies

$$(\hat{R} + \hat{A})\hat{F}^* = \hat{B} . \tag{104}$$

The optimal force f_1^* is found from (91).

(iv) The optimal shape $u^*(x) = \sum_{i=1}^{m} f_i^* g_m(x|x_i)$.

Since the optimal shape u* is a linear combination of Green's functions which satisfy (94), it will have no component in the subspace of the rigid body mode. If the desired shape $\psi(x)$ does have such a component, that is if (ψ, v_1) is not zero, the optimal shape will approximate the shape

$$\psi(x) - \langle\psi, v_1\rangle\, v_1(x) \; . \tag{105}$$

That is, it will approximate the desired shape minus its component in the subspace spanned by $v_1(x)$.

As an example, Figure 3.5 displays the desired shape $\psi(x) = \ell x - x^2$, the shape which approximates $\frac{3}{4}\,\ell x - x^2$, and the optimal shape plus the missing rigid body mode component $\frac{1}{4}\,\ell x$.

Those components of the desired shape in the subspace spanned by rigid body modes must be added by the attitude control system. A shape control system constrained to satisfy the boundary conditions cannot affect these components.

3.4 The Shape Estimation Problem

To illustrate the shape estimation algorithm we consider a simply supported beam of length ℓ and unknown shape $u(x)$, which satisfies

$$\frac{d^4u}{dx^4} = f(x) \quad \text{on} \qquad 0 \le x \le \ell, \tag{106}$$

and

$$u(0) = u''(0) = 0 \qquad u(\ell) = u''(\ell) = 0 \tag{107}$$

The function $f(x)$ represents minor model inaccuracies or random disturbances acting on the beam.

Assume sensors at positions x_i, $0 < x_1 < \ldots < x_m < \ell$, produce observations

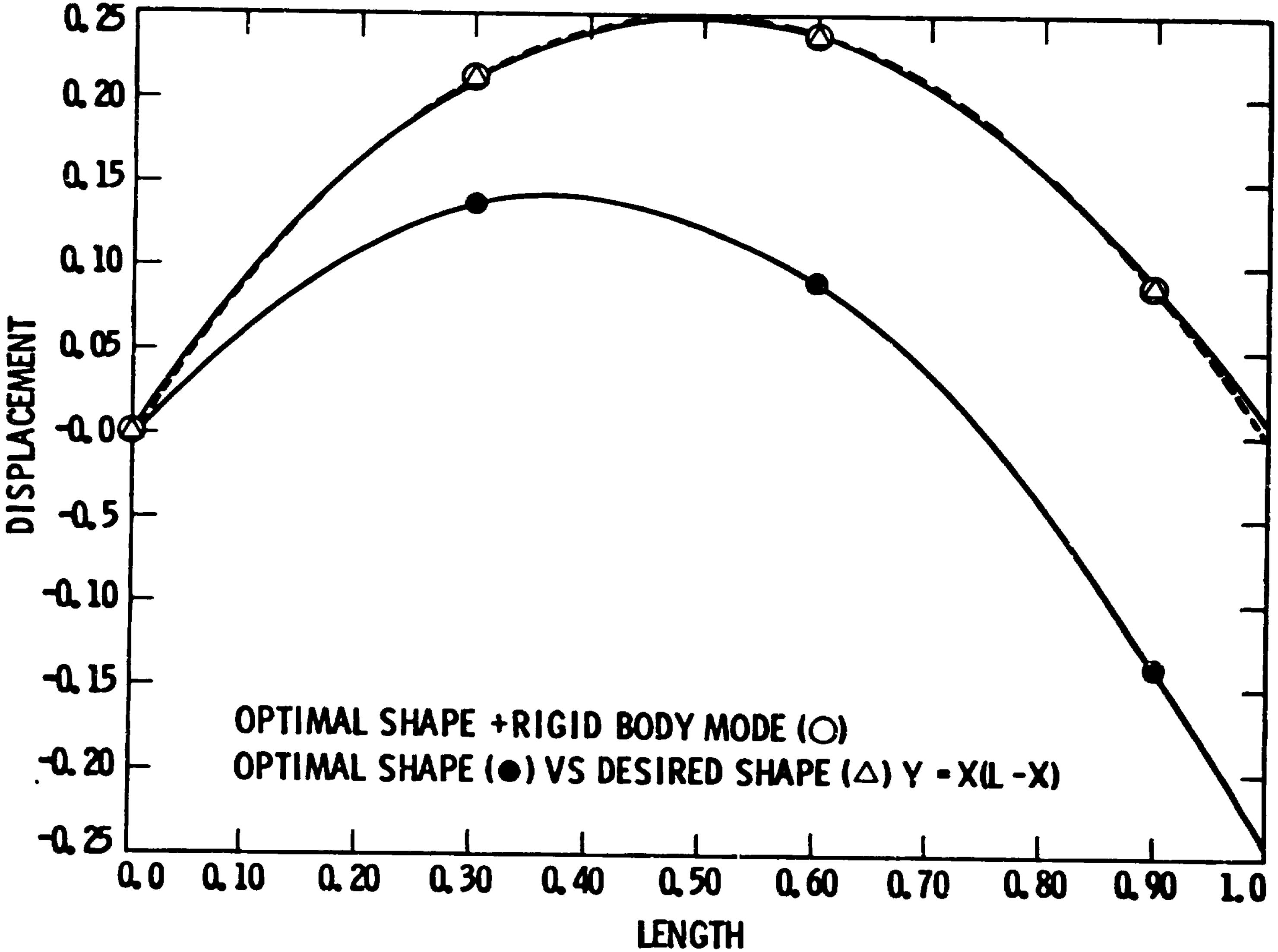


Figure 3.5 Shape Control with Rigid Body Modes

$$y_i = u(x_i) + \zeta_i \; , \qquad 1 \leq i \leq m \; . \tag{108}$$

As a measure of the accuracy of shape estimates we define the criterion

$$J(f,u) = \frac{1}{2} \sum_{i=1}^{m} (y_i - u(x_i))^2 \, r_i^{-1} + \frac{1}{2} \int_0^{\ell} f^2(x) \, dx \; . \tag{109}$$

The object is to determine the function f* which together with the solution u* of (106-107) minimizes (109) over all possible pairs (f,u).

The solution of (106-107) is given by

$$u(x) = \int_0^{\ell} g(x|\xi) \, f(\xi) \, d\xi \tag{110}$$

where $g(x|\xi)$ is the Green's function (81). We substitute (110) into the criterion (109); resulting in the criterion

$$J(f) = \frac{1}{2} \sum_{i=1}^{m} r_i^{-1} \left(y_i - \int_0^{\ell} g(x_i|\xi) f(\xi) \, d\xi\right)^2 + \frac{1}{2} \int_0^{\ell} f(\xi)^2 \, d\xi \; . \tag{111}$$

The estimation problem has reduced to one of minimizing (111) without constraints. A necessary condition for J to have a minimum at f* is that the Frechet differential

$$\partial J(f,h) = \sum_{i=1}^{m} r_i^{-1} \left(y_i - \int_0^{\ell} g(x_i|\xi) f^*(\xi) d\xi\right)\left(-\int_0^{\ell} g(x_i|\xi) h(\xi) d\xi\right) + \int_0^{\ell} f^*(\xi) h(\xi) d\xi = 0$$

for all admissible variations h. This implies

$$f^*(\xi) = \sum_{i=1}^{m} r_i^{-1} \, g(x_i|\xi)(y_i - u^*(x_i)) \; . \tag{112}$$

Then

$$u^*(x) = \sum_{i=1}^{m} r_i^{-1}(y_i - u^*(x_i)) \int_0^{\ell} g(x|\xi) g(x_i|\xi) \, d\xi \; . \tag{113}$$

Let

$$X = (u^*(x_1) \; \ldots \; u^*(x_m))^T$$

and

$$Y = (y_1 \; \ldots \; y_m) \; .$$

Evaluation of (113) at $x = x_j$ and regrouping of terms yield the following necessary condition for the vector X:

$$(I + AR^{-1})\, X = A\, R^{-1}\, Y \tag{114}$$

where A is the matrix of coefficients (85), and R^{-1} is the diagonal matrix with diagonal entries r_i^{-1}.

The Shape Estimation Algorithm

(i) Compute the elements of the matrix A given by (85), and define X, R, Y.

(ii) Solve the system (114) for the vector X.

(iii) The optimal error estimates are given by (112) and $\zeta_i = y_i - u^*(x_i)$, $1 \le i \le m$.

(iv) The optimal shape estimate is given by (113).

This algorithm is equally valid for the static beam with other boundary conditions, provided the appropriate Green's function is used.

Figure 3.6 displays the optimal shape estimate versus the actual shape

$$\sin\left(\frac{\pi x}{\ell}\right) + \frac{1}{2}\left(\frac{2\pi x}{\ell}\right),$$

for three exact observations at $\frac{1}{4}\ell$, $\frac{1}{2}\ell$, and $\frac{3}{4}\ell$.

3.5 Approximations

The approximations presented in section 2.6 take the following form on the domain $[0,\ell]$ of the x axis:

$$g(x|\xi) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k}\, \phi_k(x)\, \phi_k(\xi) \tag{115}$$

$$a_{ij} = \int_0^\ell g(x|x_i)\, g(x|x_j)\, dx = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2}\, \phi_k(x_i)\, \phi_k(x_j) \tag{116}$$

$$b_j = \int_0^\ell g(x|x_j)\, \psi(x)\, dx = \sum_{k=1}^{\infty} \phi_k(x_j)\, \langle \phi_k, \psi \rangle \tag{117}$$

ACTUAL SHAPE $\sin \frac{\pi x}{\ell} + 1/2 \sin \frac{2\pi x}{\ell}$

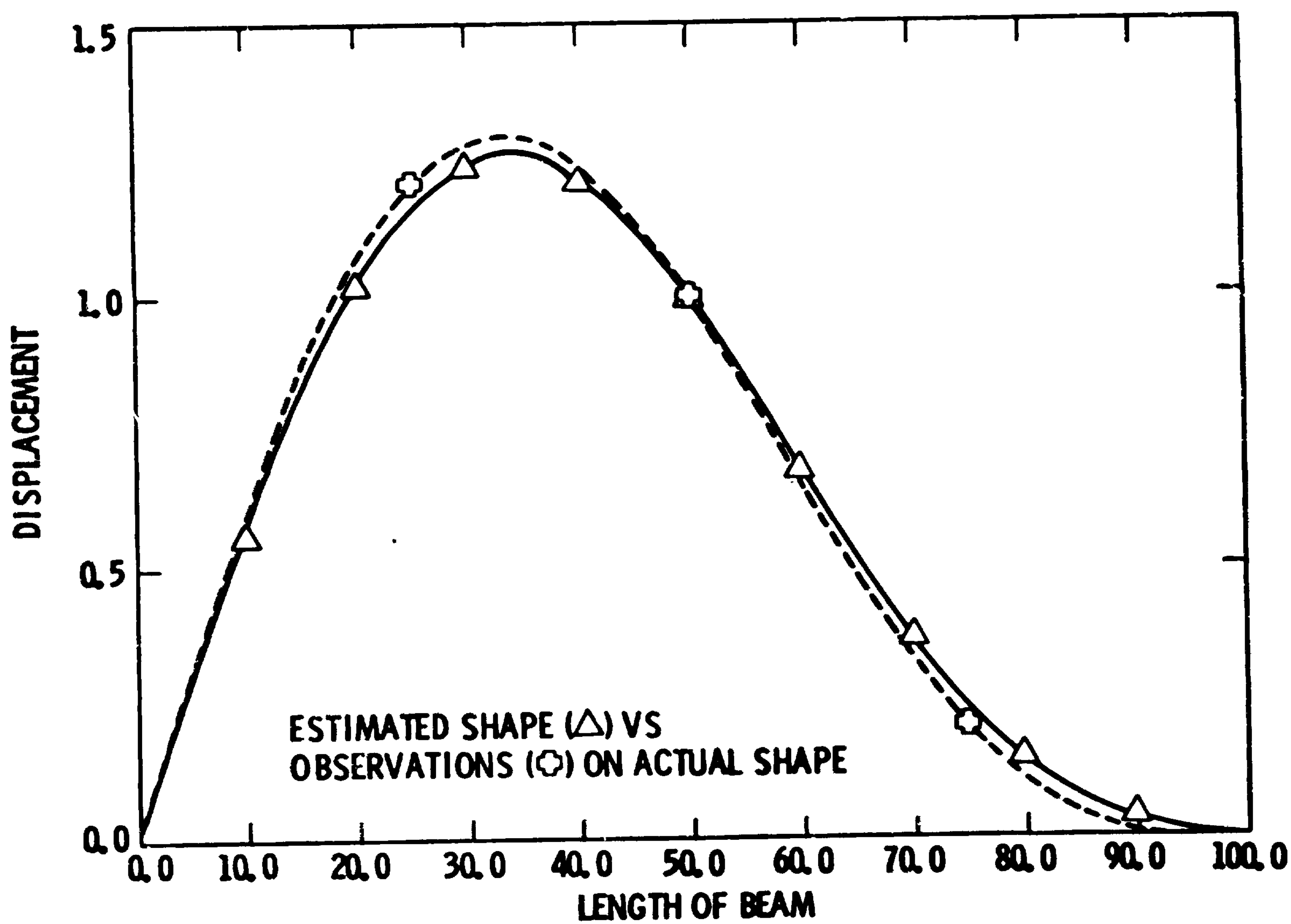


Figure 3.6 Shape Estimation Results for the Simply Supported Beam with Three Sensors

where $\{x_i\}$ are the actuator or sensor positions, λ_k are the non-zero eigenvalues, and ϕ_k are the corresponding normalized eigenfunctions of the associated boundary value problems.

Thus for the simply supported beam the approximations based on the first term of each expansion are given by

$$\hat{a}_{ij} = 2 \frac{\ell^7}{\pi^8} \sin\left(\frac{\pi x_i}{\ell}\right) \sin\left(\frac{\pi x_j}{\ell}\right) \tag{118}$$

$$\hat{b}_i = 2 \frac{\ell^3}{\pi^4} \sin\left(\frac{\pi x_i}{\ell}\right) \left(\int_0^\ell \psi(x) \sin\left(\frac{\pi x}{\ell}\right) dx\right) . \tag{119}$$

For the pinned-free beam

$$\hat{a}_{ij} = \frac{\ell^7}{\mu^8} \left[\frac{\sin\left(\frac{\mu x_i}{\ell}\right)}{\cos \mu} + \frac{\sinh\left(\frac{\mu x_i}{\ell}\right)}{\cosh \mu}\right] \left[\frac{\sin\left(\frac{\mu x_j}{\ell}\right)}{\cos \mu} + \frac{\sinh\left(\frac{\mu x_j}{\ell}\right)}{\cosh \mu}\right] \tag{120}$$

where $\mu = 3.927$ satisfies $\tan \mu = \tanh \mu$.

$$\hat{b}_i = \frac{\ell^3}{\mu^4} \left[\frac{\sin\left(\frac{\mu x_i}{\ell}\right)}{\cos \mu} + \frac{\sinh\left(\frac{\mu x_i}{\ell}\right)}{\cosh \mu}\right] \int_0^\ell \psi(x) \left(\frac{\sin \frac{\mu x}{\ell}}{\cos \mu} + \frac{\sinh\left(\frac{\mu x}{\ell}\right)}{\cosh \mu}\right) dx \tag{121}$$

(the normalizations are approximate).

Approximate algorithms constructed from the first term in the eigenfunction expansions were included in the simulations of the examples in this chapter. The graphs of approximate vs. optimal results were indistinguishable. Numerical results are included in the program outputs in Appendix B.

It is misleading to generalize from the approximations for the one-dimensional case, for which satisfactory approximations result using only the first term. The expansions (118-121) telescope rapidly, because the magnitude of the eigenvalues increases rapidly. The frequencies ω_n of large space structures increase relatively slowly ($\lambda_n = \omega_n^2$), as can be observed in the output of the shape control program for a large space antenna, in Appendix C. For multidimensional structures many more modes (eigenfunctions) must be used.

Chapter 4. Shape Control of Structures Governed by Partial Differential Equations

4.1 Introduction

In Chapter 3 static shape control and estimation problems for one dimensional cases were solved using Green's function techniques. In this chapter corresponding results for structures defined on multidimensional domains, governed by partial differential equations, are presented. It will be observed that the solutions are very similar to those for one dimensional domains. The major difference is that it becomes difficult to determine the analytical form of the Green's function, so that expressions in terms of eigenfunction expansions must be used.

We consider as examples the shape distortion of membranes and plates which in equilibrium position lie in a plane. A membrane, such as a drumhead, or the mesh of an antenna, is distinguished from a plate by the absence of bending resistance. The restoring forces of a membrane are due exclusively to tension whereas plates have bending stiffness. Consequently, membranes may be considered to be governed by the harmonic operator ∇^2, while plates are governed by the biharmonic operator $\nabla^4 = \nabla^2(\nabla^2)$.

This distinction between second and fourth order dynamics is analogous to the modeling distinction between a string and a flexible beam in the one dimensional case.

For convenience, in this chapter we consider only systems without rigid body modes.

4.2 The Boundary Value Problem and Green's Function for a Membrane

Under suitable assumptions the shape distortion of a membrane is modeled by the differential equation

$$\nabla^2 u = f(P) \; , \qquad P \in \Omega \tag{122}$$

where ∇^2 is the Laplacian operator and certain known physical constants have been incorporated into the forcing function f. Eq. (122) is known as Poisson's equation.

We will choose the boundary conditions so that conditions (6-8) are satisfied for the operator $\nabla^2 = L$. We will then discuss the determination of the Green's function $g(P|Q)$, and exhibit the solutions to the control and estimation problems for the unit disk. Finally we will exhibit approximate solutions using the eigenfunction expansions (70-72).

Green's theorem for the Laplacian operator takes the usual form

$$\int_\Omega (v \, \nabla^2 \omega - \omega \, \nabla^2 \, v) dP = \int_\Omega (v \frac{\partial \omega}{\partial n} - \omega \frac{\partial v}{\partial n}) \, ds \; . \tag{123}$$

If we impose either of the boundary conditions $u(P) = 0$ or $\frac{\partial u}{\partial n} = 0$ for $P \epsilon \Gamma$, the right side of (123) will be zero for functions ω and v which satisfy the boundary condition, and the operator ∇^2 will be self-adjoint.

For convenience we eliminate the latter boundary condition, since the homogeneous system

$$\nabla^2 u = f \; , \; \frac{\partial u}{\partial n} = 0 \tag{124}$$

has the non-trivial solution $u \equiv C$.

The Green's Function

The Green's function for the system

$$\nabla^2 u = f \; , \quad u(P) = 0 \qquad P \epsilon \Gamma \tag{125}$$

satisfies

$$\nabla^2 g(x,y,\xi,\eta) = \delta(x-\xi) \;\; \delta(y-\eta) \tag{126}$$

in rectangular coordinates $P(x,y)$, $Q(\xi,y)$ and

$$\nabla^2 g(r,\theta,\rho,\phi) = \frac{\delta(r-\rho)\ \delta(\theta-\phi)}{r} \tag{127}$$

in the polar coordinates $P = re^{i\theta}$, $Q = \rho e^{i\phi}$. In both cases $g \equiv 0$ on Γ.

The function $\gamma = \frac{1}{2\pi}\log R$, where R is the distance $\overline{QP}$, can be shown to satisfy $\nabla^2\gamma = \delta(P|Q)$. It is called the <u>free space</u> solution since it is not required to satisfy the boundary conditions.

Thus the Green's function is given by

$$g(P|Q) = \frac{1}{2\pi} \log R + \hat{g}(P|Q) \tag{128}$$

where $\hat{g}(P|Q)$ satisfies

$$\nabla^2\hat{g} = 0 \text{ on } \Omega,\ \hat{g} = -\frac{1}{2\pi}\log R \text{ on } \Gamma. \tag{129}$$

The theory of analytic functions may be applied on convenient regions to determine $\hat{g}$, hence also to determine g. For Ω equal to the unit circle $|z| \leq 1$,

$$g(r,\theta,\rho,\phi) = \frac{1}{4\pi} \log \left[\frac{r^2 - 2r\rho\ \cos(\theta-\phi) + \rho^2}{1 - 2r\rho\ \cos(\theta-\phi) + r^2V^2}\right] \tag{130}$$

for P,Q in polar coordinates [7].

Remark 4.1: Through the use of conformal mapping it is possible to determine the Green's function for some other regions, but in general it is not possible to determine the exact function g.

4.3 The Control Problem for ∇^2 on the Unit Disk

The control problem for the Laplacian on the unit disk corresponds to the problem of controlling the shape of a circular net or drumhead to a desired shape $\psi(r,\theta)$ by means of pointwise forces. Thus we desire to determine the set of forces $\{f_j\}$ at positions $P_j = \rho_j\ e^{i\phi_j}$, $1 \leq j \leq m$

which together with the solution $u(r,\theta)$ of

$$\nabla^2 u(r,\theta) = \sum_{j=1}^{m} f_j \frac{\delta(r-\rho_j)\ \delta(\theta-\phi_j)}{r} \tag{131}$$

$$u(1,\theta) = 0 \tag{132}$$

minimizes the performance criterion

$$J(F,u) = \frac{1}{2} \sum_{j=1}^{m} f_j^{\ 2} r_j + \frac{1}{2} \int_0^{2\pi} \int_0^1 [\psi(r,\theta) - u(r,\theta)]^2 r\ dr\ d\theta \tag{133}$$

over all possible sets (u,F), where

$$F = (f_1 \ \dots \ f_m)^T \ . \tag{134}$$

The optimal shape for the problem (131-132) is given by

$$\begin{aligned} u^*(r,\theta) &= \sum_{j=1}^{m} f_j^* \ g(r,\theta,\rho_j,\phi_j) \\ &= \frac{1}{4\pi} \sum_{j=1}^{m} f_j^* \log \left[\frac{r^2 - 2r\rho_j \cos(\theta-\phi_j) + \rho_j^{\ 2}}{1 - 2r\rho_j \cos(\theta-\phi_j) + r^2\rho_j^{\ 2}}\right] \end{aligned} \tag{135}$$

and the vector of optimal forces F^* satisfies $(R+A)F^* = B$, where $R = (R_{ij})$ and $A = (A_{ij})$ are m x m matrices such that

$$R_{ij} = r_i \ \delta(i-j)$$

and

$$A_{ij} = \int_0^{2\pi} \int_0^1 g(r,\theta,\rho_i,\phi_i) \ g(r,\theta,\ \rho_j,\phi_j) \ r dr d\theta \tag{136}$$

and $B = (b_i)$ is an m dimensional vector such that

$$b_i = \int_0^{2\pi} \int_0^1 \psi(r,\theta) \ g(r,\theta,\rho_i,\phi_i) \ r dr d\theta . \tag{137}$$

4.4 The Estimation Problem

The corresponding estimation problem for ∇^2 on the unit disk is, given the shape observations

$$y_i = u(\rho_i,\phi_i) + \zeta_i , \qquad 1 \leq i \leq m , \tag{138}$$

at positions $P_i = \rho_i e^{i\phi_i}$, to determine the error function $f(r,\theta)$ and corresponding shape function $u(r,\theta)$ which satisfy

$$\nabla^2 u(r,\theta) = f(r,\theta) , \ u(1,\theta) = 0 \tag{139}$$

and minimize the criterion

$$J(F,u) = \frac{1}{2} \sum_{i=1}^{m} (y_i - u(\ _i,\phi_i)^2 r_i^{-1} + \frac{1}{2} \int_0^{\pi} \int_0^1 f^2(r,\theta) r \, dr \, d\theta . \tag{140}$$

The results of Section 2.4 yield the optimal error estimates

$$\zeta_i^* = y_i - u^*(\rho_i,\phi_i)$$

$$f^*(r,\theta) = \frac{1}{4\pi} \sum_{i=1}^{m} r_i^{-1} \zeta_i^* \log \left[\frac{r^2 - 2r\rho_i \cos(\theta-\phi_j) + \rho_j^2}{1 - 2r\rho_j \cos(\theta-\phi_j) + r^2\rho_j^2} \right] \tag{142}$$

where the vector $X = (u^*(P_1) \ \dots \ u^* \ (P_m))^T$ satisfies

$$(I + AR^{-1}) X = AR^{-1}Y . \tag{143}$$

The matrices R and A are as in (136) and Y is the vector of observations $(y_i \ \dots \ y_m)^T$. The corresponding optimal shape estimate is then given by

$$u^*(r,\theta) = \sum_{i=1}^{m} [r_i^{-1} \zeta_i^* \int_0^{\pi} \int_0^1 g(r,\theta,\rho,\phi) \, g(r,\theta,\rho_i,\phi_i) \, \rho d\rho d\phi] . \tag{144}$$

4.5 Approximate Solutions

For simplicity it may be desirable to compute approximations to the solution (135-137) and (141-143) using eigenfunction expansions. The eigenvalues and (normalized) eigenfunctions corresponding to

$$\nabla^2 \Phi(r,\theta) = \lambda\Phi(r,\theta), \ \Phi(1,\theta) = 0$$

are

$$\Phi_{on}(r) = \sqrt{\frac{1}{\pi}} \ \frac{J_o(\lambda_{on}^{1/2} r)}{J_1(\lambda_{on}^{1/2})} \qquad n = 1,2, \ldots$$

$$\vdots$$

corresponding to the eigenvalues λ_{on} which satisfy

$$J_o(\lambda_{on}^{1/2}) = 0 , \qquad n = 1,2, \ldots$$

and

$$\Phi_{mnc}(r,\theta) = \sqrt{\frac{2}{\pi}} \ \left(\frac{J_m(\lambda_{mn}^{1/2} r)}{J_{m+1}(\lambda_{mn}^{1/2})}\right) \quad \cos m\theta$$

$$\Phi_{mns}(r,\theta) = \sqrt{\frac{2}{\pi}} \ \left(\frac{J_M(\lambda_{mn}^{1/2} r)}{J_{m+1}(\lambda_{mn}^{1/2})}\right) \quad \sin m\theta$$

$$1 \leq m, n < \infty$$

corresponding to the eigenvalues λ_{mn} which satisfy

$$J_m(\lambda_{mn}^{1/2}) = 0 ,$$

where J_i, $0 \leq i < \infty$ are, of course, the Bessel functions.

Thus, a first approximation to the forces $\{f_i\}$ in the control law, using the eigenvalue $\lambda_{oo} = (2.405)^2$ and eigenfunction

$$\Phi_{oo}(r) = \sqrt{\frac{1}{\pi}} \ \frac{J_0(2.405\ r)}{J_1(2.405)}$$

satisfies

$$(R + \hat{A})\hat{F} = \hat{B} , \tag{145}$$

where R is as before, $\hat{F} = (\hat{f}_1 \ldots \hat{f}_m)^T$ is the vector of approximate forces, and $\hat{A}$ and $\hat{B}$ are the approximate matrix and vector with coefficients

$$\hat{a}_{ij} = \frac{1}{\lambda_{oo}^{2}} \Phi_{oo}(\rho_i) \ \Phi_{oo}(\rho_j) \tag{146}$$

$$\hat{b}_i = \frac{1}{\lambda_{oo}} \Phi_{oo}(\rho_j) \langle\Phi_{oo},\psi\rangle, \qquad 1 \leq i, j \leq m . \tag{147}$$

The shape corresponding to the approximate forces in (145) is given by

$$\hat{u}*(r,\theta) = \sum_{i=1}^{m} \hat{f}_i \, g(r,\theta,\rho_i,\phi_i) \tag{148}$$

since the Green's function still represents the response to a unit force at $P_i = \rho_i e^{i\phi_i}$.

Using the same approximations (146) for the matrix A, the pointwise shape estimation vector X may be approximately computed from

$$(I + \hat{A}R^{-1})\hat{X} = \hat{A}\, R^{-1}Y \tag{149}$$

where $\hat{X} = (\hat{u}(Q_1) \ \ldots \ \hat{u}(Q_m))$ is the approximation to X, and R^{-1} and Y are as in (136)(138). The approximate estimates are then given by

$$\hat{\zeta}_i = y_i - \hat{u}\,(P_i)\ , \qquad 1 \leq i \leq m\ , \tag{150}$$

$$\hat{f}(r) = \frac{1}{\lambda_{oo}} \sum_{i=1}^{m} r_i^{-1}\, \hat{\zeta}_i\, \Phi_{oo}(r)\, \Phi_{oo}(\rho_i) \tag{151}$$

$$\hat{u}(r) = \frac{1}{\lambda_{oo}^{2}} \sum_{i=1}^{m} r_i^{-1}\, \hat{\zeta}_i\, \Phi_{oo}(r)\, \Phi_{oo}(\rho_i) \tag{152}$$

Approximations of greater accuracy may be obtained by including the next largest eigenvalues and their corresponding eigenfunctions.

4.6 The Static Vibration of a Plate - The Boundary Value Problem and Green's Function

The static vibrations of a plate may be modeled by the partial differential equation

$$\nabla^4 u = f(P), \qquad P\varepsilon\Omega \tag{153}$$

where $\nabla^4 = \nabla^2(\nabla^2)$ is the biharmonic operator, and again certain physical constants have been included in the forcing function f for simplicity.

We wish again to choose the boundary conditions such that the boundary value problem is self-adjoint. Green's theorem for the operator ∇^4 takes the form

$$\int_\Omega (v\nabla^4\omega - \omega\nabla^4 v)dP = \int_\Gamma [v \frac{\partial}{\partial n} (\nabla^2\omega) - \omega \frac{\partial}{\partial n} (\nabla^2 v) + (\nabla^2 v)(\frac{\partial\omega}{\partial n}) - (\nabla^2\omega)\frac{\partial v}{\partial n}] . \qquad (154)$$

The problem of boundary conditions for plate vibrations is much more difficult than for the membrane. A useful discussion of boundary conditions is contained in [4].

The Simply Supported Rectangular Plate

Consider a uniform rectangular plate on the domain $\Omega = \{(x,y) \mid 0 \leq x \leq a, 0 \leq y \leq b\}$. The boundary conditions for a simply supported edge are

$$u = 0 \text{ and } \frac{\partial^2 u}{\partial n^2} + \frac{\nu}{R}\frac{\partial u}{\partial n} = 0 \qquad (155)$$

where n is the normal vector to the edge and R is the radius of curvature. For a straight edge $R = \infty$. Furthermore, since u is constant along the edge, $\frac{\partial u}{\partial s} = 0$. Thus

$$\nabla^2 u = \frac{\partial^2 u}{\partial n^2} + \frac{1}{R}\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial s^2} = \frac{\partial^2 u}{\partial n^2} \qquad (156)$$

and the boundary conditions for a simply supported straight edge are

$$u = \nabla^2 u = 0 . \qquad (157)$$

Clearly the conditions (157) make the right side of (154) equal to zero. Thus ∇^4 is self-adjoint for the simply supported rectangle.

The Green's Function

The Green's function for the simply supported rectangle should satisfy

$$\nabla^4 g(x,y,\xi,\eta) = \delta(x-\xi)\ \delta(y-\eta) \qquad (158)$$

$$g = \nabla^2 g = 0 \quad \text{on } x = 0 , a \quad \text{and } y = 0, b . \qquad (159)$$

The free space solution $\gamma(P|Q)$ which satisfies $\nabla^4\gamma = \delta(P|Q)$ is

$$\gamma(P|Q = \frac{-1}{8\pi} r^2 \log r \tag{160}$$

where r represents the distance $\overline{PQ}$. This is proved in Appendix A. Thus, the Green's function

$$g(P|Q) = \gamma(P|Q) + \hat{g}(P|Q) \tag{161}$$

where the function $\hat{g}$ satisfies $\nabla^4 g(P|Q) = 0$, plus boundary conditions such that g satisfies (157).

The function g in (161) is no longer necessarily harmonic, as was the function in (129). It must in addition satisfy two sets of boundary conditions. Thus it is much more difficult to determine the exact function $g(P|Q)$ for a given set of boundary conditions. The Green's functions and solutions to the shape control and estimation problems will therefore be exhibited in terms of eigenfunction expansions.

4.7 Control Problem for the Operator ∇^4

On the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, we desire to determine the set of forces $\{f_i\}$ at positions $P_i = (x_i, y_i)$, $1 \leq i \leq m$, which together with the solution $u(x,y)$ of

$$\nabla^4 u = \sum_{i=1}^{m} f_i \; \delta(x-x_i) \; \delta(y-y_i) \tag{162}$$

$$u = \nabla^2 u \quad \text{on } x = 0, a \text{ and } y = 0, b \tag{163}$$

minimize the performance criterion

$$J(F,u) = \frac{1}{2} \sum_{i=1}^{m} f_i^{\,2} r_i + \frac{1}{2} \int_0^a \int_0^b (\psi(x,y) - u(x,y))^2 \, dydx \tag{164}$$

overall admissible pairs (F,u).

The optimal shape for the problem (162-164) is given by

$$u^*(x,y) = \sum_{i=1}^{m} f_i^{\,*} \; g(x,y, x_i, y_i) \tag{165}$$

where the vector of optimal forces $F^* = (f_1^* \ldots f_m^*)$ satisfies

$$(R + A)F^* = B \tag{166}$$

The m x m matrices R^{-1} and A have coordinates

$$R_{ij} = r_i \, \delta(i-j) \tag{167}$$

$$a_{ij} = \int_0^a \int_0^b g(x,y,x_i,y_i)\, g(x,y,x_j,y_j)\, dy\, dx \tag{168}$$

and the vector B has coordinates

$$b_i = \int_0^a \int_0^b \psi(x,y)\, g(x,y,x_i,y_i)\, dy\, dx. \tag{169}$$

Since a complete analytical form for the Green's function is not known, we use the eigenfunctions

$$\phi_{k\ell}(x,y) = \frac{2}{\sqrt{ab}} \sin \frac{k\pi x}{a} \sin \frac{\ell \pi y}{b} \tag{170}$$

and corresponding eigenvalues

$$\lambda_{k\ell} = \pi^4 \left[\left(\frac{k}{a}\right)^2 + \left(\frac{\ell}{b}\right)^2\right]^2 \tag{171}$$

to represent the solutions (166-169). Thus

$$u^*(x,y) = \sum_{i=1}^{m} \sum_{k,\ell=1}^{\infty} 4 f_i \frac{\sin \frac{k\pi x}{a} \sin \frac{k\pi x_i}{a} \sin \frac{\ell\pi y}{b} \sin \frac{\ell\pi y_i}{b}}{\pi^4 ab \left[\left(\frac{k}{a}\right)^2 + \left(\frac{\ell}{b}\right)^2\right]^2} \tag{172}$$

where the forces f_i satisfy (166), and the coefficients of the matrix A and vector B in (167-168) are given by

$$a_{ij} = \sum_{k,\ell=1}^{\infty} \frac{4}{ab\lambda_{k\ell}^2} \left(\sin \frac{k\pi x_i}{a} \sin \frac{k\pi x_j}{a} \sin \frac{\ell\pi y_i}{b} \sin \frac{\ell\pi y_j}{b}\right) \tag{173}$$

$$b_i = \sum_{k,\ell=1}^{\infty} \frac{2}{\sqrt{ab}\,\lambda_{k\ell}} \left(\sin \frac{\pi x_i}{a} \sin \frac{\ell\pi y_i}{b}\right) \langle \phi_{k\ell}, \psi \rangle \tag{174}$$

where

$$\langle \phi_{k\ell}, \psi \rangle = \sqrt{\frac{4}{ab}} \int_0^a \int_0^b \psi(x,y) \left(\sin \frac{k\pi x}{a}\right) \left(\sin \frac{\ell\pi y}{b}\right) dy\, dx \tag{175}$$

Approximations are available by taking the first few terms in k and ℓ.

4.8 The Estimation Problem for ∇^4

The shape estimation problem for a rectangular plate is given the shape observations

$$Y_i = u(x_i,y_i) + \zeta_i \, , \qquad 1 \le i \le m \, , \tag{176}$$

to determine the error function f(x,y) and corresponding shape function u(x,y) which satisfy

$$\nabla^4 u(x,y) = f(x,y) \, , \tag{177}$$

$$u = \nabla^2 u = 0 \quad \text{for } x = 0, a \text{ and } y = 0, b$$

and minimize the criterion

$$J(f,u) = \frac{1}{2} \sum_{i}^{m} (Y_i - u\,(x_i,y_i))^2 r_i^{-1} + \frac{1}{2} \int_0^a \int_0^b f^2(x,y)\, dy\, dx \, . \tag{178}$$

The necessary condition for an optimal solution is that the vector $X = (u^*(x_1,y_1) \, . \, . \, . \, u^*(x_m,y_m))$ satisfy

$$(I + A\, R^{-1}\, X) = A\, R^{-1}\, Y \tag{179}$$

where $u^*(x_i,y_i)$ is the optimal shape estimate at the point (x_i,y_i), the matrices A and R are defined by (167-168), and $Y = (Y_1 \, \ldots \, Y_m)$. The optimal noise estimates are

$$\zeta_i^* = Y_i - u^*(x_i,y_i) \, , \qquad 1 \le i \le m \, , \tag{180}$$

$$f^*(x,y) = \sum_{i=1}^{m} r_i^{-1} \zeta_i^* \, G(x,y,x_i,y_i) \tag{181}$$

and the optimal shape estimate is

$$u^*(x,y) = \sum_{i=1}^{m} [r_i^{-1} \zeta_i^* \int_0^a \int_0^b g\,(x,y,\xi,\eta)\; g\,(x_i,y_i,\xi,\eta)\; d\eta\, d\xi \, . \tag{182}$$

To compute the vector X in (179) we use (173) for the elements of the matrix A. Then

$$f^*(x,y) = \sum_{i=1}^{m} \sum_{k,\ell=1}^{\infty} \frac{4}{ab\lambda_{k\ell}} \left[\sin\frac{k\pi x}{a} \sin\frac{k\pi x_i}{a} \sin\frac{\ell\pi y}{b} \sin\frac{\ell\pi y_i}{b}\right] . \tag{183}$$

Finally, applying the expansion (72) to the optimal shape estimate (182),

$$u^*(x,y) = \sum_{i=1}^{m} \sum_{k,\ell=1}^{\infty} \left[r_i^{-1}\zeta_i^* \frac{4}{ab\lambda_{k\ell}^{2}} \sin\frac{k\pi x}{a} \sin\frac{k\pi x_i}{a} \sin\frac{\ell\pi y}{b} \sin\frac{\ell\pi y_i}{b}\right] \tag{184}$$

Again, approximations are obtained by taking the first few terms in k and ℓ.

4.9 Conclusions

Green's function techniques have been applied to the solution of shape control and estimation problems which have associated boundary value problems involving partial differential equations, in a manner analogous to those involving ordinary differential equations. In the case of a multidimensional domain, however, precise knowledge of the analytical form of the Green's function is usually not available. Solutions may be expressed in terms of eigenfunction expansions.

Although this chapter deals with systems which do not have rigid body modes, the techniques and solutions bear such a resemblance to those of the one dimensional case that an extension to systems with rigid body modes follows readily.

Chapter 5. Static Shape Control for Multidimensional Large Space Structures

5.1 Introduction

This chapter addresses the problems of static shape control and shape determination for multidimensional structures. Chapters 2-4 have addressed these problems for scalar shape functions, representing displacement in one direction, defined on one or multidimensional domains. However, large space structures are modeled as multidimensional states, representing translations and/or rotations in three dimensional space.

We again use an integral operator approach based on assumptions of linear self-adjoint dynamics and boundary conditions. As might be expected, algorithms which are similar in appearance arise.

However, there are important differences in interpretation and procedure. These include matrix, rather than scalar, differential and integral operators, controls and observations applied to only a part of the state, and the necessity for using approximate eigenfunctions provided by experimental or numerical methods, since the exact operators and corresponding eigenfunctions are usually not known. The algorithms derived in this chapter will be adapted to the use of modes from a dynamic finite element model, and illustrated by simulated results, in Chapter 6.

Procedure

In section 5.2 we define the multidimensional linear boundary value problem for a large space structure, and discuss the existence of solutions. We then define Green's functions for a multidimensional boundary value problem, both with and without rigid body modes, and derive solutions to the boundary value problem for both cases.

In section 5.3 we define and solve the shape control problem for a large space structure. We discuss examples of the constraints imposed on the control forces by the presence of rigid body modes. In section 5.4 we define and solve the shape determination problem.

We present eigenfunction expansions for the more general multidimensional terms in the algorithms, which involve Green's functions, in section 5.5. A summary and conclusions are stated in Section 5.6.

5.2 The Model and the Green's Function

Consider a multidimensional system represented by the n dimensional state $U(P)$, defined on a simply connected domain $\Omega \epsilon R^{\ell}$. Suppose the system is governed by linear dynamics

$$L\,U = F \qquad \text{for } P \in \Omega \tag{185}$$

where L is an n x n matrix of differential operators. $F(P)$ is an n dimensional vector function, or distribution, defined on Ω, which represents forces or torques acting on the system.

Suppose the system satisfies k_o linear boundary conditions

$$B_i(U) = 0\ , \qquad 1 \leq i \leq k_o\ , \qquad \text{for } P \in \Gamma \tag{186}$$

where Γ is the boundary of Ω. We will assume the boundary value problem (185-186) is self-adjoint, that is that $L^* = L$ and

$$\langle LU,\ V\rangle = \langle U,\ LV\rangle \tag{187}$$

where U and V are any two admissible functions which satisfy the boundary conditions and $\langle U,V\rangle$ is the inner product

$$\langle U,V\rangle = \int_{\Omega} U^T(P)\ V(P)\ dP\ . \tag{188}$$

We will also require the usual vector inner product

$$\langle X,Y\rangle = X^T Y = Y^T X\ . \tag{189}$$

We will use the norms induced by (188-189) and the weighted seminorm

$$||X||_R^2 = \langle X,X\rangle_R = X^T RX \ . \tag{190}$$

X and Y are vectors in the same space and R is a symmetric square matrix of appropriate dimension such that $R \geq 0$.

The reasons for the model formulation (185-186) become apparent when one considers an LSS (large space structure) antenna. The domain consists of the subset of three dimensional space occupied by the undistorted ideal shape, a perfect paraboloid. The state might be three dimensional also, representing vector displacements of points on the distorted antenna from their ideal positions. Boundary conditions represent a pinned antenna which may not rotate or translate as a rigid body in any direction, a free-free antenna which may rotate or translate along any of the three axes, or conditions between these two extremems.

Other state representations are possible. It may be convenient to consider a six dimensional state which represents translations and rotations of a point about the three axes. This is the case if torques are to be as control mechanisms, in addition to translational forces. A torque can be considered an impulsive force applied to a rotational coordinate of the state.

Solutions of Boundary Value Problems

We consider under what circumstances solutions to boundary value problem (185-186) exist, and what form the solutions take if they do exist.

We will apply the following alternative theorem for boundary value problems:

Theorem 5.1: Consider the boundary value problem

$$LU = F,\ B_i(U) = 0\ , \qquad 1 \leq i \leq k_o\ , \tag{191}$$

its corresponding homogeneous problem

$$LU = 0\ ,\ B_i(U) = 0\ , \qquad 1 \le i \le k_o\ , \tag{192}$$

and the related homogeneous adjoint problem

$$L^*V = 0\ ,\ B_i^*(V) = 0\ , \qquad 1 \le i \le k_o\ . \tag{193}$$

L is an $n \times n$ matrix of linear differential operators, L^* is its adjoint, U and V are vector functions defined on the simply connected domain Ω, and B_i and B_i^* are adjoint linear boundary operators defined on Γ, the boundary of Ω.

Then: (a) if the problem (192) has only the trivial solution $U \equiv 0$, so does the problem (193), and (191) has a unique solution.

(b) if (192) has s independent solutions $U_1,\ldots,U_s$, then (193) has s independent solutions $V_1,\ldots,V_s$, and (191) has solutions if and only if

$$\langle V_i, F\rangle = \int_\Omega V_i^T(P)\ F(P)\ dP = 0\ , \qquad 1 \le i \le s\ . \tag{194}$$

If the conditions (194) are satisfied, the general solution of (191) has the form

$$U(P) = \hat{U}(P) + \sum c_i\ U_i(P) \tag{195}$$

where $\hat{U}$ is a particular solution of (191), the c_i are constants and U_i, $1 \le i \le s$ are the solutions of (192).

For discussions and proof of alternative theorems see [2].

We have assumed the linear operator L and boundary conditions $B_i = 0$, $1 \le i \le k$, are such that the boundary value problem (191) is self-adjoint, that is that $L^* = L$ and $B_i^* = B_i$, $1 \le i \le k$. Thus (192) and (193) are equivalent for our purposes.

To observe the form the solutions actually take, we define Green's functions for the cases (a) and (b) of Theorem 5.1.

Green's Functions

We first consider case (a) of Theorem 5.1, that the homogeneous boundary value problem has only the solution $U \equiv 0$. This is equivalent to the physical assumption that the system has no rigid body modes.

Define the n vector functions $G_j(P|Q)$, $1 \leq j \leq n$, to satisfy

$$L\, G_j(P|Q) = e_j\, \delta(P-Q)$$
$$= e_j\, \delta(P^1-q^1)\ \ldots\ \delta(P^\ell-q^\ell) \qquad (196)$$

$$B_i(G_j) = 0\ , \qquad 1 \leq i \leq k_o\ , \qquad P \in \Gamma \qquad (197)$$

The unit vector e_j has zeros in all coordinates except the jth, where it has the value one. The points $P(p^1 \ldots p^\ell)^T$ and $Q(q^1 \ldots q^\ell)$ in (196) lie in Ω.

$G_j(P|Q)$ represents the response of the system to a unit impulsive force applied to the jth coordinate of the structure at the point Q.

Define $G(P|Q)$ to be the nxn matrix function with columns G_j. $G(P|Q)$ is the desired Green's function for the boundary value problem (191). The ijth coordinate G_{ij} $(P|Q)$ represents the response of the ith coordinate of the state at P to a unit impulsive force applied to the jth coordinate of the state at Q. We may write

$$LG(P|Q) = I_n\, \delta(P-Q) \qquad (198)$$

$$B_i(G) = 0\ , \qquad 1 \leq i \leq k_o\ , \qquad (199)$$

if it is understood that the boundary conditions in (199) are to be applied to each column of G individually,

The property derived in the next theorem will be useful when writing the solution of (191) in terms of the Green's function G.

<u>Theorem 5.2</u> Let $G(P|Q)$ be the function defined by (196-197). Then $G(P|Q)=G^T(Q|P)$.

Proof: For the moment we drop the assumption that the boundary value problem (191) is self-adjoint. Let $G_j(P|Q)$ and $H_i(P|R)$ be functions defined on Ω such that

$$L\, G_j(P|Q) = e_j\, \delta(P-Q)\ ,\quad B_v(G_j) = 0, \qquad 1 \leq v \leq k\ .$$

$$L^*\, H_i(P|R) = e_i\, \delta(P-R) \quad B_v^*(H_i) = 0\ , \quad 1 \leq v \leq k\ .$$

Since G_j and H_i satisfy adjoint boundary value problems,

$$<G_j,\ L^*H_i> = <LG_j,\ H_i>\ , \qquad 1 \leq i\ ,\ j \leq n.$$

Thus

$$\int_\Omega G_j^T(P|Q)\ e_i\ \delta(P-R)dP = \int_\Omega e_j^T\ \delta(P-Q)H_i(P|R)dP\ .$$

Evaluation of the integrals yields

$$G_j^T(R|Q)\ e_i = e_j^T\ H_i(Q|R)\ . \tag{200}$$

$$G_{ij}(R|Q) = H_{ji}(Q|R)$$

But now we recall that $L^*=L$ and $B_i^*=B_i$. Thus $H_{ji}(Q|R) = G_{ji}(Q|R)$. Substitution into (200) yields

$$G_{ij}(R|Q) = G_{ji}(Q|R), \qquad 1 \leq i, \quad j \leq n\ .\ \#$$

We now seek the solution to the boundary value problem (191), assuming case (a) of Theorem 5.1. Let $U(P)$ be a solution of (191). Then

$$<U, L\, G_j> = \int_\Omega U^T(P)\ e_j\ \delta(P-Q)\ dP = U_j(Q)$$

where the U_j is the jth coordinate of U. By Green's theorem

$$<L\, G_j,\ U> = <G_j,\ LU> = \int_\Omega G_j^T(P|Q)\ F(P)\ dP.$$

Thus,

$$U_j(Q) = \int_\Omega G_j^T(P|Q)\ F(P)\ dP, \qquad 1 \leq j \leq n\ .$$

If we apply this argument to all coordinates $1 \leq j \leq n$, we have

$$U(Q) = \int_\Omega G^T(P|Q)\ F(P)\ dP = \int_\Omega G(Q|P)\ F(P)\ dP$$

by Theorem 5.2. A change of variables yields the solution

$$U(P) = \int_\Omega G(P|Q)\ F(Q)\ dQ\ . \tag{201}$$

The Modified Green's Function

We now consider case (b) of Theorem 5.1. We assume the boundary value problem (191) has s independent solutions $V_1,\ldots,V_s$, which we assume are orthonormal. If they are not, a Gram-Schmidt orthogonalization process can be applied to generate an orthonormal set.

Define the following vector functions:

$$L\ G_j(P|Q) = [\delta(P-Q) - \sum_{i=1}^{s} V_i(P)\ V_i^T(Q)]\ e_j \tag{202}$$

$$B_i(G_j) = 0\ , \qquad 1 \le i \le k\ , \tag{203}$$

where e_j is the jth column of the nxn identity matrix.

Note that the right hand side of (202) has zero components in the space spanned by the functions $\{V_i\}$, that is that its inner product with these functions is zero. Thus by Theorem 5.1 a solution, in the distributional sense, to these problems exists.

The solutions G_j which satisfy (202) are not uniquely determined, since the addition of any linear combination of solutions to the homogeneous problem (192) yields another solution. Thus we are free to impose another condition. We require that

$$\langle G_j, V_i \rangle = 0\ , \qquad 1 \le i \le s\ ,\ \ 1 \le j \le n\ . \tag{204}$$

Mathematically this means we seek the solutions to (202-203) of minimum norm, those which lie in the orthogonal complement of the nullspace of the operation L - the space spanned by the solutions $\{V_i\}$. Thus the functions G_j have no components in the direction of the rigid body modes. Solutions of (202-204) are unique.

Let $G(P|Q)$ be the n x n matrix function whose columns are the functions G_j which satisfy (202-204), that is $G = [G_1 \mid \ldots \mid G_n]$. Then G satisfies

$$\begin{aligned} LG &= I_n\, \delta(P-Q) - \sum_{1}^{s} V_i(P)(V_i\ (Q))^T \\ B_i(G) &= 0\ , \qquad 1 \le i \le k\ . \\ \langle G, V_i\rangle &= 0\ , \qquad 1 \le i \le s\ . \end{aligned} \tag{205}$$

$G(P|Q)$ is called the <u>modified</u> Green's function for the system (191), assuming case (b) of Theorem 5.1. The property derived in Theorem 5.2 may also be shown to be true for modified Green's functions.

We seek a solution U to the boundary value problem (191) for case (b) of Theorem 5.1. We assume

$$\langle F, V_i\rangle = 0\ , \qquad 1 \le i \le s\ , \tag{206}$$

since without these conditions a solution does not exist. We will apply Green's theorem to the inner product <u, LG>. From (205)

$$\begin{aligned} \langle U, LG\rangle &= \int_\Omega U^T(P)[I_n\ \delta(P-Q) - \sum_{i=1}^{s} V_i(P)\ V_i^T(Q)]\ dP \\ &= U^T(Q) - \sum_{1}^{s} (\int_\Omega U^T(P)\ V_i(P)\ dP)\ V_i(Q)^T\ . \end{aligned}$$

But

$$\begin{aligned} \langle U,V_i\rangle &= \int_\Omega U^T(P)\ V_i(P)\ dP \\ &= \text{some constant } c_i. \end{aligned}$$

Thus

$$\langle U, LG\rangle = U^T(Q) - \sum c_i\ V_i(Q)^T \tag{207}$$

On the other hand, because the boundary value problem is self-adjoint

$$\langle U, LG\rangle = \langle LU, G\rangle = \int_\Omega F^T(P)\ G(P|Q)\ dP\ . \tag{208}$$

Equating (207) & (208) and taking the transpose, we have

$$U(Q) = \sum c_i V_i(Q) + \int_\Omega G^T(P|Q)\, F(P)\, dP .$$

We apply Theorem 5.2 and a change of variables:

$$U(P) = \int_\Omega G(P|Q)\, F(Q)\, dQ + \sum c_i V_i(P) . \tag{209}$$

As one might expect from Theorem 5.1, the solution includes an arbitrary linear combination of rigid body modes, or solutions to the homogeneous problem.

Remark 5.1 Naturally if a force is applied which does not satisfy the constraints (206), the system will still respond, but the boundary conditions will be violated. The conditions (206) usually translate physically into conditions that net forces or torques in one or more directions must be zero.

Without loss of generality, we can define a coordinate system with respect to the space vehicle itself. In the case of the antenna we define the xy plane tangent to the hub of the antenna and the z axis along the axis of the paraboloid. We may fix the x axis along a particular rib. With the coordinate system so defined, we may ignore the rigid body modes, since rotations and translations of the antenna as a rigid body occur with respect to another coordinate system.

We can then consider the solution of (191) to be

$$U(P) = \int_\Omega G(P|Q)\, F(Q)\, dQ \tag{210}$$

where $G(P|Q)$ is the modified Green's function which satisfies (202-204).

5.3 The Shape Control Problem

Static shape control forces may be applied to some or all of the coordinates of the multidimensional state. Thus we define the following control problem:

Let $\psi(P)$ be the desired shape of the space structure. Determine the set of control vectors F_i of (predetermined) dimension $n(i)$, $1 \leq i \leq m$, such that the resulting shape $U(P)$, which satisfies the dynamics

$$LU = \sum_{i=1}^{m} C_i F_i \ \delta(P-P_i) \tag{211}$$

and boundary conditions

$$B_i(U) = 0 \ , \qquad 1 \leq i \leq k_o \ , \tag{212}$$

most closely approximates the desired shape ψ on Ω. The measure of best approximation is that the set $(F_1^*,\ldots,F_m^*, U^*)$ minimize the performance criterion

$$J = \frac{1}{2} \sum_{1}^{m} ||F_i||_{R_i}^{\ 2} + \frac{1}{2} \int_{\Omega} ||\psi(P) - U(P)||_{W(P)}^{2} \ d) \ . \tag{213}$$

over all possible sets which satisfy (211-212).

The constant $n \times n$ (i) matrices C_i distribute the control vector F_i over the coordinates of the state U at P_i. $\delta(P-P_i)$ is the dirac delta function for the multidimensional point P_i.

The $n(i) \times n(i)$ matrices R_i are symmetric and $R_i \geq 0$.

$W(P)$ is a piecewise continuous symmetric positive definite matrix defined on Ω.

We first assume the homogeneous system

$$LU = 0 \ , \ B_i(U) = 0 \ , \qquad 1 \leq i \leq k_o, \tag{214}$$

has only the solution $U \equiv 0$.

We apply the solution derived in section 5.2 to the boundary value problem (211-212):

$$\begin{aligned} U(P) &= \int_{\Omega} G(P|Q)[\sum_{1}^{m} C_i \ F_i \ \delta(Q-P_i)] \ dQ \\ &= \sum_{i=1}^{m} G(P|P_i) \ C_i \ F_i \end{aligned} \tag{215}$$

where $G(P|Q)$ is the appropriate Green's function. We substitute (215) into the criterion (213), which becomes a functional depending solely on the discrete unknowns F_i.

$$J = \frac{1}{2}\sum_{1}^{m} ||F_i||_{R_i}^2 + \frac{1}{2}\int_\Omega ||\psi(P) - \sum_{i=1}^{m} G(P|P_i)\, C_i F_i||_W^2 \, dP \, . \tag{216}$$

We seek the minimum of J with respect to the constant vectors F_j:

$$\frac{\partial J}{\partial F_j} = F_j^T R_j + \int_\Omega [\psi(P) - \sum_{i=1}^{m} G(P|P_i)C_iF_i]^T W(P)\, [-G(P|P_j)C_j]\, dP \tag{217}$$

$$= 0 \, , \qquad 1 \leq j \leq m \, .$$

Thus

$$R_jF_j + \sum_{i=1}^{m} C_j^T \left(\int_\Omega G(P_j|P)G(P|P_i)\, dP\right) C_iF_i$$

$$= C_j^T \int_\Omega G(P_j|P)W(P)\, \psi(P)\, dP \qquad \text{for } 1 \leq j \leq m \, . \tag{218}$$

Let $N = \sum_{i=1}^{m} n(i)$. (219)

Let R be the block diagonal square matrix with diagonal blocks $R_1,\ldots,R_m$.

Let A be the N x N matrix of $n(i)$ by $n(j)$ blocks A_{ij}, where

$$A_{ij} = C_i^T\left(\int_\Omega G(P_i|P)\, W(P)\, G(P|P_j)\, dP\right) C_j \, . \tag{220}$$

Let D be the N dimensional vector

$$D = [D_1^T \ldots D_m^T]^T \tag{221}$$

where

$$D_j = C_j^T \int_\Omega G(P_j|P)\, W(P)\, \psi(P)\, dP \, .$$

Let F be the N dimensional vector of unknown control forces:

$$F = (F_1^T \dots F_m^T)^T . \tag{222}$$

Then the vector F* of the optimal control forces satifies

$$(R + A)\ F^* = D . \tag{223}$$

Once the vector F* has been determined, the optimal shape U*(P) is given by

$$U^*(P) = \sum_{i=1}^{m} G(P|P_i)\ C_i F_i^* . \tag{224}$$

The Shape Control Problem for Systems with Rigid Body Modes

We now assume that the homogeneous boundary value problem has s solutions $V_1(P),\dots,V_s(P)$. We let $G(P|Q)$ denote the modified Green's function which satisfies (205).

In order for a solution (211,212) to exist, the right hand side of (211) must satisfy the additional set of constraints (206). That is

$$\langle V_i, \sum_{j=1}^{m} C_j F_j\ \delta(P-P_j)\rangle = 0 , \qquad 1 \le i \le s ,$$

which by definition is the set of conditions

$$\int V_i^T(P) \sum_{j=1}^{m} C_j F_j\ \delta(P-P_j)\ dP = 0 , \quad 1 \le i \le s .$$

Evaluating the integral yields

$$\sum_{j=1}^{m} V_i^T(P_j)\ C_j F_j = 0 , \qquad 1 \le i \le s . \tag{225}$$

The shape control problem is now to find the set of forces $\{F_i\}$ and shape U(P) which satisfy (211-212) (225), and minimize the criterion (213) over all possible sets.

Examples of Constraints

Example 5.1: A three dimensional static with three rigid body modes.

Suppose the state U(P) is three-dimensional, representing the displacement vector of the actual position of the structure corresponding to the point P from its ideal position, and the antenna has three rigid body modes representing translations along any of the three axes. An orthogonal basis for the space spanned by these rigid body modes is

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Note that if U(P) is a three-dimensional state then

$$U(P) + \sum_{i=1}^{3} c_i V_i$$

does represent a translation of that state.

The constraints (225) become

$$\sum_{j=1}^{m} (C_j F_j)^i = 0 \qquad 1 \le i \le 3 . \tag{226}$$

where $(C_j F_j)^i$ is the ith coordinate of $C_j F_j$. This is equivalent to the condition that the net force applied in any direction of the state U over all the points P_i is zero. If the sum of the forces in any direction is zero, no net acceleration is applied to the structure as a whole, which is in keeping with the free boundary conditions.

Example 5.2: A six-dimensional state.

If torques are to be applied as part of the control scheme it may be convenient to consider a six-dimensional state, the first three components of which represent displacements as before, and the second three components of which represent rotations.

A torque is an impulsive force applied to a rotational coordinate. Suppose that the system has six rigid body modes, representing constant translations or rotations from an ideal position. A basis for the space of rigid body modes is $(1\ 0\ 0\ 0\ 0\ 0)^T$, $(0\ 1\ 0\ 0\ 0\ 0)^T$, $(0\ 0\ 1\ 0\ 0\ 0)^T$, $(0\ 0\ 0\ 1\ 0\ 0)^T$, $(0\ 0\ 0\ 0\ 1\ 0)^T$ and $(0\ 0\ 0\ 0\ 0\ 1)^T$. The later three vectors represent unit rotations about the three axes.

The constraints (225) again become

$$\sum_{j=1}^{m} (C_j F_j)^i = 0 , \qquad 1 \le i \le 6 . \tag{227}$$

These constraints represent the fact that the net sum of forces or torques applied to any coordinate of the state must be zero, a requirement which guarantees zero translational or rotational acceleration applied to the state.

Example 5.3: A three-dimensional state with six rigid body modes.

Suppose for computational convenience we wish to consider a three-dimensional state, but the vehicle is allowed to both rotate and translate along three axes as a rigid body. One basis for the six rigid body modes is

$$V_1 = (1\ 0\ 0)^T \quad V_2 = (0\ 1\ 0)^T \quad V_3 = (0\ 0\ 1)^T$$

$$V_4(P) = T_1 P \qquad V_5(P) = T_2 P \qquad V_6(P) = T_3 P$$

where

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad T_2 = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

and

$$T_3 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

T_1, T_2 and T_3 represent rotations by an angle θ about the x, y, and z axes respectively.

The first three constraints yield the same conditions as in example 5.1. The last three constraints yield

$$\sum_{j=1}^{m} P_j^T \, T_i^T \, C_j F_j = 0 \qquad 1 \leq i \leq 3 \, . \tag{228}$$

For the rigid body mode $V_4(P) = T_1 P$ this is

$$\sum_{j=1}^{m} [p_j^{\,1}, (p_j^{\,2} \cos\theta - p_j^{\,3} \sin\theta), (p_j^{\,2} \sin\theta + p_j^{\,3} \cos\theta)] \, C_j F_j = 0. \tag{229}$$

This expression is the requirement that the sum of the forces applied times the displacements at the points where the forces are applied must be zero. But this is just one of the constraints which resulted from rotational rigid body modes in example 5.2: the sum of the torques must be zero.

It is easily seen that the condition that the sum of the torques be zero for each coordinate is satisfied if the constraints (228) are satisfied. Thus, the constraints for six rigid body modes are the same, however the state vector is defined.

The procedure for finding the set of optimal control vectors for systems with rigid body modes is as follows:

i) Substitute the solution (215) into the criterion (213)

ii) Solve the constraints (225) for some of the control vectors in terms of the others.

iii) Substitute the expressions derived in (ii) into the criterion J, which now becomes a function of fewer control vectors.

iv) Minimize J with respect to this smaller set of control vectors. The minimization process will result in a system of linear equations which, when solved, yield the identity of the optimal set of these vectors. The other control vectors may then be determined from (ii).

The pinned-free beam in section 3.3 was an example of this procedure in the case of a one-dimensional state.

5.4 The Shape Determination Problem

The desired shape ψ in the control problem in the last section will be based on the difference between the estimated shape and the ideal parabolic shape. The estimated shape must be computed from observations of some or all of the components of the state, taken at a number of predetermined points along the structure.

Thus we seek to determine the estimates of the noise vector $F(P)$ and shape function $U(P)$, defined on Ω, based on the observations

$$Y_i = C_i\, U(P_i) + Z_i\ , \qquad 1 \leq i \leq m\ , \tag{230}$$

which minimize the performance criterion

$$J = \frac{1}{2} \sum_{i=1}^{m} \| Y_i - C_i\, U(P_i) \|^2_{R_i^{-1}} + \frac{1}{2} \int_\Omega \| F(P) \|_{W^{-1}(P)}\, dP \tag{231}$$

over all admissible sets $\{U,F\}$ which satisfy

$$LU = F,\ P \in \Omega \quad \text{and } B_i(U) = 0 \qquad 1 \leq i \leq k_o\ , \quad P \in \Gamma\ . \tag{232}$$

The constant matrices C_i are $n(i) \times n$, the $n(i)$ dimensional vectors Z_i represent noise or inaccuracies in the observations Y_i, $W(P)$ is a continuous positive definite matrix on Ω, and R_i are $n(i) \times n(i)$ constant positive definite matrices.

We will assume the boundary value problem (232) has no rigid body modes. The estimation algorithm for systems with rigid body modes is the same, with the exception of the fact that the rigid body modes themselves cannot be estimated. The derivation of this fact follows as in section 2.5.

We will evalaute the solution (210) of the boundary value problem (232) at the points P_i, $1 \le i \le m$, and substitute into the criterion J.

$$u(P_i) = \int_\Omega G(P_i|Q)\ F(Q)\ dQ\ . \tag{233}$$

$$J = \frac{1}{2}\sum_1^m ||Y_i - C_i \int_\Omega G(P_i|Q)\ F(Q)\ dQ||^2_{R_i^{-1}} + \frac{1}{2}\int_\Omega ||F(Q)||^2_{W^{-1}}\ dQ\ . \tag{234}$$

The criterion is now solely a function of the continuous unknown vector function F(Q). To minimize J with respect to F we find the Frechet derivative $\partial J(F,H)$, where H is any admissible variation, and set it equal to zero.

$$\partial J(F,H) = \sum_1^m [Y_i - C_i \int_\Omega G(P_i|Q)\ F(Q)\ dQ]^T R_i^{-1}[-C_i \int_\Omega G(P_i|Q)\ H(Q)\ dQ]$$

$$+ \int_\Omega F(Q)^T\ W^{-1}(Q)\ H(Q)\ dQ = 0\ .$$

If we transpose the equation, factor out H and recall (233), we have

$$\int_\Omega H^T(Q)\ W^{-1}(Q)\ [F(Q) + \sum_{i=1}^m G(Q|P_i)\ C_i^T R_i^{-1}(C_i\ u(P_i) - Y_i)] = 0$$

Since this must be true for all admissible variations H, we have

$$F(Q) = W(P) \sum_{i=1}^m G(Q|P_i)\ C_i^T R_i^{-1}\ (Y_i - C_i\ U(P_i))\ . \tag{235}$$

We still do not know the optimal estimate of F at this point, because the estimates $C_i\ U(P_i)$ are still not known. We substitute (235) into (233).

$$U(P_j) = \int_\Omega G(P_j|Q)\ W(Q)\ [\sum_{i=1}^m G(Q|P_i) C_i^T R_i\ (Y_i - C_i\ U(P_i))\ . \tag{236}$$

Then we have, for $1 \le j \le m$,

$$U(P_j) + \sum_{i=1}^m (\int_\Omega G(P_j|Q)\ W(Q)\ G(Q|P_i)\ dQ)\ C_i^T R_i\ C_i\ U(P_i)$$

$$= \sum_{i=1}^m \int_\Omega G(P_j|Q)\ Q(Q)\ G(Q|P_i)\ dQ)\ C_i^T R_i\ Y_i\ . \tag{237}$$

We will solve this set of m matrix equations for the vectors $C_j\ U(P_j)$, $1 \le j \le m$. Multiply both sides on the left by C_j. Again define $N = \sum_{i=1}^m n(i)$. (Recall that n(i) is the dimension of Y_i.)

Let A be the matrix of m blocks by m blocks, where the jth block

$$A_{ji} = C_j\left(\int_\Omega G(P_j|Q)\ Q(Q)\ G(Q|P_i)\ dQ\right) C_i^T R_i \tag{238}$$

is an n(j) by n(i) matrix. Thus A is N x N.

Let R^{-1} be the N x N block diagonal matrix with blocks

$$R_{ij}^{-1} = R_i^{-1}\ \delta(i-j)\ . \tag{239}$$

Let $U^*(P_i)$ be the optimal estimate of the shape function U at P_i, and let $\bar{U}$ be the N dimensional vector formed by "stacking" the n(i) dimensional vectors $C_i\ U^*(P_i)$ (240)

Let Y be the N dimensional vector

$$(Y_1^T \ldots Y_m^T)^T. \tag{241}$$

Then the vector $\bar{U}$ satisfies the system of linear equations

$$(I_N + AR^{-1})\ \bar{U} = A\ R^{-1}\ Y\ . \tag{242}$$

Once the vector $\bar{U}$ is known, the optimal estimate F* of the noise vector F is given by

$$F^*(P) = W(P) \sum_{i=1}^{m} G(P|P_i)\ C_i^T\ R_i^{-1}\ (Y_i - C_i U^*(P_i))\ . \tag{243}$$

The optimal shape estimate U*(P) is then given by

$$U^*(P) = \sum_{i=1}^{m} \left(\int_\Omega G(P|Q)\ W(Q)\ G(Q|P_i) dQ\right)\ C_i^T\ R_i^{-1}(Y_i - C_i U^*(P_i)). \tag{244}$$

5.5 Approximations

In this section approximations will be presented, which involve eigenfunctions corresponding to the static boundary value problems (211-212) (232) which parallel those in section 2.6.

However, most finite element models for large space structures are dynamic, rather than time-invariant. Therefore, in the next chapter, approximations will be developed for the use of eigenfunctions from the dynamic model corresponding to (185-186).

It was demonstrated in Theorem 5.2, section 5.2, that $G(P|Q)$ was symmetric. We will also assume that $G(P|Q)$ is a Hilbert-Schmidt kernel, that is that

$$\int_{\Omega} ||G(P|Q)||^2 \, dPdQ < \infty . \tag{245}$$

Let K be the integral operator with the Green's function as kernel. Then for F(P) in the domain of K,

$$Kf = \int_{\Omega} G(P|Q) \, f(Q) \, dQ . \tag{246}$$

Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \ \ldots$ be the non-zero eigenvalues of K, and $\phi_1, \phi_2, \ldots$ be the associated eigenfunctions, such that

$$K \, \phi_i = \mu_i \, \phi_i . \tag{247}$$

The non-zero eigenvalues $\{\mu_i\}$ of K are the inverses of the non-zero eigenvalues of L, and the eigenfunctions $\{\phi_i\}$ are also the corresponding eigenfunctions of L.

We will assume the eigenfunctions $\{\phi_i\}$ have been normalized with respect to the inner product (188).

From integral operator theory we have the following expansion for the Green's function:

$$G(P|Q) = \sum_{i=1}^{\infty} \mu_i \, \phi_i(P) \, \phi_i^T(Q) \tag{248}$$

If we assume, as in Chapter 2, that W is the identity (matrix) on Ω, we have the following expansions:

$$\int_{\Omega} G(P|Q) \, G(Q|R) \, dQ = \sum_i \mu_i^2 \, \phi_i(P) \, \phi_i^T(R) \tag{249}$$

and

$$\int_{\Omega} G(P|Q) \, \psi(Q) \, dQ = \sum_i \mu_i \, \phi_i(P) \, \langle\phi_i,\psi\rangle . \tag{250}$$

These expressions are generalizations to the multidimensional case of the approximations offered in Chapter 2.

If W is not the identity matrix, (249-250) become

$$\int_\Omega G(P|Q)\ W(Q)\ G(Q|R)\ dQ = \sum_i \sum_j \mu_i\ \mu_j\ \phi_i(P)\ \phi_j^T(Q)\ \langle\phi_i, \phi_j\rangle_W \tag{251}$$

and

$$\int_\Omega G(P|Q)\ W(Q)\ \psi(Q)\ dQ = \sum_{i=1}^{\infty} \mu_i\ \phi_i(P)\ \langle\phi_i, \psi\rangle_W\ . \tag{252}$$

5.6 Summary and Conclusions

Procedures for static shape control and determination of multidimensional large space structures were derived in this chapter, under the assumptions that the structures were continuous, governed by linear self-adjoint boundary value problems, and that the control forces are applied and observations taken at a number of predetermined points along the structure. Approximate optimal control functions and shape estimates, in terms of eigenfunctions corresponding to the static model, were presented.

As one would expect, the problem formulations and solutions for multidimensional states bear a strong resemblance to those for scalar state formulations derived in Chapter 2. This is due to the commonality among linear self-adjoint systems.

However, there are significant differences in interpretation and procedure. The differential and integral operators become matrix operators rather than scalar. Observations and control forces may now be applied to parts of the state, on to linear combinations of state components, rather than to all of the state. The additional constraints imposed in the case of rigid body modes must be interpreted and handled with more care.

Finally, it is now nearly impossible to know the differential and integral operators, or their eigenfunctions, with analytical precision. Approximations <u>must</u> be supplied using eigenfunctions computed experimentally or by a numerical method such as the finite element method.

Chapter 6. Finite Element Models: A Large Space Antenna

6.1 Introduction

In Chapter 5 static shape determination and shape control algorithms were derived for a multidimensional model defined on a multidimensional domain, the situation most likely to correspond to large space structures. It was assumed the structural models satisfied static self-adjoint linear boundary value problems of the form

$$L\ U(P) = F(P)\ ,\ B_i\ U(P) = 0\ , \qquad 1 \leq i \leq k_o\ , \tag{253}$$

where U(P) represents an n dimensional state vector of displacements at the point $P \in \Omega$, L is an n x n matrix of differential operators and B_i, $1 \leq i \leq k_o$, are linear boundary operators defined on the boundary Γ of Ω.

Terms in the solution algorithms for the static shape estimation and control problems involved the Green's function, or impulse coefficient, of the associated boundary value problem. Since it is highly unlikely that the precise Green's function for such a problem is known, approximations to these terms by means of expansions involving the eigenvalues and eigenfunctions which satisfy the corresponding eigenvalue problem

$$L\ \phi_j = \lambda_j \phi_j, \quad B_i \phi_j = 0, \qquad 1 \leq i \leq k_o\ , \tag{254}$$

were presented.

However, it is likely that the most convenient eigenfunctions will be those supplied by a finite element model, which approximate those for the dynamical boundary value problem

$$M(P)\ \frac{\partial^2 U(P,t)}{\partial t^2} + L\ U(P,t) = F(P,t)\ ,\ P \in \Omega, \tag{255}$$

$$B_i\ U(P,t) = 0\ , \qquad 1 \leq i \leq k_o\ ,\ P \in \Gamma \tag{256}$$

associated with the static problem (253). These eigenfunctions satisfy

$$L\,\hat{\phi}_j(P) - \lambda_j\,M(P)\,\hat{\phi}_j(P) = 0, \qquad P \in \Omega, \tag{257}$$

$$B_i\,\hat{\phi}_j(P) = 0, \qquad 1 \le i \le k_o,\ P \in \Gamma, \tag{258}$$

and are orthonormal with respect to the norm induced by the weighted inner product

$$\langle U,V\rangle_M = \int_\Omega U^T(P)\,M(P)\,U(P)\,dP \tag{259}$$

rather than the usual inner product

$$\langle U,V\rangle = \int_\Omega U^T(P)\,V(P)\,dP\ . \tag{260}$$

This chapter investigates the modifications necessary for the use of the eigenfunctions $\{\hat{\phi}_j\}$ which satisfy (257-258), rather than those for the static problem.

The finite element method is outlined in section 6.2. In section 6.3 eigenfunction approximations are derived for terms which involved the static Green's function, using eigenfunctions for the dynamic problem. In comparison, we solve the discrete static control problem in section 6.4 in order to demonstrate the remarkable consistency between the discrete and continuous solutions.

Finally in section 6.5 we present specific examples of algorithms for multidimensional shape determination and control, which are illustrated by simulations using an available finite element model of a large space antenna. Tables and plots of results are included at the end of the chapter.

For convenience, only the case that there are no rigid body modes, or non-trivial solutions of the unforced (homogeneous) boundary value problem, will be considered. The extension of these results to the case of system does have rigid body modes is obvious.

6.2 The Finite Element Model

The finite element method is a modification of the Rayleigh-Ritz procedure for solving self-adjoint boundary value problems. The Rayleigh-Ritz method will be described briefly first. It is based on two principles:

1) The unique solution of the self-adjoint boundary value problem which governs a system is equivalent to the unique function in a certain class which minimizes an integral, or functional, which usually represents the energy of the system.

Examples of such equivalences are the following:

Example 6.1: The solution of the system of linear equations $A x^* = b$, where A is a symmetric matrix, is equivalent to the unique vector x^* which minimizes the functional $J(x) = \frac{1}{2} \langle Ax,x\rangle - \langle b,x\rangle$.

This equivalence is equally applicable if A is a self-adjoint linear operator. [10]

Example 6.2: The function $y \in C^2[0,1]$ is the unique solution of the boundary value problem

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\, y = f(x)\ , \qquad 0 \leq x \leq 1, \tag{261}$$

$$y(0) = y(1) = 0 \tag{262}$$

if and only if y is the unique function in $C_o^2[0,1]$ which minimizes the integral

$$J(u) = \int_0^1 \{p(x)[u'(x)]^2 + q(x)[u(x)]^2 - 2f(x)\, u(x)\}\, dx\ . \tag{263}$$

(Ref. [11]).

2) The second principle of the Rayleigh-Ritz method is that the functional J is not minimized over all appropriate functions (in example 6.2, for example, J is minimized over those functions in $C_o^2[0,1]$ which satisfy (262)). It is minimized over a smaller set consisting of linear

combinations of certain basis functions $\phi_1,\ldots,\phi_n$, referred to as coordinate functions, which are defined on the region and satisfy the boundary conditions.

Thus, the solution of the linear boundary value problem becomes the question of determining the set of constants $c_1,\ldots,c_n$ such that the function $f = \sum_{i=1}^{n} c_i\phi_i$ minimizes J overall such sets, a finite dimensional problem. In effect we are finding the best approximation of the solution to the original problem in terms of the functions ϕ_i. The trick in the Rayleigh-Ritz method is to find a sequence of suitable functions $\{\phi_i\}$ such that as n goes to infinity the functions $f_n = \sum_{i=1}^{n} c_i\phi_i$ converge to the solution of the boundary value problem. Frequently used sets $\{\phi_i\}$ are piecewise linear polynomials and cubic splines.

The finite element method is a modification of the Rayleigh-Ritz method for more complicated structures, which cannot be described accurately by as simple an equation as (261). The domain of the structure is divided into smaller regions, or elements, which are interconnected at a discrete number of nodal points.

The displacements of the structure at the nodal points form the unknown constants. The displacements at one node represent translations, rotations or higher order terms in one or several dimensions. Within an element, a set of displacement functions is chosen to define displacements between the nodal points in terms of the displacements at them. These functions correspond to the coordinate functions of the Rayleigh-Ritz method.

A state vector X representing the displacements at all the nodal points is formed. The usual order in the vector is that all displacements for the first node are first, followed by all displacements for the second node, and so on.

The object of the finite element method is to determine the state vector, or displacements, which will yield the closest approximation to the actual displacement pattern of the structure. The derivation of the equation that this vector satisfies is an application of the following principle which is analogous to the first principle of the Rayleigh-Ritz method.

Hamilton's Principle: Let $L = T-V$ be the Lagrangian of a system, where T is the total kinetic energy and V is the potential energy. Then the actual path of the system in time, $X(t)$, renders the integral

$$\int_{t_1}^{t_2} L(X,\dot{X},t)\, dt$$

stationary with respect to all possible neighboring paths the system may take between times t_1 and t_2. Therefore the Frechet differential

$$\partial J(X,H) = \frac{d}{d\alpha} \int_{t_1}^{t_2} L(X + \alpha H,\ \dot{X} + \alpha H,\ t)dt \,\Big|_{\alpha=0} = 0$$

for all admissible variations H. This is a classical problem in the Calculus of Variations, which leads to the Euler-Lagrange equations for the system:

$$L_x(X,\dot{X},t) - \frac{d}{dt} L_{\dot{x}}(X,\dot{X},t) = 0 \qquad (264)$$

(Ref. [3], p. 181).

For dynamic finite element models

$$T = \frac{1}{2} \dot{X}^T M\dot{X} \quad \text{and} \quad V = \frac{1}{2} X^T KX \ . \qquad (265)$$

M and K are square symmetric matrices and M is positive definite.

The mass matrix M arises out of an analysis of the inertial forces acting at the nodes. The coefficients M_{ij} of M are referred to as mass influence coefficients, which relate the accelerations at the nodes to the resulting inertial forces. M_{ij} is the force at coordinate i due to a unit acceleration at coordinate j. The total inertial forces acting on the system may be expressed in vector form by $F_I = M\ddot{X}$.

The stiffness matrix K arises out of an analysis of the elastic force relationships at the nodes. The stiffness influence coefficient k_{ij} represents the force at the coordinate i due to a unit displacement of coordinate j. In vector form the elastic forces acting on the system X may be written $F_s = K X$. The stiffness matrix K in the discrete system corresponds to the linear operator L in the continuous systems (253) or (255).

The coefficients of M and K are computed by integrations over each element using the coordinate functions.

If the Euler-Lagrange equations (264) are evaluated for the finite element model the following equations result:

for a conservative system: $M\ddot{X} + KX = 0$ (266)

and, if a vector of nonconservative (outside) forces F(t) is acting on the system: $M\ddot{X} + KX = F(t)$. (267)

In a static system $\ddot{X} = 0$, which yields a system of linear equations as a necessary condition for the state X:

$$KX = F \ . \tag{268}$$

The final step in the finite element method is to solve (267) or (268) for X, the vector of nodal displacements, given a known force vector F.

The system (267) is self-adjoint if and only if the weighted inner produce

$$\langle X,Y\rangle_M = X^T MY = Y^T MX \tag{269}$$

is used. Consequently there exists a complete set of eigenvectors (modes) $\{\phi_i\}$, $1 \le i \le N_o$, where N_o is the dimension of the state X, and corresponding eigenvalues $\{\hat{\lambda}_i\}$, such that

$$\hat{\lambda}_i M \hat{\phi}_i = K \hat{\phi}_i \ , \quad 1 \le i \le N_o . \tag{270}$$

The eigenvalue $\hat{\lambda}_i = \omega_i^2$, where ω_i is the frequency corresponding to the mode, or eigenvector, $\hat{\phi}_i$.

Eigenvectors corresponding to different eigenvalues are orthogonal under the norm (269). We assume they have been normalized with respect to that norm. The solution of (267) is given by

$$X(t) = \sum_{i=1}^{N_o} C_i(t)\,\hat{\phi}_i \tag{271}$$

where $C_i(t)$ satisfied $\ddot{C}_i + \omega_i^2 C_i = \langle F, \hat{\phi}_i \rangle_M$.

Thus, given a known vector F of non-conservative forces the solution of (269) is expressed in terms of the eigenvectors $\hat{\phi}_i$ and frequencies which satisfy (270). These are the modes and frequencies supplied by the finite element method, which must be used to approximate the static shape control and determination algorithms.

Because of computational limitations, only a fraction of the total number of modes are actually computed.

The solution of (268) is discussed in section 6.4.

In summary, the basic steps of the finite element method are as follows:

<u>Summary of the Finite Element Method</u>

i) The domain is divided into a number of elements, which are interconnected at a discrete number of nodal points.

ii) A state vector X is formed, representing the displacements of which knowledge is desired at each node. Displacements within an element are expressed in terms of coordinate functions. The unknown constants in the displacement functions are the displacements at surrounding nodes.

iii) The mass matrix M and stiffness matrix K are computed. The state vector X there satisfies $M\ddot{X} + KX = F$, for dynamical systems, or $KX = F$, for static systems, where F is a vector representing outside forces acting on the system.

iv) The modes $\{\hat{\phi}_i\}$ and frequencies $\{\omega_i\}$ which satisfy $\omega_i^2 M \hat{\phi}_i = K \hat{\phi}_i$ are then computed. Solutions to the model may be expressed in terms of these modes.

The Lumped Mass Method

A simplification of the finite element method, the lumped mass method, is frequently used for models of large space structures at JPL. The entire mass of the structure is assumed to be concentrated at the nodal points, which are interconnected by massless segments. Thus no coordinate functions need be defined. The mass matrix is a diagonal matrix, with identical entries for all translations corresponding to the same node, and zeros for rotations or higher order terms [12].

6.3 Approximations from the Dynamic Model

Given the eigenfunctions $\{\hat{\phi}_i\}$ which satisfy (257-8) and are orthonormal with respect to the weighted norm (259), we wish to generate approximations to terms in the shape control and determination algorithms. The eigenfunctions (270) supplied by the finite element method are discrete approximations to those of (257-8).

If the Green's function $G(P|Q)$ is not known, we require approximations for the following quantities:

$$G(P|Q) \tag{272}$$

$$\int_\Omega G(P|Q)\ W(Q)\ G(Q|R)\ dQ \tag{273}$$

$$\int_\Omega G(P|Q)\ W(Q)\ \psi(Q)\ dQ \tag{274}$$

where $\psi(Q)$ is a known function and $W(Q)$ is a symmetric positive definite matrix.

We will first assume that we have available the continuous eigenfunctions for which the finite element method provides approximations. For convenience from this point forward we drop the hats on these eigenfunctions, which satisfy the following properties:

$$L\ \phi_j(P) = \lambda_j\ M(P)\ \phi_j,\quad P \in \Omega \tag{275}$$

$$B_i\phi_j(P) = 0\ ,\qquad 1 \leq i \leq k_o\ ,\quad P \in \Gamma \tag{276}$$

$$<\phi_j, \phi_i>_m = \int_\Omega \phi_j^T(P)\ M(P)\ \phi_i(P)\ dP = \delta(i-j)\ . \tag{277}$$

Properties (275) and (277) easily yield the following property:

$$<\phi_j, L\phi_i> = \int_\Omega \phi_j^T(P)\ L\ \phi_i(P)\ dP = \lambda_j\ \delta(i-j)\ . \tag{278}$$

The application of the Green's function (198-9) to solve the boundary value problem (275-6) yields

$$\phi_j(P)=\lambda_j \int_\Omega G(P|Q)\ M(Q)\ \phi_j(Q)\ dQ\ . \tag{279}$$

If there are no eigenfunctions ϕ_i corresponding to the eigenvalue $\lambda = 0$, that is, if the nullspace of the operator L is only the zero vector, the functions $\{\phi_i\}$ form a complete set for all functions in a suitable class which satisfy the boundary conditions.

If there are eigenfunctions corresponding to $\lambda = 0$, the modified Green's Function defined in Chapter 5 has no component in the nullspace which is spanned by these functions. Therefore in either case the column vector $G_j(P|Q)$ can be expanded in terms of the eigenfunctions ϕ_i corresponding to non-zero eigenvalues:

$$G_j(P|Q) = \sum_i \phi_i(P)\ \gamma_{ji}(Q) \tag{280}$$

where $\gamma_{ji}(Q)$ are continuous scalar functions defined on Ω. If we define $\gamma_j(Q) = (\gamma_{j1}(Q)\ \ldots\ \gamma_{jn}(Q))$, then

$$G(P|Q) = \sum_j \phi_j(P)\ \gamma_j(Q)\ . \tag{281}$$

In order to determine γ_j, we multiply both sides of (281) on the left by $\phi_i^T(P)\ M(P)$ and integrate over Ω.

$$\int_\Omega \phi_i^T(P)\ M(P)\ G(P|Q)\ dP = \sum_j \int_\Omega \phi_i^T(P)\ M(P)\ \phi_j(P)\ \gamma_j(Q)\ dP\ .$$

If we apply the orthogonality relationship (277):

$$\int_\Omega \phi_i^T(P)\ M(P)\ G(P|Q)\ dP = \gamma_i(Q)\ . \tag{282}$$

If (282) is compared with (279) it is observed that $\gamma_i(Q) = \frac{1}{\lambda_i}\ \phi_i^T(Q)$, and

$$G(P|Q) = \sum_i \frac{1}{\lambda_i}\ \phi_i(P)\ \phi_i^T(Q)\ , \tag{283}$$

where the sum is over the non-zero eigenvalues and eigenfunctions of the system (255-6).

We use the expression (283) to find expansions for (273-4):

$$\int_\Omega G(P|Q)\ W(Q)\ \psi(Q)\ dQ = \sum_i \frac{1}{\lambda_i} \int_\Omega \phi_i(P)\ \phi_i^T(Q)\ W(Q)\ \psi(Q)\ dQ$$

$$= \sum_i \frac{1}{\lambda_i} \phi_i(P) \int \phi_i^T(Q)\ W(Q)\ \psi(Q)\ dQ$$

$$= \sum_i \frac{1}{\lambda_i} \phi_i(P) <\phi_i,\psi>_W \ . \tag{284}$$

Finally we evaluate expression (273):

$$\int_\Omega G(P|Q)\ W(Q)\ G(Q|R)\ dQ$$

$$= \int_\Omega (\sum_i \frac{1}{\lambda_i} \phi_i(P)\phi_i^T(Q))\ W(Q)\ (\sum_j \frac{1}{\lambda_i} \phi_j(Q)\ \phi_j^T(R))\ dQ$$

$$= \sum_i \sum_j \frac{1}{\lambda_i\lambda_j} \phi_i(P)\ \phi_j^T(R) <\phi_i, \phi_j>_W \tag{285}$$

In the event that the matrix $W(P)$ is chosen to be the mass matrix $M(P)$, the relation (285) becomes

$$\int_\Omega G(P|Q)\ M(Q)\ G(Q|R)\ dQ = \sum_i \frac{1}{\lambda_i^2} \phi_i(P)\ \phi_i^T(Q)\ . \tag{286}$$

The expressions for (272-4) in terms of eigenfunctions for the dynamic problem are very similar to those in terms of eigenfunctions for the static problem. The major difference is the loss of orthogonality with respect to an unweighted inner product.

C-2

Section 6.4 The Discrete Control Problem

It is satisfying, although not unexpected, to note the resemblence between the solutions of the shape control and determination problems for continuous and discrete models. For example, the discrete control problem analogous to the problem (211-212) is as follows:

Let X be an N_o dimensional state vector representing the displacements for a sequence of nodal points along a structure. Let Y represent the vector of desired displacements. Suppose m scalar forces F_j are to be applied to coordinates r(j) of the vector X in order to achieve the desired "shape" Y. Then the control problem is to determine the control vector X which is the solution of

$$KX = CF \tag{287}$$

and minimizes the criterion

$$J = \frac{1}{2} ||F||_R^2 + \frac{1}{2} ||X-Y||_M^2 \tag{288}$$

over all pairs (F,X) which satisfy (287).
C is an N_o x m matrix with entries $C_{ij} = \delta(i - r(j))$. R is a symmetric constant m x m matrix, and M is the mass matrix of the corresponding dynamical model. Since we are considering systems without rigid body modes, there are no nontrivial solutions of KX = 0. Thus K is non-singular, and the solution of (287) is given by

$$X = K^{-1} CF \tag{289}$$

when F is known.

Finding K^{-1} is analogous to finding the inverse of the operator L, that is to finding the Green's function such that the solution of LU = F plus boundary conditions may be expressed as

$$U = L^{-1} F = \int_\Omega G(P|Q) F(Q) dQ .$$

As in the continuous case, while it is easy to refer to K^{-1} in theory, in practice the system dimension N_o is on the order of 10^3, so it is desirable to find a means of approximating K^{-1} rather than actually computing it.

We substitute (289) into the criterion J:

$$J = \frac{1}{2}\,(F^T RF) + \frac{1}{2}(K^{-1}CF - Y)^T\, M(K^{-1}CF - Y). \tag{290}$$

We minimize (290) with respect to the unknown vector F:

$$\frac{\partial J}{\partial f} = F^T R + (K^{-1}CF - Y)^T\, M\, K^{-1}\, C = 0\ .$$

This equation results in the following necessary condition for F:

$$(R + C^T K^{-1} M K^{-1} C)\ F = C^T\, K^{-1}\, M\, Y\ . \tag{291}$$

Once F is known from this m dimensional system of equations, the optimal shape X is given by (289).

Since it is awkward to compute K^{-1}, we seek eigenfunction expansions for it, and the terms $K^{-1}\, M\, K^{-1}$ and $K^{-1}\, M\, Y$. We assume we have available the eigenfunctions and eigenvalues of the corresponding dynamical system $M\ddot{X} + KX = F$, which satisfy (270), together with the orthogonality conditions

$$\langle \phi_i, \phi_j \rangle_M = \delta(i-j) \tag{292}$$

and

$$\langle \phi_i, \phi_j \rangle_K = \phi_i^T\, K\, \phi_j = \lambda_i\, \delta(i-j)\ . \tag{293}$$

Let Φ be the N_o by N_o matrix $[\phi_1 \mid \ldots \mid \phi_{N_o}]$. Then

$$\Phi^T\, K\, \Phi = \Lambda \tag{294}$$

where Λ is the diagonal matrix with diagonal entries λ_i, $1 \le i \le N_o$. Thus

$$K = (\Phi^{-1})\ \Lambda\ \ \Phi^{-1}$$

and

$$K^{-1} = \Phi \Lambda^{-1} \Phi^T = \sum_{i=1}^{N_o} \frac{\phi_i \phi_i^T}{\lambda_i} . \tag{295}$$

Furthermore

$$K^{-1} M K = \left(\sum_{i}^{N_o} \frac{1}{\lambda_i} \phi_i \phi_i^T\right) M\left(\sum_{j}^{N_o} \frac{1}{\lambda_j} \phi_j \phi_j^T\right)$$

$$= \sum_{i=1}^{N_o} \sum_{j=1}^{N_o} \frac{1}{\lambda_i \lambda_j} \phi_i \phi_j^T \langle \phi_i, \phi_j \rangle_M$$

$$= \sum_{i=1}^{N_o} \frac{1}{\lambda_i^2} \phi_i \phi_i^T , \tag{296}$$

and

$$K^{-1} M Y = \sum_{i=1}^{N_o} \frac{1}{\lambda_i} \phi_i \langle \phi_i , Y \rangle_M . \tag{297}$$

Note the marked resemblence between the discrete expressions (295-7) the analogous expressions (283-4)(286) in the continuous problem.

6.5 Applications to a Large Space Antenna

In this section we present actual algorithms for the shape determination and control of a large space structure, subject to the choice of certain constants, which may be varied in simulations. The algorithms are illustrated using eigenfunctions and frequencies provided by a finite element model of a large space antenna, which has been developed at JPL.

The model is constructed by the lumped mass method described at the end of section 6.2. It assumes 18 ribs and 882 nodal points located on 14 concentric circular cross-sections of the mesh. The ribs are assumed to be very stiff in comparison with the mesh. The hub of the antenna is assumed to be firmly fixed to the bus of a more massive spacecraft, so that there are no rigid body modes.

Available data on the model includes the rest coordinates in R^3, which represent the positions of the nodes on the ideal shape $U°$, the masses at each node, and 33 modes and frequencies.

We will restate the problems and algorithms to incorporate two subtle refinements necessary for the application to a large space antenna.

The first arises from the fact that the mode shapes and Green's function represent displacements of the antenna from its ideal shape. The actual antenna shape is the sum of its ideal, or rest shape $U°$, a perfect paraboloid, and its displacement. Thus the Green's function represents the displacement of the antenna from its ideal shape due to a unit impulsive force at one point.

The second refinement is that shape estimation is accomplished first, and the resulting shape estimate U^* is used as the desired shape in the control problem. Once the forces necessary to control the ideal shape to the shape estimate are determined, the negative of those forces will bring the estimated shape to the optimal corrected shape.

After the full algorithms are stated, we state the corresponding approximations used in the simulations, which are based on the expansions developed in section 6.3.

The results of the simulations include tables representing comparisons of results for varying choices of control and observation positions, number of modes in the approximations, weighting matrices and choices of actual distorted shapes. Plots of the first eleven mode shapes, and the actual distorted shape, estimated shape and corrected shape for various initially distorted antenna.

The computer program listing and output for the shape control of a large space antenna are found in Appendix C.

The Shape Estimation Problem

Consider an n dimensional space structure, the shape U(P) of which satisfies the following linear self-adjoint boundary value problem on the ℓ dimensional domain Ω with boundary Γ:

$$LU(P) = F(P), \qquad P \in \Omega \tag{298}$$

$$B_j\, U(P) = 0\,, \qquad 1 \leq j \leq k_o, \quad P \in \Gamma. \tag{299}$$

L is an nxn matrix of linear differential operators, which is related to the stiffness of the structure. B_j, $1 \leq j \leq k_o$, are linear homogeneous boundary conditions. F is a vector function of unknown disturbances.

The shape estimation problem is to determine the unknown disturbance function F* and shape function U*, based on the m observation vectors

$$Y_i = C_i\, U(P_i) + Z_i, \qquad 1 \leq i \leq m, \tag{300}$$

which satisfy (298-9) and minimize the performance criterion (301) over all possible pairs (F,U) which satisfy (298-9). The vectors Z_i represent noise in the observations.

$$J = \frac{1}{2} \sum_{i=1}^{m} || Z_i ||^2_{R_i^{-1}} + \frac{1}{2} \int_\Omega || F(P) ||^2_{W^{-1}(P)} \, dP \,. \tag{301}$$

R_i and W are symmetric positive definite weighting matrices of appropriate dimensions.

The Static Shape Control Problem

Given the optimal shape estimate U*(P), the shape control problem is to determine the set of m control forces $\hat{F}_i$, applied at the positions P_i, which together with the resulting shape $\hat{U}(P)$ which satisfies

$$LU(P) = \sum_{i=1}^{m} C_i \, F_i \, \delta(P-P_i), \qquad P \in \Omega \tag{302}$$

$$B_i \, U(P) = 0, \qquad 1 \leq j \leq k, \quad P \in \Gamma \tag{303}$$

minimizes the criterion

$$\hat{J} = \frac{1}{2} \sum \; || F_i ||^2_{\hat{R}_i} + \frac{1}{2} \int_\Omega || U(P) - U^*(P) ||^2_{\hat{W}(P)} \, dP \tag{304}$$

over all possible sets $\{U,\{F_i\}\}$ which satisfy (302-3). The matrices $\hat{R}_i$ are positive semidefinite and the matrix $\hat{W}$ is positive definite.

The forces F_i, $1 \leq i \leq m$, when applied to the positions P_i of the ideal shape U°, will produce the closest approximation to U* obtainable by the pointwise application of forces at those positions. Consequently, because the system is linear, the application of the negatives of the forces $\hat{F}_i$ to positions P_i on the estimated shape U* will produce the optimal shape correction of U* to the desired shape U°.

The Shape Determination Algorithm

Assume the positions P_i, observations Y_i and their directions determined by C_i are known. Choose the weighting matrices R_i and W in the criterion (301). Then

i) Compute the block matrices A_{ij}, $i \leq i, j \leq m$, given by

$$A_{ij} = C_i^T \left(\int_\Omega G(P_i|P)\, W(P)\, G(P|P_j)\, dP\right) C_j \tag{305}$$

where $G(P|Q)$ is the associated Green's function for the system.

ii) Form the matrix A whose block coordinates are A_{ij} and the diagonal block matrix R^{-1} whose diagonal blocks are R_i^{-1}. Form the vector Y by "stacking" the observations Y_i.

iii) Compute the solution $\bar{U}$ of the system

$$[I + AR^{-1}]\, \bar{U} = A\, R^{-1}\, Y \tag{306}$$

The vector $\bar{U}$ contains the optimal pointwise shape estimates $C_i\, U^*(P_i)$.

iv) The estimate of the continuous optimal shape distortion ΔU^* is given by

$$\Delta U^*(P) = \sum_{i=1}^{m} \left(\int_\Omega G(P|Q)\, W(Q)\, G(Q|P_i) dQ\right) C_i^T\, R_i^{-1}\, (Y_i - C_i\, U^*(P_i)) \tag{307}$$

The optimal shape estimate is $U^* = U^\circ + \Delta U^*$.

The Optimal Shape Control Algorithm

Assume again that the positions P_i and matrices C_i, $\hat{R}_i$ and $\hat{W}$ have been chosen. Assume also that the desired shape or optimal shape estimate U^* is available. Then

i) Compute the block matrices A_{ij} given by (305) and the vector elements D_j given by

$$D_j = C_j^T \int_\Omega G(P_j|P)\, U^*(P)\, dP\ . \tag{308}$$

ii) Form the block matrix whose block components are A_{ij}, $1 \leq i, j \leq m$. Form the block diagonal matrix $\hat{R}$ whose diagonal elements are $\hat{R}_i$, and the vector D by "stacking" the vectors D_j.

iii) Solve the system (309) of linear equations for the vector of optimal forces $\hat{F}$.

$$(R + A)\ \hat{F} = D \tag{309}$$

iv) The optimal shape correction resulting from the application of these forces at the points P_i is

$$\Delta\hat{U} = \sum_{i=1}^{m} G(P|P_i)\ C_i F_i^* \ . \tag{310}$$

If the negative of the forces $\hat{F}_i$ is applied to the shape estimate U^*, the resulting shape is $U^* - \Delta\hat{U}$, the optimal corrected shape.

Approximate Algorithms

We assume the weighting matrices W and $\hat{W}$ are chosen to be the mass matrix of the dynamical model which corresponds to the static model (253). The eigenfunctions ϕ_k and frequencies ω_k for that model satisfy

$$\omega_k^{\ 2}\ M\ \phi_k = L\ \phi_k$$

Then an approximate Green's function, based on the first n_m modes, is given by

$$G(P|Q) = \sum_{k=1}^{n_m} \frac{1}{\omega_k^{\ 2}}\ \phi_k(P)\ \phi_k^{\ T}(Q) \ . \tag{311}$$

Furthermore, the elements A_{ij} and D_j in the shape control and determination algorithms are given by

$$A_{ij} = C_i^{\ T}\Big(\sum_{k=1}^{n_m} \frac{1}{\omega_R^{\ 4}}\ \phi_k(P_i)\ \phi_k^{\ T}(P_j)\Big)C_j \tag{312}$$

and

$$D_j = C_j^{\ T} \sum_{k=1}^{n_m} \frac{1}{\omega_k^{\ 2}}\ \phi_k(P_j)\ \langle\phi_k,\ U^*\rangle_M \ . \tag{313}$$

Substitution of (311) into the expression (307) for the optimal shape estimate U* yields

$$U^*(P) = \sum_{i=1}^{m} \sum_{k=1}^{n_m} \frac{1}{\omega_k^4} \phi_k(P)\, \phi_k^T(P_i)\, C_i^T\, R_i^{-1}\, (Y_i - C_i U^*(P_i)) \tag{314}$$

Thus the coefficient of the mode $\phi_k(P)$ in the approximate shape estimate is

$$\frac{1}{\omega_k^4} \sum_{i=1}^{m} (C_i \phi_k(P_i))^T\, R_i (Y_i - C_i U^*(P_i))\ . \tag{315}$$

These computed estimated modal coefficients may be compared to the actual coefficients of the known distorted shape. Representative comparisons may be found in the tables at the end of this chapter.

Substitution of expression (311) into the expression (310) of the optimal shape correction $\Delta\hat{U}$ yields

$$\Delta\hat{U} = \sum_{i=1}^{m} \sum_{k=1}^{n_m} \frac{1}{\omega_k^2} \phi_k(P)\, \phi_k^T(P_i)\, C_i \hat{F}_i \tag{316}$$

Thus the coefficient of the mode $\phi_k(P)$ in the optimal shape correction $\Delta\hat{U}$ is

$$\frac{1}{\omega_k^2} \sum_{i=1}^{m} \phi_k(P_i)\, C_i \hat{F}_i \tag{317}$$

Comparisons of these terms with the actual coefficients are also found in the tables.

Results of the Simulations

The tables 6.1-6.3 at the end of this chapter exhibit representative results for the following choices of variables. Figures 6.4-6.10 illustrate the results of shape determination and control simulation for selected distorted shapes.

Control/Observation Positions: The control and observation points were chosen colocated both in position and direction. Since conventional stability questions do not arise in static problems, colocation serves the convenience of the programmer, but is not necessary for accuracy.

Either nine or eighteen points were chosen on a given circle. Thus they were located on every rib or every other rib on the circle. The second, fifth, eighth and eleventh circles were tried. [Table 6.1]

The forces/observations were chosen to be all in the x direction (C_i = (1 0 0)), the y direction (C_i = (0 1 0)), the z direction (C_i = (0 0 1)), or both in the x and y directions at each point (C_i = (1 1 0)). Table 6.2 compares results for the same shape and varying numbers of points and directions. The results for the z direction are not included (see remarks below).

Modes: The number of modes used in the approximations was either 7 or 11. Plots of the first eleven modes are contained in Figure 6.1 - 6.3.

Weighting Matrices: The weighting matrix W(P) was chosen to be the mass matrix M of the finite element model. This is a natural choice when using modes from the same model, since the inner product for the space spanned by the modes is weighted by M.

The weights R_i and $\hat{R}_i$ are scalars in these simulations. They are chosen to be the same number R in both the control and estimation problems. The criteria was that R be as small as possible, while large enough that the matrix (RI + A) is invertible. The correct choice of R varies from circle to circle, but appears to be half-way in order of magnitude from the minimum and maximum elements of the matrix A.

Observations: A good test of an estimation algorithm is its performance when given exact observations of a known shape distortion. This provides a means of comparison of the accuracy of the results. The program was provided with the modal coefficients of several known distorted shapes, from which it computed exact observations. It uses the exact observations in the shape estimation algorithm.

It should be remembered when observing results that the mode shapes represent <u>displacements</u> of the antenna from its natural or ideal shape U^*. Thus if ΔU represents the combination of modes in the distorted shape of the antenna, the actual shape in $U^* + \Delta U = U$.

Results

1) As long as the value of the weighting factor R is chosen small enough, it does not appear to matter on which circle the observations are chosen. [Table 6.1] There is one exception: the innermost circle may not be used. Because of the assumption that the hub is fixed, the values of all the modes on this circle are zero.

2) Good results are obtained from observation/control forces applied only in the x direction, or equivalently only in the y direction. Thus if observations and/or control forces may be applied in these, or in radial directions, satisfactory results can be obtained. [Table 6.2]

On the other hand, when observations/control forces were applied <u>in the z direction</u>, results were very poor (and are not included in the tables). Examination of the modes reveals two reasons: The first is that in the lower order modes there is very little displacement in the z direction. This is due to the assumption that the ribs are very stiff in comparison with the mesh, so the lower order modes consist of ribs being pinched together at some points and spaced apart in others. (Figures 6.1-6.3). The second reason is that the changes in the z direction do not vary much on the same circle. Control/observation points on two circles simultaneously were tried, but results, although better, were still poor.

For a fixed number of observations, slightly better results are obtained if they are taken at different points in one direction, rather than in several directions at fewer points [Tables 6.2 and 6.3].

3) More control/observation points than modes should be used. Aside from the fact that this is easily observed from the data, it is a matter of common sense. Both problems involve the determination of the coefficients of each of the modes. One must have at least as many pieces of independent data as one has unknowns.

However, it is estimated that there will be from 50 to 150 observations taken of LSS antenna. Since it is unlikely that 150 modes will be, or could be, used in the modeling, this restriction does not actually pose a problem.

Table Symbols

ϕ_i The ith mode.

U° The rest shape, or ideal shape, of the antenna.

ΔU The modal displacements of the actual distorted shape.

U The actual distorted shape: $U = U^\circ + \Delta U$.

ΔU^* The modal displacements of the shape estimate.

U^* The estimated shape: $U^* = U^\circ + \Delta U^*$.

$\Delta\hat{U}$ The modal displacements of the shape resulting from the application of the control forces.

$\hat{U}$ The antenna shape resulting from the application of control forces: $\hat{U} = U^\circ + \Delta\hat{U}$.

Table 6.1

Estimation/Control Points on Circles of Different Radii

9 observation points, x direction, 7 modes. Actual Shape = $U° + 10\phi_1 + 5\phi_2 + 5\phi_3 + 5\phi_4 + 5\phi_5$

	Actual Coeff.	Second Circle $R = 10^{-12}$		Fifth Circle $R = 10^{-10}$		Eight Circle $R = 10^{-9}$		Eleventh Circle $R = 10^{-8}$	
	ΔU	ΔU^*	$\Delta\hat{U}$	ΔU^*	$\Delta\hat{U}$	ΔU^*	$\Delta\hat{U}$	ΔU^*	$\Delta\hat{U}$
ϕ_1	10.	10.000	9.998	9.998	9.997	10.000	9.999	9.997	9.997
ϕ_2	5.	5.000	4.999	5.001	5.002	5.002	5.001	4.999	4.997
ϕ_3	5.	4.997	4.995	4.998	4.996	4.999	4.998	4.996	4.993
ϕ_4	5.	4.998	4.997	4.997	4.996	4.999	4.997	4.996	4.994
ϕ_5	5.	5.004	5.004	4.999	5.000	5.000	5.001	5.000	5.001
ϕ_6	0	.000	.001	.004	.005	.002	.003	.003	.003
ϕ_7	0.	-.001	-.002	.001	-.000	.001	.000	-.001	-.003

Table 6.2

Results of Shape Estimation/Control for Observations/Controls

On the Fifth Circle, with 7 or 11 modes used in approximations, and 9 or 18 points in the x, y or x and y direction. The actual shape is $U° + 10\phi_2 + 10\phi_4 + 5\phi_6$. $R = 10^{-10}$.

	Actual Coeff.	9 pts, x dir. 7 modes		9 pts, y dir. 7 modes		9 pts, x dir. 11 modes		18 obs: 9x, 9y 11 modes		18 pts, x dir. 11 modes	
	U	ΔU^*	$\Delta\hat{U}$	ΔU^*	$\Delta\hat{U}$	ΔU^*	$\Delta\hat{U}$	ΔU^*	$\Delta\hat{U}$	ΔU^*	$\Delta\hat{U}$
ϕ_1	0.	-.003	-.003	.003	.004	-1.447	-1.538	-.004	-.004	-.003	-.004
ϕ_2	10.	10.0	10.0	10.0	9.999	9.291	9.261	9.999	10.0	10.001	10.0
ϕ_3	0.	-.001	-.001	-.002	-.001	-1.965	-2.047	-.003	-.003	.004	.004
ϕ_4	10.	9.995	9.994	9.998	9.997	8.275	8.200	10.0	9.999	9.997	9.996
ϕ_5	0.	-.002	-.003	.004	.004	1.186	1.238	-0.	-0.	.001	.001
ϕ_6	5.	5.004	5.003	5.004	5.004	6.028	6.074	5.0	4.996	4.996	4.996
ϕ_7	0.	-.003	-.003	-.003	-.002	- 1.843	-1.926	-.003	-.004	-.002	-.002
ϕ_8	0.	0.	0.	0.	0.	-1.122	-1.149	-.033	-.002	-.006	-.007
ϕ_9	0.	0.	0.	0.	0.	.190	.194	-.073	-.002	.007	.007
ϕ_{10}	0.	0.	0.	0.	0.	-1.001	-1.079	.051	.002	.011	.011
ϕ_{11}	0.	0.	0.	0.	0.	-.004	-.004	.070	-.003	-.006	-.006

Table 6.3

Measurements and Controls applied to both x and y directions at 9 points vs. x direction only at 18 points, on the fifth circle

Actual Shape: $U^\circ + 10\phi_1 + 10\phi_4 + 5\phi_8 + 5\phi_{10}$

11 modes used in approximations. $R = 10^{-10}$.

Actual Coefficient		9 x, 9 y		18 x	
	U°	U^*	$\hat{U}$	U^*	$\hat{U}$
ϕ_1	10.	10.010	10.010	10.0	9.994
ϕ_2	0.	.000	.000	-.001	-.001
ϕ_3	0.	.004	.004	-.000	-.001
ϕ_4	10.	10.011	10.010	9.996	9.994
ϕ_5	0.	-.006	-.006	.004	.005
ϕ_6	0.	-.003	-.003	.001	.001
ϕ_7	0.	.007	.007	.001	.002
ϕ_8	5.	5.255	5.111	4.996	4.994
ϕ_9	0.	-.551	-.559	.003	.003
ϕ_{10}	5.	4.644	4.791	5.003	5.003
ϕ_{11}	0.	.488	.530	.000	.000

For a fixed number of observations, it appears better to take them at different points in the same direction, rather than to take observations of several directions at fewer points.

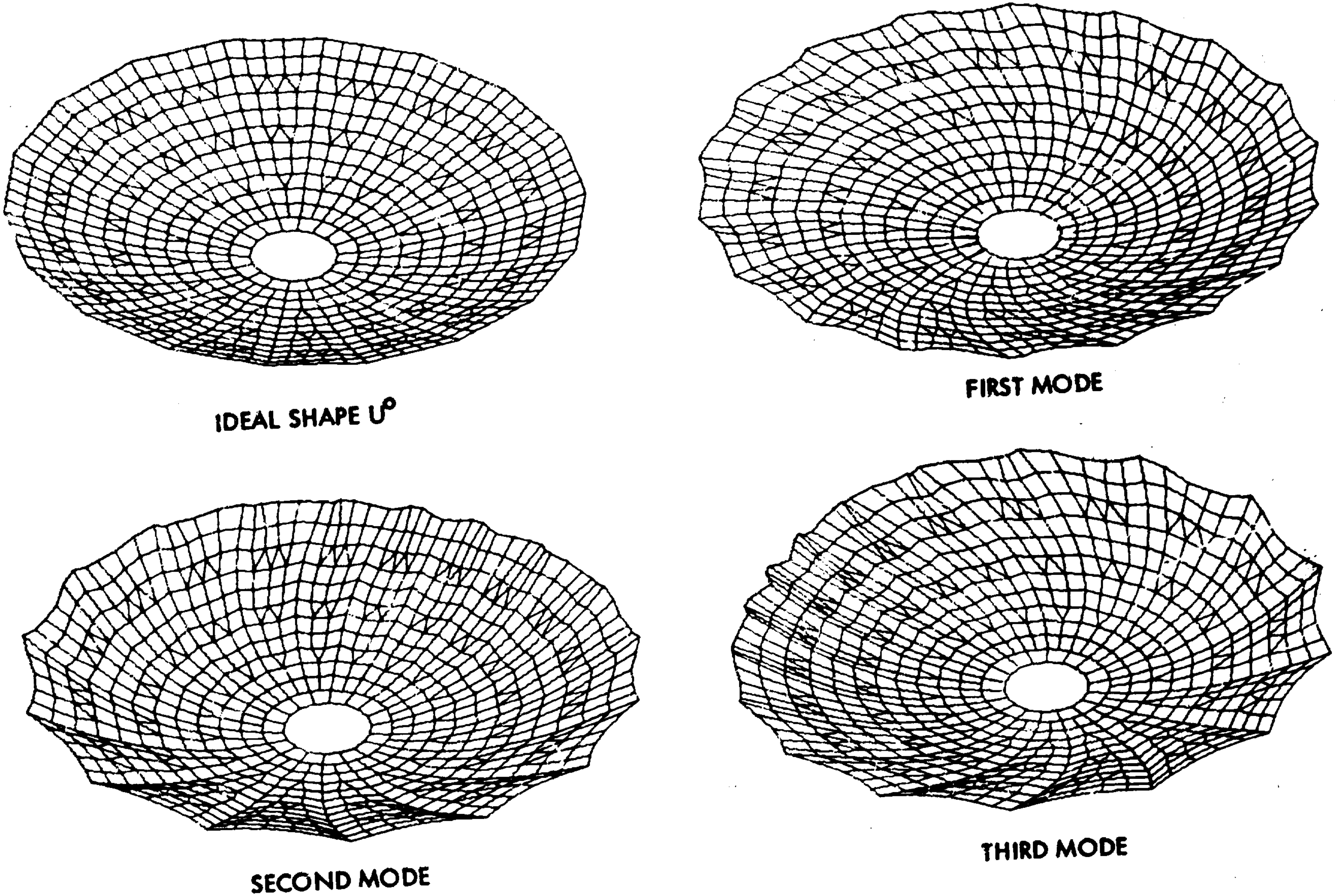


Figure 6.1 Ideal Shape, First, Second, and Third Modes

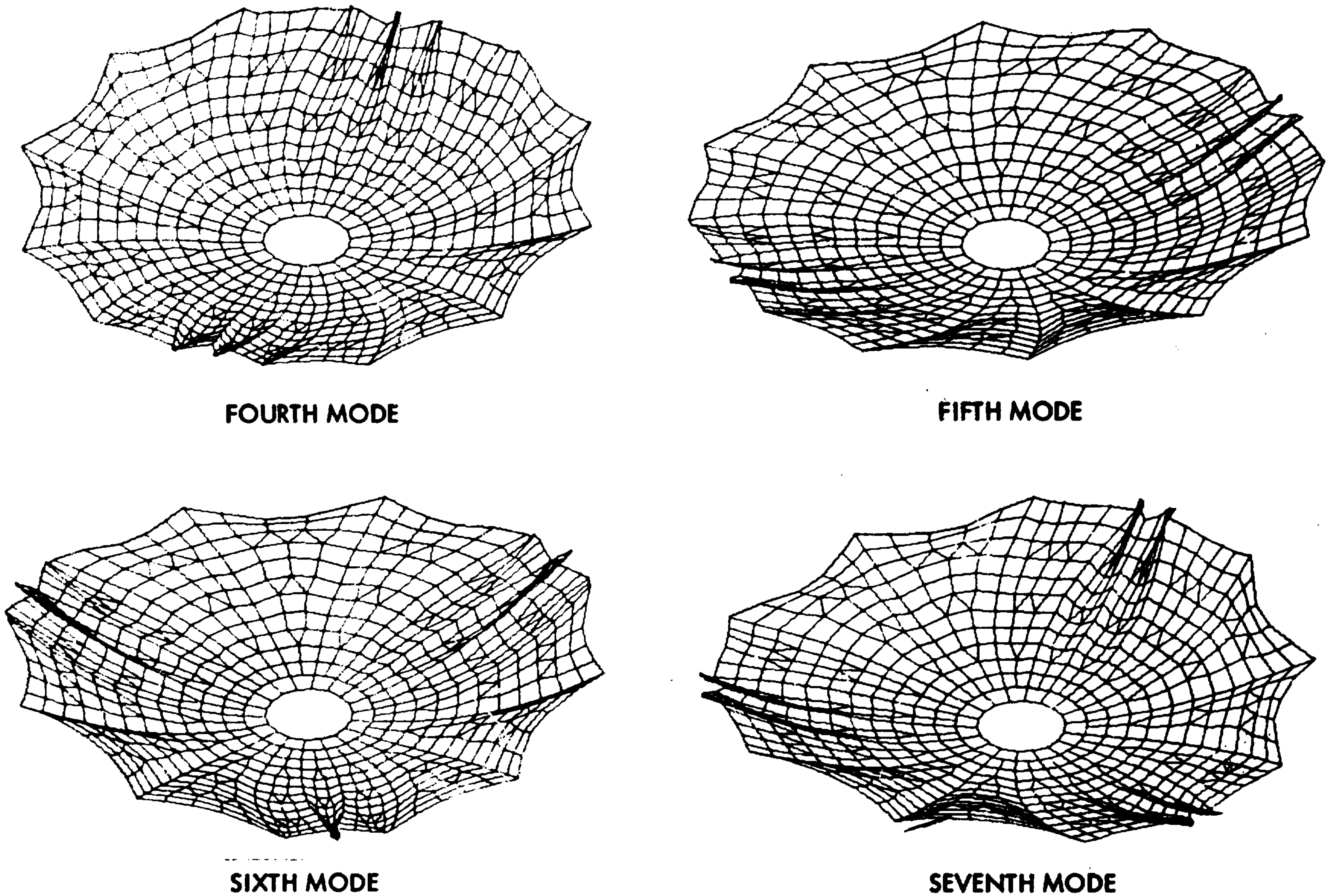


Figure 6.2 Fourth, Fifth, Sixth, and Seventh Modes

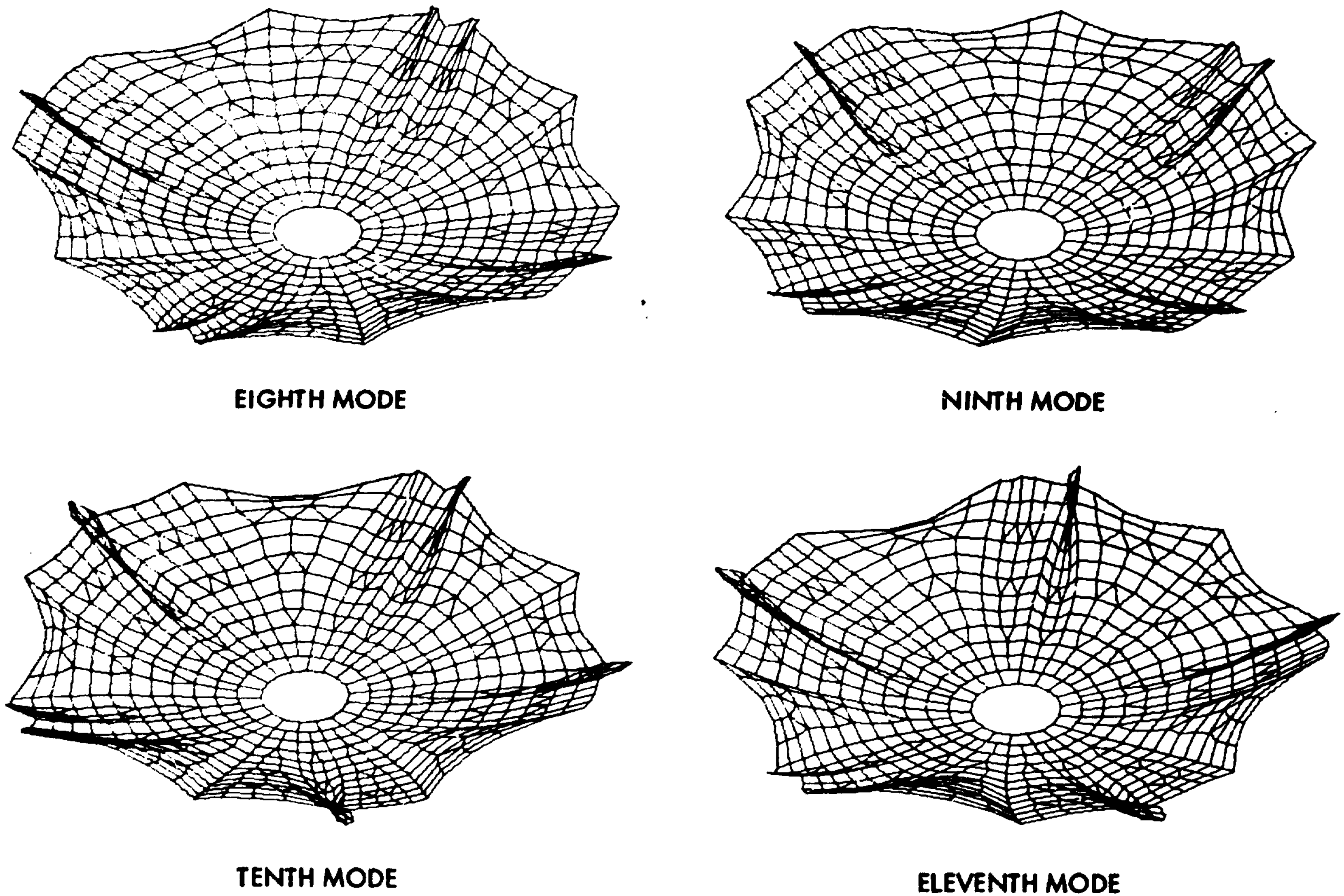


Figure 6.3 Eighth, Ninth, Tenth, and Eleventh Modes

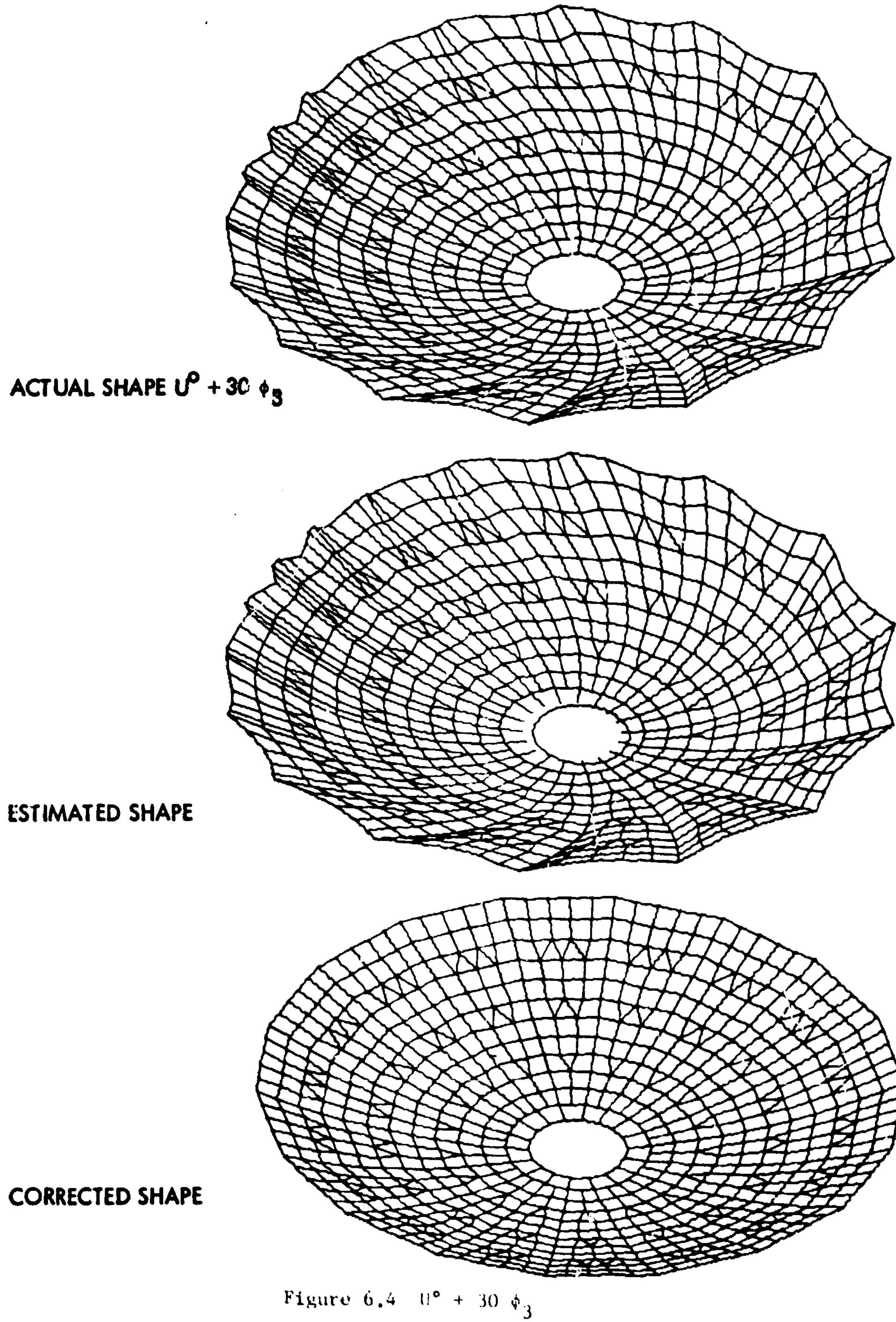
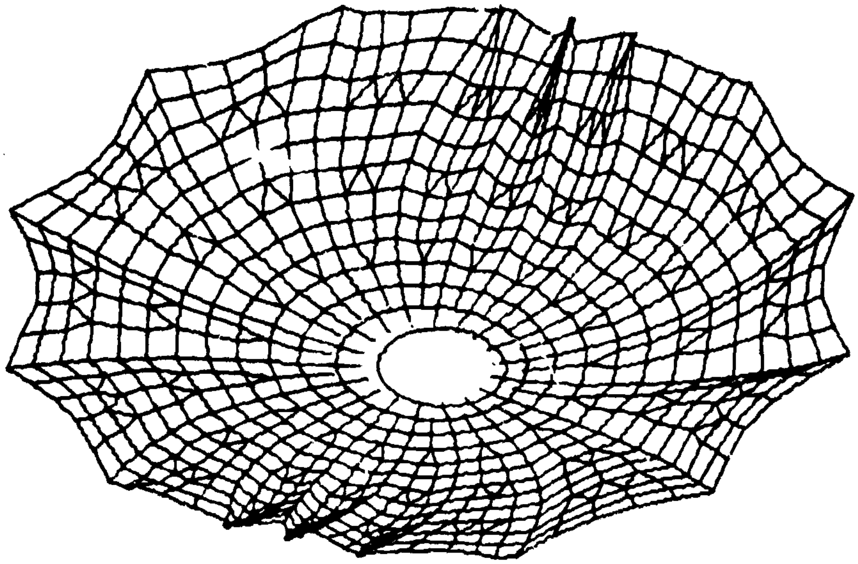
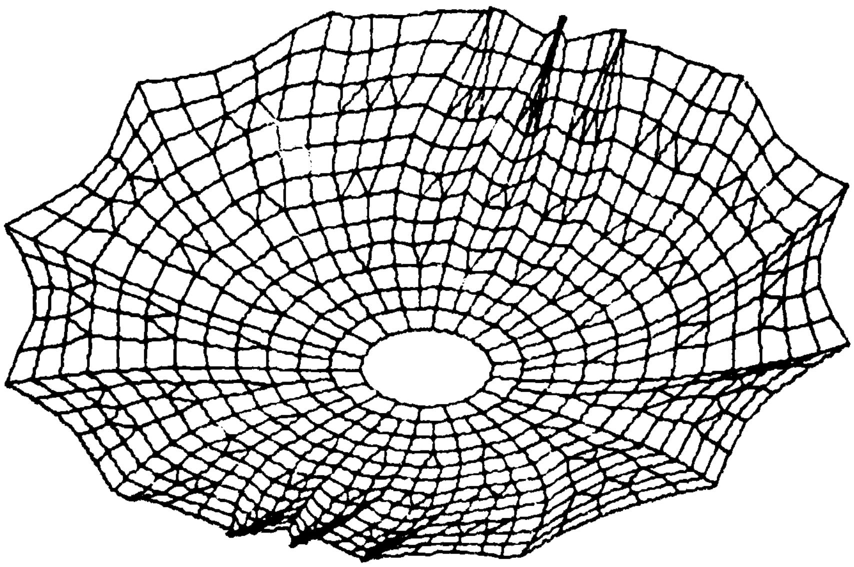


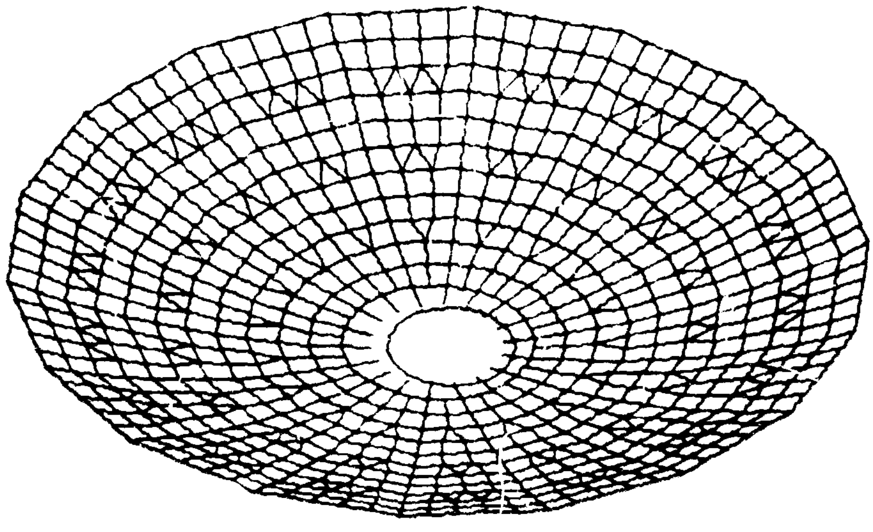
Figure 6.4 U^{o} + 30 ϕ_3



ACTUAL SHAPE $U^{o} + 25\phi_4$

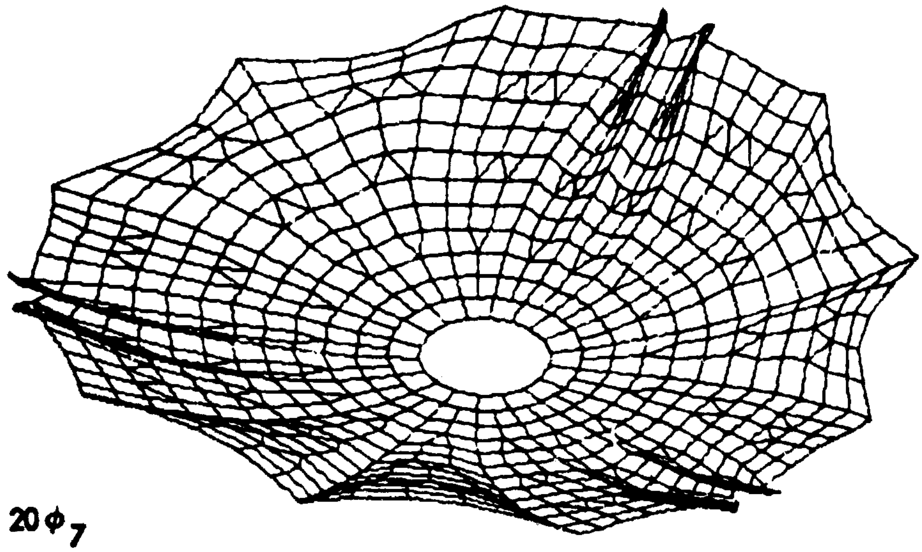


ESTIMATED SHAPE

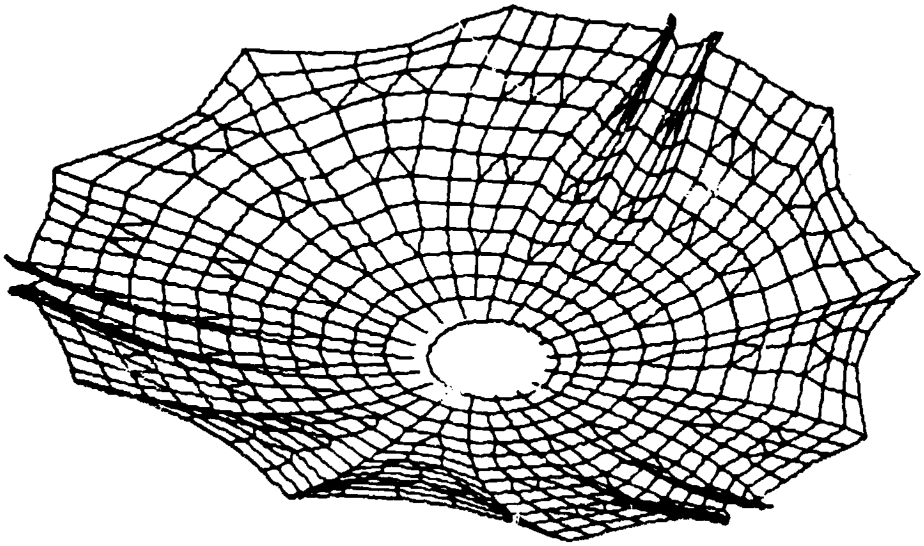


CORRECTED SHAPE

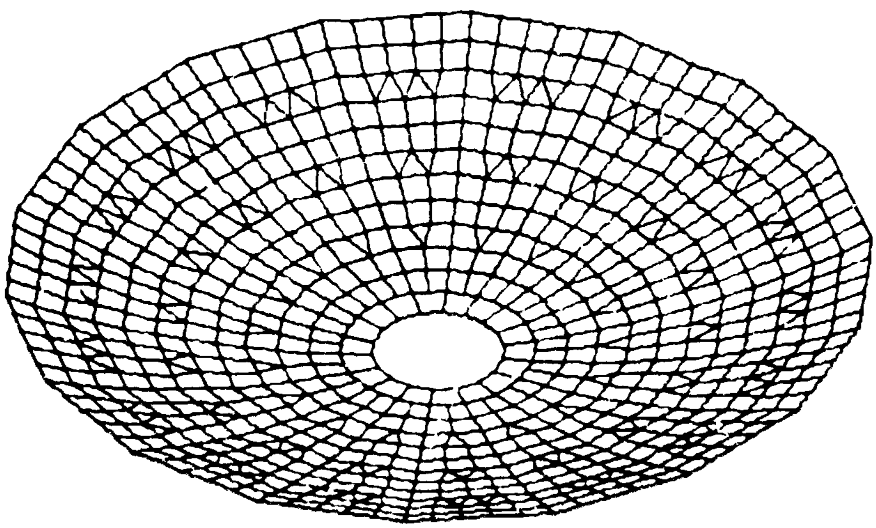
Figure 6.5 $U^{o} + 25\ \phi_4$



ACTUAL SHAPE $U^{0} + 20\phi_7$

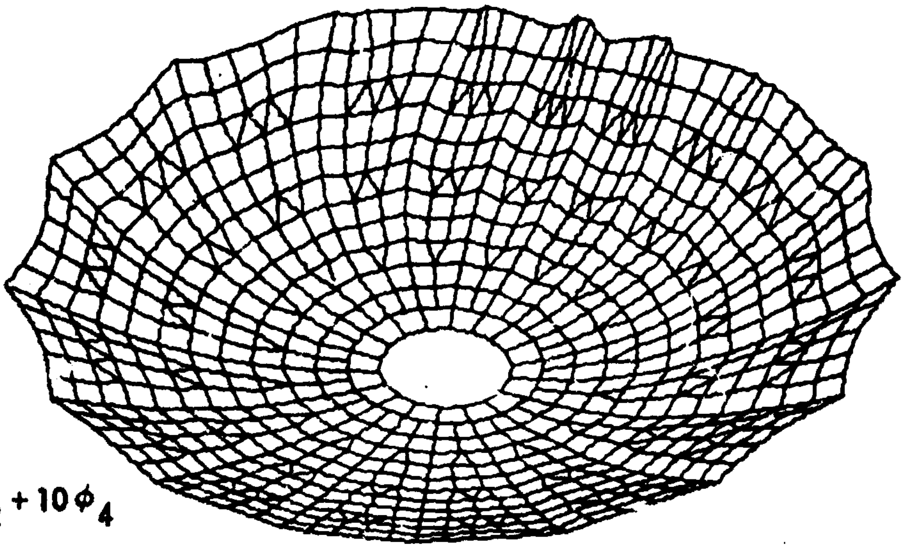


ESTIMATED SHAPE

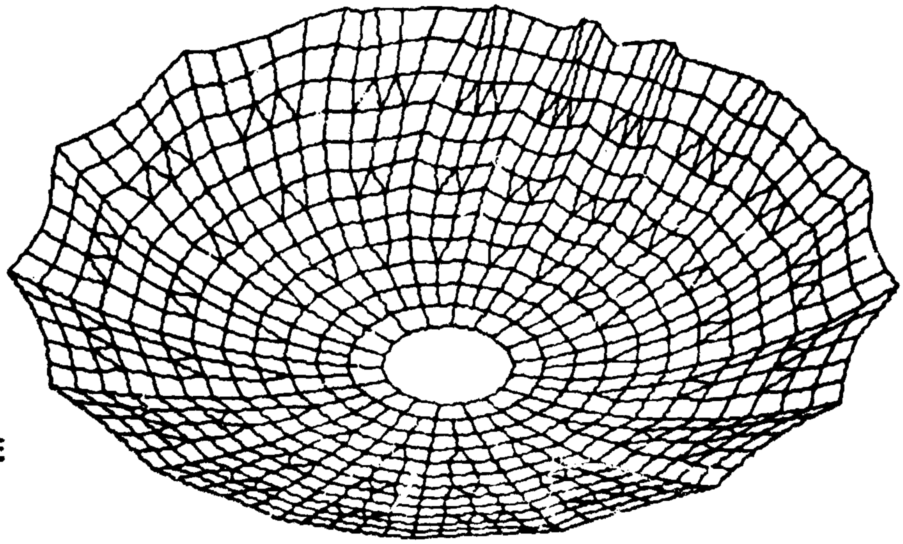


CORRECTED SHAPE

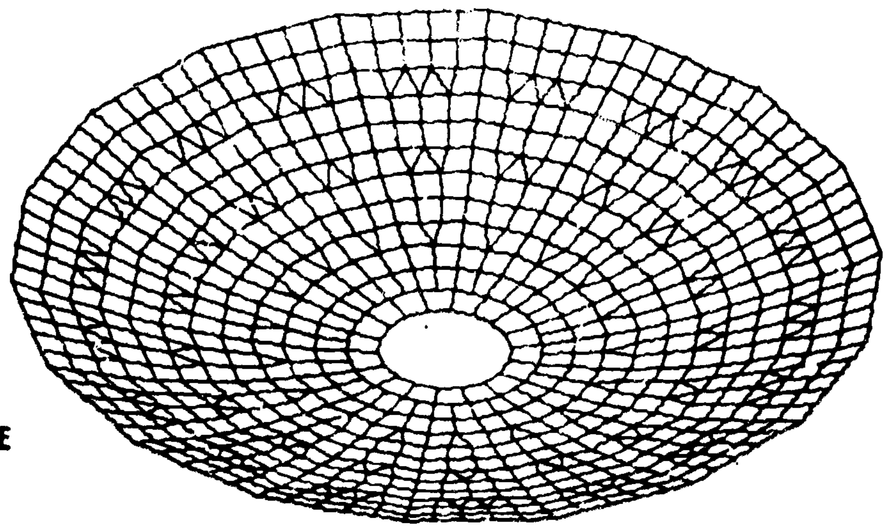
Figure 6.6 $U^{\circ} + 20\ \phi_7$



ACTUAL SHAPE $U^{o} + 15\phi_2 + 10\phi_4$

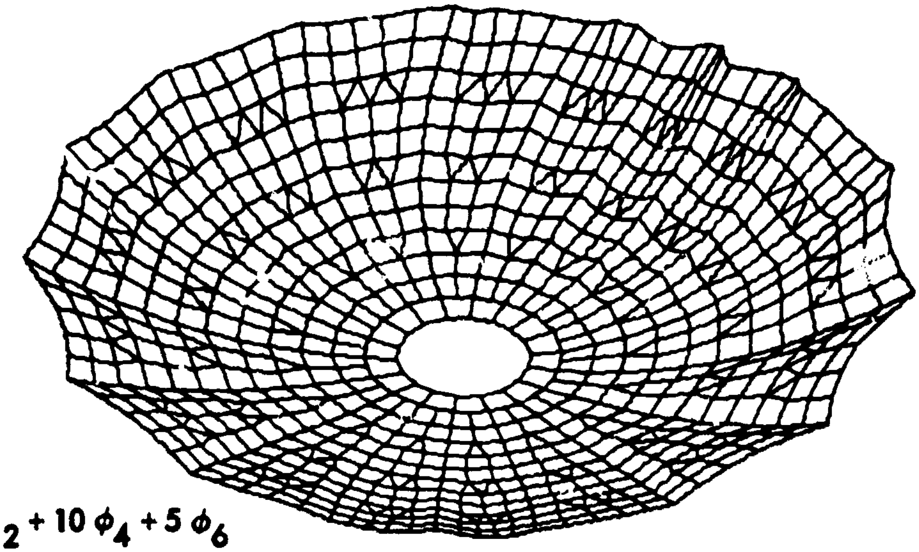


ESTIMATED SHAPE

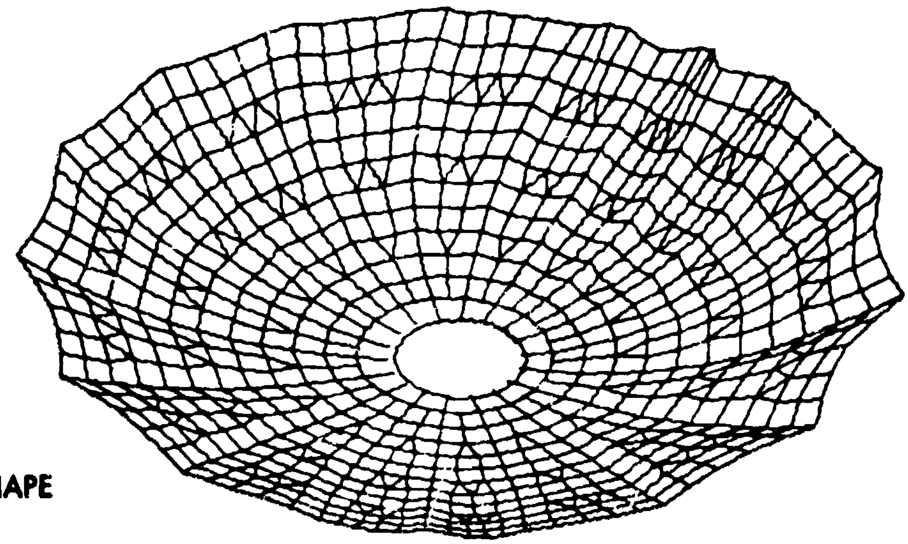


CORRECTED SHAPE

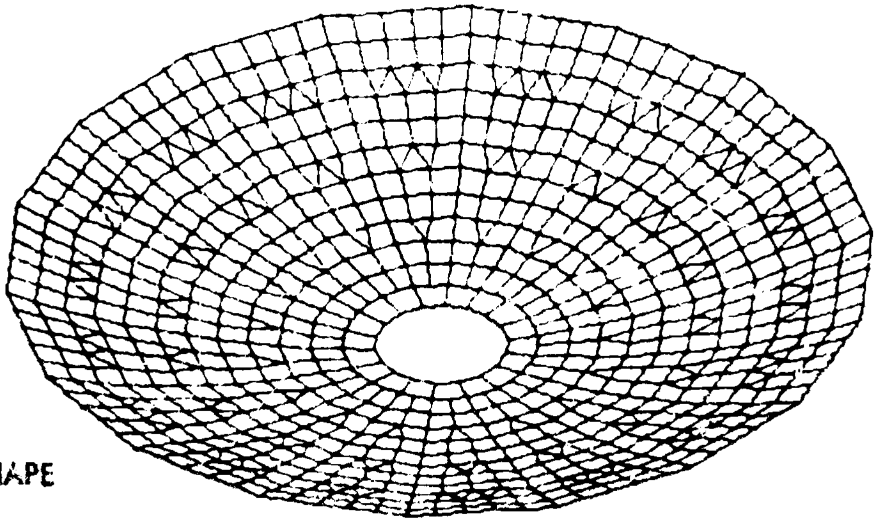
Figure 6.7 $U^{o} + 15\ \phi_2 + 10\ \phi_4$



ACTUAL SHAPE $U^{o} + 10\phi_2 + 10\phi_4 + 5\phi_6$

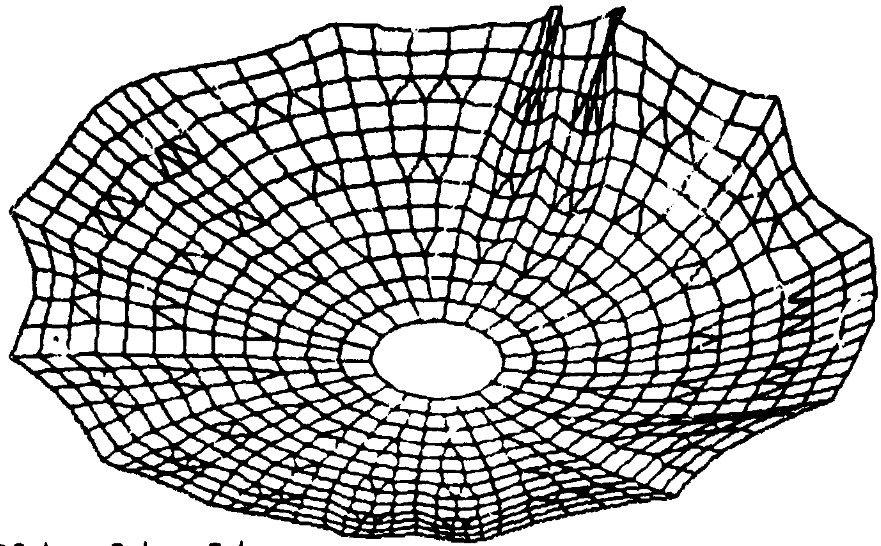


ESTIMATED SHAPE

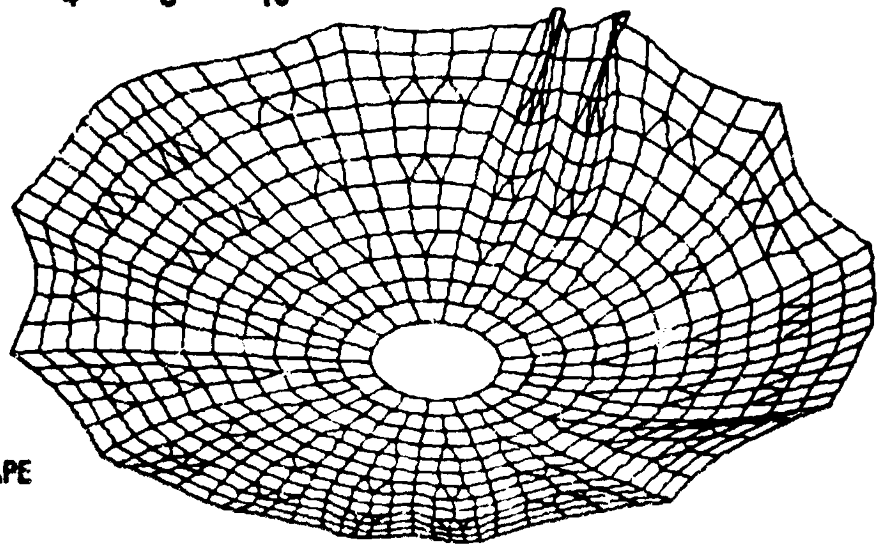


CORRECTED SHAPE

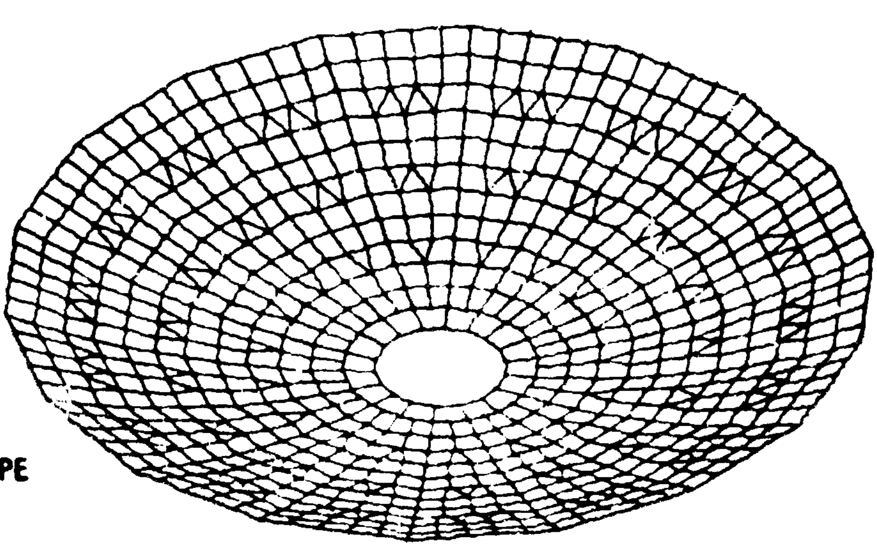
Figure 6.8 $U^{o} + 10\phi_2 + 10\phi_4 + 5\phi_6$



ACTUAL SHAPE $U^{o} + 10\phi_1 + 10\phi_4 + 5\phi_8 + 5\phi_{10}$

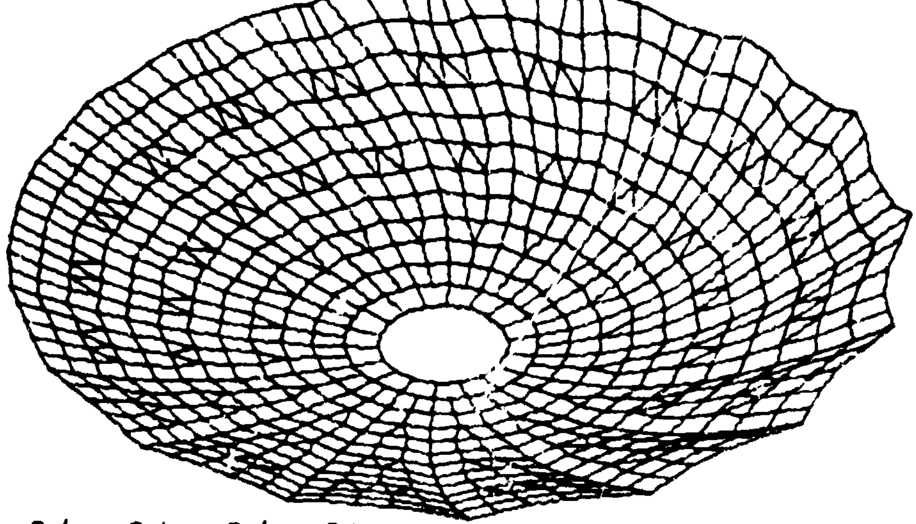


ESTIMATED SHAPE

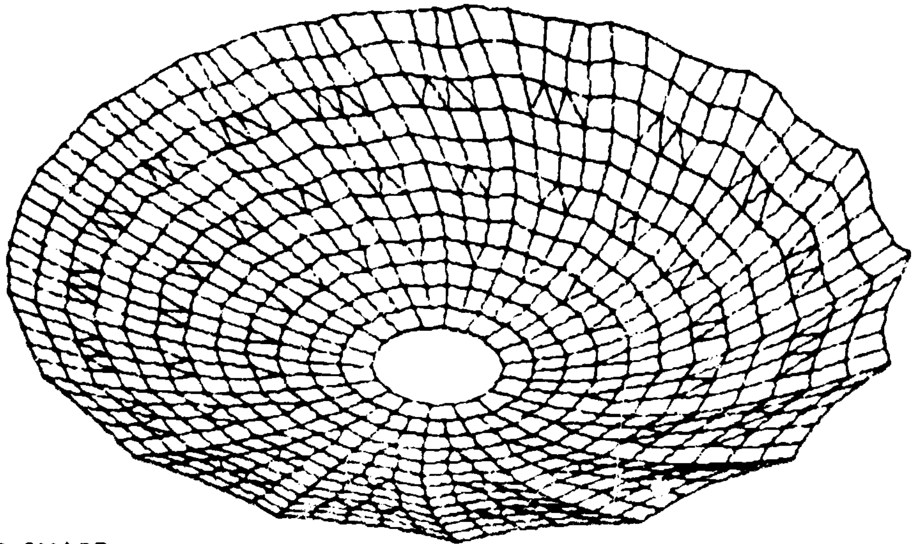


CORRECTED SHAPE

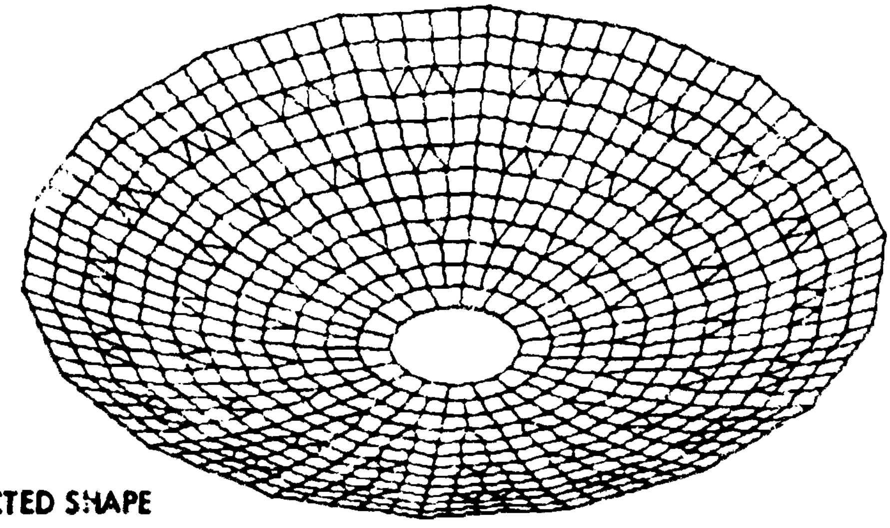
Figure 6.9 $U^{\circ} + 10\,\phi_1 + 10\,\phi_4 + 5\,\phi_8 + 5\,\phi_{10}$



ACTUAL SHAPE $U^o + 10\phi_1 + 5\phi_2 + 5\phi_3 + 5\phi_4 + 5\phi_5$



ESTIMATED SHAPE



CORRECTED SHAPE

Figure 6.10 $U^o + 10\phi_1 + 5\phi_2 + 5\phi_3 + 5\phi_4 + 5\phi_5$

Chapter 7. Conclusions and Future Work

It is possible to accurately determine and control the static shape of a large space structure by means of a number of control devices and sensor measurements at discrete points along the structure.

An integral operator approach to the continuous-discrete optimization problems of static shape estimation and control proves ideal for these problems. Solutions reduce to the solution of linear equations of dimension less than or equal to the number of observations, or control forces.

Elements of the linear equations involve the Green's function, or influence coefficient, of the structure, which represents the response of the structure to a force at one point. In the event that the Green's function cannot be computed analytically, approximations based on modal expansions have been presented, involving modes either from the static or associated dynamics model, which may be computed experimentally, or numerically.

The distinction between the shape control system and the attitude control, orbit and stationkeeping system arises in connection with the rigid body modes of the structure. The rigid body modes represent translations and/or rotations in space of the structure as a whole, clearly a concern of the attitude control, orbit and stationkeeping systems.

On the other hand, the rigid body modes are indetectable to the shape control system. Furthermore, a shape control system may not apply a net force in the direction of a rigid body mode, to correct it, since this would violate the boundary assumptions upon which shape control forces are computed. The latter restriction places additional constraints on the shape control forces in the case that rigid body modes are possible.

The use of modal expansions for terms in the shape control and determination algorithms invites the inevitable trade-off between accuracy and computational difficulty. If a few modes are used and the structural distortion involves significant components in higher order modes, the shape control and determination schemes will not be accurate. On the other hand, the use of many modes increases the necessary storage, time and expense of computation. A compensating factor is that while dynamic shape control must be accomplished on board the spacecraft, and within a short response time, static shape control may be accomplished by ground computers over a much longer period of time. Thus, the use of modal approximations may not present a difficulty.

Future Work

The solutions of both the shape determination and control problems depend on the solutions of linear systems which have dimensions on the order of the number of observations or control forces applied. It is estimated that actual large space antennae will require from 50 to 150 observation points for static control. It is therefore desirable to develop a geometric scanning algorithm, which would successively process data sets of antenna sections in an adaptive manner.

Despite the fact that linearity and self-adjointness are common engineering assumptions, it is probable that large space structures will not always have these characteristics. It is anticipated that the integral equation techniques used here will be applied to an iterative technique for the solution of non-linear problems, and that it will be adapted for the solution of non-self-adjoint problems.

Appendix A. Some Mathematical Background

A.1 A Little Distribution Theory

We should give some consideration to what is meant mathematically by a solution to (13-14) or (24-25).

A classical or strict solution to an nth order differential equation $Lu = f$ is an n times differentiable function y which "satisfies" the differential equation: $Ly = f$ on $[a,b]$.

Clearly it is not possible for a function to be both n times differentiable and to exhibit delta function behavior in a combination of its derivatives.

A rigorous development of the theory of solutions of equations of the type (13) may be found in distribution theory:

Distribution theory was developed to provide a rigorous framework for "functions" such as the delta function. One cannot deduce from the definition

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 \,, \tag{318}$$

or

$$\int_{-\infty}^{\infty} \delta(x)\, \phi(x)\, dx = \phi(0) \,, \tag{319}$$

or even that such expressions are meaningful. Thus a pointwise definition of the δ function does not characterize it.

On the other hand, if the δ function is defined by (319), the other information about it can be deduced. Thus $\delta(x)$ is defined <u>by its action on other functions through the inner product</u>

$$\langle \delta, \phi \rangle = \int_{-\infty}^{\infty} \phi(x)\, \delta(x)\, dx = \phi(0) \,.$$

In distribution theory this concept is extended to an entire collection of generalized functions, or distributions. Rather than characterizing distributions by pointwise values, they are defined by their "action" on a specific class of functions, called test functions. Test functions are infinitely differentiable functions on R^ℓ which vanish outside of some bounded domain. Eligible test functions for boundary value problems on the interval $[a,b]$ must vanish outside of $[a,b]$. For problems defined on Ω, the test functions must vanish outside of Ω.

On one dimensional domains, a distribution t "acts" on a test function through the inner product

$$<t, \phi> = \int_{\infty}^{\infty} t(x)\ \phi(x)\ dx \ .$$

Two distributions t_1 and t_2 are equal if $<t_1,\phi> = <t_2,\phi>$ for all eligible test functions ϕ.

The derivative of a distribution t is defined by $<t',\phi> = <t, -\phi'>$. The nth derivative is defined by $<t^{(n)},\phi> - <t_1\ (-1)^n \frac{d}{dx^n}\ \phi>$. Note that again the definition describes actions on test functions rather than some pointwise behavior.

If $\Omega \subset R^\ell$, we denote a partial differential operator on Ω by

$$D^K = \frac{\partial^{k_1+\ldots k_\ell}}{\partial x_1^{k_1}\ldots x_\ell^{k_\ell}}$$

where K is the vector $(k_1,\ldots,k_\ell)$ and $|K| = k_1+\ldots+k_\ell$. As an example of this notation, if $\ell = 3$, a point in R^3 is denoted by (x_1,x_2,x_3), and $K = (2, 0, 5)$, then

$$D^K = \frac{\partial^7}{\partial x_1^2\ \partial x_3^5} \ .$$

In R^{ℓ} a distribution T acts on a test function Φ through the inner product

$$\langle T,\Phi\rangle = \int_{\Omega} T(P)\, \Phi(P)\, dP \ .$$

Again, two distribution T_1 and T_2 are equal if $\langle T_1,\Phi\rangle = \langle T_2,\Phi\rangle$ for all eligible test functions Φ, and the derivatives of T are defined by

$$\langle D^K T, \Phi\rangle = (-1)^{|K|} \langle T, D^K\Phi\rangle \ .$$

By this new definition of the derivative, since test functions are infinitely differentiable, distributions are infinitely differentiable.

Finally, the distribution T is a generalized solution of LU = F if $\langle LT,\Phi\rangle = \langle F,\Phi\rangle$ for all test functions Φ. This removes the problem with finding solutions to (13), that is, how a function may be n times differentiable and yet have delta function behavior in a combination of its derivatives.

If T corresponds to a pointwise defined function which satisfies LT = F but is not sufficiently differentiable it is called a weak solution. If T corresponds to a function which is sufficiently differentiable so that the differential operations in LT = F may be performed in the classical sense, T is a classical solution, or strict solution. Classical solutions are easily shown to be generalized (distributional) solutions, so none of these solutions is lost by appealing to distribution theory.

Examples

A.1) $X \frac{dt}{dx} = 0$ has the classical solution t = C. It also has the weak solution t = H(x) (the heavy side step function).

A.2) $X^2 \frac{dt}{dx} = 0$ has the generalized or distributional solution $t = \delta(x)$, which is neither a weak solution nor a strict solution.

A.3) Green's functions, which are solutions of $Lt = \delta(x-\xi)$ are weak solutions, since they may be defined pointwise but lack sufficient differentiability to be strict solutions.

The use of the alternative theorem 6.1, and the assumption of the existence of complete orthonormal eigenfunction expansions which are the basis of the approximations, depend on the assumption that the operators L and K be defined in Hilbert spaces. The Hilbert spaces which can accommodate members such as the delta function are known as Sobolev spaces. An excellent treatment of Sobolev spaces is contained in [9].

A.2 The Free Space Solution of $\nabla^4\gamma = -\delta(P|Q)$

The equation

$$\nabla^4 \gamma = -\delta(P|Q) \tag{320}$$

represents the response of a plate in free space at the point P to a unit negative impulsive force at Q.

Theorem: A fundamental solution of (320) is given by

$$\gamma(x,y,\xi,\eta) = \frac{1}{8\pi} R^2 \log R \tag{321}$$

where R is the distance $\overline{PQ}$.

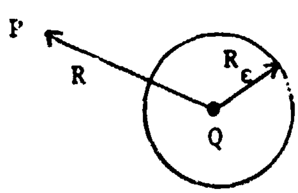
Proof: We wish to show that (321) defines a solution in the distributional sense. Thus it is necessary to show that

$$\langle\nabla^4\gamma,\phi\rangle = \langle\gamma,(\nabla^4)^*\phi\rangle = -\phi(Q)$$

for all test functions ϕ, where the inner produce $\langle u,v\rangle$ in free space is

$$\langle u,v\rangle = \int_{R^2} u(P)\, v(P)\, dP\ .$$

Let R_ε be a circle of radius ε about Q.



The function (321) is continuous except for a removable singularity at $R = 0$. Thus it is locally integrable and

$$\int_{R^2} \gamma(P) \ (\nabla^4\phi(P))dP = \lim_{\varepsilon\to o} \int_{R^2-R_\varepsilon} \gamma(P) \ (\nabla^4\phi(P))dP.$$

We apply Green's theorem, making use of the fact that ϕ vanishes for sufficiently large R to eliminate the surface integral at infinity. Thus

$$\int_{R^2-R_\varepsilon} \gamma(P) \ (\nabla^4\phi(P))dP = \int_{R^2-R_\varepsilon} \nabla^4\gamma(P) \ \phi(P) \ dP$$

$$- \int_{\partial R_\varepsilon} [\frac{\partial}{\partial n} (\nabla^2\phi) - \phi \frac{\partial}{\partial n} (\nabla^2\gamma)]ds$$

$$- \int_{\partial R_\varepsilon} [(\nabla^2\gamma) \frac{\partial\phi}{\partial n} - \nabla^2\phi(\frac{\partial\gamma}{\partial n})]ds.$$

On R^2-R_ε, $\nabla^4\gamma = 0$. The first integral on the right is zero. On the boundary of R_ε, $ds = \varepsilon d\theta$ and $\frac{\partial}{\partial n} = -\frac{\partial}{\partial R}$.

Therefore $\int_{R^2-R_\varepsilon} \gamma(P) \ \nabla^4\phi(P)dP$

$$= \varepsilon \int_0^{2\pi} [\frac{\partial}{\partial R} (\nabla^2\phi) - \phi \frac{\partial}{\partial R} (\nabla^2\gamma)] \ d\theta$$

$$+ \varepsilon \int_0^{2\pi} [\nabla^2\gamma \ (\frac{\partial\phi}{\partial R}) - \nabla^2\phi \ (\frac{\partial\gamma}{\partial R})]d\theta \tag{322}$$

Now $\frac{\partial\gamma}{\partial R} = \frac{r}{4\pi} (\log r + \frac{1}{2})$,

$$\nabla^2\gamma = \frac{1}{2n} \ (\log r + 1) ,$$

and

$$\frac{\partial}{\partial r} (\nabla^2\gamma) = \frac{1}{2\pi r} .$$

Furthermore, the test function ϕ has continuous derivatives of all orders which have compact support in R^2. Hence ϕ and any of the derivatives are bounded on all of R^2. Thus

$$\left|\frac{\partial}{\partial R}(\nabla^2\phi)\right| \leq M_1, \quad |\nabla^2\phi| \leq M_2 \text{ and } \left|\frac{\partial}{\partial R}\right| \leq M_3 \text{ in } R^2 .$$

We apply these relations to the elements of (322):

$$\left| \varepsilon \int_0^{2\pi} \gamma \frac{\partial}{\partial R}(\nabla^2\phi) d\theta \right| \leq M_1 \varepsilon^3 \log \varepsilon \, (2\pi) = o(\varepsilon).$$

$$\left| \varepsilon \int_0^{2\pi} \nabla^2\gamma \left(\frac{\partial\phi}{\partial R}\right) d\theta \right| \leq \frac{\varepsilon}{2\pi} (\log \varepsilon + 1) M_3 (2\pi) = o(\varepsilon)$$

$$\left| \varepsilon \int_0^{2\pi} \nabla^2\phi \left(\frac{\partial\gamma}{\partial R}\right) d\theta \right| \leq M_2 \left(\frac{\varepsilon}{4\pi}\right)^2 \left(\log \varepsilon + \frac{1}{2}\right) (2\pi) = o(\varepsilon) .$$

Finally,

$$- \varepsilon \int_0^{2\pi} \phi(R) \frac{\partial}{\partial R} (\nabla^2\gamma) d\theta = \frac{-1}{2\pi} \int_0^{\pi} \phi(\theta) \, d\theta.$$

Taking the limit as $\varepsilon \to 0$, only the last term provides a contribution. We can conclude

$$\int_R \gamma(\nabla^4\phi) \, dQ = - \phi(Q) . \qquad \#$$

Appendix B. The Flexible Beam Program Listings and Output

B.1 The Simply Supported Beam Control Program

B.2 The Pinned-Free Beam Control Program

B.3 The Simply Supported Beam Estimation Program

B.1 The Simply Supported Beam Control Program Listing

```
      COMMON START,HSTAR,HMIN,HMAX,ERMAX,XL,KEY,XZ(10) ,NM
      REAL A(10,10),B(10),        WORK(50),AO(10,10)
      REAL Z(50),U(50),PSI(50),DEL
      REAL AA(10,10),BB(10),UA(50)
      REAL YZ(10),YY(100)
      REAL Q(10)
      DATA YZ/10*0./
      DATA Q/10*1.E8/
      DATA NB,NDA,NDB/ 1,10,10/
      DATA Z(1),U(1),PSI(1),UA(1)/4*0. /
C
C
C     ******INTRODUCTION******
C
C     THIS PROGRAM COMPUTES THE OPTIMAL DISCRETE FORCES
C     FOR THE SHAPE CONTROL PROBLEM FOR THE SIMPLY SUPPORTED
C     BEAM, AND GRAPHS THE RESULTING SHAPE VS THE DESIRED SHAPE.
C     THE QUADRATIC COST IS ALSO COMPUTED.
C
C
C     PLEASE DEFINE THE FOLLOWING VARIABLES.
C     XL IS THE LENGTH OF THE BEAM.
C     NM IS THE NUMBER OF ACTUATORS.
C     NP IS THE NUMBER OF POINTS ALONG THE BEAM AT WHICH YOU
C     WISH THE GRAPHS TO BE PLOTTED.
C     NP IS LESS THAN OR EQUAL TO 50.
C     XZ(I), I=1,...,NM ARE THE ACTUATOR POSITIONS.
C     BE CERTAIN XZ(I) IS BETWEEN 0. AND XL.
C     Q(I), I=1,...,NM ARE THE WEIGHTS ON THE FORCES F(I) SQUARED
C     IN THE QUADRATIC COST CRITERION.
C     IF Q(I)=0,ALL I, THE MATRIX Q+A MAY BE SINGULAR, RESULTING
C     IN NO SOLUTION.
C     RECOMMEND Q=  1.E-5 * XL**7
C
C
C     PLEASE CHOOSE ONE OF THE FOLLOWING OPTIONS.
C
C     JOPT=1     ONLY EXACT OPTIMAL FORCES WILL BE CONSIDERED.
C     JOPT=2     EXACT OPTIMAL FORCES AND FIRST APPROXIMATIONS
C                BASED ON EIGENFUNCTION EXPANSIONS ARE TO BE CONSIDERED.
C
C
C     THE DESIRED SHAPE IS THE PARABOLA   Y=LENGTH*X - X*X
C
C
C     PLEASE CHOOSE ONE OF THE FOLLOWING OPTIONS.
C     KOPT DETERMINES WHAT GRAPHS ARE GENERATED.
C
C     KOPT=1     NO GRAPHS. THE OPTIMAL FORCES, SHAPE AND COST
C                WILL BE PRINTED.
```

```
C     KOPT=2     EXACT OPTIMAL SHAPE VS. DESIRED SHAPE.
C     KOPT=3     DESIRED VS. APPROXIMATE SHAPES
C     KOPT=4     EXACT, APPROXIMATE, AND DESIRED SHAPES ON ONE GRAPH.
C     KOPT=5     BOTH 2 AND 3.
C     KOPT=6     BOTH 4 AND 5.
C
C
      NM=2

      KOPT=4
      JOPT=2
      IF(KOPT.GT.2) JOPT=2
      PI=3.14159
      NP=20
      XL=100.
      DEL=XL/NP
      NP=NP+1
      WRITE(6,1) XL
    1 FORMAT(////1X,25HTHE LENGTH OG THE BEAM IS,F10.2)
      WRITE(6,2)
    2 FORMAT(////)
C
C     THE FOLLOWING VARIABLES ARE NECESSARY FOR THE JPL QUADRATURES
C     SUBROUTINE.
C
      START=0.
      HSTAR=.01*XL
      HMIN=.001*XL
      HMAX=.05*XL
      ERMAX=1.E-4
      KEY=0
C
C
C     THESE CONSTANTS ARE NECESSARY FOR THE PLOTTING SUBROUTINES.
C
      NG=1
      TIC5='X'
      NT5=-1
      NT1=2
      NT2=2
      NT3=2
      TIC1='O'
      TIC2='.'
      TIC3='/'
      XLEN=8.
      YLEN=6.
      DO 495 IN=1,3
      NM=IN
      DO 7 NXZ=1,IN
      XZ(NXZ)=NXZ*XL/(IN+1)
    7 CONTINUE
      CALL VOUT(G,NM,17,17HOTHE WEIGHTS G(I))
      CALL VOUT(XZ,NM,33,33HOTHE VECTOR OF ACTUATOR POSITIONS)
C
C     HERE THE EXACT A MATRIX AND B VECTOR ARE COMPUTED.
C
C
```

```
C     THE SUBROUTINE BVEC COMPUTES THE EXACT VECTOR B.
C
      CALL BVEC(B)
      DO 50 I=1,NM
      X=XZ(I)
      C1=X*(X-2.*XL)
      DO 50 J=I,NM
      Y=XZ(J)
C
      C2=Y*Y-2.*XL*Y
      C3=X*X+Y*Y
      C4=X*X*Y*Y
C
      A(I,J)=((X-XL)*(Y-XL)/(36.*XL*XL))* ((X**7)/7.+(X**5)*(C1+C2)/5.+
     1 (X**3)*C1*C2/3.)+(X*Y/(36.*XL*XL))*((XL**7-Y**7)/7.+XL*Y**6-XL**7
     2 +.2*(XL**5-Y**5)*(13.*XL*XL+C3)-(XL**4-Y**4)*(3.*XL**3+XL*C3)
     3 +(XL**3-Y**3)*(1./3.)*(5.*XL*XL*C3+4.*XL**4+C4) -(XL*(L-Y*Y)*(XL*
     4 C4+(XL**3)*C3)+(XL-Y)*(XL*XL*C4))  +  (X/(36.*XL**2))*(Y-XL)*((Y
     5 **7-X**7)/7.-XL*.5*(Y**6-X**6)+.2*(Y**5-X**5)*(C2+2.*XL*XL+X*X)-
     6 .25*(Y**4-X**4)*(3.*XL*C2+XL*X*X)+(Y**3-X**3)*C2*(X*X+2.*XL*XL)/
     73.-(Y*Y-X*X)*.5*(XL*X*X*C2))
   50 CONTINUE
      DO 51 J=2,NM
      JJ=J-1
      DO 51 I=1,JJ
      A(J,I)=A(I,J)
   51 CONTINUE
      WRITE(6,2)
      CALL MOUT(A,NDA,NM,NM,19,19H0THE EXACT A MATRIX)
      CALL VOUT(B,NM,13,13H0THE B VECTOR)
      DO 60 I=1,NM
      DO 60 J=1,NM
      AQ(I,J)=A(I,J)
   60 CONTINUE
      DO 61 I=1,NM
      AQ(I,I)=AQ(I,I) + Q(I)
   61 CONTINUE
      CALL MOUT(AQ,NDA,NM,NM,21,21H0THE EXACT MATRIX Q+A)
C
C     SOR IS A JPL LINEAR EQUATION SOLVING SUBROUTINE.
C
      CALL SOR(AQ,NDA,NM,B,NDB,NB,$30,WORK)
      CALL VOUT(B,NM,25,25H0VECTOR OF OPTIMAL FORCES)
      GO TO 40
   30 WRITE(6,31)
   31 FORMAT(15X,26H0MATRIX IS NEARLY SINGULAR)
      GO TO 500
   40 IF(JOPT.EQ.1) GO TO 170
C
C
C     HERE THE APPROXIMATE VALUES OF A AND B ARE COMPUTED.
C
      Z1=2.*(XL**7)/(PI**8)
C
      DO 150 I=1,NM
      X=XZ(I)
      DO 150 J=1,NM
```

```
      Y=XZ(J)
      AA(I,J)=Z1*(SIN(PI*X/XL))*SIN(PI*Y/XL)
  150 CONTINUE
      WRITE(6,2)
      CALL MOUT(AA,NDA,NM,NM,32,32HOFIRST APPROXIMATION TO A MATRIX)
C
C     THE SUBROUTINE BAP COMPUTES AN APPROXIMATE VECTOR B.
C
      CALL BAP(BB)
      CALL VOUT(BB,NM,32,32HOFIRST APPROXIMATION TO B VECTOR)
      DO 160 I=1,NM
      DO 160 J=1,NM
      AQ(I,J)=AA(I,J)
  160 CONTINUE
      DO 165 I=1,NM
      AQ(I,I)=AQ(I,I) + Q(I)
  165 CONTINUE
      CALL MOUT(AQ,NDA,NM,NM,27,27HOTHE APPROXIMATE MATRIX A+Q)
C
C     SGR IS A JPL LINEAR EQUATION SOLVING SUBROUTINE.
C
      CALL SGR(AQ,NDA,NM,BB,NDB,NB,$30,WORK)
      CALL VOUT(BB,NM,29,29HOVECTOR OF APPROXIMATE FORCES)
  170 CALL COST(B,Q,NM,C)
      WRITE(6,171) C
  171 FORMAT(///,7X,17HTHE EXACT COST IS, E15.5   )
      IF(JOPT.EQ.1) GO TO 175
      CALL COST(BB,Q,NM,C)
      WRITE(6,172) C
  172 FORMAT(///,1X,23HTHE APPROXIMATE COST IS , E15.5)
      WRITE(6,194)
  175 IF(JOPT.EQ.1) WRITE(6,195)
      WRITE(6,2)
C
C     HERE THE SHAPES ARE COMPUTED.
C
C     Z(I) IS THE X VALUE OF THE ITH POINT ON A GRAPH.
C     U(I) IS THE Y VALUE OF THE ITH POINT ON THE GRAPH OF THE
C     OPTIMAL SHAPE.
C     UA(I) IS THE Y VALUE OF THE ITH POINT ON THE GRAPH OF
C     APPROXIMATE SHAPE.
C     PSI(I) IS THE Y VALUE OF THE ITH POINT ON THE GRAPH OF
C     DESIRED SHAPE.
      DO 200 K=2,NP
      Z(K)=(K-1) * DEL
      X=Z(K)
      PSI(K)=XL*X-X*X
      U(K)=0.
      UA(K)=0.
      DO 190 I=1,NM
      IF(X.GT.XZ(I)) GO TO 180
      G=(XZ(I)-XL)*X*(X*X-2.*XZ(I)*XL+XZ(I)**2)
      GO TO 185
  180 G=(X-XL)*XZ(I)*(XZ(I)**2-2.*XL*X+X*X)
  185 U(K)=U(K)+G*B(I)/(6.*XL)
      IF (JOPT.NE.2) GO TO 190
      UA(K)=UA(K) + G*BB(I)/(6.*XL)
```

```
190   CONTINUE
      IF(JOPT-1) 191,191,192
191   WRITE(6,196) X,PSI(K),U(K)
      GO TO 200
192   WRITE(6,196) X,PSI(K),U(K),UA(K)
194   FORMAT(1H0,3X,8HPOSITION,4X,13HDESIRED SHAPE,07X,5HSHAPE,2X,
     1 13HAPPROX. SHAPE)
195   FORMAT(1H0,3X,8HPOSITION,4X,13HDESIRED SHAPE,07X,5HSHAPE)
196   FORMAT(/  ,1X,F10.2,3E15.5)
200   CONTINUE
      NP2=2*NP
      DO 215 I=1,NP
      YY(I)=PSI(I)*1.25
      YY(I+NP)=U(I)*1.25
215   CONTINUE
      GO TO (500,300,350,250,300,250),KOPT
C
250   CALL BGNPLT
      CALL PLFORM('_IN_IN',XLEN,YLEN)
      CALL PLSCAL(Z,NP,NG,YY,NP2,NG)
      CALL PLABEL('THE  IMPLY SUPPORTED BEAM',25,'LENGTH IN METERS',16,
     1 'DISPLACEMENT',12)
      CALL PLGRAF
      CALL PLAXIS(-2,XLEN,0.)
      CALL PLCURV(Z,U,NP,NT1,TIC1)
      CALL PLCURV(Z,PSI,NP,NT2,TIC2)
      CALL PLCURV(Z,UA,NP,NT3,TIC3)
      CALL PLCURV(X2,Y2,NM,NT5,TIC5)
      CALL PLTEXT(2.5,5.5,.1,0.,30HACTUATOR POSITIONS MARKED BY X,30,1)
      CALL PLTEXT(1.5,5.0,.10,0.,49HDESIRED(*) VS OPTIMAL(O) VS APPROXIM
     1ATE(/) SHAPES,49,1)
      GO TO (270,271,272,273,274,275,276),NP
276   CALL PLTEXT(3.2,7.0,.1,0.,15HSEVEN ACTUATORS,15,1)
      GO TO 260
270   CALL PLTEXT(3.4,7.0,.1,0.,12HONE ACTUATOR,12,1)
      GO TO 260
271   CALL PLTEXT(3.3,7.0,.1,0.,13HTWO ACTUATORS,13,1)
      GO TO 260
272   CALL PLTEXT(3.2,7.0,.1,0.,15HTHREE ACTUATORS,15,1)
      GO TO 260
273   CALL PLTEXT(3.3,7.0,.1,0.,14HFOUR ACTUATORS,14,1)
      GO TO 260
274   CALL PLTEXT(3.3,7.0,.1,0.,14HFIVE ACTUATORS,14,1)
      GO TO 260
275   CALL PLTEXT(3.3,7.0,.1,0.,13HSIX ACTUATORS,13,1)
260   IF(KOPT.EQ.4) GO TO 490
      CALL ADVPLT
300   CALL BGNPLT
      CALL PLFORM('_IN_IN',XLEN,YLEN)
      CALL PLSCAL(Z,NP,NG,YY,NP2,NG)
      CALL PLABEL('THE SIMPLY SUPPORTED BEAM',25,'LENGTH IN METERS',16,
     1 'DISPLACEMENT',12)
      CALL PLGRAF
      CALL PLAXIS(-2,XLEN,0.)
      CALL PLCURV(Z,U,NP,NT1,TIC1)
      CALL PLCURV(Z,PSI,NP,NT2,TIC2)
      CALL PLCURV(X2,Y2,NM,NT5,TIC5)
```

```
      CALL PLTEXT(2.5,5.5,.1,0.,30HACTUATOR POSITIONS MARKED BY X,30,1)
      CALL PLTEXT(2.5,5.0,.10,0.,31HDESIRED(*) VS OPTIMAL(C) SHAPES,31,
     1 1)
      GO TO (320,321,322,323,324,325,326),NP
  326 CALL PLTEXT(3.2,7.0,.1,0.,15HSEVEN ACTUATORS,15,1)
      GO TO 330
  320 CALL PLTEXT(3.4,7.0,.1,0.,12HONE ACTUATOR,12,1)
      GO TO 330
  321 CALL PLTEXT(3.3,7.0,.1,0.,13HTWO ACTUATORS,13,1)
      GO TO 330
  322 CALL PLTEXT(3.2,7.0,.1,0.,15HTHREE ACTUATORS,15,1)
      GO TO 330
  323 CALL PLTEXT(3.3,7.0,.1,0.,14HFOUR ACTUATORS,14,1)
      GO TO 330
  324 CALL PLTEXT(3.3,7.0,.1,0.,14HFIVE ACTUATORS,14,1)
      GO TO 330
  325 CALL PLTEXT(3.3,7.0,.1,0.,13HSIX ACTUATORS,13,1)
  330 IF(KOPT.EQ.2) GO TO 490
      CALL ADVPLT
  350 CALL BGNPLT
      CALL PLFORM('LIN_LIN',XLEN,YLEN)
      CALL PLSCAL(Z,NP,NG,YY,NP2,NG)
      CALL PLABEL('THE SIMPLY SUPPORTED BEAM',25,'LENGTH IN METERS',16,
     1 'DISPLACEMENT',12)
      CALL PLGRAF
      CALL PLAXIS(-2,XLEN,0.)
      CALL PLCURV(Z,PSI,NP,NT2,TIC2)
      CALL PLCURV(Z,UA,NP,NT3,TIC3)
      CALL PLCURV(XZ,YZ,NM,NT5,TIC5)
      CALL PLTEXT(2.5,5.5,.1,0.,30HACTUATOR POSITIONS MARKED BY X,30,1)
      CALL PLTEXT(2.3,5.0,.10,0.,35HDESIRED(*) VS APPROXIMATE(/) SHAPES,
     1 35,1)
      GO TO (370,371,372,373,374,375,376),NM
  376 CALL PLTEXT(3.2,7.0,.1,0.,15HSEVEN ACTUATORS,15,1)
      GO TO 490
  370 CALL PLTEXT(3.4,7.0,.1,0.,12HONE ACTUATOR,12,1)
      GO TO 490
  371 CALL PLTEXT(3.3,7.0,.1,0.,13HTWO ACTUATORS,13,1)
      GO TO 490
  372 CALL PLTEXT(3.2,7.0,.1,0.,15HTHREE ACTUATORS,15,1)
      GO TO 490
  373 CALL PLTEXT(3.3,7.0,.1,0.,14HFOUR ACTUATORS,14,1)
      GO TO 490
  374 CALL PLTEXT(3.3,7.0,.1,0.,14HFIVE ACTUATORS,14,1)
      GO TO 490
  375 CALL PLTEXT(3.3,7.0,.1,0.,13HSIX ACTUATORS,13,1)
  490 CONTINUE
      CALL ADVPLT
  495 CONTINUE
      CALL ENDPLT
C    DONT FORGET TO REPLACE 490 CALL ENDPLT
  500 STOP
      END
```

END OF COMPILATION: NO DIAGNOSTICS.

732 LTF:0.041 SUPS:7.070

```
      SUBROUTINE BAP(B)
      COMMON START,HSTAR,HMIN,HMAX,ERMAX,XL,KEY,XZ(10),NM
      REAL B(10)
C
C     THIS SUBROUTINE COMPUTES AN APPROXIMATE B VECTOR FOR
C     THE SIMPLY SUPPORTED BEAM.
C
C     INTEGRATIONS ARE PERFORMED BY THE JPL QUADRATURES SUBROUTINE.
C     ROMBS AND ROM2 ARE PART OF THAT SUBROUTINE.
C
C     PSI IS THE DESIRED SHAPE.
C
      PI=3.14159
      DO 50 I=1,NM
      CALL ROMBS(START,XL,X,FOFX,HSTAR,HMIN,HMAX,ERMAX,ANS,K,KEY)
   10 PSI=XL*X-X*X
      FOFX=PSI*SIN(PI*X/XL)
      CALL ROM2
      IF(K.EQ.1) GO TO 10
      B(I)=ANS*SIN(PI*XZ(I)/XL)*2.*(XL**3)/(PI**4)
   50 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE BVEC(B)
      COMMON START,HSTAR,HMIN,HMAX,ERMAX,XL,KEY,XZ(10),NM
      REAL B(10)
C
C     THIS SUBROUTINE COMPUTES THE EXACT B VECTOR FOR THE
C     SIMPLY SUPPORTED BEAM.
C
C     INTEGRATIONS ARE PERFORMED BY THE JPL QUADRATURES SUBROUTINE.
C     ROMBS AND ROM2 ARE PART OF THAT SUBROUTINE.
C
C     PSI IS THE DESIRED SHAPE.
C
      DO 50 I=1,NM
      Z=XZ(I)
      CALL ROMBS(START,XL,X,FOFX,HSTAR,HMIN,HMAX,ERMAX,ANS,K,KEY)
   10 IF(X.GT.Z) GO TO 15
      G=(Z-XL)*X*(X*X-2.*Z*XL+Z*Z)
      GO TO 20
   15 G=(X-XL)*Z*(Z*Z-2.*XL*X+X*X)
   20 G=G/(6.*XL)
      PI=3.14159
      PSI=X*XL-X*X
      FOFX=PSI*G
      CALL ROM2
      IF(K.EQ.1) GO TO 10
      B(I)=ANS
   50 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE COST(B,Q,NM,C)
      COMMON START,HSTAR,HMIN,HMAX,ERMAX,XL,KEY,XZ(10)
      REAL B(10),Q(10)
C
C     THIS SUBROUTINE COMPUTES THE QUADRATIC COST FUNCTIONAL
C     FOR A SET OF NM FORCES B(I) AT POSITIONS XZ(I) ALONG
C     A SIMPLY SUPPORTED BEAM
C
C     COST=(1/2)*(SUM(Q(I)*B(I)**2) + INTEGRAL((U(X)-PSI(X))**2))
C
C     WHERE U IS THE OPTIMAL SHAPE AND PSI IS THE DESIRED SHAPE
C
      C=0.
      DO 5 I=1,NM
      C=C+Q(I)*(B(I)**2)
    5 CONTINUE
      CALL ROMBS(START,XL,X,FOFX,HSTAR,HMIN,HMAX,ERMAX,ANS,K,KEY)
   10 SHAPE=0.
      DO 200 I=1,NM
      IF(X.GT.XZ(I)) GO TO 180
      G=(XZ(I)-XL)*X*(X*X-2.*XZ(I)*XL+XZ(I)**2)
      GO TO 190
  180 G=(X-XL)*XZ(I)*(XZ(I)**2-2.*XL*X+X*X)
  190 SHAPE=SHAPE+G*B(I)/(6.*XL)
  200 CONTINUE
      PI=3.14159
      PSI=X*XL-X*X
      FOFX=.5*(SHAPE-PSI)**2
      CALL ROM2
      IF(K.EQ.1)GO TO 10
      C=.5*C + ANS
      RETURN
      END
```

Output

```
THE LENGTH OG THE GEAM IS    100.00

THE WEIGHTS Q(I)
  1 TO    3     1.0000000+08   1.0000000+08   1.0000000+08

THE VECTOR OF ACTUATOR POSITIONS
  1 TO    3     2.5000000+01   5.0000000+01   7.5000000+01



THE EXACT A MATRIX

                 COL    1        COL    2        COL    3
ROW    1        1.0623027+10   1.4902127+10   1.0458322+10
ROW    2        1.4902127+10   2.1061304+10   1.4902119+10
ROW    3        1.0458322+10   1.4902119+10   1.0622956+10

THE B VECTOR
  1 TO    3     1.8737839+09   2.6475730+09   1.8737792+09

THE EXACT MATRIX Q+A

                 COL    1        COL    2        COL    3
ROW    1        1.0723027+10   1.4902127+10   1.0456322+10
ROW    2        1.4902127+10   2.1161304+10   1.4902119+10
ROW    3        1.0458322+10   1.4902119+10   1.0722956+10

VECTOR OF OPTIMAL FORCES
  1 TO    3     5.2136838-02   5.1635507-02   5.2134565-02



FIRST APPROXIMATION TO A MATRIX

                 COL    1        COL    2        COL    3
ROW    1        1.0539096+10   1.4904545+10   1.0539125+10
ROW    2        1.4904543+10   2.1078222+10   1.4904563+10
ROW    3        1.0539125+10   1.4904583+10   1.0539153+10

FIRST APPROXIMATION TO B VECTOR
  1 TO    3     1.8729539+09   2.6467587+09   1.8729590+09

THE APPROXIMATE MATRIX A+Q

                 COL    1        COL    2        COL    3
ROW    1        1.0639096+10   1.4904543+10   1.0539125+10
ROW    2        1.4904543+10   2.1178222+10   1.4904563+10
ROW    3        1.0539125+10   1.4904583+10   1.0639153+10

VECTOR OF APPROXIMATE FORCES
  1 TO    3     4.4323496-02   6.2662374-02   4.4323645-02
```

THE EXACT COST IS .62204+06

THE APPROXIMATE COST IS .63505+06

POSITION	DESIRED SHAPE	SHAPE	APPROX. SHAPE
5.00	.47500+03	.40412+03	.40207+03
10.00	.90000+03	.79850+03	.79459+03
15.00	.12750+04	.11734+04	.11664+04
20.00	.16000+04	.15191+04	.15137+04
25.00	.18750+04	.18257+04	.18212+04
30.00	.21000+04	.20848+04	.20823+04
35.00	.22750+04	.22919+04	.22923+04
40.00	.24000+04	.24438+04	.24470+04
45.00	.24750+04	.25373+04	.25428+04
50.00	.25000+04	.25692+04	.25756+04
55.00	.24750+04	.25373+04	.25428+04
60.00	.24000+04	.24438+04	.24470+04
65.00	.22750+04	.22919+04	.22923+04
70.00	.21000+04	.20848+04	.20823+04
75.00	.18750+04	.18257+04	.18212+04
80.00	.16000+04	.15190+04	.15137+04
85.00	.12750+04	.11734+04	.11664+04
90.00	.90000+03	.79850+03	.79459+03
95.00	.47500+03	.40412+03	.40207+03
100.00	.00000	.00000	.00000

@SYM,P PUNCHS,198/648,89PLTD

@PLOT,P

B.2 The Pinned-Free Beam Control Program Listing

```
      COMMON START,HSTAR,HMIN,HMAX,ERMAX,XL,KEY,XZ(10),NM
      REAL A(10,10),B(10)
      REAL Y(100) ,UA(50)
      REAL AB(10,10),AD(10,10),BIG(10),Q(10)
      REAL AA(10,10),PHI(10),BA(10),FA(10)
      REAL ABA(10,10),BBA(10),AQA(10,10)
      DIMENSION WORK(100)
      REAL F(10),U(50),PSI(50),X(50)
      REAL PHI2(10)
      DATA Q/10*0.0/
C
C     THIS PROGRAM COMPUTES THE OPTIMAL DISCRETE FORCES
C     FOR THE SHAPE CONTROL PROBLEM FOR THE JPL FLEXIBLE
C     BEAM, AND GRAPHS THE RESULTING SHAPE VS THE DESIRED SHAPE.
C
C     PLEASE DEFINE THE FOLLOWING VARIABLES.
C
C     XL IS THE LENGTH OF THE BEAM.
C     NM IS THE NUMBER OF ACTUATORS.
C     NM MUST BE GREATER THAN OR EQUAL TO 2.
C     NP IS THE NUMBER OF POINTS ALONG THE BEAM AT WHICH YOU
C     WISH THE GRAPHS TO BE PLOTTED.
C     NP IS LESS THAN OR EQUAL TO 50.
C     XZ(I), I=1,...,NM ARE THE ACTUATOR POSITIONS.
C     BE CERTAIN XZ(I) IS BETWEEN 0. AND XL.
C     Q(I), I=1,...,NM ARE THE WEIGHTS ON THE FORCES F(I) SQUARED
C     IN THE QUADRATIC COST CRITERION.
C
C
C     PLEASE CHOOSE ONE OF THE FOLLOWING OPTIONS.
C
C     JOPT=1     ONLY EXACT OPTIMAL FORCES WILL BE CONSIDERED.
C     JOPT=2     EXACT OPTIMAL FORCES AND FIRST APPROXIMATIONS
C                BASED ON EIGENFUNCTION EXPANSIONS ARE TO BE CONSIDERED.
C
C
C     THE DESIRED SHAPE IS THE PARABOLA  Y= LENGTH * 3/4 X - X*X.
C
C
C     PLEASE CHOOSE ONE OF THE FOLLOWING OPTIONS.
C
C     KOPT DETERMINES WHAT GRAPHS ARE GENERATED.
C
C     KOPT=1     NO GRAPHS. THE OPTIMAL FORCES, SHAPE AND COST
C                WILL BE PRINTED.
C     KOPT=2     EXACT OPTIMAL SHAPE VS. DESIRED SHAPE.
C     KOPT=3     DESIRED VS. APPROXIMATE SHAPES
```

```
C     KOPT=4    EXACT, APPROXIMATE, AND DESIRED SHAPES ON ONE GRAPH.
C     KOPT=5    BOTH 2 AND 3.
C     KOPT=6    BOTH 4 AND 5.
C
C
      KOPT=6
      JOPT=2
      IF(KOPT.GT.2) JOPT=2
      NM=2
      XL=100.
      NP=20
      DO 1 I=1,NM
      XZ(I)=I*.5 *XL
1     CONTINUE
      WRITE(6,2) XL
2     FORMAT(////,1X,25HTHE LENGTH OF THE BEAM IS,F10.2)
      CALL VOUT(XZ,NM,33,33H0THE VECTOR OF ACTUATOR POSITIONS)
      DEL=XL/NP
      NP=NP+1
C
C     THESE CONSTANTS ARE NECESSARY FOR THE PLOTTING SUBROUTINES.
C
      XLEN=8.
      YLEN=6.
      NG=1
      NT=6
      TIC1='*'
      TIC2='0'
      TIC3='/'
      TIC4='$'
C
C     THESE CONSTANTS ARE NECESSARY FOR THE MATRIX INVERSION ROUTINE SOR.
C
      NDA=10
      NDB=1
      NB=1
      X1=XZ(1)
      M=NM-1
C
C     THE FOLLOWING VARIABLES ARE NECESSARY FOR THE JPL QUADRATURES
C     SUBROUTINE.
C
      START=0.
      HSTAR=.01*XL
      HMIN=XL*1.E-4
      HMAX=.05*XL
      ERMAX=1.E-5
      KEY=0
C
C
C     HERE THE EXACT LITTLE A MATRIX AND B VECTOR ARE COMPUTED.
C
      CALL AMAT(A)
      CALL BVEC(B)
      CALL MOUT(A,NDA,NM,NM,20,20H0THE LITTLE A MATRIX)
      CALL VOUT(B,NM,20,20H0THE LITTLE B VECTOR)
C
```

```
C     HERE THE BIG A MATRIX AND B VECTOR ARE COMPUTED.
C
        DO 75 I=2,NM
        BIG(I-1)=B(I)-XZ(I)*B(1)/X1
        DO 75 J=2,NM
        AB(I-1,J-1)=Q(1)*XZ(I)*XZ(J)/(X1*X1)+A(I,J)-A(1,J)*XZ(I)/X1-A(I,1)
     1*XZ(J)/X1 +A(1,1)*XZ(I)*XZ(J)/(X1*X1)
 75     CONTINUE
        CALL MOUT(AB,NDA,M,M,17,17H0THE BIG A MATRIX)
        CALL VOUT(BIG,M,17,17H0THE BIG B VECTOR)
        CALL VOUT(Q,NM,28,28H0FOR THIS WEIGHTING VECTOR Q)
C
C     HERE THE EXACT WEIGHTED MATRIX A+Q IS COMPUTED.
C
        DO 80 I=1,M
        DO 80 J=1,M
        AQ(I,J)=AB(I,J)
 80     CONTINUE
        DO 85 I=1,M
        AQ(I,I)=AQ(I,I)+Q(I+1)
 85     CONTINUE
        CALL MOUT(AQ,NDA,M,M,24,24H0THE MATRIX BIG A PLUS Q)
C
C     NOW WE SOLVE FOR THE EXACT OPTIMAL FORCES F2 TO FM.
C
        CALL SOR(AQ,NDA,M,BIG,NDE,NB,$90,WORK)
        GO TO 95
 90     WRITE(6,91)
 91     FORMAT(////,10X,25HMATRIX IS NEARLY SINGULAR)
        GO TO 500
 95     CALL VOUT(BIG,M,20,20H0THE FORCES F2 TO FM)
C
C     WE COMPUTE THE ENTIRE VECTOR OF OPTIMAL FORCES.
C
        F(1)=0.
        DO 100 I=1,M
        F(1)=F(1)-BIG(I)*XZ(I+1)/X1
        F(I+1)=BIG(I)
 100    CONTINUE
        CALL VOUT(F,NM,29,29H0THE VECTOR OF OPTIMAL FORCES)
C
        IF(JOPT.EQ.1) GO TO 175
C
C     ******THE APPROXIMATIONS******
        DO 105 I=1,10

        Q(I)=(XL**7)*1.E-7
 105    CONTINUE
C
        V=3.927
        V2=7.069
C
C     V AND V2 SATISFY TAN V = TANH V.
C     THE FIRST EIGENVALUE IS (V/XL)**4.
C     THE SECOND EIGENVALUE IS (V2/XL) ** 4.
C
C     FIRST COMPUTE THE EIGENFUNCTION VALUES AT XZ(I).
```

```
C
      DO 110 I=1,NM
      ARG=V*XZ(I)/XL
      ARG2=V2*XZ(I)/XL

      PHI(I)=(-1.4142)*SIN(ARG)+(1.9695E-2)*(EXP(ARG)-EXP(-ARG))
      PHI2(I)=(1.4148)*SIN(ARG2)+(.8511E-3)*(EXP(ARG2)-EXP(-ARG2))
  110 CONTINUE
C
C     THE APPROXIMATE LITTLE A MATRIX.
C
      DO 120 I=1,NM
      DO 120 J=1,NM
      AA(I,J)=((XL**7)/(V**8))*PHI(I)*PHI(J)

      AA(I,J)=AA(I,J)+((XL**7)/(V2**8))*PHI2(I)*PHI2(J)
  120 CONTINUE
C
C     NOW THE APPROXIMATE LITTLE B VECTOR.
C
      DO 130 I=1,NM
      KEY=0
      CALL ROMBS(START,XL,T,FOFT,MSTAR,HMIN,HMAX,ERMAX,ANS,K,KEY)
   10 WANT=.75*XL*T-T*T
      ARG=V*T/XL
      P=(-1.4142)*SIN(ARG)+.019695*(EXP(ARG)-EXP(-ARG))
      FOFT=WANT*P
      CALL ROM2
      IF (K.EQ.1) GO TO 10
      BA(I)=(XL**3)*PHI(I)*ANS/(V**4)
  130 CONTINUE
      CALL VOUT(PHI,NM,15,15H0THE PHI VECTOR)
      CALL MOUT(AA,NDA,NM,NM,32,32H0THE APPROXIMATE LITTLE A MATRIX)
      CALL VOUT(BA,NM,32,32H0THE APPROXIMATE LITTLE B VECTOR)
C
C     HERE WE COMPUTE THE BIG APPROXIMATE A AND B.
C
      DO 140 I=2,NM
      BBA(I-1)=BA(I)-XZ(I)*BA(1)/X1
      DO 140 J=2,NM
      ABA(I-1,J-1)=Q(1)*XZ(I)*XZ(J)/(X1*X1)+AA(I,J)-AA(1,J)*XZ(I)/X1
     1 -AA(1,I)*XZ(J)/X1+AA(1,1)*XZ(I)*XZ(J)/(X1*X1)
  140 CONTINUE
      CALL MOUT(ABA,NDA,M,M,24,24H0THE BIG APPROX A MATRIX)
      CALL VOUT(BBA,M,24,24H0THE BIG APPROX B VECTOR)
C
C     HERE THE APPROXIMATE WEIGHTED MATRIX BIG A + Q IS COMPUTED.
C
      DO 150 I=1,M
      DO 150 J=1,M
      AQA(I,J)=ABA(I,J)
  150 CONTINUE
      DO 155 I=1,M
      AQA(I,I)=AQA(I,I)+Q(I+1)
  155 CONTINUE
      CALL MOUT(AQA,NDA,M,M,31,31H0THE APPROX MATRIX BIG A PLUS Q)
      CALL SOR(AQA,NDA,M,BBA,NDB,NB,$90,WORK)
```

```
  160 CALL VOUT(BBA,M,32,32HOTHE APPROXIMATE FORCES F2 TO FM)
      FA(1)=0.
      DO 170 I=1,M
      FA(1)=FA(1)-BBA(I)*XZ(I+1)/X1
      FA(I+1)=BBA(I)
  170 CONTINUE
      CALL VOUT(FA,NM,31,31HOTHE APPROXIMATE FORCE VECTOR F)
      GO TO 180
  175 WRITE(6,178)
      GO TO 185
  178 FORMAT(////,3X,8HPOSITION,4X,13HDESIRED SHAPE,2X,13HOPTIMAL SHAPE)
  179 FORMAT(////,3X,8HPOSITION,4X,13HDESIRED SHAPE,2X,13HOPTIMAL SHAPE,
     1 2X,13HAPPROX. SHAPE)
  180 WRITE(6,179)
C
C     HERE WE COMPUTE THE SHAPES.
C
  185 DO 210 I=1,NP
      X(I)=(I-1)*DEL
      T=X(I)
      PSI(I)=.75*T*XL-T*T
      U(I)=0.
      UA(I)=0.
      DO 205 J=1,NM
      Z=XZ(J)
      H=T*Z*(33.*XL/140.+(Z*Z+T*T)/(4.*XL)-(Z**4+T**4)/(40.*XL**3) )
      IF(T.GT.Z) GO TO 195
      G=H-(Z*Z*T*.5+(T**3)/6.)
      GO TO 200
  195 G=H-(T*T*Z*.5+(Z**3)/6.)
  200 U(I)=U(I)+G*F(J)
      IF(JOPT.EQ.1) GO TO 205
      UA(I)=UA(I)+G*FA(J)
  205 CONTINUE
      IF (JOPT.EQ.1) GO TO 208
      WRITE(6,206) T,PSI(I),U(I),UA(I)
  206 FORMAT(/,1X,F10.2,6E15.5)
      GO TO 210
  208 WRITE(6,206) T,PSI(I),U(I)
  210 CONTINUE
C
C     Y IS FOR SCALING PURPOSES.
C
      NP2=2*NP
      DO 215 I=1,NP
      Y(I)=PSI(I)
      Y(I+NP)=U(I)
  215 CONTINUE
C
C    HERE WE GENERATE THE PLOTS.
C
      GO TO (500,300,350,250,300,250), KOPT
  250 CALL BGNPLT
      CALL PLFORM('LINLIN',XLEN,YLEN)
      CALL PLSCAL(X,NP,NG,Y,NP2,NG)
      CALL PLABEL('THE FLEXIBLE BEAM EXPERIMENT',28,'LENGTH',6,'DISPLACE
     1MENT',12)
```

```
      CALL PLGRAF
      CALL PLAXIS(-2,XLEN,0. )
      CALL PLCURV(X,PSI,NP,NT,TIC1)
      CALL PLCURV(X,U,NP,NT,TIC2)
      CALL PLCURV(X,UA,NP,NT,TIC3)
      CALL PLTEXT(1.5,.50,.10,0.,49HDESIRED(*) VS OPTIMAL(O) VS APPROXIM
     1ATE(/) SHAPES,49,1)
      GO TO (271,272,273,274,275), M
  271 CALL PLTEXT(3.3,7.0,.1,0.,13HTWO ACTUATORS,13,1)
      GO TO 280
  272 CALL PLTEXT(3.2,7.0,.1,0.,15HTHREE ACTUATORS,15,1)
      GO TO 280
  273 CALL PLTEXT(3.3,7.0,.1,0.,14HFOUR ACTUATORS,14,1)
      GO TO 280
  274 CALL PLTEXT(3.3,7.0,.1,0.,14HFIVE ACTUATORS,14,1)
      GO TO 280
  275 CALL PLTEXT(3.3,7.0,.1,0.,13HSIX ACTUATORS,13,1)
  280 IF(KOPT.EQ.4) GO TO 490
      CALL ADVPLT
  300 CALL BGNPLT
      CALL PLFORM('LINLIN',XLEN,YLEN)
      CALL PLSCAL(X,NP,NG,Y,NP2,NG)
      CALL PLABEL('THE FLEXIBLE BEAM EXPERIMENT',28,'LENGTH',6,'DISPLACE
     1MENT',12)
      CALL PLGRAF
      CALL PLAXIS(-2,XLEN,0. )
      CALL PLCURV(X,U,NP,NT,TIC2)
      CALL PLCURV(X,PSI,NP,NT,TIC1)
      CALL PLTEXT(2.5,.50,.10,0.,31HDESIRED(*) VS OPTIMAL(O) SHAPES,31,
     1 1)
      GO TO (321,322,323,324,325),M
  321 CALL PLTEXT(3.3,7.0,.1,0.,13HTWO ACTUATORS,13,1)
      GO TO 330
  322 CALL PLTEXT(3.2,7.0,.1,0.,15HTHREE ACTUATORS,15,1)
      GO TO 330
  323 CALL PLTEXT(3.3,7.0,.1,0.,14HFOUR ACTUATORS,14,1)
      GO TO 330
  324 CALL PLTEXT(3.3,7.0,.1,0.,14HFIVE ACTUATORS,14,1)
      GO TO 330
  325 CALL PLTEXT(3.3,7.0,.1,0.,13HSIX ACTUATORS,13,1)
  330 IF(KOPT.EQ.2) GO TO 490
      CALL ADVPLT
  350 CALL BGNPLT
      CALL PLFORM('LINLIN',XLEN,YLEN)
      CALL PLSCAL(X,NP,NG,Y,NP2,NG)
      CALL PLABEL('THE FLEXIBLE BEAM EXPERIMENT',28,'LENGTH',6,'DISPLACE
     1MENT',12)
      CALL PLGRAF
      CALL PLAXIS(-2,XLEN,0. )
      CALL PLCURV(X,UA,NP,NT,TIC3)
      CALL PLCURV(X,PSI,NP,NT,TIC1)
      CALL PLTEXT(2.5,.50,.10,0.,35HDESIRED(*) VS APPROXIMATE(/) SHAPES,
     1 35,1)
      GO TO (371,372,373,374,375),M
  371 CALL PLTEXT(3.3,7.0,.1,0.,13HTWO ACTUATORS,13,1)
      GO TO 490
  372 CALL PLTEXT(3.2,7.0,.1,0.,15HTHREE ACTUATORS,15,1)
```

```
      GO TO 490
  373 CALL PLTEXT(3.3,7.0,.1,0.,14HFOUR ACTUATORS,14,1)
      GO TO 490
  374 CALL PLTEXT(3.3,7.0,.1,0.,14HFIVE ACTUATORS,14,1)
      GO TO 490
  375 CALL PLTEXT(3.3,7.0,.1,0.,13HSIX ACTUATORS,13,1)
  490 CALL ENDPLT
  500 STOP
      END

      SUBROUTINE AMAT(A)
      COMMON START,HSTAR,HMIN,HMAX,ERMAX,XL,KEY,X2(10),NM
      REAL A(10,10)
      DO 50 J=1,NM
      DO 50 I=J,NM
      XI=X2(I)
      XJ=X2(J)
      CALL ROMBS(START,XL,X,FOFX,HSTAR,HMIN,HMAX,ERMAX,ANS,K,KEY)
   10 HI=X*XI*(33.*XL/140.+(XI*XI+X*X)/(4.*XL)-(XI**4+X**4)/(40.*XL**3))
      HJ=X*XJ*(33.*XL/140.+(XJ*XJ+X*X)/(4.*XL)-(XJ**4+X**4)/(40.*XL**3))
      IF(X.GT.XI) GO TO 15
      GI=HI-(XI*XI*X/2.+(X**3)/6.)
      GO TO 16
   15 GI=HI-(XI*X*X/2.+(XI**3)/6.)
   16 IF(X.GT.XJ) GO TO 20
      GJ=HJ-XJ*XJ*X/2.-(X**3)/6.
      GO TO 21
   20 GJ=HJ-X*X*XJ/2.-(XJ**3)/6.
   21 FOFX=GI*GJ
      CALL ROM2
      IF(K.EQ.1) GO TO 10
      A(J,I)=ANS
   50 CONTINUE
      DO 60 I=2,NM
      II=I-1
      DO 60 J=1,II
      A(I,J)=A(J,I)

   60 CONTINUE
      RETURN
      END

      SUBROUTINE BVEC(B)
      COMMON START,HSTAR,HMIN,HMAX,ERMAX,XL,KEY,X2(10),NM
      REAL B(10)
      DO 50 I=1,NM
      Z=X2(I)
      CALL ROMBS(START,XL,X,FOFX,HSTAR,HMIN,HMAX,ERMAX,ANS,K,KEY)
   10 G=X*Z*(33.*XL/140.+(Z*Z+X*X)/(4.*XL)-(Z**4+X**4)/(40.*XL**3))
      IF(X.GT.Z) GO TO 15
      G=G-(Z*Z*X*.5+(X**3)/6.)
      GO TO 20
   15 G=G-(X*X*Z*.5+(Z**3)/6.)
   20 PSI=.75*X*XL-X*X
      FOFX=PSI*G
      CALL ROM2
      IF(K.EQ.1) GO TO 10
      B(I)=ANS
   50 CONTINUE
      RETURN
      END
```

Output

```
THE LENGTH OF THE BEAM IS     100.00

THE VECTOR OF ACTUATOR POSITIONS
  1 TO   2    5.0000000+01   1.0000000+02

THE LITTLE A MATRIX

                   COL    1          COL    2
ROW    1         2.4259911+09   -4.1531314+09
ROW    2        -4.1531314+09    7.1462926+09

THE LITTLE B VECTOR
  1 TO   2    5.4873537+08  -9.4246017+08

THE BIG A MATRIX

                   COL    1
ROW    1         3.3462782+10

THE BIG B VECTOR
  1 TO   1   -2.0399309+09

FOR THIS WEIGHTING VECTOR Q
  1 TO   2    0.0000000        0.0000000

THE MATRIX BIG A PLUS Q

                   COL    1
ROW    1         3.3462782+10

THE FORCES F2 TO FM
  1 TO   1   -6.0961186-02

THE VECTOR OF OPTIMAL FORCES
  1 TO   2    1.2192237-01  -6.0961186-02

THE PHI VECTOR
  1 TO   2   -1.1690310+00   1.9992210+00

THE APPROXIMATE LITTLE A MATRIX

                   COL    1          COL    2
ROW    1         2.4204669+09   -4.1487150+09
ROW    2        -4.1487150+09    7.1311938+09

THE APPROXIMATE LITTLE B VECTOR

  1 TO   2    5.4816966+08  -9.3747769+08

THE BIG APPROX A MATRIX

                   COL    1
ROW    1         3.3447921+10

THE BIG APPROX B VECTOR
  1 TO   1   -2.0338170+09

THE APPROX MATRIX BIG A PLUS Q

                   COL    1
ROW    1         3.3457921+10

THE APPROXIMATE FORCES F2 TO FM
  1 TO   1   -6.0787339-02
```

THE APPROXIMATE FORCE VECTOR F
1 TO 2 1.2157462-01 -6.0787309-02

POSITION	DESIRED SHAPE	OPTIMAL SHAPE	APPROX. SHAPE
.00	.00000	.00000	.00000
5.00	.35000+03	.2[illegible]067+03	.25993+03
10.00	.65000+03	.51672+03	.51226+03
15.00	.90000+03	.75154+03	.74939+03
20.00	.11000+04	.96649+03	.96373+03
25.00	.12500+04	.11510+04	.11477+04
30.00	.13500+04	.12973+04	.12936+04
35.00	.14000+04	.13980+04	.1394[illegible]+04
40.00	.14000+04	.14453+04	.14412+04
45.00	.13500+04	.14316+04	.14276+04
50.00	.12500+04	.13494+04	.13456+04
55.00	.11000+04	.11935+04	.11901+04
60.00	.90000+03	.96903+03	.96626+03
65.00	.65000+03	.66359+03	.66164+03
70.00	.35000+03	.34451+03	.34383+03
75.00	.00000	-.39668+02	-.39575+02
80.00	-.40000+03	-.46229+03	-.46097+03
85.00	-.85000+03	-.91537+03	-.91276+03
90.00	-.13500+04	-.13913+04	-.13873+04
95.00	-.19000+04	-.18825+04	-.18771+04
100.00	-.25000+04	-.23813+04	-.23745+04

B.3 The Simply Supported Beam Estimation Program Listing

```
      REAL Q(10),B(10),A(10,10),X2(10),AG(10)
      REAL PSI(50),AG(10,10),WORK(100),U(50),Z(50),S(10,50)
      REAL SCL(500)
      DATA B(1),B(2),B(3) /.1875, .25, .1875 /
C
C     ******INTRODUCTION******
C
C     THIS PROGRAM PERFORMS SHAPE ESTIMATION FOR THE
C     SIMPLY SUPPORTED BEAM.  UPON INPUT OF NM DISCRETE
C     OBSERVATIONS B(I) OF DISPLACEMENT AT POSITIONS X2(I),
C     AN OPTIMAL ESTIMATE OF THE SHAPE OF THE BEAM
C     WITH RESPECT TO THE CRITERION
C
C     (1/2)*(SUM((B(I)**2)*(1/Q(I)))+INTEGRAL((U-F)**2))
C
C     WHERE U(X) IS THE SHAPE AND F(X) IS THE NOISE IN THE
C     DYNAMIC MODEL   (DEL**4)U = F(X).
C
C     PLEASE DEFINE THE FOLLOWING VARIABLES.
C
C     XL IS THE LENGTH OF THE BEAM.
C     NM IS THE NUMBER OF OBSERVATIONS.
C     X2(I), I=1,....,NM ARE THE POSITIONS ALONG THE BEAM AT WHICH
C     OBSERVATIONS ARE TAKEN.
C     Q(I) ARE THE **INVERSES** OF THE WEIGHTS ON THE OBSERVATIONS IN
C     THE PERFORMANCE CRITERION.C
C     NP IS THE NUMBER OF POINTS ON EACH CURVE TO BE PLOTTED.
C
      PI=3.14159
      XL=1.
      NM=3
      NUA=10
      NUB=1
      NB=1
      NP=20
      DEL=XL/NP
      NP=NP+1
      NP2=2*NP
      DO 1 I=1,NM
      Q(I)=(XL**7)*1.E-7
      X2(I)=.25*I
    1 CONTINUE
      WRITE(6,299)
  299 FORMAT(1H1)
      WRITE(6,2)
    2 FORMAT(/////)
      WRITE(6,3)XL
    3 FORMAT(1X,25HTHE LENGTH OF THE BEAM IS,F10.2)
      CALL VOUT(X2,NM,26,26HOTHE OBSERVATION POSITIONS)
      CALL VOUT(B,NM,17,17HOTHE OBSERVATIONS)
C
C     COMPUTE THE MATRIX A
C
      DO 5 I=1,NM
      X=X2(I)
      C1=X*(X-2.*XL)
      DO 5 J=1,NM
      Y=X2(J)
      C2=Y*Y-2.*XL*Y
      C3=X*X+Y*Y
      C4=X*X*Y*Y
      A(I,J)=((X-XL)*(Y-XL)/(36.*XL*XL))* ((X**7)/7.+(X**5)*(C1+C2)/5.+
```

```
     1 (X**3)*C1*C2/3.)+(X*Y/(36.*XL*XL))*((XL**7-Y**7)/7.+XL*Y**6-XL**7
     2 +.2*(XL**5-Y**5)*(13.*XL*XL+C3)-(XL**4-Y**4)*(3.*XL**3+XL*C3)
     3 +(XL**3-Y**3)*(1./3.)*(5.*XL*XL*C3+4.*XL**4+C4) -(XL*XL-Y*Y)*(XL*
     4 C4+(XL**3)*C3)+(XL-Y)*(XL*XL*C4))  +  (X/(36.*XL**2))*(Y-XL)*((Y
     5 **7-X**7)/7.-XL*.5*(Y**6-X**6)+.2*(Y**5-X**5)*(C2+2.*XL*XL+X*X)-
     6 .25*(Y**4-X**4)*(3.*XL*C2+XL*X*X)+(Y**3-X**3)*C2*(X*X+2.*XL*XL)/
     73.-(Y*Y-X*X)*.5*(XL*X*X*C2))
    5  CONTINUE
       DO 6 I=2,NM
       II=I-1
       DO 6 J=1,II
       A(I,J)=A(J,I)
    6  CONTINUE
       CALL MOUT(A,NDA,NM,NM,13,13HOTHE MATRIX A)
C
C     HERE WE COMPUTE A*B.
C
       DO 20 I=1,NM
       AB(I)=0.
       DO 20 J=1,NM
       AB(I)=AB(I)+ A(I,J)*B(J)
   20  CONTINUE
       CALL VOUT(AB,NM,15,15HOTHE VECTOR A*B)
C
C     HERE COMPUTE A+Q.
C
       DO 25 I=1,NM
       DO 25 J=1,NM
       AQ(I,J)=A(I,J)
   25  CONTINUE
       DO 30 I=1,NM
       AQ(I,I)=AQ(I,I) + Q(I)
   30  CONTINUE
       CALL MOUT(AQ,NDA,NM,NM,15,15HOTHE MATRIX A+Q)
C
C     HERE WE SOLVE FOR THE OPTIMAL SHAPE AT POSITIONS XZ.
C
       CALL SGR(AQ,NDA,NM,AB,NDB,NB,$35,WORK)
       CALL VOUT(AB,NM,24,24HOOPTIMAL SHAPE POSITIONS)
       GO TO 40
   35  WRITE(6,36)
   36  FORMAT(1H0,1X,36H****** MATRIX NEARLY SINGULAR ******)
       GO TO 500
C
C     NOW WE COMPUTE THE OPTIMAL SHAPE.
C
   40  DO 45 I=1,NP
       Z(I)=(I-1)*DEL
   45  CONTINUE
       DO 50 I=1,NM
       DO 50 J=2,NP
       IF(XZ(I).GT.Z(J)) GO TO 51
       X=XZ(I)
       Y=Z(J)
       GO TO 52
   51  X=Z(J)
       Y=XZ(I)
```

```
 52       C2=Y*Y-2.*XL*Y
          C1=X*(X-2.*XL)
          C3=X*X+Y*Y
          C4=X*X*Y*Y
          S(I,J)=((X-XL)*(Y-XL)/(36.*XL*XL))* ((X**7)/7.+(X**5)*(C1+C2)/5.+
         1 (X**3)*C1*C2/3.)+(X*Y/(36.*XL*XL))*((XL**7-Y**7)/7.*XL*Y**6-XL**7
         2 +.2*(XL**5-Y**5)*(13.*XL*XL+C3)-(XL**4-Y**4)*(3.*XL**3+XL*C3)
         3 +(XL**3-Y**3)*(1./3.)*(5.*XL*XL*C3+4.*XL**4+C4) -(XL*XL-Y*Y)*(XL*
         4 C4+(XL**3)*C3)+(XL-Y)*(XL*XL*C4))  +  (X/(36.*XL**2))*(Y-XL)*((Y
         5 **7-X**7)/7.-XL*.5*(Y**6-X**6)+.2*(Y**5-X**5)*(C2+2.*XL*XL+X*X)-
         6 .25*(Y**4-X**4)*(3.*XL*C2+XL*X*X)+(Y**3-X**3)*C2*(X*X+2.*XL*XL)/
         73.-(Y*Y-X*X)*.5*(XL*X*X*C2))
 50       CONTINUE
          WRITE(6,56)
 56       FORMAT(///,1X,12HTHE MATRIX S)
          WRITE(6,58)(XZ(I), I=1,NM)
 57       FORMAT(/,1X,F10.2,10E15.5)
 58       FORMAT(//,11X,10E15.5)
          U(1)=0.
          DO 53 I=2,NP
          WRITE(6,57) Z(I),(S(J,I), J=1,NM)
          U(I)=0.
          DO 60 J=1,NM
          U(I)=U(I)+(B(J)-AB(J))*S(J,I)/Q(J)
 60       CONTINUE
          WRITE(6,2)
          WRITE(6,66)
          DO 65 I=1,NP
          PSI(I)=XL*Z(I)-Z(I)**2
          SCL(I)=U(I)*1.1
          SCL(I+NP)=PSI(I)*1.1
          WRITE(6,67) Z(I),PSI(I),U(I)
 65       CONTINUE
 67       FORMAT(/,F10.2,4E15.5)
 66       FORMAT(2X,8HPOSITION,3X,12HACTUAL SHAPE,3X,12HESTIM. SHAPE)
          XLEN=8.
          YLEN=6.
          NG=1
          TIC1='*'
          TIC2='E'
          TIC3='+'
          NT1=2
          NT2=0
          NT3=-1
          CALL BGNPLT
          CALL PLFORM('LINLIN',XLEN,YLEN)
          CALL PLSCAL(Z,NP,NG,SCL,NP2,NG)
          CALL PLABEL('SHAPE ESTIMATION FOR THE SIMPLY SUPPORTED BEAM',
         1 46,'LENGTH OF THE BEAM',18,'DISPLACEMENT',12)
          CALL PLGRAF
          CALL PLAXIS(-2,XLEN,0.)
          CALL PLCURV(XZ,AB,NM,NT3,TIC1)
          CALL PLCURV(Z,U,NP,NT1,TIC2)
          CALL PLCURV(Z,PSI,NP,NT2,TIC3)
          CALL PLTEXT(1.4,.5,.1,.0,53HESTIMATED SHAPE(E) VS OBSERVATIONS(*)
         1ON ACTUAL SHAPE,53,1)
          GO TO (70,71,72,73,74,75), NM
```

```
70    CALL PLTEXT(3.4,7.0,.1,.0,12HONE ACTUATOR,12,1)
      GO TO 80
71    CALL PLTEXT(3.3,7.0,.1,0.,13HTWO ACTUATORS,13,1)
      GO TO 80
72    CALL PLTEXT(3.2,7.0,.1,0.,15HTHREE ACTUATORS,15,1)
      GO TO 80
73    CALL PLTEXT(3.3,7.0,.1,0.,14HFOUR ACTUATORS,14,1)
      GO TO 80
74    CALL PLTEXT(3.3,7.0,.1,0.,14HFIVE ACTUATORS,14,1)
      GO TO 80
75    CALL PLTEXT(3.3,7.0,.1,0.,13HSIX ACTUATORS,13,1)
80    CALL ENDPLT
500   STOP
      END
```

```
THE LENGTH OF THE BEAM IS       1.00

THE OBSERVATION POSITIONS
  1 TO   3    2.5000000-01   5.0000000-01   7.5000000-01

THE OBSERVATIONS
  1 TO   3    1.8750000-01   2.5000000-01   1.8750000-01

THE MATRIX A

                 COL   1        COL   2        COL   3
ROW   1      1.0623015-04   1.4902122-04   1.0456313-04
ROW   2      1.4902122-04   2.1061301-04   1.4902125-04
ROW   3      1.0458313-04   1.4902125-04   1.0622984-04

THE VECTOR A*B
  1 TO   3    7.6782793-05   1.0858621-04   7.6782742-05

THE MATRIX A+Q

                 COL   1        COL   2        COL   3
ROW   1      1.0633015-04   1.4902122-04   1.0458313-04
ROW   2      1.4902122-04   2.1091301-04   1.4902125-04
ROW   3      1.0456313-04   1.4902125-04   1.0632984-04

OPTIMAL SHAPE POSITIONS
  1 TO   3    1.8421271-01   2.5452762-01   1.8421149-01
```

Output

THE MATRIX S

	.25000+00	.50000+00	.75000+00
.05	.23560-04	.32959-04	.23071-04
.10	.46559-04	.65109-04	.45591-04
.15	.66353-04	.95661-04	.670[illegible]1-04
.20	.86411-04	.12386-03	.86845-04
.25	.10623-03	.14902-03	.10458-03
.30	.12137-03	.17051-03	.11980-03
.35	.13346-03	.18761-03	.13213-03
.40	.14222-03	.20048-03	.14125-03
.45	.14744-03	.20821-03	.14693-03
.50	.14902-03	.21081-03	.14902-03
.55	.14693-03	.20821-03	.14744-03
.60	.14125-03	.20046-03	.14222-03
.65	.13213-03	.18781-03	.13346-03
.70	.11980-03	.17051-03	.12137-03
.75	.10458-03	.14902-03	.10623-03
.80	.86845-04	.12386-03	.88411-04
.85	.67021-04	.95663-04	.68351-04
.90	.45591-04	.65109-04	.46558-04
.95	.23071-04	.32959-04	.23580-04
1.00	.26560-09	.58935-09	.10315-06

POSITION	ACTUAL SHAPE	ESTIM. SHAPE
.00	.00000	.00000
.05	.47500-01	.41587-01
.10	.90000-01	.81910-01
.15	.12750+00	.11979+00
.20	.16000+00	.15416+00
.25	.18750+00	.18421+00
.30	.21000+00	.20929+00
.35	.22750+00	.22902+00
.40	.24000+00	.24319+00
.45	.24750+00	.25169+00
.50	.25000+00	.25453+00
.55	.24750+00	.25169+00
.60	.24000+00	.24320+00
.65	.22750+00	.22903+00
.70	.21000+00	.20932+00
.75	.18750+00	.18423+00
.80	.16000+00	.15418+00
.85	.12750+00	.11978+00
.90	.90000-01	.81901-01
.95	.47500-01	.41590-01
1.00	.74506-08	.15969-04

Appendix C. The Large Space Antenna Computer Program and Output

ORIGINAL PAGE IS
OF POOR QUALITY

C.1 The Large Space Antenna Computer Program Listing

```
          DIMENSION MOH(20),FREQ(50)
          DIMENSION X(882),Y(882),Z(882),SLUGS(882)
          DIMENSION VECTOR(2646),U(882),V(882),W(882)
          DIMENSION OCHK(18),D(18),F(18),A(18,18),AA(18,18)
          DIMENSION PHI(11,18)
          DIMENSION BETA(11)
          DIMENSION ALPHA(11), COEF(11), WORK(100)
          DIMENSION YSTAR(18),AY(18),USTAR(18)
          INTEGER IPT(18),JPT(18)
          INTEGER JSEQ(2428), ISEQ(882)
          EQUIVALENCE (U(1),VECTOR(1)),(VECTOR(883),V(1))
          EQUIVALENCE (W(1),VECTOR(1765))
    C
    C
    C     ****************************************************************
    C
    C
    C     STATIC SHAPE ESTIMATION AND CONTROL OF A LARGE SPACE ANTENNA.
    C
    C
    C     ****************************************************************
    C
    C
    C     THIS PROGRAM ESTIMATES AND CONTROLS THE STATIC DISTORTION OF A
    C     LARGE SPACE ANTENNA, USING REST COORDINATES, MODES AND FREQUENCIES
    C     SUPPLIED BY A FINITE ELEMENT MODEL.
    C
    C     THE MODEL INCLUDES 18 RIBS, THE X, Y, AND Z COORDINATES FOR 882
    C     POINTS, OR NODES, LOCATED ON 14 CONSECUTIVE CIRCLES.
    C
    C     IT IS ASSUMED THAT THE HUB OF THE ANTENNA IS RIGIDLY ATTACHED,
    C     SO THAT THERE ARE NO RIGID BODY MODES.
    C     THERE ARE 33 FREQUENCIES AND CORRESPONDING EIGENFUNCTIONS
    C     (MODES)  FOR THIS MODEL .
    C
    C     THIS PROGRAM READS FROM THE TEMPORARY FILE FIXFDANT, WHICH IS
    C     CREATED FROM THE TAPE A1960 BY RUNNING A PRELIMINARY PROGRAM
    C     CREATED BY VEJAYARAGHAVAN (VEJAY) ALWAR.(REF. JPL EM 347-80-434,
    C     AUGUST 14, 1980, BY VEJAY ALWAR.
    C
    C     THE ANALYSIS ON WHICH THIS PROGRAM IS BASED IS FOUND IN
    C     JPL EM 347-112, APRIL 6, 1981, 'ADAPTING STATIC SHAPE CONTROL/
    C     DETERMINATION ALGORITHMS FOR THE USE OF MODES SUPPLIED BY A
    C     FINITE ELEMENT MODEL' .  BY CONNIE WEEKS.
    C
    C
    C     THE FOLLOWING VARIABLES MUST BE DEFINED...
    C
    C
    C     THE VECTOR ALPHA(I), I=1,MM, ARE THE COEFFICIENTS OF THE MODES
    C     IN THE ACTUAL DISTORTED SHAPE, WHICH IS TO BE ESTIMATED AND
    C     CORRECTED.
    C
    C
    C     THE OBSERVATIONS AND CONTROL POSITIONS ARE ASSUMED TO BE COLOCATED
    C     AND APPLIED TO THE SAME DIRECTIONS AT EACH POINT.
```

```
C      THUS IF A FORCE IS APPLIED IN THE X DIRECTION AT THE NODE 99,
C      IT IS ASSUMED THERE IS AN OBSERVATION OF THE   DIRECTION AT NODE
C      99, AND CONVERSELY.
C
C
C     NM IS THE NUMBER OF MODES IN OUR APPROXIMATIONS.
C
C     NPT IS THE NUMBER OF FORCES TO BE APPLIED.
C
C     IPT(I),I=1,....,NPT IS THE NUMBER OF THE NODAL POINT
C     TO WHICH THE ITH FORCE IS APPLIED.
C
C     JPT(I),I=1,....,NPT INDICATES THE DIRECTION OF THE ITH FORCE.
C     JPT(I) IS 1,2 OR 3.  JPT(I)=1 MEANS THE FORCE IS IN THE X DIRECTION
C     JPT(I)=2,3 MEANS THE FORCE IS IN THE Y DIRECTION OR Z DIRECTION.
C
C     EXAMPLE.....TO APPLY 4 FORCES, ONE EACH IN THE X AND Y DIRECTIONS
C     AT NODE 14, ONE IN THE Z DIRECTION AT NODE 10, AND ONE IN THE Y
C     DIRECTION AT NODE 39,  NPT=4, IPT=(14,14,10,39) AND JPT=(1,2,3,2).
C
C
C      ALL ELEMENTS OF THE DIAGNAL WEIGHTING MATRIX R ARE CHOSEN TO BE
C      THE SAME VALUE RR.  ON INPUT...
C      RR IS THE INITIAL VALUE OF THE DIAGNAL ELEMENTS OF THE WEIGHTING
C      MATRIX R.
C      IN THIS PROGRAM R IS THE INCREMENT BY WHICH RR INCREASES.
C      THE CRITERION BY WHICH RR IS DETERMINED ... THE SMALLEST
C      VALUE FOR WHICH THE MATRIX  A+R  IS INVERTIBLE.
C
C
C      SET IOPT EQUAL TO 1 IF NO PLOTS ARE DESIRED.
C
C
C      *****************************************************************
C
C
C      THESE VARIABLES ARE NOT INPUT, THEY HAVE BEEN DEFINED OR WILL
C      BE DEFINED IN THE PROGRAM. ..........
C
C      EXACT OBSERVATIONS OF THE DISTORTED SHAPE WILL BE COMPUTED
C      AND STORED IN THE VECTOR YSTAR.
C      YSTAR IS THE VECTOR OF OBSERVATIONS AT THE POINTS IPT(I), I=1,NPT.
C
C      USTAR WILL CONTAIN THE OPTIMAL SHAPE ESTIMATES AT THE POINTS
C      IPT(I).
C
C      COEF IS THE VECTOR OF ESTIMATED COEFFICIENTS OF THE MODES
C      IN THE DISTORTED SHAPE.
C
C      F IS THE VECTOR OF OPTIMAL  CONTROL FORCES.
C      F(I) IS APPLIED TO NODE IPT(I) IN THE DIRECTION  JPT(I).
C
C      BETA IS THE VECTOR OF MODAL COEFFICIENTS RESULTING FROM
C      THE APPLICATION OF THE OPTIMAL CONTROL FORCES, WHICH WILL BE
C      COMPUTED BASED ON THE OPTIMA SHAPE ESTIMATE STORED IN THE VECTOR
C      COEF.
C
```

```
C
C     ISEQ IS THE NUMBERING OF THE NODES.
C     NC IS THE NUMBER OF PLOT COMMANDS.
C     JSEQ IS THE SEQUENCE OF PLOT COMMANDS.
C
C
C      THE LAST CIRCLE HAS BEEN DELETED FROM THE PLOTS
C      BECAUSE OF BAD DATA POINTS ON THAT CIRCLE IN SOME MODES.
C
C      IF IT IS DESIRED THAT THE LAST CIRCLE BE PLOTTED, SET NC=2428
C      IN THE DATA STATEMENT,  IN THE SUBROUTINE DRAW,  AND REMOVE
C      THE DO LOOP INVOLVING 25 CONTINUE.
C
C    HDR IS THE HEADING ON THE FILE.
C    NODES IS THE NUMBER OF NODES=882.
C    NFREQ IS THE NUMBER OF FREQUENCIES=50.
C
C
C      ****************************************************************
C
C
C      DATA CARDS...
C
       DATA ALPHA/10.,2*0.,10.,3*0.,5.,0.,5.,0./
       DATA NC,NM,NPT/2140,11,18/
C
C      FIFTH CIRCLE
C
       DATA IPT/127,130,133,136,139,142,145,148,151,154,157,160,163,166,
     1 169,172,175,178/
       DATA JPT/18*1/
       ISTG=0
       IOPT=0
       RR=1.0E-10
       R=9.*R
C
C      READ PLOT COMMAND SEQUENCE
C
       READ(5,21)(JSEQ(I),I=1,NC)
 21    FORMAT(20I4)
C
C      HERE WE DELETE THE LAST CIRCLE FROM THE PLOT COMMANDS.
C
       DO 25 KK=1,NC
       JS=IABS(JSEQ(KK))
       IF(JS.LT.775) GO TO 25
       JSEQ(KK)=-JS
 25    CONTINUE
C
C     DEFINE NODE SEQUENCE ISEQ(I)...
C
       DO 29 I=1,882
       ISEQ(I)=I
 29    CONTINUE
C
       IF(IOPT.EQ.1) GO TO 930
       CALL PLOTS
```

```
      CALL PLOT(6.0,4.0,-3)
      CALL FACTOR(.0068)
  930 CONTINUE
C
C
      DO 9 IN=1,NM
      COEF(IN)=0.
      BETA(IN)=0.
    9 CONTINUE
      DO 99 JN=1,NPT
      DCHK(JN)=0.
      YSTAR(JN)=0.
      AY(JN)=0.
      USTAR(JN)=0.
   99 CONTINUE
C
      REWIND 45
      READ(45)(HDR(K),K=1,20)
      READ(45)NODES,NFREQ
      NCHECK=NODES*3
C
C   X(I),Y(I),Z(I) ARE THE COORDINATES OF NODE I.
C
      READ(45)(X(I),Y(I),Z(I),I=1,NODES)
C
C   SLUGS(I) IS THE MASS AT NODE I.
C
      READ(45)(SLUGS(I),I=1,NODES)
      READ(45)(FREQ(I),I=1,NFREQ)
      WRITE(6,10)(HDR(K),K=1,20)
   10 FORMAT(1H1,//,15X,20A4,5X,'FREQUENCIES',//)
      WRITE(6,12)(FREQ(K),K=1,NFREQ)
   12 FORMAT(5X,6E15.8,/)
      WRITE(6,14)NODES
   14 FORMAT(//,40X,'NO. OF NODES = ',I4)
      WRITE (6,15)
      WRITE (6,16)
   15 FORMAT(//,2X,'POSITIONS AND DIRECTIONS OF CONTROL/OBSERVATION POIN
     1TS')
   16 FORMAT(//,2X,'NODE',2X,'DIRECTION')
      DO 20 I=1,NPT
      JS=JPT(I)
      IF(JS-2) 17,18,19
   17 WRITE(6,22) IPT(I)
      GO TO 20
   18 WRITE(6,23) IPT(I)
      GO TO 20
   19 WRITE(6,24) IPT(I)
   20 CONTINUE
   22 FORMAT(/,2X,I4,6X,'X')
   23 FORMAT(/,2X,I4,6X,'Y')
   24 FORMAT(/,2X,I4,6X,'Z')
C
      DO 100 KF=1,NM
C
C
C   NTRNS IS A CHECK TO SEE THAT THE TAPE IS BEING READ PROPERLY.
```

```
C     NTRNS = 3(882)=2646.
C     KFR IS THE MODE (EIGENVECTOR) NUMBER.
C     FR IS THE FREQUENCY.
C
C
         READ(45)KFR,FR,NTRNS,(VECTOR(K),K=1,NTRNS)
         IF(NCHECK.NE.NTRNS)GO TO 125
C
C        PHI(I,J), I=1,NM AND J=1,NPT HOLDS THE VALUE OF MODE I
C        AT NODE IPT(J) IN THE DIRECTION JPT(J).
C
         DO 35 I=1,NPT
         J=IPT(I)
         JS=JPT(I)
         IF (JS-2) 30,31,32
30       PHI(KF,I)=U(J)
         GO TO 33
31       PHI(KF,I)=V(J)
         GO TO 33
32       PHI(KF,I)=W(J)
33       CONTINUE
35       CONTINUE
         DO 50 I=1,NODES
C
C       HERE WE COMPUTE THE KNOWN DISTORTED SHAPE.
C
         X(I)=X(I)+ALPHA(KF)*U(I)
         Y(I)=Y(I)+ALPHA(KF)*V(I)
         Z(I)=Z(I)+ALPHA(KF)*W(I)
50       CONTINUE
100      CONTINUE
         IF(IOPT.EQ.1) GO TO 105
C
C       HERE WE PLOT THE KNOWN DISTORTED SHAPE.
C
C
C        DRAW IS A SUBROUTINE CREATED BY G. RODRIGUEZ TO PLOT THREE
C        DIMENSIONAL SURFACES.    IT CALLS THE SUBROUTINE TRANS.
C
         CALL DRAW(X,Y,Z,JSEQ,ISEQ)
         CALL FACTOR(1.0)
         CALL PLOT(10.0,0.0,-3)
         CALL FACTOR(.0068)
105      CONTINUE
         CALL MOUT(PHI,NM,NM,NPT,15,15HOTHE MATRIX PHI)
C
C
C        HERE WE COMPUTE THE MATRIX A AND THE VECTOR OF EXACT OBSERVATIONS
C        YSTAR.
C
C
         DO 190 I=1,NPT
         D(I)=0.
         DO 190 J=1,NPT
         A(I,J)=0.
190      CONTINUE
         DO 200 IK=1,NM
```

```
          DO 200 I=1,NPT
          COEF(I)=0.
          YSTAR(I)=YSTAR(I)+ALPHA(IK)*PHI(IK,I)
          DO 200 J=1,NPT
          A(I,J)=A(I,J)+PHI(IK,I)*PHI(IK,J)/(FREQ(IK)**4)
    200   CONTINUE
          CALL MOUT(A,NPT,NPT,NPT,13,13HOTHE MATRIX A)
          CALL VOUT(YSTAR,NPT,30,30HOTHE VECTOR OF OBSERVATIONS Y*)
    C
    C     COMPUTATION OF THE PRODUCT A(YSTAR)=AY .
    C
          DO 202 I=1,NPT
          DO 202 J=1,NPT
          AY(I)=AY(I)+A(I,J)  * YSTAR(J)
    202   CONTINUE
          CALL VOUT(AY, NPT, 14,14HOTHE VECTOR AY)
    C
    C     HERE WE ADD THE R MATRIX TO THE A MATRIX.
    C
    C
          DO 203 I=1,NPT
          A(I,I)=A(I,I)+RR
    203   CONTINUE
    204   FORMAT(//,2X,'R=  ',E15.8)
    205   CALL MOUT(A,NPT,NPT,NPT,15,15HOTHE MATRIX A+R)
          WRITE(6,204) RR
          DO 210 I=1,NPT
          USTAR(I)=AY(I)
          DO 210 J=1,NPT
          AA(I,J)=A(I,J)
    210   CONTINUE
    C
    C
    C     HERE WE HOPE TO SOLVE THE SYSTEM (R+A)U=AY* .
    C    SOR IS A JPL LINEAR EQUATION SOLUTION ROUTINE.
    C
    C
          CALL SOR(AA,NPT,NPT,USTAR ,NPT,1,$290,WORK)
          CALL VOUT(USTAR,NPT,28,28HOVECTOR OF OPTIMAL ESTIMATES)
          DO 215 I=1,NPT
          DO 215 J=1,NPT
          DCHK(I)=DCHK(I)+A(I,J)*USTAR(J)
    215   CONTINUE
          CALL VOUT(DCHK,NPT,19,19HOTHE VECTOR (A+R)UC)
          DO 220 I=1,NM
          FOR=(FREQ(I)**4)*RR
          DO 220 J=1,NPT
          COEF(I)=COEF(I)+(YSTAR(J)-USTAR(J))*PHI(I,J)/FOR
    220   CONTINUE
          CALL VOUT(ALPHA,NM,34, 34HOTHE VECTOR OF ACTUAL COEFFICIENTS)
          CALL VOUT(COEF,NM,37,37HOTHE VECTOR OF ESTIMATED COEFFICIENTS)
    C
    C     NOW WE COMPUTE THE ESTIMATED SHAPE.
    C
    C
          REWIND 45
          READ(45)(NOR(K),K=1,20)
```

```
      READ(45)NODES,NFREQ
      NCHECK=NODES*3
      READ(45)(X(I),Y(I),Z(I),I=1,NODES)
      READ(45)(BLUBS(I),I=1,NODES)
      READ(45)(FREQ(I),I=1,NFREQ)
      DO 240 KF=1,NM
      READ(45)KFR,FR,NTRNS,(VECTOR(K),K=1,NTRNS)
      IF(NCHECK.NE.NTRNS)GO TO 125
      DO 240 I=1,NODES
      X(I)=X(I)+COEF(KF)*U(I)
      Y(I)=Y(I)+COEF(KF)*V(I)
      Z(I)=Z(I)+CUEF(KF)*W(I)
240   CONTINUE
      IF (IOPT.EQ.1) GO TO 300
C
C     HERE WE PLOT THE ESTIMATED SHAPE.
C
      CALL DRAW(X,Y,Z,JSEQ,ISEQ)
      CALL FACTOR(1.0)
      CALL PLOT(10.,0.0,-3)
      CALL FACTOR(.0068)
C
C
      GO TO 300
250   WRITE (6,251)
      ISIG=ISIG+1
251   FORMAT(15X,26HOMATRIX IS NEARLY SINGULAR)
      WRITE(6,252)
252   FORMAT(//,15X,5HSHIT.)
      IF(ISIG.GT.6) GO TO 400
      WRITE(6,253)
253   FORMAT(//,15X,23HREDEFINE THE MATRIX A+R)
      DO 260 I=1,NPT
      A(I,I)=A(I,I) + R
260   CONTINUE
      RR=10.*RR
      R=9.*RR
      GO TO 205
C
300   CONTINUE
C     HERF WE COMPUTE THE VECTOR D IN THE CONTROL PROBLEM.
      DO 310 I=1,NM
      DO 310 J=1,NPT
      D(J)=D(J)+PHI(I,J)*COEF(I)/(FREQ(I)**2)
310   CONTINUE
      CALL VOUT(D,NPT,13,13HOTHE VECTOR D)
315   CALL MOUT(A,NPT,NPT,NPT,15,15HOTHE MATRIX A+R)
      WRITE (6,204) RR
      DO 320 I=1,NPT
      F(I)=D(I)
      DCHK(I)=0.
      DO 320 J=1,NPT
      AA(I,J)=A(I,J)
320   CONTINUE
C
C
C    HERE WE HOPE (FERVENTLY)  TO SOLVE THE MATRIX (A+R)F=D.
```

```
      C     SOR IS A JPL LINEAR EQUATION SOLUTION ROUTINE.
      C
      C
             CALL SOR(AA,NPT,NPT,F,NPT,1,S350,WORK)
             CALL VOUT(F,NPT,25,25HOVECTOR OF OPTIMAL FORCES)
             DO 325 I=1,NPT
             DO 325 J=1,NPT
             DCHK(I)=DCHK(I)+A(I,J)*F(J)
        325  CONTINUE
             CALL VOLT(DCHK,NPT,18,18HOTHE VECTOR (A+R)F)
             DO 330 I=1,NM
             FQ=FREQ(I)**2
             DO 330 J=1,NPT
             BETA(I)=BETA(I)+F(J)*PHI(I,J)/FQ
        330  CONTINUE
             CALL VOUT(BETA, NM, 39, 39HOMODAL COEFFICIENTS FROM CONTROL FORCES
            1)
             CALL VOUT(ALPHA,NM,34,34HOTHE VECTOR OF ACTUAL COEFFICIENTS)
      C
      C
      C      NOW WE COMPUTE THE SHAPE ADJUSTMENT.
      C
      C
             DO 335 I=1,NM
             COFF(I)=ALPHA(I)-BETA(I)
        335  CONTINUE
             REWIND 45
             READ(45)(MON(K),K=1,20)
             READ(45)NODES,NFREQ
             NCHECK=NODES*3
             READ(45)(X(I),Y(I),Z(I),I=1,NODES)
             READ(45)(SLUGS(I),I=1,NODES)
             READ(45)(FREQ(I),I=1,NFREQ)
             DO 340 KF=1,NM
             READ(45)KFR,FR,NTRNS,(VECTOR(K),K=1,NTRNS)
             IF(NCHECK.NE.NTRNS) GO TO 125
             DO 340 I=1,NODES
             X(I)=X(I)+COEF(KF)*U(I)
             Y(I)=Y(I)+COEF(KF)*V(I)
             Z(I)=Z(I)+COEF(KF)*W(I)
        340  CONTINUE
             IF (IOPT.EQ.1) GO TO 500
      C
      C      HERE WE PLOT THE CORRECTED SHAPE.
      C
             CALL DRAW(X,Y,Z,JSEQ,ISEQ)
             CALL FACTOR(1.0)
             CALL PLOT(10.0,0.0,-3)
             CALL FACTOR(.0068)
             GO TO 400
        350  WRITE(6,251)
             ISIG=ISIG+1
             IF(ISIG.GT.10) GO TO 400
             WRITE(6,253)
             DO 360 I=1,NPT
             A(I,I)=A(I,I)+R
        360  CONTINUE
```

```
                RR=10.*RR
                R=9.*RR
                GO TO 315
          C
          C
          125   CONTINUE
          400   CONTINUE
          500   CONTINUE
                IF(IOPT.EQ.1) GO TO 510
                CALL PLOT(10.,0.,999)
          510   CONTINUE
                STOP
                END
```

```
ND OF COMPILATION:        NO  DIAGNOSTICS.
  CTP:.067   SUPS:5.380
```

```
          C
                SUBROUTINE DRAW(UX,UY,UZ,SEQ,SEQ1)
          C
          C     DRAW IS A SUBROUTINE CREATED BY G. RODRIGUEZ TO PLOT THREE
          C     DIMENSIONAL SURFACES.    IT CALLS THE SUBROUTINE TRANS.
          C
                PARAMETER NP=882, NC=2140
                REAL UX(NP),UY(NP),UZ(NP)
                INTEGER SEQ(NC),FLAG,SEQ1(NP)
                DO 10 I=1,NC
                   J=IABS(SEQ(I))
                DO 5 K=1,NP
                IF(SEQ1(K).EQ.J) KK=K
          5     CONTINUE
                   FLAG=2
                   IF(SEQ(I).LT.0) FLAG=3
                X = UX(KK)
                Y=UY(KK)
                Z=UZ(KK)
                CALL TRANS(X,Y,Z,XP,YP)
                CALL PLOT(XP,YP,FLAG)
             10 CONTINUE
                RETURN
                END
```

```
                SUBROUTINE TRANS(X,Y,Z,XP,YP)
                REAL X,Y,Z,XP,YP
                THETA=30.0
                DR=3.1416/180.0
                XP=(X-Y)*COS(THETA*DR)
                YP=(X+Y)*SIN(THETA*DR)+Z
                RETURN
                END
```

Output

CLAMPED WRAP-RIB REFLECTOR (15 M.DIAM.,18 RIBS)(MESH MODES SUPPRESSED) FREQUENCIES

.23380053+02	.23506397+02	.23506397+02	.23866139+02	.23866139+02	.24606121+02
.24606121+02	.25051175+02	.25051175+02	.25718166+02	.25718166+02	.60649112+02
.60702706+02	.60702706+02	.60713087+02	.60713087+02	.60728870+02	.60728870+02
.60748248+02	.60748248+02	.60768828+02	.60768828+02	.92883509+02	.92902211+02
.92902211+02	.92955820+02	.92955820+02	.93037964+02	.93037964+02	.93138743+02
.93138743+02	.93246012+02	.93246012+02			

NO. OF NODES = 682

POSITIONS AND DIRECTIONS OF CONTROL/OBSERVATION POINTS

NODE	DIRECTION
127	Z
130	Z
133	Z
136	Z
139	Z
142	Z
145	Z
148	Z
151	Z
154	Z
157	Z
160	Z
163	Z
166	Z
169	Z
172	Z

175 X

178 X

THE MATRIX PHI

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6	COL 7	COL 8
ROW 1	1.2831248-07	-1.0899605-01	-2.0484570-01	-2.7598793-01	-3.1384195-01	-3.1384199-01	-2.7598806-01	-2.0484569-01
ROW 2	1.3478150-04	4.3696770-04	9.9555796-02	2.5111246-01	3.8819193-01	4.3652883-01	3.8362800-01	2.5024235-01
ROW 3	-4.8923803-05	-1.5389032-01	-2.7154582-01	-2.9796210-01	-2.2077970-01	-7.6112728-02	6.8347540-02	1.4500654-01
ROW 4	1.5276443-04	-4.0518568-02	1.1985485-01	3.5028730-01	4.2682763-01	2.5572863-01	-3.0730317-02	-2.0184042-01
ROW 5	2.2373823-04	1.4770635-01	2.6171161-01	1.6644222-01	-1.1098914-01	-3.5929662-01	-3.8650992-01	-2.0522571-01
ROW 6	3.2220858-04	1.5208018-01	1.4562971-01	-1.8859184-01	-4.3774423-01	-2.2358217-01	1.8826944-01	2.8572405-01
ROW 7	1.7780292-04	-1.5013493-03	2.4585995-01	3.3563675-01	5.2584169-03	-3.7642160-01	-3.3581455-01	-3.7532675-03
ROW 8	6.8355308-05	1.8544752-04	2.7886328-01	1.3079359-01	-3.7585623-01	-2.7939509-01	2.4866187-01	2.4559368-01
ROW 9	4.1717327-04	1.3067874-01	4.9246443-02	-3.5840261-01	-2.1738908-01	3.3190726-01	2.9273966-01	-1.4096012-01
ROW 10	1.3205106-05	3.3071253-02	2.5868209-01	-2.0402260-01	-3.1595747-01	3.8159150-01	1.7376900-01	-2.6757124-01
ROW 11	-4.2714470-04	-1.4551093-01	1.0830026-01	3.1800889-01	-2.9115114-01	-2.6071609-01	3.3549107-01	8.3936704-02

	COL 9	COL 10	COL 11	COL 12	COL 13	COL 14	COL 15	COL 16
ROW 1	-1.0899629-01	-1.2831242-07	1.0899605-01	2.0484570-01	2.7598793-01	3.1384195-01	3.1384199-01	2.7598806-01
ROW 2	9.8790633-02	1.3478150-04	4.3696770-04	9.9555796-02	2.5111246-01	3.8419193-01	4.3652883-01	3.8362800-01
ROW 3	1.1799466-01	-4.8923805-05	-1.5389032-01	-2.7154582-01	-2.9796210-01	-2.2077970-01	-7.6112728-02	6.8347540-02
ROW 4	-1.5244798-01	-1.5276443-04	4.0518568-02	-1.1985485-01	-3.5028730-01	-4.2682763-01	-2.5572863-01	3.0730317-02
ROW 5	-1.4760722-02	-2.2373823-04	-1.4770635-01	-2.6171161-01	-1.6644222-01	1.1098914-01	3.5929662-01	3.8659924-01
ROW 6	7.7912461-02	3.2220858-04	1.5208018-01	1.4562971-01	-1.8859184-01	-4.3774423-01	-2.2358217-01	1.8826944-01
ROW 7	1.3056245-01	1.7780292-04	-1.5013493-03	2.4585995-01	3.3563675-01	5.2584169-03	-3.7642160-01	-3.3581455-01
ROW 8	-5.0959618-02	-6.8355308-05	-1.8544752-04	-2.7886328-01	-1.3079359-01	3.7585623-01	2.7939509-01	-2.4866187-01
ROW 9	-1.4179093-01	-4.1717327-04	-1.5067874-01	-4.9246403-02	3.5840261-01	2.1738908-01	-3.3190726-01	-2.9273966-01
ROW 10	-1.9485739-02	1.3205106-05	3.3071253-02	2.5868209-01	-2.0402260-01	-3.1595747-01	3.4150150-01	1.7376900-01
ROW 11	-1.4793162-01	-4.2714470-04	-1.4551093-01	1.0830026-01	3.1800889-01	-2.9115114-01	-2.6071609-01	3.3549107-01

	COL 17	COL 18
ROW 1	2.0484589-01	1.0899629-01
ROW 2	2.5024235-01	9.8790633-02
ROW 3	1.4500654-01	1.1799466-01
ROW 4	2.0184082-01	1.5244798-01
ROW 5	2.0522571-01	1.4760722-02
ROW 6	2.8572405-01	7.7912461-02
ROW 7	-3.7532675-03	1.3096265-01
ROW 8	-2.4559368-01	5.0959618-02
ROW 9	1.4096012-01	1.4179093-01
ROW 10	-2.6757123-01	-1.9485739-02
ROW 11	8.3936706-02	-1.4793162-01

THE MATRIX A

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6	COL 7	COL 8
ROW 1	1.5844735-12	5.4761670-10	5.8237686-10	-2.1845972-10	-8.6296151-11	2.5371603-10	-8.9876578-11	-9.8551298-11
ROW 2	5.4761670-10	3.6533901-07	3.7978711-07	-5.7243202-08	-7.5506597-08	1.0410551-07	-9.6352437-09	-4.5781471-08
ROW 3	5.8237686-10	3.7978711-07	1.2833204-06	9.6081389-07	-1.2209381-07	-1.4169078-07	1.7052277-07	-1.3540132-08
ROW 4	-2.1845972-10	-5.7243202-08	9.6081389-07	2.5094763-06	1.0717591-06	-1.6506603-07	-1.6783281-07	1.7051888-07
ROW 5	-8.6296151-11	-7.5506597-08	-1.2209381-07	1.0717591-06	5.0122986-06	1.6735057-06	-1.6585641-07	-1.4169084-07

ROW 6	2.5371603-10	1.0314551-07	-1.4169478-07	-1.6590803-07	1.6735457-06	3.0122873-06	1.4717556-06	-1.2297258-07
ROW 7	-8.9876578-11	-9.6352437-09	1.7052277-07	-1.6783281-07	-1.6589561-07	1.4717556-06	2.3294475-06	9.6080859-07
ROW 8	-8.8551294-11	-4.5741673-08	-1.3540132-08	1.7051888-07	-1.4169044-07	-1.2297258-07	9.6080859-07	1.2832877-06
ROW 9	6.3685800-11	1.9037563-08	-4.5739264-08	-9.6338040-09	1.0313570-07	-7.5499925-08	-5.7219068-08	3.7978246-07
ROW 10	1.8650063-13	6.2862930-11	-9.2413100-11	-8.3777641-11	2.5592184-10	-9.7450693-11	-2.3259088-10	5.9813454-10
ROW 11	6.2862930-11	2.3908487-08	-1.5967387-08	-4.8527387-08	6.9862079-08	1.1501538-08	-9.0876637-08	4.8893162-08
ROW 12	-9.2413100-11	-1.5967387-08	8.8881806-08	-4.0526336-08	-1.0360401-07	1.3136986-07	1.8841085-08	-1.2672526-07
ROW 13	-8.3777641-11	-4.8527387-08	-4.0526336-08	1.5527593-07	-6.2122581-08	-1.3955139-07	1.5565938-07	1.8840310-08
ROW 14	2.5592184-10	6.9862079-08	-1.0360401-07	-6.2122581-08	1.9820257-07	-7.0650973-08	-1.3954933-07	1.3136534-07
ROW 15	-9.7450693-11	1.1501538-08	1.3136986-07	-1.3955139-07	-7.0650973-08	1.9820018-07	-6.2121012-08	-1.0359880-07
ROW 16	-2.3259088-10	-9.0876637-08	1.8841085-08	1.5565938-07	-1.3954933-07	-6.2121012-08	1.5326970-07	-4.0523963-08
ROW 17	5.9813454-10	4.8893162-08	-1.2672526-07	1.8840310-08	1.3136534-07	-1.0359880-07	-4.0523963-08	8.4430721-08
ROW 18	5.4577944-10	2.3425704-08	4.8890842-08	-9.0869357-08	1.1499599-08	6.9855147-08	-4.8521433-08	-1.5965304-08

	COL 9	COL 10	COL 11	COL 12	COL 13	COL 14	COL 15	COL 16
ROW 1	6.3685800-11	1.8650063-13	6.2862930-11	-9.2413100-11	-8.3777641-11	2.5592184-10	-9.7450693-11	-2.3259088-10
ROW 2	1.9037563-08	6.2862930-11	2.3908487-08	-1.5967387-08	-4.8527387-08	6.9862079-08	1.1501538-08	-9.0876637-08
ROW 3	-4.5739264-08	-9.2413100-11	-1.5967387-08	8.8881806-08	-4.0526336-08	-1.0360401-07	1.3136986-07	1.8841085-08
ROW 4	-9.6338040-09	-8.3777641-11	-4.8527387-08	-4.0526336-08	1.5527593-07	-6.2122581-08	-1.3955139-07	1.5565938-07
ROW 5	1.0313570-07	2.5592184-10	6.9862079-08	-1.0360401-07	-6.2122581-08	1.9820257-07	-7.0650973-08	-1.3954933-07
ROW 6	-7.5499925-08	-9.7450693-11	1.1501538-08	1.3136986-07	-1.3955139-07	-7.0650973-08	1.9820018-07	-6.2121012-08
ROW 7	-5.7219068-08	-2.3259088-10	-9.0876637-08	1.8841085-08	1.5565938-07	-1.3954933-07	-6.2121012-08	1.5326970-07
ROW 8	3.7978246-07	5.9813454-10	4.8893162-08	-1.2672526-07	1.8840310-08	1.3136534-07	-1.0359880-07	-4.0523963-08
ROW 9	3.6331767-07	5.4577944-10	2.3425704-08	4.8890842-08	-9.0869357-08	1.1499599-08	6.9855147-08	-4.8521433-08
ROW 10	5.4577944-10	1.5464735-12	5.4761670-10	5.8237686-10	-2.3845972-10	-8.6296151-11	2.5371603-10	-8.9876578-11
ROW 11	2.3425704-08	5.4761670-10	3.6333901-07	3.7978711-07	-5.7243202-08	-7.5506597-08	1.0314551-07	-9.6352437-09
ROW 12	4.8890842-08	5.8237686-10	3.7978711-07	1.2833204-06	9.6081389-07	-1.2299381-07	-1.4169478-07	1.7052277-07
ROW 13	-9.0869357-08	-2.3845972-10	-5.7243202-08	9.6081389-07	2.3294763-06	1.4717591-06	-1.6590803-07	-1.6783281-07
ROW 14	1.1499599-08	-8.6296151-11	-7.5506597-08	-1.2299381-07	1.4717591-06	3.0122984-06	1.6735457-06	-1.6589561-07
ROW 15	6.9855147-08	2.5371603-10	1.0314551-07	-1.4169478-07	-1.6590803-07	1.6735457-06	3.0122873-06	1.4717556-06
ROW 16	-4.8521433-08	-8.9876578-11	-9.6352437-09	1.7052277-07	-1.6783281-07	-1.6589561-07	1.4717556-06	2.3294475-06
ROW 17	-1.5965304-08	-8.8551294-11	-4.5741673-08	-1.3540132-08	1.7051888-07	-1.4169044-07	-1.2297258-07	9.6080859-07
ROW 18	2.3903865-08	6.3685800-11	1.9037563-08	-4.5739264-08	-9.6338040-09	1.0313570-07	-7.5499925-08	-5.7219068-08

	COL 17	COL 18
ROW 1	5.9813454-10	5.4577944-10
ROW 2	4.8893162-08	2.3425704-08
ROW 3	-1.2672526-07	4.8890842-08
ROW 4	1.8840310-08	-9.0869357-08
ROW 5	1.3136534-07	1.1499599-08
ROW 6	-1.0359880-07	6.9855147-08
ROW 7	-4.0523963-08	-4.8521433-08
ROW 8	8.4430721-08	-1.5965304-08
ROW 9	-1.5965304-08	2.3903865-08
ROW 10	-8.8551294-11	6.3685800-11
ROW 11	-4.5741673-08	1.9037563-08
ROW 12	-1.3540132-08	-4.5739264-08
ROW 13	1.7051888-07	-9.6338040-09
ROW 14	-1.4169044-07	1.0313570-07
ROW 15	-1.2297258-07	-7.5499925-08
ROW 16	9.6080859-07	-5.7219068-08
ROW 17	1.2832877-06	3.7978246-07
ROW 18	3.7978246-07	3.6331767-07

THE VECTOR OF OBSERVATIONS Y4
1 TO 8 1.9367295-01 -1.5286627+00 1.8378183+00 7.7680659-01 -2.3272116+00 -2.7[illegible]-01 -9.7502945-01 -8.1767509+00

9 TO 16	-2.9464695+00	-1.8046785-03	1.6595792+00	7.4900261-01	-2.4170746+00	-8.3234307-01	3.6854163+00	2.7127198+00
17 TO 18	1.5910346+00	2.7718121+00						

THE VECTOR BY

1 TO 8	2.0985491-09	5.0640909-07	3.2679906-06	-1.9257898-06	-6.0729749-06	-8.7539256-06	-6.2491125-06	-7.5382262-06
9 TO 16	-2.4871993-06	-2.0166721-09	9.9658128-07	-1.0266533-07	-6.7991737-06	-1.6866288-06	1.6308179-09	1.8801938-09
17 TO 18	3.9176151-06	1.1903012-06						

THE MATRIX A*B

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6	COL 7	COL 8
ROW 1	1.0154687-10	5.4761670-10	5.8237686-10	-2.3845972-10	-8.6296151-11	2.5371603-10	-8.9876578-11	-8.8551294-11
ROW 2	5.4761670-10	3.6345909-07	3.7978711-07	-5.7243202-08	-7.5506597-08	1.0314551-07	-9.6352437-09	-4.5741673-08
ROW 3	5.8237686-10	3.7978711-07	1.2834204-06	9.6081389-07	-1.2299381-07	-1.4169478-07	1.7052277-07	-1.3540132-08
ROW 4	-2.3845972-10	-5.7243202-08	9.6081389-07	2.3295763-06	1.4717591-06	-1.6590803-07	-1.6783281-07	1.7051888-07
ROW 5	-8.6296151-11	-7.5506597-08	-1.2299381-07	1.4717591-06	3.0123986-06	1.6735457-06	-1.6589961-07	-1.4169044-07
ROW 6	2.5371603-10	1.0314551-07	-1.4169478-07	-1.6590803-07	1.6735457-06	3.0123873-06	1.4717556-06	-1.2297298-07
ROW 7	-8.9876578-11	-9.6352437-09	1.7052277-07	-1.6783281-07	-1.6589961-07	1.4717556-06	2.3295879-06	9.6080859-07
ROW 8	-8.8551294-11	-4.5741673-08	-1.3540132-08	1.7051888-07	-1.4169044-07	-1.2297298-07	9.6080859-07	1.2833877-06
ROW 9	6.3685800-11	1.9037563-08	-4.5739264-08	-9.6338040-09	1.0313570-07	-7.5499925-08	-5.7219068-08	3.7978244-07
ROW 10	1.8690063-13	6.2862930-11	-9.2413100-11	-8.3777641-11	2.5592188-10	-9.7450693-11	-2.3259088-10	5.9813454-10
ROW 11	6.2862930-11	2.3908487-08	-1.5967387-08	-8.8527347-08	6.9862079-08	1.1501538-08	-9.0876637-08	4.8893162-08
ROW 12	-9.2413100-11	-1.5967387-08	8.8441806-08	-4.0526336-08	-1.0360401-07	1.3136986-07	1.8841085-08	-1.2672526-07
ROW 13	-8.3777641-11	-8.8527347-08	-4.0526336-08	1.5327593-07	-6.2122581-08	-1.3955139-07	1.5565938-07	1.8840310-08
ROW 14	2.5592188-10	6.9862079-08	-1.0360401-07	-6.2122581-08	1.9820257-07	-7.0650973-08	-1.3954933-07	1.3136534-07
ROW 15	-9.7450693-11	1.1501538-08	1.3136986-07	-1.3955139-07	-7.0650973-08	1.9820018-07	-6.2121012-08	-1.0399880-07
ROW 16	-2.3259088-10	-9.0876637-08	1.8841085-08	1.5565938-07	-1.3954933-07	-6.2121012-08	1.5326970-07	-4.0523963-08
ROW 17	5.9813454-10	4.8893162-08	-1.2672526-07	1.8840310-08	1.3136534-07	-1.0399880-07	-4.0529963-08	8.8434721-08
ROW 18	5.4577984-10	2.3425704-08	4.8890842-08	-9.0869357-08	1.1499599-08	6.9855187-08	-8.8521433-08	-1.5965304-08

	COL 9	COL 10	COL 11	COL 12	COL 13	COL 14	COL 15	COL 16
ROW 1	6.3685800-11	1.8690063-13	6.2862930-11	-9.2413100-11	-8.3777641-11	2.5592188-10	-9.7450693-11	-2.3259088-10
ROW 2	1.9037563-08	6.2862930-11	2.3908487-08	-1.5967387-08	-8.8527347-08	6.9862079-08	1.1501538-08	-9.0876637-08
ROW 3	-4.5739264-08	-9.2413100-11	-1.5967387-08	8.8441806-08	-4.0526336-08	-1.0360401-07	1.3136986-07	1.8841085-08
ROW 4	-9.6338040-09	-8.3777641-11	-8.8527347-08	-4.0526336-08	1.5327593-07	-6.2122581-08	-1.3955139-07	1.5565938-07
ROW 5	1.0313570-07	2.5592188-10	6.9862079-08	-1.0360401-07	-6.2122581-08	1.9820257-07	-7.0650973-08	-1.3954933-07
ROW 6	-7.5499925-08	-9.7450693-11	1.1501538-08	1.3136986-07	-1.3955139-07	-7.0650973-08	1.9820018-07	-6.2121012-08
ROW 7	-5.7219068-08	-2.3259088-10	-9.0876637-08	1.8841095-08	1.5565938-07	-1.3954933-07	-6.2121012-08	1.5326970-07
ROW 8	3.7978244-07	5.9813454-10	4.8893162-08	-1.2672526-07	1.8840310-08	1.3136534-07	-1.0399880-07	-4.0523963-08
ROW 9	3.6341747-07	5.4577984-10	2.3425704-08	4.8890842-08	-9.0869357-08	1.1499599-08	6.9855187-08	-8.8521433-08
ROW 10	5.4577984-10	1.0154687-10	5.4761670-10	5.8237686-10	-2.3845972-10	-8.6296151-11	2.5371603-10	-8.9876578-11
ROW 11	2.3425704-08	5.4761670-10	3.6343900-07	3.7978711-07	-5.7243202-08	-7.5506597-08	1.0314551-07	-9.6352437-09
ROW 12	4.8890842-08	5.8237686-10	3.7978711-07	1.2834204-06	9.6081389-07	-1.2299381-07	-1.4169478-07	1.7052277-07
ROW 13	-9.0869357-08	-2.3845972-10	-5.7243202-08	9.6081389-07	2.3295763-06	1.4717591-06	-1.6590803-07	-1.6783281-07
ROW 14	1.1499599-08	-8.6296151-11	-7.5506597-08	-1.2299381-07	1.4717591-06	3.0123986-06	1.6739447-06	-1.6589961-07
ROW 15	6.9855187-08	2.5371603-10	1.0314551-07	-1.4169478-07	-1.6590803-07	1.6735457-06	3.0123873-06	1.4717556-06
ROW 16	-8.8521433-08	-8.9876578-11	-9.6352437-09	1.7052277-07	-1.6783281-07	-1.6589961-07	1.4717556-06	2.3295879-06
ROW 17	-1.5965304-08	-8.8551294-11	-4.5741673-08	-1.3540132-08	1.7051888-07	-1.4169044-07	-1.2297298-07	9.6080859-07
ROW 18	2.3903865-08	6.3685800-11	1.9037563-08	-4.5739264-08	-9.6338040-09	1.0313570-07	-7.5499925-08	-5.7219068-08

	COL 17	COL 18
ROW 1	5.9813454-10	5.4577984-10
ROW 2	4.8893162-08	2.3425704-08
ROW 3	-1.2672526-07	4.8890842-08
ROW 4	1.8840310-08	-9.0869357-08

ROW	5	1.3136534-07	1.1499599-08
ROW	6	-1.0359880-07	6.9895147-08
ROW	7	-6.0923963-08	-8.8521433-08
ROW	8	8.4434721-08	-1.5965304-08
ROW	9	-1.5965304-08	2.3903869-08
ROW	10	-8.8551294-11	8.3685800-11
ROW	11	-4.5741673-08	1.9037563-08
ROW	12	-1.3540132-08	-4.5739264-08
ROW	13	1.7051888-07	-9.6338040-09
ROW	14	-1.4169084-07	1.0313570-07
ROW	15	-1.2297258-07	-7.5499925-08
ROW	16	9.6080859-07	-5.7219068-08
ROW	17	1.2833877-06	3.7978246-07
ROW	18	3.7978246-07	3.6341767-07

R= .10000000-09

VECTOR OF OPTIMAL ESTIMATES

1 TO 8	1.9378825-03	-1.3272162+00	1.8369060+00	3.7705291-01	-2.3270264+00	-2.7069810-01	-9.7531889-01	-8.1757794+00
9 TO 16	-2.9477296+00	-1.8065925-03	1.7381831+00	7.4910560-01	-2.4161861+00	-8.3378389-01	3.6871077+00	2.7114884+00
17 TO 18	1.5016131+00	2.7711929+00						

THE VECTOR (A*R)U*

1 TO 8	2.0985491-09	5.0640908-07	3.2679904-06	-1.9297896-06	-6.9729747-06	-8.7535258-06	-6.2491125-06	-7.9382259-06
9 TO 16	-2.4871994-06	-2.0146722-09	9.9658128-07	-1.0266540-07	-6.7991736-06	-1.6866267-06	1.6344175-05	1.6801939-05
17 TO 18	3.9176152-06	1.1003012-06						

THE VECTOR OF ACTUAL COEFFICIENTS

1 TO 8	1.0000000+01	0.0000000	0.0000000	1.0000000+01	0.0000000	0.0000000	0.0000000	5.0000000+00
9 TO 11	0.0000000	5.0000000+00	0.0000000					

THE VECTOR OF ESTIMATED COEFFICIENTS

1 TO 8	9.9956750+00	-1.0210872-03	-1.8748681-04	9.9964799+00	8.2074919-03	7.1460009-04	5.9470934-04	4.9957830+00
9 TO 11	2.7091501-03	5.0032493+00	3.4660101-04					

THE VECTOR D

1 TO 8	3.3309976-04	-2.4506863-03	2.5368726-03	5.9837175-04	-3.6299730-03	-8.9455632-04	-2.3262182-03	-7.3593563-03
9 TO 16	-5.2224792-03	-3.1310058-06	2.9513282-03	1.3770841-03	-5.6851474-03	-1.1526736-03	6.0579116-05	8.9537850-03
17 TO 18	3.3110892-03	4.9275328-03						

THE MATRIX A*R

	COL 1	COL 2	COL 3	COL 4	COL 5	COL 6	COL 7	COL 8
ROW 1	1.0154647-10	5.4761670-10	5.8237686-10	-2.3885972-10	-8.6296151-11	2.5371603-10	-8.9876578-11	-8.8551294-11
ROW 2	5.4761670-10	3.6343900-07	3.7978711-07	-5.7243202-08	-7.5506597-08	1.0314551-07	-9.6352637-09	-8.5741673-08
ROW 3	5.8237686-10	3.7978711-07	1.2834204-06	9.6081389-07	-1.2299381-07	-1.4169478-07	1.7052277-07	-1.3540132-08
ROW 4	-2.3885972-10	-5.7243202-08	9.6081389-07	2.3299763-06	1.8717591-06	-1.6590805-07	-1.6783281-07	1.7051888-07
ROW 5	-8.6296151-11	-7.5506597-08	-1.2299381-07	1.8717591-06	5.0124986-06	1.6735057-06	-1.6589561-07	-1.4169084-07
ROW 6	2.5371603-10	1.0314551-07	-1.4169478-07	-1.6590805-07	1.6735057-06	5.0123873-06	1.8717556-06	-1.2297258-07
ROW 7	-8.9876578-11	-9.6352637-09	1.7052277-07	-1.6783281-07	-1.6589561-07	1.8717556-06	2.3299875-06	9.6080859-07
ROW 8	-8.8551294-11	-8.5741673-08	-1.3540132-08	1.7051888-07	-1.4169084-07	-1.2297258-07	9.6080859-07	1.2833877-06
ROW 9	6.3685800-11	1.9037563-08	-4.5739264-08	-9.6338040-09	1.0313570-07	-7.5499925-08	-5.7219068-08	3.7978246-07
ROW 10	1.8650063-13	6.2862930-11	-9.2413100-11	-8.4777641-11	2.5592184-10	-9.7450693-11	-2.3259088-10	5.9813054-10
ROW 11	6.2862930-11	2.3908087-08	-1.5967387-08	-8.8527787-08	6.2962079-08	1.1501558-09	-9.0876637-08	8.8893162-08
ROW 12	-9.2413100-11	-1.5967387-08	8.4841806-10	-8.2526336-09	-1.0360801-07	1.3136534-07	1.8841085-08	-1.2672524-07

ROW 13	-8.3777641-11	-8.8527347-08	-4.0526336-08	1.5327593-07	-6.2122581-08	-1.3955139-07	1.5565938-07	1.8840310-08
ROW 14	2.5592184-10	6.9862079-08	-1.0360401-07	-6.2122581-08	1.9820257-07	-7.0650973-08	-1.3954933-07	1.3136534-07
ROW 15	-9.7450693-11	1.1501538-08	1.3136986-07	-1.3955139-07	-7.0650973-08	1.9820018-07	-6.2121012-08	-1.0359880-07
ROW 16	-2.3259088-10	-9.0876637-08	1.8841085-08	1.5565938-07	-1.3954933-07	-6.2121012-08	1.5326970-07	-8.0523963-08
ROW 17	5.9413454-10	8.8893162-08	-1.2672526-07	1.8840310-08	1.3136534-07	-1.0359880-07	-8.0523963-08	8.8834721-08
ROW 18	5.4577944-10	2.3425704-08	4.8890842-08	-9.0869357-08	1.1499599-08	6.9855147-08	-8.8521433-08	-1.5965304-08

	COL 9	COL 10	COL 11	COL 12	COL 13	COL 14	COL 15	COL 16
ROW 1	6.3485890-11	1.8650063-13	6.2862930-11	-9.2413100-11	-8.3777641-11	2.5592180-10	-9.7490693-11	-2.3259088-10
ROW 2	1.9037563-08	6.2862930-11	2.3908487-08	-1.5967387-08	-8.8527347-08	6.9862079-08	1.1501538-08	-9.0876637-08
ROW 3	-4.5739264-08	-9.2413100-11	-1.5967387-08	8.4461806-08	-4.0526336-08	-1.0360401-07	1.3136986-07	1.8841085-08
ROW 4	-9.6338040-09	-8.3777641-11	-8.8527347-08	-4.0526336-08	1.5327593-07	-6.2122581-08	-1.3955139-07	1.5565938-07
ROW 5	1.0313570-07	2.5592184-10	6.9862079-08	-1.0360401-07	-6.2122581-08	1.9820257-07	-7.0650973-08	-1.3954933-07
ROW 6	-7.5499925-08	-9.7450693-11	1.1501538-08	1.3136986-07	-1.3955139-07	-7.0650973-08	1.9820018-07	-6.2121012-08
ROW 7	-5.7219068-08	-2.3259088-10	-9.0876637-08	1.8841085-08	1.5565938-07	-1.3954933-07	-6.2121012-08	1.5326970-07
ROW 8	3.7978246-07	5.9813454-10	8.8893162-08	-1.2672526-07	1.8840310-08	1.3136534-07	-1.0359880-07	-4.0523963-08
ROW 9	3.6341767-07	5.4577944-10	2.3425704-08	4.8890842-08	-9.0869357-08	1.1499599-08	6.9855147-08	-8.8521433-08
ROW 10	5.4577944-10	1.0154847-10	5.4761670-10	5.8237486-10	-2.3845972-10	-8.6296151-11	2.5371603-10	-8.9876574-11
ROW 11	2.3425704-08	5.4761670-10	3.6343900-07	3.7978711-07	-5.7243202-08	-7.5506597-08	1.0314551-07	-9.6352437-09
ROW 12	4.8890842-08	5.8237486-10	3.7978711-07	1.2834204-06	9.6081389-07	-1.2299381-07	-1.4169478-07	1.7052277-07
ROW 13	-9.0869357-08	-2.3845972-10	-5.7243202-08	9.6081389-07	2.9299763-06	1.4717591-06	-1.6590805-07	-1.6783281-07
ROW 14	1.1499599-08	-8.6296151-11	-7.5506597-08	-1.2299381-07	1.4717591-06	5.0123986-06	1.6734497-06	-1.6589541-07
ROW 15	6.9855147-08	2.5371603-10	1.0314551-07	-1.4169478-07	-1.6590805-07	1.6735457-06	3.0129873-06	1.4717596-06
ROW 16	-8.8521433-08	-8.9876574-11	-9.6352437-09	1.7052277-07	-1.6783281-07	-1.6589541-07	1.4717596-06	2.3295479-06
ROW 17	-1.5965304-08	-8.8551294-11	-8.5741673-08	-1.3540132-08	1.7051888-07	-1.4169044-07	-1.2297258-07	9.6040859-07
ROW 18	2.3903865-08	6.3685800-11	1.9037563-08	-4.5739264-08	-9.6338040-09	1.0313570-07	-7.5499925-08	-5.7219068-08

	COL 17	COL 18
ROW 1	5.9813454-10	5.4577944-10
ROW 2	8.8893162-08	2.3425704-08
ROW 3	-1.2672526-07	4.8890842-08
ROW 4	1.8840310-08	-9.0869357-08
ROW 5	1.3136534-07	1.1499599-08
ROW 6	-1.0359880-07	6.9855147-08
ROW 7	-8.0523963-08	-8.8521433-08
ROW 8	8.8434721-08	-1.5965304-08
ROW 9	-1.5965304-08	2.3903865-08
ROW 10	-8.8551294-11	6.3685800-11
ROW 11	-8.5741673-08	1.9037563-08
ROW 12	-1.3540132-08	-4.5739264-08
ROW 13	1.7051888-07	-9.6338040-09
ROW 14	-1.4169044-07	1.0313570-07
ROW 15	-1.2297258-07	-7.5499925-08
ROW 16	9.6040859-07	-5.7219068-08
ROW 17	1.2833877-06	3.7978246-07
ROW 18	3.7978246-07	3.6341767-07

R= .10000000-09

VECTOR OF OPTIMAL FORCES

1 TO 8	2.7121840+00	-1.0433663+00	3.5725431+02	4.0772821+03	-3.4888524+03	-5.1803657+02	2.6842801+03	-6.1305980+03
9 TO 16	-4.5708115+03	-2.5146010+00	7.6366151+03	2.6150642+03	-8.1058915+03	6.4556704+02	4.3074721+03	-2.7790098+03
17 TO 18	3.2561610+03	1.2501745+03						

THE VECTOR (A*H)F

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  1 TO   8    3.3309974-04  -2.0506864-03   2.5368725-03   5.9837163-04  -3.6299733-03  -8.9855457-04  -2.3262142-03  -7.3593563-03
  9 TO  16   -5.2224793-03  -3.1310058-06   2.9513282-03   1.3774260-03  -3.6851475-03  -1.1526734-03   6.0574117-03   8.9937884-03
 17 TO  18    3.3110892-03   4.9275325-03

 MODAL COEFFICIENTS FROM CONTROL FORCES
  1 TO   8    9.9934233+00  -1.1650024-03  -5.7685975-04   9.9982013+00   5.4148100-03   9.2828274-04   1.9568229-08   8.9981190+00
  9 TO  11    2.7027756-03   5.0027378+00   3.3932924-04

 THE VECTOR OF ACTUAL COEFFICIENTS
  1 TO   8    1.0000000+01   0.0000000      0.0000000      1.0000000+01   0.0000000      0.0000000      0.0000000      5.0000000+00
  9 TO  11    0.0000000      5.0000000+00   0.0000000

     1 PLOTS SENT TO SITEID: PLTF
 ADDRESS: 125    / H
DATA IGNORED - IN CONTROL MODE

OFTU                                                  NITO
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Bibliography

1) "Large Space Structures - Challenge of the Eighties," Special Section Astronautics and Aeronautics, October 1978, Vol. 16, No. 10, pp. 22-51.

2) Stakgold, I., Green's Functions and Boundary Value Problems, 1st ed., Wiley, New York, 1979.

3) Luenberger, David, Optimization by Vector Space Methods, Wiley, New York, 1969.

4) Meirovitch, Leonard, Analytical Methods in Vibrations, MacMillan, London, 1967.

5) Leissa, A.W., "The Free Vibration of Rectangular Plates," Journal of Sound and Vibration, Vol. 31(3), 1973, pp. 257-293.

6) Roach, G.F., Green's Functions: Introductory Theory with Applications, Van Nostrand, London, 1970.

7) Hildebrand, Francis B., Advanced Calculus for Applications, Second Edition, Prentice-Hall, N.J., 1976.

8) Sobolev, S.L., Partial Differential Equations of Mathematical Physics, Addison-Wesley Publishing Company, Reading, Massachusetts, 1964.

9) Aubin, J.P., Applied Functional Analysis, John Wiley & Sons, New York, 1979.

10) Blum, E., Numerical Analysis and Computation: Theory and Practice, Addison Wesley, Reading, Mass. 1972.

11) Burden, R., Faires, J., Reynolds, A., Numerical Analysis, Prindle, Weber and Schmidt, Boston, Mass. 1978.

12) Clough, R. and Penzien, J., Dynamics of Structures, McGraw Hill, 1975.

13) Zienkiewicz, O.C., The Finite Element Method in Engineering Science, McGraw-Hill, London 1971.