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# INTRODUCTION TO ORBITAL FLIGHT PLANNING (I)

edited by  
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(NASA-CR-165052) INTRODUCTION TO ORBITAL  
FLIGHT PLANNING (I) (Texas Southern Univ.)  
140 p HC A07/MF A01 CSCL 22A

N82-14106

Unclas  
G3/12 08698

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### **Acknowledgements**

The primary contributor to this monograph is Eugene L. Davis, a former NASA employee at JSC, whose course notes provide its basis. He is responsible for both the basic exercises and the approach. Seeing that his problem solving and learning developmental strategies were similar to mine, I have assisted in evolving the form of his information in putting together this monograph, and, along with Dr. Bell, have supplemented and extended it.

Della D. Bell, an Associate Professor of Mathematics at Texas Southern University, contributed significantly in simplifying the task assignments to make them more adaptable for entry level college students. Her vast and extensive teaching experience and modular development experience was particularly beneficial in the determination of both the content and the methods of the learning strategies. It is only through her efforts that this work has been completed.

Work in the development of this material has been supported by the National Aeronautics and Space Administration, through Grant NAG 9-6. That support is gratefully acknowledged. I wish to acknowledge further the assistance on this project of Frank N. Barnes, Lockheed Engineering and Management Company. I wish also to acknowledge the Texas Southern University administration, without whom this project could have been completed. Thanks are due also to Jacquelyn A. Alford, an educator and research specialist, who has read this material and offered many helpful suggestions.

## **Preface**

This material is designed for students interested in the area of space flight planning, who after training, may serve as flight planning aides. Flight planning aides are individuals who will assist engineers and scientists in the planning of orbital space missions. With the development of the NASA space shuttle and its earlier successful missions, the time will come shortly when orbital flight missions will become a routine and nearly continuous activity. Space station construction is likely to be the first significant activity of the shuttle, along with the deployment of strategic and scientific experiments. Extensive commercial use of orbital flight is expected to follow.

With orbital flight becoming much more common, the need for routine flight planning activities increases. There are significant tasks in the planning of flights that can be performed by individuals with less than an engineer's background. Those tasks will be performed by the flight planning aide.

The objective in the training procedure for a flight planning aide is to provide an output, in a short period of time, that is capable of performing engineering-type calculations and analyses. To perform useful and significant calculations, it is not necessary that one know the details and background of all operations. That is true for tasks other than calculations. While it is hoped that calculators and computers do not retard us in ability to add, multiply and reason, they do allow many of us who bore to error with the task of addition to become painlessly highly accurate. Some of us are curious about process and insist upon a thorough understanding. The flight planning aide student will be capable of performing certain operations without understanding their bases, and hopefully have a curiosity that will be required for learning extension. It is on that basis that this training material has been put together in a directed learning approach.

For this material, a calculator is necessary. Calculators and computers, along with the pencil, will certainly be the professional tools of the flight planning aide. Use of the calculator, common now in the general population, is emphasized first in the workbook in order to proceed quickly toward programming and eventual use of the computer. A strong objection might be made on the emphasis put in handling a particular calculator, the Hewlett-Packard 67. That we have done. Any programmable calculator with a good guide to its use, however, will be appropriate for the given exercises. Texas Instruments has such a calculator, though it uses algebraic logic, and there are others. Although we have described the HP keyboard and made specific reference to the HP handbook, any calculator may be used with the understanding that its operations must be thoroughly understood, through use of its calculator guide or owner's handbook. The first requirement for the flight planning aide student, therefore, is to be able to learn the use of the calculator. A strong mathematical inclination is also thought necessary.

But we have attempted to integrate learning activities. Flight planning vocabulary and orbital flight relations are introduced in the beginning and are used while facility with problem solving and the first major objective, use of the calculator, is learned. To do this as we have, primary calculator instruction must come from the manufacturer or elsewhere. The learning tasks proceed through the solving of problems, as relatively more complex problems are quickly introduced into the calculator and programming exercises.

Teaching beginning students through such a problem sequence has long been a fascination of mine, but difficult to effect in traditional college sequential course curriculum. An information scattering technique is used, one I use in lecture and believe to be quite useful in inputting data to the learner that is not to be immediately used. But too much non-useful written information can be inhibitive. This written material contains only purposeful information with the hope that curiosities will be aroused to the point of asking questions and seeking further knowledge. The teaching-learning philosophy of Mr. Davis, in teaching such a course in flight planning, has been quite similar to my own.

H.E. Blackwell, Ph.D.

*June 30, 1981*

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# A. HAND CALCULATOR USE AND INTRODUCTION TO PROBLEM ANALYSIS

## A.0. INTRODUCTION

The flight planning aide is expected to solve many problems involving numerical calculations. Some problems might be done easily using paper and pencil algorithms and simple basic computations. Other problems will be more difficult to analyze and calculate, and may be handled most efficiently through use of a calculator. Since high accuracy will be necessary, calculator use allows lengthy, cumbersome calculations to be handled with speed and accuracy. Take full advantage of the power of the calculator.

## A.1. CALCULATOR LOGIC AND SIMPLE OPERATIONS

Three types of logic are used on calculators: Reverse Polish Notation (RPN), Algebraic Logic (alg) and Arithmetic Logic (Arith). Those calculators using RPN are characterized by a key marked **Enter** or **Save**. The algebraic and arithmetic calculators are distinguished by the order in which some operations are done. The key sequences below show how each type of calculator would be used in a simple operation such as:  $(4 + 5) \times 3$ .

RPN	Alg	Arith
4	4	4
enter †	+	+
5	5	5
+	=	×
3	×	3
×	3	=
ANS 27	=	ANS
	ANS	

These notes are written for a Programmable Calculator which uses RPN.



The Hewlett Packard HP-67 has been chosen. Note two conventions we will follow: operations will be written in calculator language, not in algebraic language, with the language written in a column, top to bottom. These conventions are motivated by practical considerations: the calculator language gives preparation for later programming activities, and the column representation enables you to write mnemonic notes on the same line to the right of each operation. The RPN calculator was chosen in order to de-emphasize the algebraic manipulations and to emphasize the calculator language.

Your initial task will be learning the use of various functions and operators on the calculator. Read the material from pages 15-24 in the HP-67 Owners Handbook and Programming Guide to get a "feel" for the calculator. Do the suggested exercises. These will provide some sense of the power of the calculator in performing simple calculations as well as some idea of what is involved in writing and running a program. Read the materials from Part 1, section 1, pp. 27-38. Note the information concerning the keyboard on page 27. Note further that each key on the keyboard can perform four different functions. One function is indicated on the flat plane of the key face; another is printed in black on the slanted face of the key; a third and fourth function is indicated by printed symbols in gold and blue below the key. There are three prefix keys; a gold one, a blue one, and a black one. By pressing one of the prefix keys before pressing a function key, you select the function printed on the slanted key face or one of the functions in gold or blue below the function key. As you read the material in section one, work through the exercises. If the need for additional problems is felt, do the following:

$$(14 - 3) + [(8 \times 5) + 4] \quad \text{ANS: 21}$$

$$\frac{[-2 + 4(1 - 6)]}{11} \quad \text{ANS: -2}$$

$$\sqrt{(4 - 2) \div (15 + 8)} + \sqrt{(6 - 5) + (4 \times 20)} \quad \text{ANS: 14}$$

In evaluating an expression with pencil and paper involving several operations, the following procedure is conventionally followed:

- (1) Perform the operations within each grouping symbol, if any, beginning with the innermost grouping symbol and working out to the entire expression.
- (2) Perform multiplications and divisions in order from left to right.
- (3) Perform additions and subtractions in order from left to right.

In using the calculator, a decision must be made on where to begin before the first key is pressed. The above procedure will help you. Please note that in general, only one operation is performed at a time.

Now proceed to read and work through section 2 on Display Control (pp. 41-51). This section explains the methods for displaying numbers in certain formats, including scientific notation, fixed point display, and engineering notation display. Work through these exercises also. Read the information on automatic switching between fixed and scientific notation if the answer is too large. Do the related exercises. Note the information in this section relating to "Keying in Exponents of Ten," "Calculator Overflow," "Error Display," and "Low Power Display." Remembering where certain types of information may be found in the handbook will facilitate its use as a quick reference.

## A.2. USE OF THE AUTOMATIC MEMORY STACK

This section of the HP-67 handbook deals with the four register automatic memory stack and the LAST X register of the HP-67 Calculator. There are four registers in the calculator that are positioned to form the automatic memory stack. These registers are labeled X, Y, Z, and T. They are "stacked" one on top of the other with the displayed X-register on the bottom. When the calculator is switched on, these four registers are cleared to zero, and the display shows 0.00 in RUN mode. This represents the contents of the X-register. Certain keys on the calculator enable the stack contents to be reviewed or to shift data within the stack for computation at any time.

When the h function key (black) and the  $\boxed{R\downarrow}$  key (roll down) are pressed, the stack contents shift down one register. If the h function key and the  $\boxed{R\uparrow}$  (roll up) key are pressed, the stack contents roll up instead of down. Pressing the h function key and the  $\boxed{\begin{matrix} x \rightarrow y \\ y \rightarrow x \end{matrix}}$  (x exchange y) key causes the contents of the x and y-registers to interchange without affecting the Z and T registers. Pressing the h function key and the  $\boxed{\begin{matrix} x \rightarrow y \\ y \leftarrow x \end{matrix}}$  key a second time in succession will restore the numbers in the X and Y register to their original position.

The g function key (blue) and the  $\boxed{STK}$  key will allow the entire stack

contents to be reviewed at any time. This is particularly useful in checking to be sure that the proper numbers have been entered into the register. When operating the g **STK** key manually, you can slow down or speed up the review of the stack contents by pressing **R/S** or any other key on the keyboard while the calculator is executing a g **STK** review.

The **CLX** (clear X, key) is used to clear the x-register to zero. No other register is affected when you press clear X. When you are entering several numbers, the **ENTER** key on the calculator is used to separate the digits of one number from another.

In addition to the four stack registers, there is a last X **LST X** register, used with the h function key. This key is used to preserve the value that was in the displayed X register before the performance of a function. The **LST X** key makes it easy to recover from keyboard mistakes such as pressing the wrong function key or keying in the wrong number. The last X **LST X** key is also useful in calculations where a number occurs more than once.

In the past few paragraphs, we have reviewed very briefly material covered in pages 33-69 of the Owners Handbook. Read these pages with calculator in hand and do the suggested exercises while reading.

### A.3. EVALUATING FORMULAS

A formula is a statement in mathematical language of a general rule or principle. It tells how the value of one unknown may be found by performing certain operations upon the variables whose values are known. The unknown will generally be placed to the left of the equality sign such that it is evaluated by performing the operations on the right.

$$\text{unknown} = [\text{variable expression}]$$

A calculation to determine a numerical value for the unknown thus involves (a) replacing the variables with their numerical values, and (b) performing the indicated operations of addition, division, etc., to arrive at

$$\text{unknown} = \text{number}$$

In the calculator exercises which follow, formulae used in astro- and aerodynamic calculations will be presented in that format, unknown on the left. Variables will be defined. While a clear understanding of the formula and its basis is not necessary at this time, some understanding of the many terms and relationships will follow through a careful reading and performance of the exercises.

In addition, constants used in the actual planning of missions will be used for the exercises in order to facilitate use of accurate and consistent numerical values. For your convenience, data from the Astrodynamical Constants Handbook, JSC-14264, is reproduced as Appendix A. The constants are written in powers of 10 notation. For example, the value of the earth's equatorial radius is presented as

$$1 \text{ E.r.} = .2092572178477690 + 008 \pm .16404 + 002 \text{ int. ft.}$$

The +008 means that the 16-digit number is multiplied by 10 to the 8th power ( $10^8$ ). The  $\pm .16404 + 002$  provides the accuracy of the given number. Written without power of 10 notation, the above constant would be

$$1 \text{ E.r.} = 20,925,721.78477690 \pm 16.404 \text{ int. ft.}$$

Plus or minus 16 ft in a measure of 21 million ft is not bad accuracy. Strictly speaking, because 16.404 is the possible error, it is impractical to maintain more than three decimal places here in the measurement. Note that the units of measure are given and check the Appendix to see that the radius is given in several other units of measure. It is important that appropriate and consistent units are used. We will discuss units and their conversion later. To input powers of 10 to the calculator see pages 48-51 of the Owners Handbook.

As a first example, consider the following information. The acceleration of gravity at the earth's surface is given by the formula

$$g = \frac{\mu}{(R_{\text{eq}})^2}$$

where

$\mu$  is the gravitational constant, in units of international feet per second (int. ft/sec<sup>2</sup>) squared

$R_{\text{eq}}$  is the earth's equatorial radius, in int. ft.

Taking  $g$  as the unknown, your exercise is to calculate its numerical value. The calculator steps or program sequence is as follows.



X

**ANS**

Int. ft.

A given exercise or problem consists of a set of words from which you must extract the set of "given" information (input) which relates to formulae to be used, and determine what you are asked to "find" or calculate (output). Relate quickly the given information to formula variables and check the units for consistency. Calculator steps are provided in the initial exercises.

•

The altitude of a satellite is the height above the earth's equatorial radius. For a satellite in a circular orbit, the altitude is the same everywhere in the orbit, by definition (see Figure 1A). The actual height varies since the earth is not truly a sphere, but we will discuss this later. The formula for finding the semimajor axis (equivalent to radius here) for a satellite in circular orbit is

$$a = H + R_{eq} \quad (\text{circular orbit})$$

where

a is the semimajor axis, in units identical to units of H and  $R_{eq}$

H is the altitude, in units identical to  $R_{eq}$

$R_{eq}$  is the earth's radius, in units of nautical miles (n.mi)

**Exercise A.4.2.** If the shuttle orbiter is in a circular orbit of 160 n.mi altitude, find the semimajor axis.

H	160	n.mi
---	-----	------

↑

$R_{eq}$		n.mi
----------	--	------

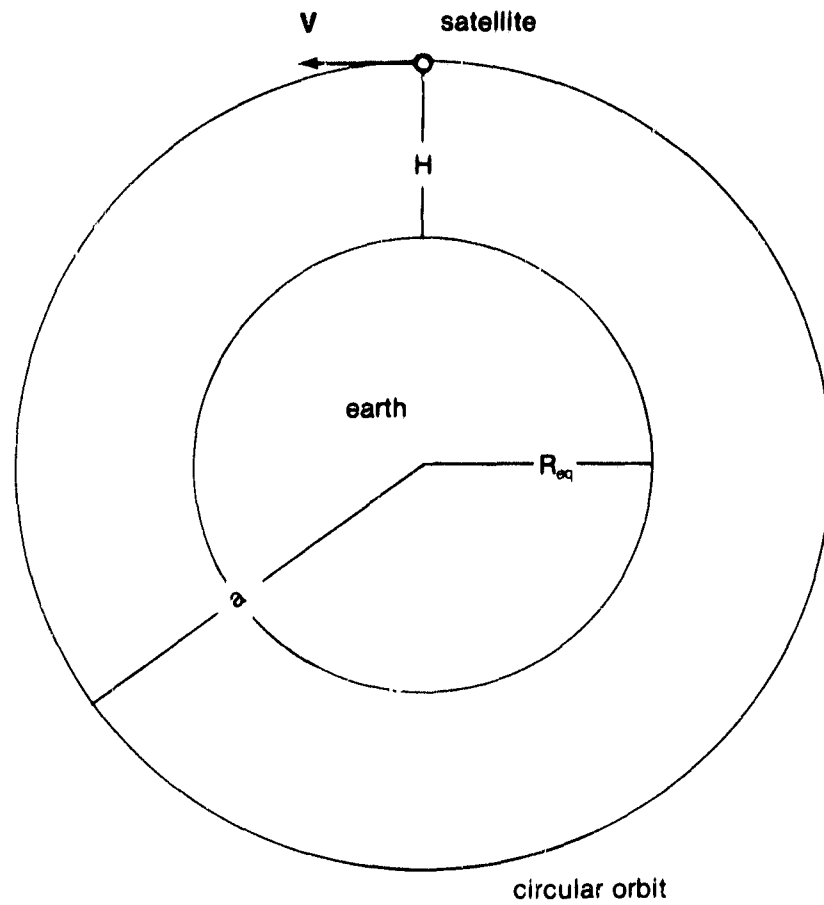
+

**ANS**

Find the semimajor axis in Int.ft.

•

The velocity of a satellite in a circular orbit is everywhere the same, and is given by



**Figure 1A.** Satellite in Circular Earth Orbit.  $R_{eq}$  is the equatorial earth radius,  $H$  the height of the satellite above the earth surface, and  $a$  is the radius of the circular orbit. Its velocity, which is horizontal or parallel to the earth surface, is labeled  $v$ .





where

- t is the time lapse, in secs.
- d is the distance in int.ft
- v is the velocity in int.ft per sec.

**Exercise A.4.4.** The distance between a tracking site near Los Angeles and a point near Baytown, Texas is 1210.83 nautical miles. How long will it take the orbiter to traverse an equivalent distance if it is in a circular orbit of 160 n.mi altitude? (see exercise A.4.2.)

d                      1210.83                      n.mi

↑

Conv                      0.60761155                       $\frac{\text{int.ft}}{\text{n.mi}}$

EEX

4

X

V                      from previous exercise                       $\left( \frac{\text{int.ft}}{\text{sec}} \right)$

+

**ANS**

Using the calculator function, express this answer in hours, minutes, and seconds.

•

The total distance around a circular orbit is equivalent to the circumference of a circle of radius a.

$$d = 2 \pi a$$

where

- d is the distance
- a is the semimajor axis

**Exercise A.4.5.** If the orbiter is in a circular orbit of 160 n. mi. altitude, how long does it take to complete one revolution in this orbit? (Remember that  $T = d/v$ .)

2  
↑  
g π  
x  
a  
x  
v  
+

ANS

### A.5. THE "EXPONENTIATION" OPERATOR

The operation  $fy^x$  has many uses; it can be used to find squares, cubes, square roots, cube roots, and so on. It is also used to raise a number in the y register to a fractional power.

•

The period of a satellite in an earth orbit is the time required to make one complete revolution in the orbit. The period is given by the formula (the orbit need not be circular)

$$P = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

where

- P is the period, in seconds
- $\mu$  is the gravitational constant, in consistent units
- a is the semimajor axis, in consistent units.

**Exercise A.5.1.** What is the Period of the orbiter when in a circular orbit of 160 n.mi. altitude?

a	160	n.mi
↑		
$R_{eq}$		n.mi

+

3

↑

2

+

$fy^x$

$g\pi$

x

2

x

$\mu$

$\frac{n.mi^3}{sec^2}$

$f \sqrt{x}$

+

ANS

Compare this answer with the one you found in exercise A.4.4.

**Exercise A.5.2.** Suppose a communications satellite is in a circular orbit with an altitude of 19322.91 n. mi. What is the period in hours, minutes, secs.?

H

19322.91

n.mi

↑

$R_{eq}$

n.mi

+

1.5

$fy^x$

$g^{\pi}$

X

2

x

$\mu$

$\frac{n.mi^3}{sec^2}$

$f\sqrt{x}$

+

Conv

3600

sec./hr.

+

f fix 4

f → H.MS

ANS

If this satellite is directly over the equator south of Houston at mid-nite, where will it be approximately one hour later?

•

The time required for a rotating body to rotate through a given angle is given by the formula

$$T = \frac{\theta}{\omega}$$

where

t is the time lapsed, in seconds

$\theta$  is the total angle traversed, in degrees or radians

$\omega$  is the angular velocity, in degrees per sec or radians per sec.

**Exercise A.5.3.** The angular velocity of the earth relative to a precessing equinox given in Appendix A is

$$\omega_p = .729211585 \text{ E} - 04 \text{ radians per sec}$$

Use the Appendix to find this value. Find the time, in hours, minutes, seconds, required for the earth to rotate  $2\pi$  radians (equivalent to  $360^\circ$ ).

$$\begin{aligned}
 & 2\pi \\
 & \uparrow \\
 & 2 \\
 & \times \\
 & \omega_p \qquad \qquad \qquad \text{rad/sec} \\
 & + \\
 & \text{Conv} \qquad \qquad \frac{1}{3600} \qquad \qquad \text{hr/sec} \\
 & \times \\
 & f \text{ FIX } 4 \\
 & f \rightarrow \text{H.MS}
 \end{aligned}$$

ANS

Compare this answer with the period of the communications satellite.

### A.6. "READING" EQUATIONS AND PROBLEMS

We will have a distinct advantage when we are able to "read" an equation and translate it into calculator language with the same ease that we read alphabetic symbols and translate those into English. Some simple rules are helpful: try to do terms within multiple parentheses "inside out" taking advantage of the temporary storage features of the shift registers. In very long complex expressions, it may be necessary to store intermediate results in one of the working storage registers provided by the calculator.

**Exercise A.6.1.** Practice order of computation using the following.

$$x = (14^2 + 196^{1/2}) (1/2)$$

$$x = (72/8)^{1/2} + (72/9)^{1/3} - 5$$

$$x = 1 + 3(6 + 2(7 + 13(7)))$$

But a major objective for us is to be able to *solve problems* related to flight planning. So definitions, concepts, principles, and ideas presented here will relate directly to actual problems that are encountered in flight planning. To assist in problem "reading," consider four phases that are involved in problem solving. First is an understanding of the verbal statement of the problem. Understanding the problem involves being able to answer these questions: What data is given? What are the unknowns? What conditions are given? For many this can be difficult.

The second phase in problem solving involves seeing how the various items are connected, how the unknown is linked to the given information in order to obtain the idea of a solution. This involves making a plan. You have a plan when you are able to determine the appropriate equations or relations and outline the calculations, computations, or constructions that will have to be made in order to arrive at a solution. Here, your plan will involve primarily the identification of the appropriate formula(s) relating the unknowns to the given data and determining the calculator process (step-by-step) necessary for evaluation of the unknown through use of the formula. Developing the plan is the main part in the solution of the problem.

The third phase involves carrying out the plan, that is to say, performing the calculations. Since the calculator will do the work, this is the simplest part. The last step is to look at the completed solution and see if it is reasonable. Does it satisfy the conditions of the problem?

Consider now the set of formula given below. They are used in determining performance characteristics of a rocket engine.

(i)  $I_{sp} = F/W_b$

(ii)  $F = W_b I_{sp}$

(iii)  $W_b = F/I_{sp}$

(iv)  $I_{sp} = c/g$

(v)  $c = g I_{sp}$

where

- F is the rocket thrust in pounds (lbs);
- $W_b$  is the fuel burned per sec, lbs/sec;
- c is the exhaust velocity of the gases issuing from the rocket nozzle, in ft./sec;
- $I_{sp}$  is the thrust developed for each lb. of fuel burned in one second.  $I_{sp}$  is called "the specific impulse" and is a property of the fuel used;
- g is the acceleration of gravity, in ft/sec<sup>2</sup>.

**Exercise A.6.2.** Each S R B at orbiter launch burns fuel at the rate of 9,330 lbs/sec. The  $I_{sp}$  of the rocket fuel used is 260 sec.

- a) What is the thrust of an SRB?
- b) What is the exhaust velocity of the burning gases?

The main engine (SSME's) each burn a Hydrogen-LOX fuel at the rate of 1 J20 lbs. per second. Each develops a thrust of 470,000 lbs.

- c) What is the  $I_{sp}$  of the Hydrogen-LOX fuel?
- d) What is the effective exhaust velocity of the burning Hydrogen-LOX fuel?

Give the dimensions for each answer. There are 2 SRB's and 3 SSME's. The total weight of the SRB's, orbiter, and external tank at launch is 4.461485 E06 lbs.

- e) What is the total thrust at launch?
- f) Will the system get off the pad?

Let us review the exercise just completed. The first two statements provided you the following:

$$W_b = 9,330 \text{ lb/sec}$$

$$I_{sp} = 260 \text{ sec}$$

You related the words "burns fuel at the rate of" with the definition of  $W_b$  given. To answer the first question, you looked at the symbol for thrust given in the definitions and determined immediately that formula (ii) was to be used. From the formula

$$F = W_b I_{sp}$$

a simple multiplication determines the answer.

$W_b$	9,300	lb/sec
↑		
$I_{sp}$	280	sec
X		
<span style="border: 1px solid black; padding: 2px;">ANS</span>	2,418,000	lb

For question (b), formula (v) was determined appropriate and calculation proceeded similarly. To answer question (f), it was necessary to conclude that the total thrust must be greater than the total weight for lift off, a reasonable conclusion one might reach without an understanding of Newton's laws of physics.

Note that in the set of five formula, there are but two primary relations. The other three are simply equivalent forms that we have provided you in order that the unknown variable be positioned on the left. For example, the first was

$$I_{sp} = \frac{F}{W_b}$$

Multiplying both sides of the equation by  $W_b$  yields

$$I_{sp} W_b = \frac{F}{\cancel{W_b}} \cancel{W_b}$$

Cancelling  $W_b$  on the right side isolates F:

$$I_{sp} W_b = F$$

This is the same equation, given in what is called an equivalent form. Now if we divide both sides of the equation by  $I_{sp}$ ,

$$\frac{\cancel{I_{sp}} W_b}{\cancel{I_{sp}}} = \frac{F}{I_{sp}}$$

or

$$W_b = \frac{F}{I_{sp}}$$

The variable  $W_b$  is isolated giving another equivalent form of the equation. We use this relation when  $W_b$  is the unknown variable to be calculated. To simplify your work in many cases, we will provide both the basic equation or formula and the useful equivalent forms.



## A.7. DIMENSIONS AND THEIR CONVERSION

Numbers obtained from most physical measurements are of two types: pure numbers, and numbers with dimensions of mass, weight, and time. Pure numbers are nearly always formed as ratios of dimensional numbers, so that dimensions cancel out in the result. Mechanical measurements yield numbers with named dimensions or units that are combinations of the basic dimensions or units of mass, length, and time. For example, the unit for force is the pound, but the basic are mass X length/time<sup>2</sup>. When evaluating a formula, we should perform a dimensional analysis if the final units are in question. For example, in the last exercise, dimensional analysis of the formula  $F = W_b t_{sp}$  would yield

$$W_b \left( \frac{\text{lbs}}{\text{sec}} \right) \times t_{sp} (\text{sec}) = F (\text{lbs})$$

A conversion constant has a ratio of dimensions associated with it. Obtaining a conversion constant is the first step in converting dimensions. Let us take time as an example. There are, of course, sixty seconds in one minute, or we may say that 60 seconds is equivalent to 1 minute.

$$60 \text{ secs} = 1 \text{ min.}$$

Dividing both sides by 1 min, we obtain

$$\frac{60 \text{ secs}}{1 \text{ min}} = \frac{1 \text{ min}}{1 \text{ min}}$$

or

$$\frac{60 \text{ secs}}{\text{min}} = 1$$

Note that on the right is the pure number 1, and on the left we have obtained a conversion constant. Since multiplication by 1 does not change a number, we may use the conversion constant as a multiplicative factor. Note also that had we divided by 60 secs, we would have obtained

$$1 = \frac{1 \text{ min}}{60 \text{ sec}}$$

the reciprocal of the above factor. To convert seconds to minutes, we multiply using this factor. To convert minutes to seconds, we use the other.

As an example, we are asked to convert to minutes  $t = 14.85$  seconds. The object is to use the conversion constant with seconds in the denominator so that it will cancel the seconds in time  $t$ .

$$t = 14.85 \text{ sec} \times \frac{1}{60} \frac{\text{min}}{\text{sec}} = 0.2475 \text{ min}$$

Or, suppose we are to convert to ft/min the velocity  $v = 14.85$  ft/sec. We do the following:

$$V = 14.85 \frac{\text{ft}}{\text{sec}} \times 60 \frac{\text{sec}}{\text{min}} = 891.00 \frac{\text{ft}}{\text{min}}$$

We have used a conversion constant in previous exercises, to convert nautical miles to international feet. From the data (in the section Equatorial Earth Radius) in Appendix A, we had obtained

$$1 \text{ E.R.} = .2092572178477690 + 008 \text{ int.ft.} = .3443930885519158 + 004 \text{ n.mi}$$

$$\frac{.2092572178477690 \text{E } 08}{.3443930885519158 \text{E } 04} \frac{\text{int. ft}}{\text{n.mi}} = 1$$

$$.607611549 \text{E } 04 \frac{\text{int. ft}}{\text{n.mi}} = 1$$

Or, there are 6076.1155 International feet in one nautical mile. Check also in Appendix A the section on "Equivalents and Conversion Factors."

Control of dimensions will be made easier if we multiply when converting and check to make certain that the dimension to be converted from always cancels out in the multiplication.

The earth's rotational velocity is

$$\omega_p = .729211585 \text{E } - 04 \text{ rad/sec}$$

There are  $2\pi$  radians in one revolution. There are 86,400 seconds in one day.

**Exercise A.7.1.** Find the angular velocity of the earth in revolutions per day.

$$\begin{aligned} & 0.7292585 \text{E } - 04 \frac{\text{rad}}{\text{sec}} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} \times \frac{86,400 \text{ sec}}{1 \text{ day}} \\ & = 1.0027379 \text{ rev/day} \end{aligned}$$

How many days per revolution? If there are 24 clock hours per day, how many hours per revolution?

The angular velocity and period of rotation of a satellite (the period is the time required to complete one revolution or  $2\pi$  radians) are related by the formulas

$$\omega_p = \frac{2\pi}{P}$$

$$P = \frac{2\pi}{\omega_p}$$

where

$P$  is the period of the earth's rotation, in dimensions of time;  
 $\omega_p$  is the angular velocity of the earth to a precessing equinox, obtained from the Appendix.

**Exercise A.7.2.** Find the period of the earth's rotation (a) in seconds, (b) in hours and decimal fractions of hours, (c) in hours, minutes, and seconds. Compare with answers obtained in previous exercises.

Astronomers call the average angular velocity of a satellite in orbit the *mean motion*. The formulas relating the period and the mean motion are

$$\eta = \frac{2\pi}{P}$$

$$P = \frac{2\pi}{\eta}$$

where

$\eta$  is the mean motion in radians per second, or degrees per minute, or similar dimensions of angle per unit time;  
 $P$  is the period in units of time.

**Exercise A.7.3.**

- Find the mean motion of the orbiter when it is in a circular orbit of 160 n.mi altitude, in units of degrees per minute (see exercise A.5.1.).
- Find the mean motion of the communication satellite of exercise A.5.2.

The average angular position in the orbit is called the *mean anomaly* by astronomers. It is found from the time and the mean motion by means of the formula

$$M = \eta (t - t_0) + M_0$$

Here

- M is the mean anomaly, in degrees,
- $\eta$  is the mean motion, in degrees per unit time,
- t is the current time, in consistent units,
- $t_0$  is the reference time, called the epoch,
- $M_0$  is the mean anomaly at epoch.

**Exercise A.7.4.** Suppose epoch is at midnight on July 18, 1981, and the mean anomaly at epoch is  $40^\circ$ . Find the mean anomaly one hour and 15 minutes after midnight for the orbiter when it is in a circular orbit of 160 n.mi altitude.

#### A.8. MISCELLANEOUS PRACTICE EXERCISES

Here are a set of practice exercises for which the formula to be used may not immediately precede the exercise. Where not given, you are to find from previous pages the appropriate formula. Follow carefully the last two calculator exercises.

**Exercise A.8.1.** The orbital elements of the sun (*pretend* that the sun is in orbit about the earth) in the plane of its motion about the earth are:

$$\begin{aligned} a &= 1.0000023 \text{ A.U. (Astronomical Units),} \\ \epsilon &= 0.01671821 \text{ (this is eccentricity).} \end{aligned}$$

- (a) Find the semimajor axis of the sun's orbit in n.mi.
- (b) Find the period of the sun (see  $\mu$  sun in Appendix).
- (c) Find the mean motion of the sun in degrees per day.

**Exercise A.8.2.** Ionized Hydrogen has a  $I_{sp}$  of about 6000 secs. Find the value of the exhaust velocity in ft./sec. If an ionized hydrogen rocket motor develops 100 lbs of thrust, what is the rate of use of hydrogen, in lbs/sec.?

**Exercise A.8.3.** Steam ( $H_2O$ ) has an  $I_{sp}$  of 600 to 1000 secs, depending upon the temperature. If steam at  $750^\circ R$  develops an  $I_{sp}$  of 800 secs, how many lbs of water is used per second by a rocket motor developing 100 lbs thrust? What is the effective exhaust velocity?

**Exercise A.8.4.** The  $F y^x$  function can be used to find the natural number  $e$  (this is not eccentricity) defined by

$$e = \left(1 + \frac{1}{x}\right)^x \text{ when } x \text{ is very large}$$

Use this to find the limiting value of  $e$ , using  $x = 100$ ,  $x = 1000$ ,  $x = 10,000$ ,  $x = 100,000$ . Compare the answers for each value of  $x$ .

•

**Exercise A.8.5.** The square root of a number can be found by a cut-and-try method by means of the formula

$$X_{n+1} = \frac{X_n^2 + N}{2X_n}$$

where

$N$  is the number,

$X_n$  is the current trial value,

$X_{n+1}$  is the new trial value.

Find the square root of 196. Suppose the first guess is 8.

8

STO 1

2

f y<sup>x</sup>

196

+

RCL 1

+

2

+

ANS

If this guess differs from the previous guess, use it to repeat the process.

**Exercise A.8.6.** Write the calculator program for the cube root of 81, if the formula is

$$X_{n+1} = \frac{2X_n^3 + N}{3X_n^2}$$

and the symbols have the same meaning as given in the previous exercise.

## A.9. CALCULATOR FUNCTIONS

For our purposes now, a function is a rule for converting one number (the number in the x-register, for example) to another number (which then appears in the x-register). Read Section 5 in the Owner's Handbook, pages 85-107. A function may simply be considered a "more complicated" operator. The squaring operator is a function, since it is a rule for converting a number for another (for example, it is the rule for converting 3 to 9, 4 to 16, and so on). While we shall assume that the rule is programmed into the calculator, and that it is exercised by pushing the right buttons, we need to worry about two things:

1. Does the number in the x-register have the right units (e.g., radians or degrees)?
2. Does the rule "work" for the number in the x-register or y-register, in the case of two number functions?

Both concerns require that we think about the problem while performing the calculations. A good rule is to decide, before starting a calculation, on the dimensions to be used, then convert all numbers during calculations. If there is a problem with this now, make all conversions before and write down the list of new numbers with their units.

The calculator has 5 "named" functions. These are

log, ln, sin, cos, tan.

The log and ln functions require that the number in the x-register be a pure number (the number can be units of radians, but not degrees, and it can be in no combination of length, time, and/or mass.) This should cause no trouble, since we assume that the formulae have been stated correctly. The number appearing in the x-register cannot be negative or zero. As examples, use the calculator to find values for:

$\log 1, \log 0, \log -1, \log -8$

$\ln 1, \ln 0, \ln -1, \ln -8.$

Note that when the rule does not "work", the calculator will announce your mistake. Other obvious examples of "illegal" numbers are: division when the divisor is zero, taking  $\sqrt{x}$  when the x-register is negative, and so on. They are "illegal" because the function is "not defined" in the mathematical sense for real numbers.

The function "ln" is called the "natural logarithm". It has as its base the natural number  $e$  (Exercise A.8.4.). Use of calculators and computers makes the logarithm to the base 10, "log," far less useful than before.

The *sin*, *cos*, and *tan* functions require that the number in the x-register be in units of *degrees* or *radians*. The calculator has a mode button  $\boxed{\text{DEG}}$  or  $\boxed{\text{RAD}}$  which tells it what to expect (see page 93 of the Owner's Handbook). To convert X degrees to radians, multiply the degree measure by  $\pi/180$ . To convert radians to degrees, multiply the radian measure by  $\frac{180}{\pi}$ . To convert an angle specified in degrees to radians on the calculator, key in the angle and press  $\boxed{\rightarrow R}$ . To convert an angle specified in radians to decimal degrees, key in the angle and press  $\boxed{D \rightarrow}$ .

**Exercise A.9.1.**

- Find the angle in degrees, minutes, and secs corresponding to 1 radian.
- Find the number of radians in  $91^\circ 40' 23''$ .
- Convert the following angles to degrees, minutes, and seconds:

32.5042 degrees  
57.2175 degrees  
.01745329 radians  
1/9 radian

**Exercise A.9.2.** Construct a table giving values of *sin*, *cos*, and *tan* for:

$0^\circ, 15^\circ, 30^\circ, 60^\circ, 75^\circ, 90^\circ, 105^\circ, 120^\circ, 150^\circ, 180^\circ,$   
 $-0^\circ, -15^\circ, -30^\circ, -60^\circ, -75^\circ, -90^\circ, -105^\circ, -120^\circ, -150^\circ, -180^\circ$

The radial distance of a satellite from the center of the earth is given by the formula

$$r = \frac{a(1 - e^2)}{1 + e \cos \Theta}$$

where

$r$  is the distance from the center of the earth, in units of length,

$a$  is the semimajor axis of the orbit, in units of length,

$e$  is the eccentricity of the orbit (dimensionless),

$\Theta$  is the true anomaly, the angle in the orbit between the satellite and the point of closest approach to the earth (perigee).

Note that the altitude is given by

$$H = r - R_{\text{eq}}$$

where

$H$  is the altitude,

$R_{\text{eq}}$  is the equatorial radius.

**Exercise A.9.3.** If an upper stage is in an elliptical orbit with a semimajor axis of 13,185.39 n.mi and an eccentricity of 0.7267, what is the altitude of the satellite when  $\theta = 0^\circ, 15^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$ ?

**Exercise A.9.4.** The point of closest approach in an elliptical orbit is called "perigee," and the farthest point is called the "apogee." Clearly, the denominator  $(1 + e \cos \Theta)$  must be the maximum, to make  $r$  smallest (at perigee), and smallest, to make  $r$  largest (apogee). At what value of  $\Theta$  does perigee and apogee occur? Note that  $(1 + e \cos \Theta)$  is maximum at the maximum value of  $\cos \Theta$ .

•

The velocity of a satellite is given by the formula

$$v = \sqrt{\mu \left( \frac{2}{r} - \frac{1}{a} \right)}$$



where

- $r$  is the radial distance from the center of the earth, units of length;
- $a$  is the semimajor axis in units of length;
- $\mu$  is the gravitational constant, in units consistent with  $r$ ,  $a$ , and the time.

**Exercise A.9.5.** Find the velocity for the upper stage in the previous two exercises for values of  $\theta = 0^\circ, 15^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$ . What is the velocity at perigee? At apogee? Give dimensions of your answer.

### A.10. INVERSE FUNCTIONS

A function is a rule for converting one number to another. The *inverse function* is the rule for converting the second number back to the first. Another way of thinking: a function "does" and the inverse "undoes." The function appears in gold on your calculator. When it has an inverse, the inverse appears directly to the right of it in blue lettering.

Most functions have inverses. A few do not. Examples of cases when the inverse fails are:

- (a) division is the inverse of multiplication, except when the number is zero;
- (b) the  $f \sqrt{x}$  is the inverse of  $g x^2$ , except when the number is negative. On the other hand  $g x^2$  is *always* the inverse of  $f \sqrt{x}$ . (Can you determine why?)

The inverse of the  $\ln$  function is  $e^x$ . As examples of its use do the following.

3	5	7
f ln	f ln	f ln
$g e^x$	$g e^x$	$g e^x$
<b>ANS</b>	<b>ANS</b>	<b>ANS</b>

3	5	1
$g e^x$	$g e^x$	$g e^x$
f ln	f ln	f ln
<span style="border: 1px solid black; padding: 2px;">ANS</span>	<span style="border: 1px solid black; padding: 2px;">ANS</span>	<span style="border: 1px solid black; padding: 2px;">ANS</span>

For a horizontal burn (that is, gravity acts at right angle (90°) to or perpendicular to the direction of thrust), the weight of fuel burned in increasing the speed of a rocket is given by the formula

$$W_{TB} = W_F \left( e^{\frac{\Delta V}{c}} - 1 \right)$$

where

- $W_{TB}$  is the total weight of fuel burned, in lbs;
- $W_F$  is the final weight of rocket and payload, after fuel has been burned, in lbs.;
- $\Delta v$  is the increase in velocity, in feet per second.  $\Delta v$  is the greek symbol pronounced "delta" and usually means "change in" or "difference";
- $c$  is the effective exhaust velocity of the burning fuel, feet per second.

**Exercise A.10.1.** The orbiter has about 45 minutes to get far enough away from an upper stage so that the exhaust particles from the burning rocket will not damage windows or tiles. To move 30 nautical miles away in 45 minutes requires that the orbiter increase its speed relative to the upper stage by 70 feet per second. If the  $I_{sp}$  of the OMS fuel (Hydrazine) is 260 secs, and the orbiter weighs 200,000 lbs after the OMS burn, how much hydrazine is required, in lbs?

ANS: 1680.8 lbs

**Exercise A.10.2.** The OMS engines and the RCS thrusters both use Hydrazine, which has an  $I_{sp}$  of 260 secs. The OMS engines develop 12,000 lbs thrust total, and the RCS jets develop about 900 lbs. thrust per jet. Two jets can be used to separate from the Upper Stage. Which saves the most fuel, the OMS or the RCS, for a  $\Delta v$  of 70 ft/sec? (ANS: Both exactly the same. Explain.)

**Exercise A.10.3** How many lbs. of OMS fuel is burned for each ft/ sec of  $\Delta v$  acquired, when the orbiter weighs 200,000 lbs? When the orbiter weighs 180,000 lbs?

The amount of fuel required to perform a maneuver is important, since this fuel must be loaded before launch, and there is a limit to how much can be loaded. The flight planners must determine if enough fuel is aboard to perform the given maneuvers.

The inverse problem is to find the  $\Delta v$  when the weight of fuel burned is known. This is given by the formula

$$\Delta v = + c \ln \left( 1 + \frac{W_{TB}}{W_F} \right)$$

where the symbols have the same meaning as before.

**Exercise A.10.4.** Find the  $\Delta v$  when 1200 lbs of Hydrazine is burned during a separation maneuver, when the orbiter weighs 200,000 lbs. Note that in all of the above exercises, the burns were made in a *horizontal* direction. If the thrust acts against (or with) the pull of gravity, the formulas for  $\Delta v$  do not apply.

The  $\Delta v$  required to change from one *circular orbit* to another close by is given by

$$V_{\text{circ}} = \sqrt{\mu} a_1^{-3/2} \frac{(a_2 - a_1)}{2}$$

where

$a_1$  is the semimajor axis of the first circular orbit,

$a_2$  is the semimajor axis of the second circular orbit,

$\mu$  is the gravitational constant, in consistent units.

This formula applies when  $(a_2 - a_1)$  is less than 100 n.mi.

**Exercise A.10.5.** The orbiter is in a 160 n.mi. orbit and a payload is in a separate circular orbit. The orbiter is 10 n.mi. below the payload, that is,

$$(a_2 - a_1) = 10 \text{ n.mi.}$$

- a) How much  $\Delta v$  is required to transfer the orbiter into the same orbit as the payload?
- b) How many lbs of Hydrazine is required?
- c) Suppose the orbiter is in the higher orbit, how does this change the answers?

**Exercise A.10.6.** If the orbiter is in the same circular orbit as a payload, but several miles behind, what maneuvers would the orbiter make to catch up?

### A.11. THE TRIGONOMETRIC FUNCTIONS AND THEIR INVERSES

The sin, cos, tan functions appear in gold letters and their inverses  $\sin^{-1}$ ,  $\cos^{-1}$ ,  $\tan^{-1}$  appear in blue letters to the right. As examples, run the following h DEG mode:

15	30	60
f sin	f cos	f tan
g sin <sup>-1</sup>	g cos <sup>-1</sup>	g tan <sup>-1</sup>
ANS	ANS	ANS

•

The distance between two points on the surface of the earth is given by the formula (See page 97 of the Owner's Handbook)

$$D = 60 \cos^{-1} (\cos (lat_1) \cos (lat_2) \cos (long_1 - long_2) + \sin (lat_1) \sin (lat_2))$$

where

- D is the distance, in nautical miles;
- lat<sub>1</sub>, lat<sub>2</sub> are the latitudes of the first and second points respectively, with southern latitudes entered as negative numbers;
- long<sub>1</sub>, long<sub>2</sub> are the longitudes of the first and second points respectively, with eastern longitudes entered as negative numbers.

**Exercise A.11.1.** Two tracking stations, one near Los Angeles and the other near Baytown, Texas have the locations:

Los Angeles	34° 20' 42" N	lat
	118° 42' 05" W	long
Baytown	30° 41' 05" N	lat
	95° 06' 32" W	long

What is the distance between the tracking stations?

•

In solving for the location of an object in orbit, Kepler introduced an Angle E, called the *eccentric anomaly*. The formulas relating the *eccentric anomaly* to the *true anomaly* are

$$E = 2 \tan^{-1} \left( \sqrt{\frac{1-e}{1+e}} \tan \frac{\Theta}{2} \right)$$

$$\Theta = 2 \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \tan E/2 \right)$$

**Exercise A.11.2.** For the orbit in exercise A.10.5., find values of E for each value of  $\Theta$ . How much do E and  $\Theta$  differ at perigee? At apogee? At  $\Theta = 90^\circ$ ?

## A.12. QUADRANT CHECKS

If the circle is divided into four equiangular parts

Part I	0°-90°
Part II	90°-180°
Part III	180°-270°
Part IV	270°-360°

each part is called a *quadrant*. Quadrants are numbered counterclockwise from I to IV.

**Exercise A.12.1.** With the above system of numbering quadrants, what is the logically consistent way to label the direction of increasing angles? Decreasing angles?

One problem with trigonometric functions is the inability to find the angle using the inverse function when the angle lies in the second or third quadrant.

Examples are

h	Deg	Mode:
---	-----	-------

45°	135°	135°	225°
sin	sin	cos	cos
sin <sup>-1</sup>	sin <sup>-1</sup>	cos <sup>-1</sup>	cos <sup>-1</sup>
ANS	ANS	ANS	ANS

The situation is that the inverse function always yields an angle called the *principal value*.

<u>Inverse Function</u>	<u>Principal Value</u>
sin <sup>-1</sup>	-90° to +90°
cos <sup>-1</sup>	0 to +180
tan <sup>-1</sup>	-90 to +90°

If we have only the value of the sum of the angle, or if we have only the value of the cosine of the angle, we are stuck with the principal value.

**Exercise A.12.2.** Using the tables constructed in Exercise A.9.2., find corresponding inverses and compare with the original angles.

If we have both the sine value and the cosine value of the angle, we can find the value of the angle in the interval 0 to 360° by using the inverse tangent function and the truth table.

Given:  $\sin \theta, \cos \theta$

Find:  $\theta = \tan^{-1} \frac{\sin \theta}{\cos \theta}$

sin θ	+	-
cos θ	+	-
+	0°	+ 360°
-	+ 180°	+ 180°

The truth table is used as follows.

1. If  $\sin \theta$  is positive, and  $\cos \theta$  is positive do nothing to  $\theta$ .
2. If  $\sin \theta$  is positive, and  $\cos \theta$  is negative, add  $180^\circ$  to  $\theta$ .
3. If  $\sin \theta$  is negative, and  $\cos \theta$  is positive, add  $360^\circ$  to  $\theta$ .
4. If  $\sin \theta$  is negative, and  $\cos \theta$  is negative, add  $180^\circ$  to  $\theta$ .

**Exercise A.12.3.** Using tables constructed in Exercise A.9.2., use the truth table to find the angles from the inverse tangent.

Another problem with trigonometric inverses is that we can never discover if the original angle was outside the interval 0 to  $360^\circ$ . As examples, show that the truth tables yield the same result for the following angles

$$\theta = 135, 495^\circ, 855^\circ,$$

$$\theta = -225, -585^\circ.$$

That is, taken  $\sin \theta$ ,  $\cos \theta$ , and find  $\theta = \tan^{-1} \frac{\sin \theta}{\cos \theta}$  and apply truth table.

There is apparently nothing we can do about this, so we adapt to it.

## A.13. EVALUATION

### A.13.1. Learning Objectives

The principal objectives of Section A were threefold: first, to teach basic computational skills through use of the calculator; second, to explore calculator capability and limitations in examining formulas and functions used in flight planning; and thirdly, to teach the basics of problem analysis, relating word sets to formulas to calculator logic.

Having completed this section and related references, you should have an understanding of:

- the logic and capability of the calculator in solving simple problems involving basic mathematical functions
- the basic components of formulas and the rules for problem solving

- the calculator functions which simplify problem solving
- basic mathematical and space flight related terminology.

### A.13.2. Outcome Measures

You should now be able to:

1. Write the calculator code or key steps in evaluation of a formula;
2. Read a simple problem statement and select given variables and those to be calculated;
3. Evaluate formulas with the calculator containing powers, roots, logarithms, exponentials, trigonometric functions, or inverse functions, or all;
4. Make use of calculator conversions, such as radians to degrees, fractions of degrees to minutes, etc.;
5. Make conversions of units of measure from one set of measurements through use of equivalent relations;
6. Demonstrate a passing acquaintance (they will become clearer as we proceed) with the following terms.

radius	altitude	axis
velocity	force	mass
angle	radian	rotation
degree	period	epoch
revolution	satellite	orbit
rocket	thrust	launch
conversion	dimension	function
formula	equation	impulse
mean motion	transfer	payload
mean anomaly	true anomaly	perigee
elliptical orbit	eccentricity	eccentric anomaly
astronomical units	semimajor axis	international feet
nautical miles	angular velocity	exhaust velocity
specific impulse	gravitational constant	



## **B. PROGRAMMING THE CALCULATOR**

### **B.0. INTRODUCTION**

You have been solving problems manually on the calculator using a step-by-step process. In this section you will begin to write programs for the calculator that will enable you to press a single key on the calculator, commanding it to execute all the steps necessary for the solution of a problem. It is not quite that simple, however.

But for you now a program will consist simply of a series of calculator keystrokes pressed to solve a problem manually. The calculator will remember these keystrokes when keyed in, then execute them in given order at the press of a single key. We introduced use of the storage key in Exercise A.8.5. Programs are extremely useful when solving problems involving use of the same process (or formulas) but containing different values for the variables.

You must learn how to construct programs, load them, run them on the calculator, and record them for future use. Some programs are simple to write, others are not. Understanding your calculator will also serve to provide the foundation for future work in programming larger computers.

### **B.1. PROGRAMMING PROCEDURE: THE JULIAN DAY NUMBER**

Review the material on pages 17-23 in the HP-67 Owner's Handbook and Programming Guide, and do the suggested exercise. This material provides an introduction to running a prerecorded program, creating, loading, and running a program, and recording of the program. Now turn to Section 6, pages 123-145, entitled Simple Programming, and do the suggested activities. Spend some time learning the calculator code, pages 129-130.

A stored program is relatively simple. It consists of code

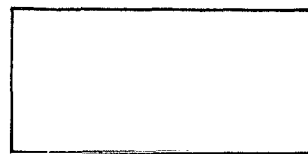
g LBL f (I) or f LBL (L),  
set of instructions,  
h RTN.

The first entry is a LABEL designation, and the last entry is a RETURN (RTN). The first identifies the beginning of program or first step of program memory through the label L (one of letter keys A through E) or label I (one of letter keys a through e), and the first RTN identifies the end of that program. These are required by the calculator control logic, otherwise the calculator could not begin or terminate.

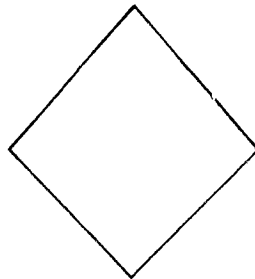
To illustrate programming construction, we will write a program for computing Julian Day Numbers from the Calendar Date. Note that writing a stored program is a form of problem-solving since we must prepare:

- (1) a list of data that will be contained in the answer, together with dimensions;
- (2) a list of input, or "givens," together with dimensions, from which the answer must be constructed;
- (3) a list of formulas connecting the given data and the answers, or output.

The plan of a solution, in programming construction, is called a "flow diagram" or flow chart. Such diagrams are useful in solving programming problems, but their use is not restricted to this area. It is preferable to some to work out the calculation process, constructing a flow diagram as they go, and then write the program. Others draw a flow diagram after the program is completed for reference purposes. The most important standard symbols used are:



for input and output  
or  
general operations



for decisions

These symbols are joined by lines, with logical flow normally from top to bottom unless indicated otherwise by arrows. Flow diagrams are discussed on pages 140-145 of the Owner's Handbook.

A "subroutine" is a section of a program that is performed as a separate entity, and stored ready for use when needed in a master program. It is often convenient to write a very complicated program as a set of subroutines or subprograms. For our purposes a subroutine will be no more than evaluation of a single formula.

While a stored program is simple, the *preparation for* and *documentation of* a program to be stored may not be simple. Unless certain rules are strictly followed, chaos may result. The procedure is as follows:

1. Define:
  - Input data
  - Output data
2. Select dimensions for:
  - Input data
  - Calculations
  - Output data
3. Write down formulas.
4. Note logic tests required, branches, and subroutines required, if any.
5. Do flow diagram of calculations.
6. Test the code through:
  - Editing
  - Using known results from other programs.
  - Exercising all paths
7. Document the program by:
  - Title
  - Date written
  - Brief description of what program does
  - Formulas used
  - Operating limits and warnings
  - Flow diagram
  - Operating instructions
  - Code and notations

For astronomical considerations time is an important factor. We can describe the positions of celestial objects in terms of time; we can describe the positions of earth orbiting bodies in terms of time. A system of Julian Day Numbers was introduced to reduce computational labor in the comparison of time spans made many years apart and to avoid time ambiguities. The Julian Date, which we will use in other calculations, is given for every day of the year in the *American Ephemeris* and *Nautical Almanac*. Its calculation, however, will be helpful. The Julian calendar epoch was chosen to be January 1, 4713 BC, and it assumes the length of a year to be exactly 365.25 days. The Julian Date may be calculated from the relation

$$JD = (365.25) y + (30.60) m + \text{Day} + 1720981.5,$$

where

$y = \text{year} - 1$ , if month is less than or equal to 2 ( $\leq 2$ ),

$y = \text{year}$ , if month is greater than 2 ( $> 2$ ),

$m = \text{month} + 13$ , if month is less than or equal to 2 ( $\leq 2$ ),

$m = \text{month} + 1$ , if month is greater than 2 ( $> 2$ ).

This relation has conditional statements: depending on the month, the calculation may go one of two ways, with variables  $y$  and  $m$  given by

$y = \text{year} - 1$

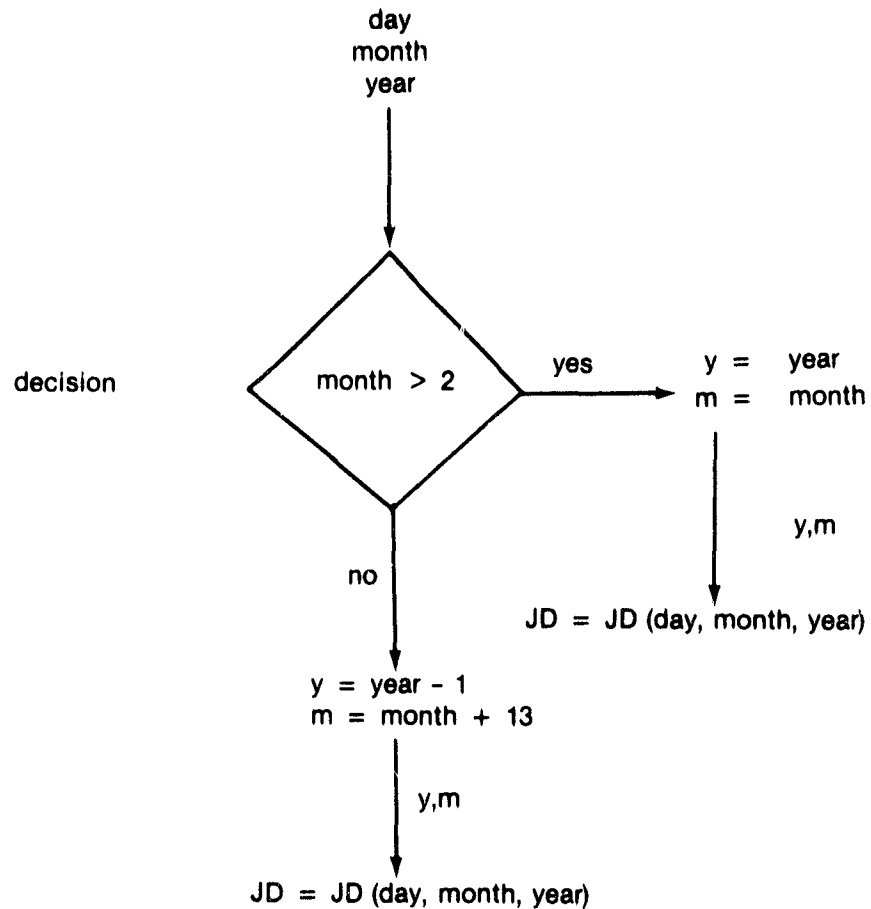
$m = \text{month} + 13$ ,

or

$y = \text{year}$ ,

$m = \text{month} + 1$ .

In a flow diagram, this would be described by



Let us now construct the program documentation. With a present calendar date (day, month, year) as input and Julian Date as output, we start with the preliminaries, program description and formula. Follow carefully this documentation. Key in the program as coded.

---

**Program Documentation (1)**  
(Title, Date, Brief Description, Formulas, Limits)

---

**Title:** Julian Day Number from Calendar Date

**Date:**

**Name:**

**Program Description:**

Program computes Julian Day Number for 00 Hrs. 00 Min. 00 Sec for input calendar date

**Equations, Variables, etc:**

$$\text{Julian Day No.} = (365.25) y + (30.6001)m + \text{Day} + 1720981.5$$

where

y = year - 1, if month less than or equal to 2 (  $\leq 2$  ),

y = year, if month greater than 2 (  $> 2$  )

m = month, + 13, if month less than or equal to 2 (  $\leq 2$  )

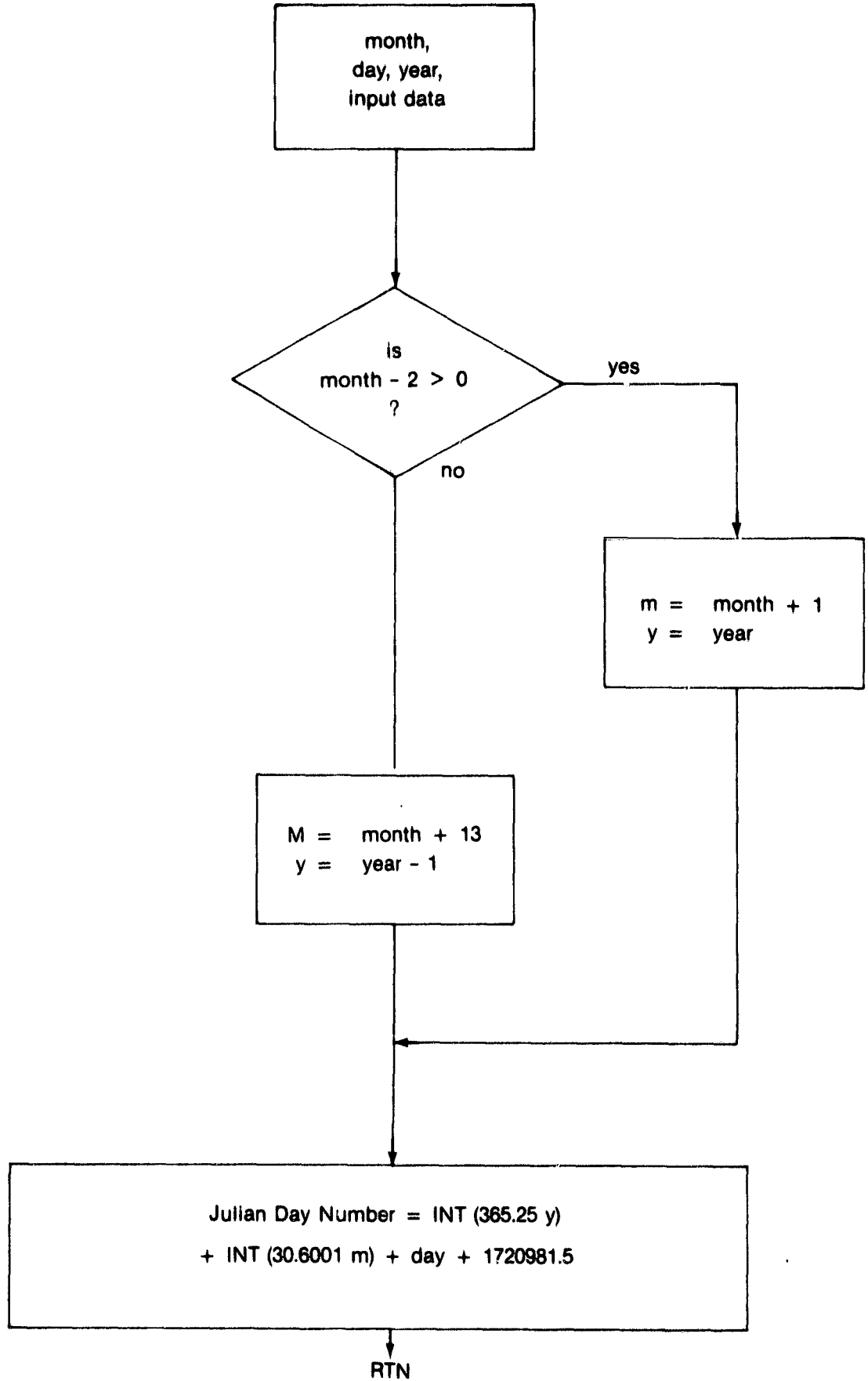
M = month + 1, if month greater than 2 (  $> 2$  )

**Operating Limits:**

Valid for calendar dates March 1, 1900 to February 28, 2100

Program Documentation (2)  
(Flow Diagram)

Title: Julian Day Number from Calendar Date



**Program Documentation (3)**  
**(Operating Instructions)**

**Title: Julian Day Number from Calendar Day**

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Key in program		<input type="text"/> <input type="text"/>	
2	Key in month		<input type="text"/> <input type="text"/>	
3	Store in register 1		STO <input type="text"/> 1	
4	Key in day		<input type="text"/> <input type="text"/>	
5	Store in register 2		STO <input type="text"/> 2	
6	Key in year		<input type="text"/> <input type="text"/>	
7	Store in register 3		STO <input type="text"/> 3	
8	Key in 1720981.5		<input type="text"/> <input type="text"/>	
9	Store in register 4		STO <input type="text"/> 4	
10	Key in 365.25		<input type="text"/> <input type="text"/>	
11	Store in register 5		STO <input type="text"/> 5	
12	Key in 30.6001		<input type="text"/> <input type="text"/>	
13	Store in register 6		STO <input type="text"/> 6	
14	Start Program		GSB <input type="text"/> 0	J.D.
			<input type="text"/> <input type="text"/>	
	For new date bypass		<input type="text"/> <input type="text"/>	
	steps 8 thru 13		<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	
			<input type="text"/> <input type="text"/>	





**Exercise B.1.1.** Write and document below a subroutine for calculating the altitude of a satellite from its orbital parameters and the true anomaly (Exercise A.9.3). Use documentation sheets which follow.

**Exercise B.1.2.** Write and document subroutines for calculating:

- a) the velocity of a satellite from its orbital parameters and the true anomaly (Exercise A.9.5),
- b) the velocity at perigee and the velocity at apogee from orbital parameters (Exercise A.9.4),
- c) the period of a satellite (Exercise A.5.1),
- d) the mean motion (Exercise A.8.1),
- e) the mean anomaly from the mean motion (Exercise A.7.4).

---

**Program Documentation (1)**

---

**Title:** Altitude of Satellite from Orbital Parameters

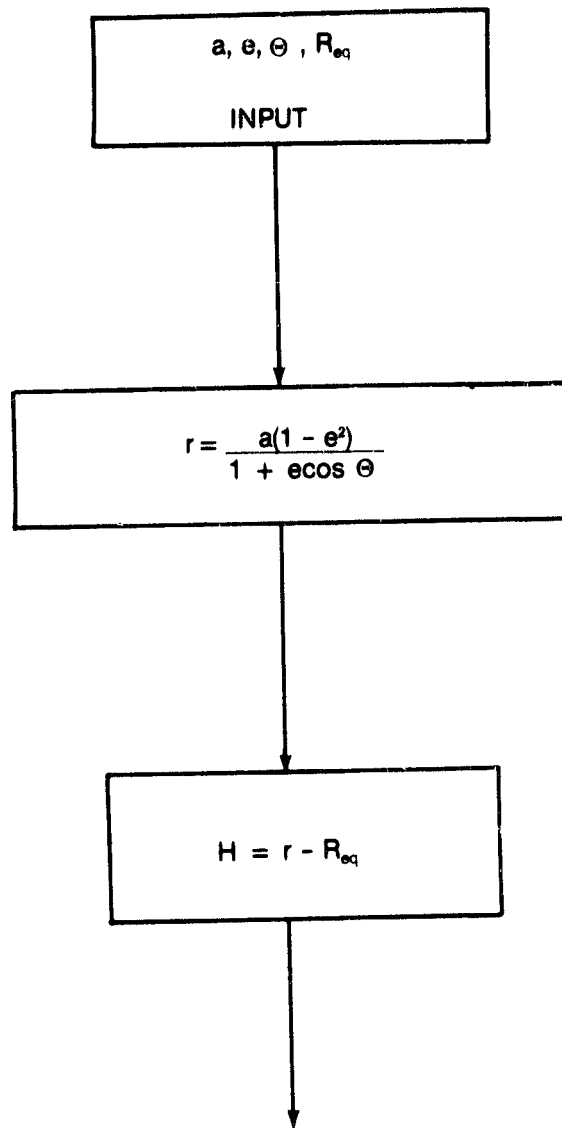
**Date:**

**Name:**

**Program Description:**

**Equations, Variables, etc:**

**Operating Limits:**







## B.2. POSITION IN PLANE OF ORBIT FOR NEARLY CIRCULAR ORBITS

For small eccentricities ( $e$ ), the true anomaly may be found from the formula,

$$\Theta = M + 2e \sin M + \frac{5}{4} e^2 \sin 2M \\ + \frac{e^3}{12} (13 \sin 3M - 3 \sin M)$$

The position vector of a satellite can be found at any time  $t$  by the following procedure.

Given:  $M_0$ ,  $t_0$ ,  $a$ ,  $\mu$ ,  $e$  and time  $t$ ,

Compute:

$$P = \frac{2 \pi a^{3/2}}{\sqrt{\mu}} \text{ (Exercise A.5.1.)}$$

$$\eta = \frac{2 \pi}{P} \text{ (Exercise A.8.2.)}$$

$$M = \eta (t - t_0) + M_0 \text{ (Exercise A.8.3.)}$$

Test:

if  $e > 0.1$  STOP with  $e$  in X-Reg.

$e < 0.1$  GTO next step.

Compute:  $\Theta$  from  $e$  and  $M$  using formula above.

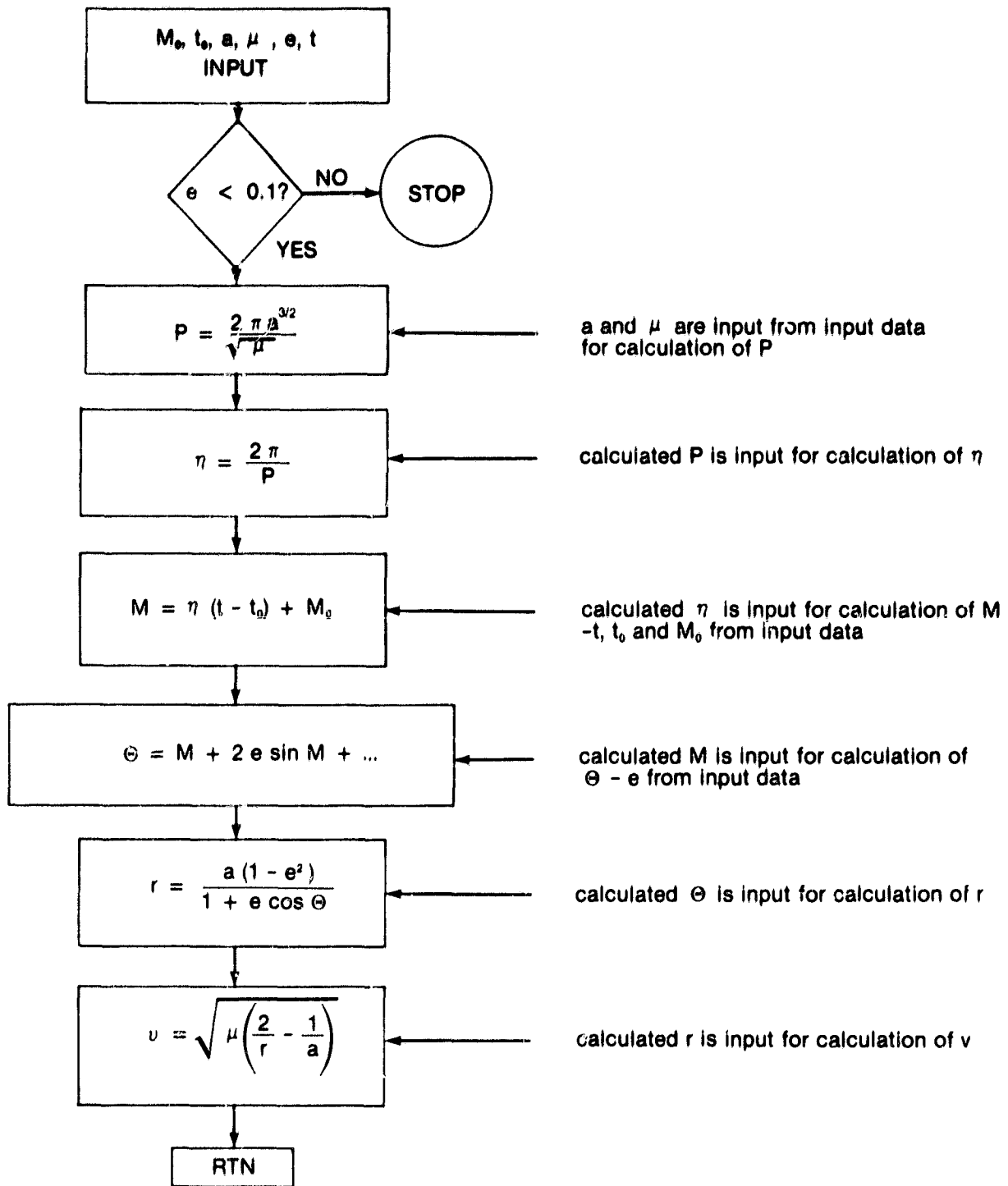
Compute:

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \text{ (Exercise A.9.3.)}$$

$$V = \sqrt{\mu \frac{2}{r} - \frac{1}{a}} \text{ (Exercise A.9.5.)}$$

The output data:  $P$ ,  $\Theta$ ,  $r$ ,  $V$  at time  $t$ .

**Exercise B.2.1.** Write and document a calculator stored program for obtaining  $\Theta$ ,  $r$ , and  $V$  at time  $t$ . (Specify dimensions of output!)





**Exercise B.2.2.** Modify the above program so that the output is the altitude  $H$ , as well as  $\Theta$ ,  $r$  and  $V$  at time  $t$ .

Measures of a distance  $r$  and an angle  $\Theta$  are called the "polar coordinates" of a point. Topics involving motion about a point can better be investigated with polar coordinates rather than by rectangular coordinates (ordered pairs of real numbers  $(x, y)$ ). Graphs of polar coordinates are generally done on polar graph paper, paper with angular and radial divisions. To make a graph we mark the point for each order pair  $(r, \Theta)$ . For example, for  $r = 1 + \cos \Theta$ , a table could be constructed from  $r$ -values calculated by inputting different values of  $\Theta$ .

$\Theta$	$0^\circ$	$45^\circ$	$90^\circ$	$120^\circ$	$150^\circ$	$180^\circ$	$210^\circ$	$240^\circ$	$270^\circ$	$300^\circ$	$330^\circ$
$r$	2.0	1.7	1.0	0.5	0.1	0	0.1	0.5	1.0	1.5	1.9

Using polar graph paper, find the point for each pair  $(r, \Theta)$ , see if a pattern develops, and then join the plotted points with a smooth curve.

**Exercise B.2.3.** Using the calculator program from the previous exercise, let

$$t_0 = 0$$

$$M_0 = 0$$

Find  $r$  and  $\Theta$  for  $t = 0$ . Input  $t$  in 15 minute intervals, draw up a table, and plot the values of  $r$  vs  $\Theta$  on polar graph paper for:

- the orbiter in a 160 n.mi. circular orbit ( $e = 0.0$ ),
- the orbiter in an orbit with  $a = 3875.1944$  n. mi. and  $e = 0.07$ ,
- the orbiter in an orbit with  $a = 4004.3676$  n.mi. and  $e = 0.1$ .

Plot all on the same graph paper. The graph will show a circular orbit and two elliptical orbits with different eccentricities.

**Exercise B.2.4.** Find the velocity for each orbit above when all are coincident (take program  $V$  output at  $\Theta$  point seen in graph). The difference in velocity is the  $\Delta v$  required to transfer from one orbit to the other at the point of coincidence.

**Exercise B.2.5.** Write a program to calculate amount of fuel in pounds when the  $I_{sp}$ ,  $\Delta v$ , and orbiter weight are given. How many pounds of hydrazine is required to transfer between any two orbits above if the orbiter weighs 200,000 lbs? (See Exercises A.6.2. and A.10.1.)

### B.3. ELLIPTICAL ORBITS

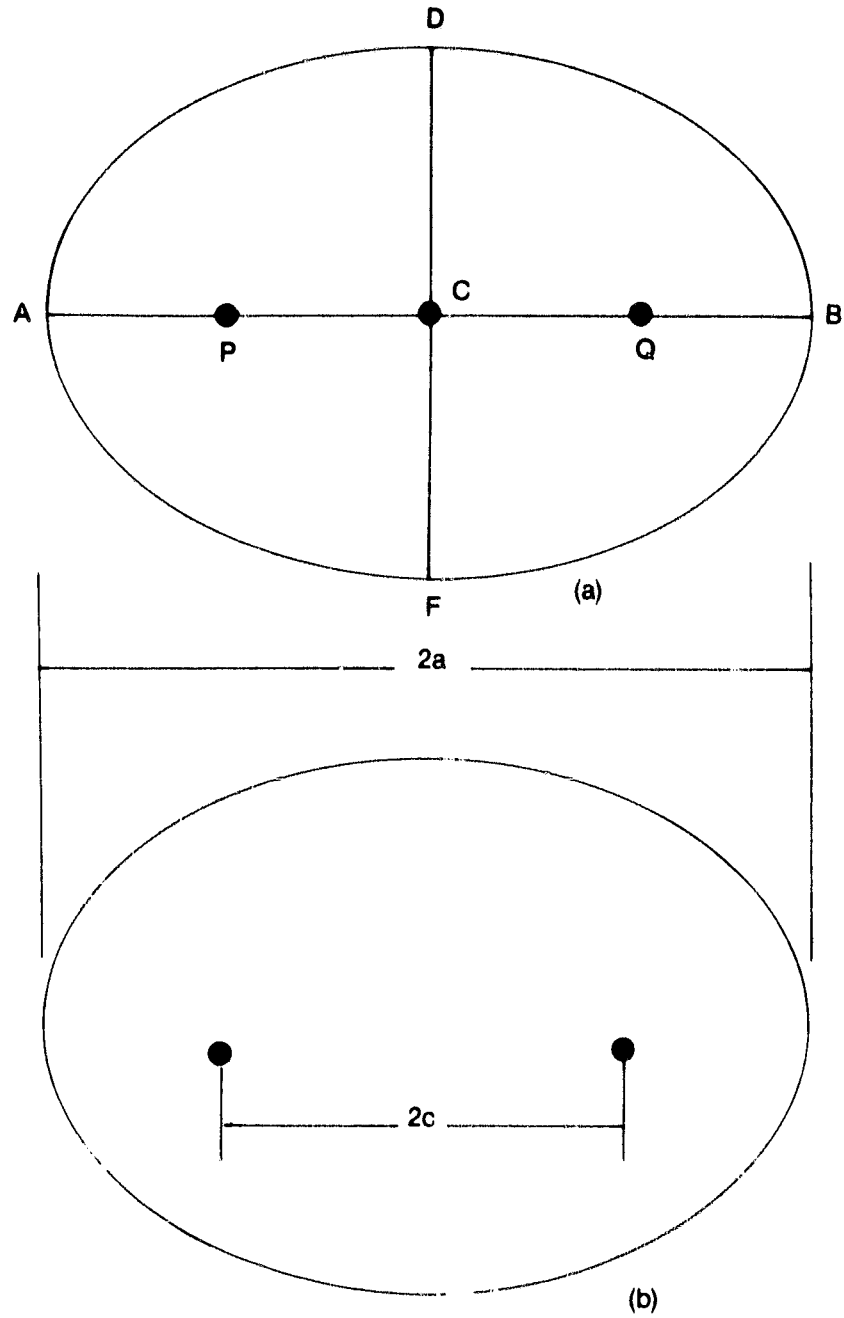
In Exercise B.2.3 a set of orbits was plotted. The first orbit was circular, but the more general path followed by an orbiting body is elliptical. By mathematical definition an ellipse represents the set of all points  $(x, y)$  in an  $x, y$  plane, the sum of whose distances from two distinct fixed points (foci) is constant. If we fasten the end of a fixed line of string to thumb tacks and draw the string taut with a pencil, then the traceable path of the pencil will be an ellipse. The thumb tacks in this case would be the focal points (foci) of the ellipse.

In Figure 1B(a), C is the center of the ellipse and points A and B are the vertices of the ellipse. The points P and Q are called the focus points, or the foci of the ellipse. In every ellipse, the foci are located along the major axis between the vertices and the center.

The length of the semimajor axis is generally denoted by the letter  $a$  as shown in Figure 1B(b). Thus the length of the entire major axis is  $2a$ . The length of the semi-minor axis is generally denoted by the letter  $b$ . The length of the segment from the center to one foci is generally denoted by the letter  $c$ ; thus the length of the segment from one foci through the center to the other foci is  $2c$ .

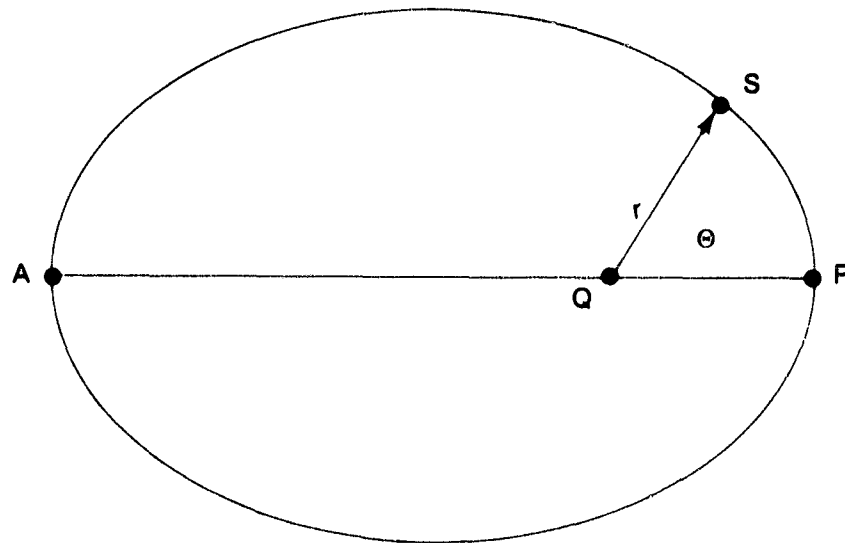
Since the foci are located along the major axis between the vertices and the center,  $c$  is greater than zero but less than  $a$  ( $0 < c < a$ ), and therefore as  $c$  becomes small approaching zero, the foci approach the center and the ellipse becomes more circular. As  $c$  approaches  $a$  in length, the foci approach the vertices and the ellipse flattens out. Thus, the shape of an ellipse can be described in terms of the relative sizes of  $a$  and  $c$ . This common measure of the shape of an ellipse is called its eccentricity, given by the ratio of  $c$  to  $a$ .

Six orbital elements are used in describing a satellite's path in space. Three of them define the orientation of the orbit with respect to a set of axes, two of them define the size and shape of the orbit, and the sixth, with the time, defines the position of the body within the orbit at that time. For a satellite of the earth traveling in an elliptic orbit, the fundamental reference plane is the equator.

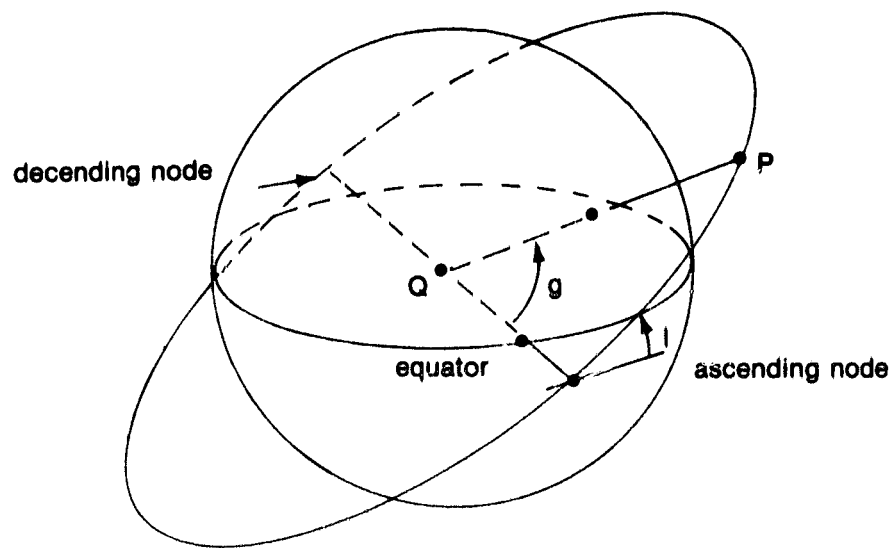


**Figure 1B.** Properties of the Ellipse. In (a),  $C$  is the center of the ellipse,  $AB$  the major axis,  $AC$  and  $CB$  the semimajor axes,  $DF$  the minor axis, and  $P$  and  $Q$  are foci of the ellipse. In (b), the distance between foci is  $2c$ : with  $a$  as the semimajor axis, the eccentricity is defined by  $e = c/a$ .

The six elements defining the orbit and the position of the body in it are the semi-major axis and eccentricity of the elliptic orbit, longitude of the ascending node (to be discussed in later sections), argument (angle) of perigee, inclination of the orbital plane toward the equator, and the time of the passage through perigee, or the mean anomaly at the initial moment of time. Perigee, apogee, and true anomaly are described in Figure 2B; argument of perigee and angle of inclination are described in Figure 3B.



**Figure 2B.** An elliptical orbit. For a satellite at position S,  $r$  is the radius vector from the center of the earth Q. P, the point of closest approach to earth, is *perigee*. A is *apogee*. PA is the *line of apsides*. The angle  $\theta$ , measured between QP and QS, is called the *true anomaly*.

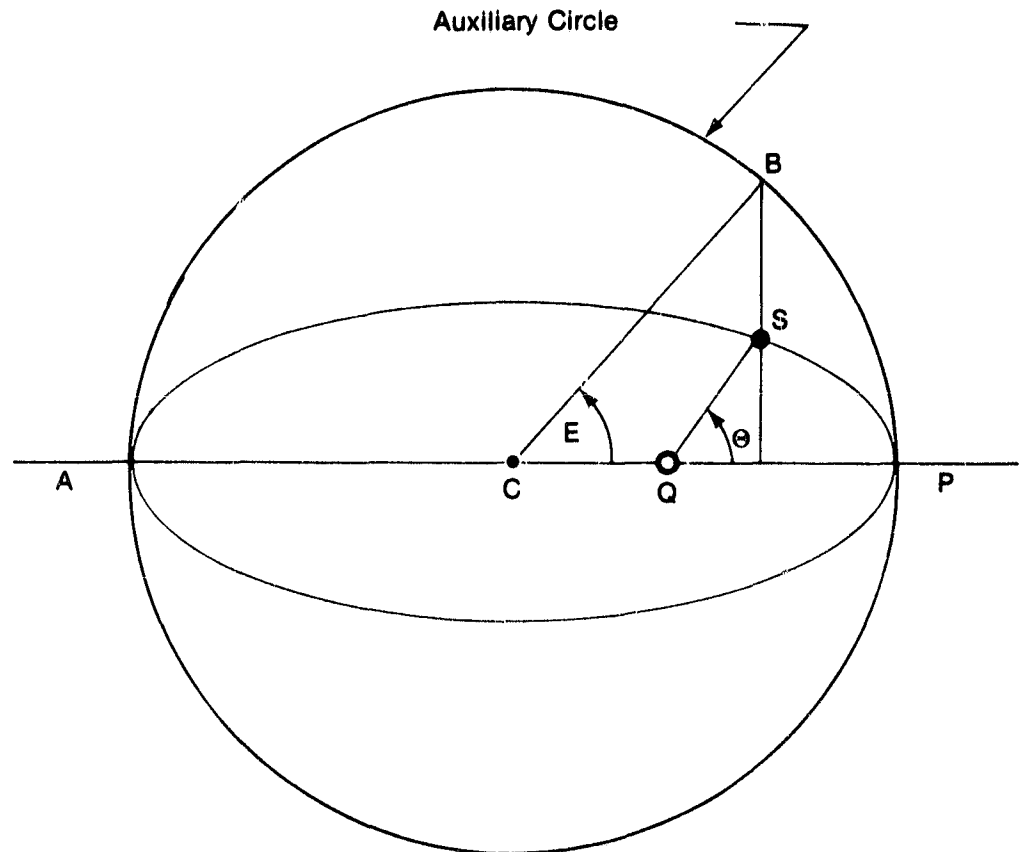


**Figure 3B.** Earth orbit. With P denoting perigee,  $g$  is the argument of perigee for an orbit with angle of inclination  $i$ .

#### **B.4. KEPLER'S EQUATION**

Johannes Kepler (1571-1630) formulated three major laws of planetary motion. The first was that the orbit of each planet is an ellipse with the sun always located at one focus. But a fundamental problem of motion for a satellite in any elliptical orbit (earth about sun or object about earth) is to find its position at any given time. Kepler worked on this problem for twenty years, then gave up.

The solution that he found and thought unsatisfactory, however, has not been improved upon in the more than 300 years since his death. It was derived from his second law which indicates how the speed of a satellite changes in its orbit. Kepler defined a new angle, the eccentric anomaly, by means of an auxiliary circle of radius equal to the semimajor axis and a geometric construction, as shown in Figure 4B.



**Figure 4B.** Eccentric Anomaly. Constructing a circle about the elliptical center C as shown, the eccentric anomaly E can be defined through the true anomaly  $\theta$ .

The formula derived is

$$M = E - e \sin E$$

where

M is the mean anomaly,

E is the eccentric anomaly,

e is the eccentricity.

If M is known, E may then be found by approximation methods. We will use that which is called the Newton-Raphson method. We use the relation

$$E_{n+1} = E_n + \frac{M - E_n + e \sin E_n}{1 - e \cos E_n}$$

where n is a number starting with zero representing the n<sup>th</sup> approximation. The 0<sup>th</sup> approximation for E is M for eccentricities smaller than 0.1. That is to say, for  $e < 0.1$ , let  $E_0 = M$ .

The first approximation for E is

$$E_1 = E_0 + \frac{M - E_0 + e \sin E_0}{1 - e \cos E_0}$$

If the difference  $E_1 - E_0$  is large, we take the second approximation for E:

$$E_2 = E_1 + \frac{M - E_1 + e \sin E_1}{1 - e \cos E_1}$$

We then check the difference  $E_2 - E_1$ . If that is large we continue with  $E_3$ , etc., until the difference  $E_{n+1} - E_n$  is so small that further calculation is unnecessary. A calculation such as this, while appearing complex and lengthy, may easily be done by the calculator through programming. We will do this in the next exercise.

Having obtained a value for E, the true anomaly can be calculated from

$$\Theta = 2 \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right)$$

with

$$\eta = \sqrt{\frac{\mu}{a^3}}$$

and

$$M^* = \eta (t - t_0) + M_0$$

Kepler's equation is solved from a given  $t$ ,  $t_0$ ,  $M_0$  and  $\eta$ . Programming for a solution is given in the following exercise. Since it is a more complex exercise, the steps will be provided. Study the flow diagram, and give attention to the approximation or iteration procedure for determining  $E$ .

**Exercise B.4.1.** Write a program to solve Kepler's equation.

Inputs:  $t$ ,  $t_0$  minutes  
 $M_0$  degrees  
 $a$  n.mi

Constants:  $\mu$  (n.mi)<sup>3</sup>/sec<sup>2</sup>  
 60 sec/min  
 180 for deg to radian

Computations:

$$\eta = \sqrt{\mu a^{-3/2}} (60) \text{ rad/min}$$

$$M = \eta (t - t_0) + M_0 (\pi / 180) \text{ rad}$$

$$E_{n+1} = E_n + \frac{M - E_n + e \sin E_n}{1 - \cos E_n}$$

$$\Theta = 2 \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \tan E/2 \right) \frac{180}{\pi} \text{ deg}$$

Storage:  $M$ ,  $e$ ,  $E_n$ ,  $\theta$

Working storage required: 4 or more

Tests: test on difference  $E_{n+1} - E_n$  and stop if difference less than  $10^{-6}$  radians



---

**Program Documentation (1)**

(Title, Date, Brief Description, Formulas, Limits)

---

**Title:** Kepler's Equation

**Date:**

**Name:**

**Program Description:** Program solves Keplers equation for  $M_0$  eccentric anomaly to obtain the true anomaly from given input data,  $t$ ,  $t_0$ ,  $M_0$ ,  $a$ , and  $\mu$

**Equations, Variables, etc:**

$$M = E - e \sin E$$

$$\eta = \sqrt{\frac{\mu}{a^3}}$$

$$M = \eta (t - t_0) + M_0$$

$$E_{n+1} = E_n + \left( \frac{M - E_n + e \sin E_n}{1 - e \cos E_n} \right)$$

$$\Theta = 2 \tan^{-1} \left( \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right)$$

**Operating Limits and Warnings:**

Use  $M$  for  $E_0$  in  $E_{n+1} = E_n + \frac{M - E_n + e \sin E_n}{1 - e \cos E_n}$

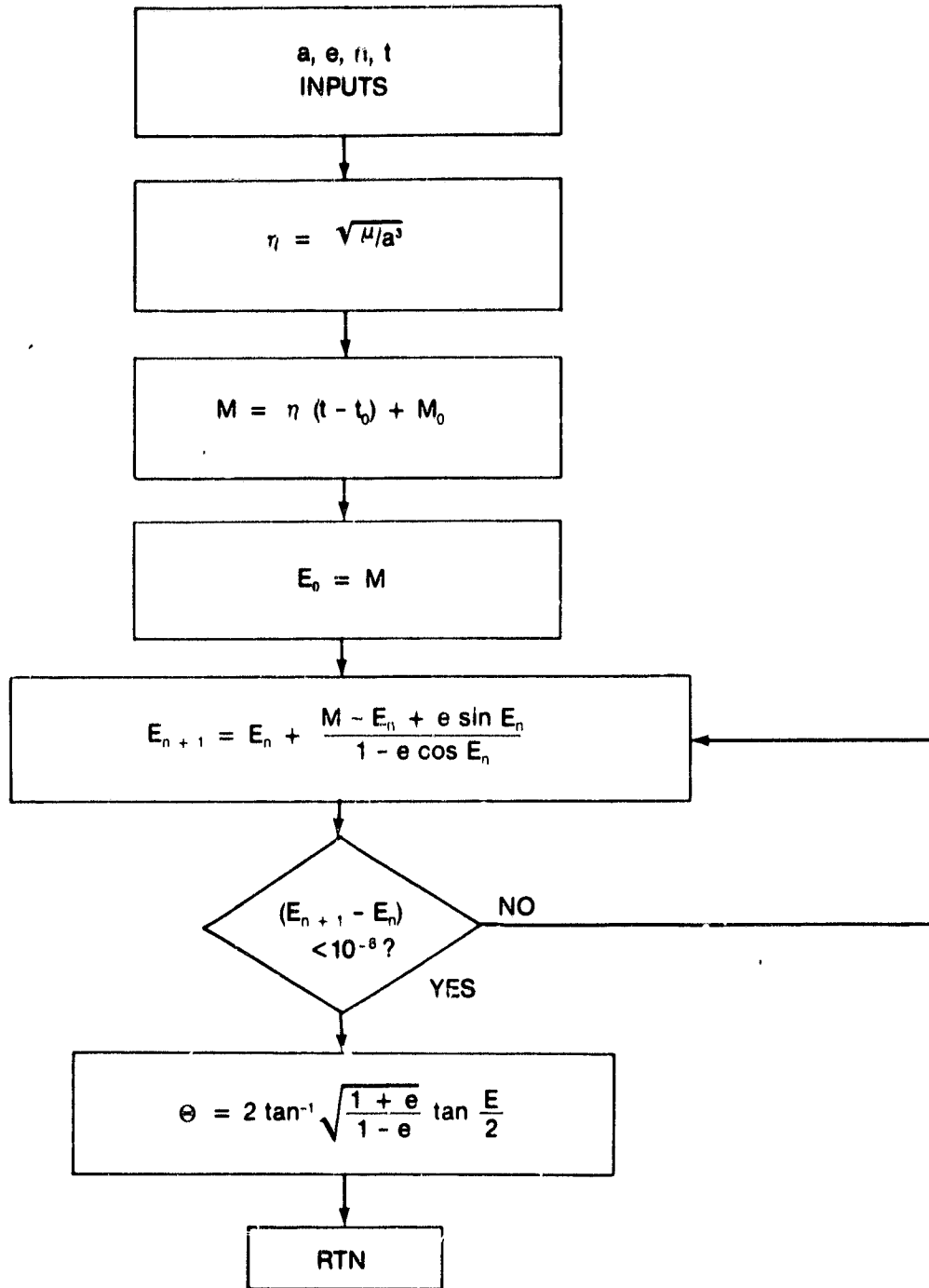
Stop calculation for  $E_{n+1}$  when  $E_{n+1} - E_n < 10^{-6}$

$$M = \eta t$$

**Program Documentation (2)**

(Title, Date, Brief Description, Formulas, Limits)

**Title: Kepler's Equation**



**Program Documentation (3)**

**Title: Kepler's Equation**

STEP	INSTRUCTIONS	INPUT DATA/UNITS	KEYS	OUTPUT DATA/UNITS
1	Key in program		<input type="text"/> <input type="text"/>	
2	Inputs: a → R,2		STO <input type="text"/> .2	
3	e → R2		STO <input type="text"/> 2	
4	$\mu$ → R.1		STO <input type="text"/> .1	
5	t → R7		STO <input type="text"/> 7	
6	$t_0$ → R8		STO <input type="text"/> 8	
7	$M_0$ → R0		STO <input type="text"/> 0	
8	0.174532925 → R6		STO <input type="text"/> 6	
9	57.2957795199 → R.4		STO <input type="text"/> .4	
10	$10^{-6}$ → R.3		STO <input type="text"/> .3	
11	Find $\eta$ : $\sqrt{\mu/a^3}$		gLBL <input type="text"/> 0	$\eta$
			STO <input type="text"/> 9	
12	Find M: $\eta (t - t_0) + M_0$		gLBL <input type="text"/> 1	M in deg.
			STO <input type="text"/> 1	
13	Convert M to radians		gLBL <input type="text"/> 2	M in rad
			STO <input type="text"/> 1	
14	Use M as $E_0$ for 1st iteration		STO <input type="text"/> 3	
15	Find $E_{n+1}$		gLBL <input type="text"/> 3	
			STO <input type="text"/> 5	
16	Use $E_{n+1}$ as $E_n$ for iteration		RCL <input type="text"/> 5	
			STO <input type="text"/> 3	
17	Repeat iterations until $(E_{n+1} - E_n) > 10^{-6}$		RCL <input type="text"/> 5	E
			STO <input type="text"/> 3	
18	Use E to find $\theta$		gLBL <input type="text"/> 4	$\theta$ in rad.
			STO <input type="text"/> 4	
19	Convert $\theta$ to degrees		STO <input type="text"/> 4	$\theta$ in deg.



**Exercise B.4.2.** Compare the values of  $\theta$  obtained from the program of Exercise B.4.1. with values of  $\theta$  obtained from the program of Exercise B.2.3. for values of  $e = 0.05$ ,  $e = 0.1$ ,  $e = 0.2$ ,  $e = 0.5$ .

**Exercise B.4.3.** Using the calculator program of Exercise B.4.1, plot values of  $r$  vs  $\theta$  on polar graph paper for the following orbits.

- (a) A circular orbit of 160 n.mi altitude ( $e = 0$ ) with  $t_0 = 0$ ,  $M_0 = 0$  for  $t$  at intervals of 15 min.
- (b) An elliptical orbit, with  $a = 13185.3858$  n.mi,  $e = 0.7266723$  and  $t_0 = 0$ ,  $M_0 = 0$  at intervals of 30 n.mi
- (c) A circular orbit, at an altitude of 19322.91 n.mi, and  $t_0 = 0$ ,  $M_0 = 180^\circ$  at intervals of 1 hour

## **B.5. EVALUATION**

### **B.5.1. Learning Objectives**

The principal objective of Section B was to teach more complex problem solving based on task analysis which involved examples of flight planning problems with arbitrary and incremental variable input. The complexity of the task was simplified through the use of the calculator programming capability.

Having completed study of this section and related references, you should have an understanding of:

- the conceptual content of problems
- the types of data needed for problem evaluation
- the use of multiple formula in problem evaluation
- the logic of flowcharting
- the methods of constructing a calculator program

### **B.5.2. Outcome Measures**

You should now be able to:

1. Identify input data and dimensions needed to solve problems,
2. Identify appropriate formulas for problem solution,
3. Flowchart a problem to solution,
4. Write a calculator program for problem solution,
5. Document and edit programs,
6. Demonstrate a familiarity with orbital terminology.

If not, GTO B.1.

## C. POSITION AND VELOCITY IN THE ORBITAL PLANE

### C.0. INTRODUCTION

The purpose of the following is to develop ways to calculate the position of a satellite in the plane of its orbit about the earth when the orbits are highly eccentric as well as circular. The given forms of input data will be typical of that encountered for satellites or other orbiting vehicles in current mission planning. Because of the importance of upper stage delivery in orbiter flight programs, we will look at the details of orbit transfers in the plane.

### C.1. LIST OF SYMBOLS AND DIMENSIONS

A listing now will be made of symbols used to denote all the variables to be employed in our calculations. Many of them have been introduced before. Here, dimensions will be given using the three basic units of mass (M), length (L), and time (T).

$v$	=	velocity, dimensions L/T
$r$	=	distance from center of earth, L
$a$	=	semimajor axis, L
$\mu$	=	gravitational constant, $L^3/T^2$
$e$	=	eccentricity, dimensionless
$r_p$	=	perigee distance from center of earth, L
$r_a$	=	apogee distance from center of earth, L
$H$	=	altitude above equatorial radius, L
$H_a$	=	altitude at apogee, L
$H_p$	=	altitude at perigee, L
$P$	=	period, T
$v$	=	velocity in orbit, L/T
$V_{\text{circ}}$	=	velocity in circular orbit, L/T
$R_{\text{eq}}$	=	equatorial radius of earth, L

- $V_R$  = radial component of orbital velocity, L/T
- $V_H$  = horizontal component of velocity, L/T
- $p$  = parameter of the ellipse
- $\eta$  = mean motion, L/T
- $\Theta$  = true anomaly
- $E$  = eccentric anomaly
- $M$  = mean anomaly
- $\gamma$  = flight path angle relative to local horizontal
- $I_{sp}$  = specific impulse, T
- $F$  = rocket thrust, ML/T<sup>2</sup>
- $W_b$  = rate of fuel burned, ML/T<sup>3</sup>
- $W_{TB}$  = total fuel burned, ML/T<sup>3</sup>
- $\Delta v$  = velocity change, L/T
- $C$  = effective exhaust velocity, L/T

## C.2. BASIC FORMULAS FOR THE ELLIPTIC ORBIT

Again, while a number of the basic formula have been introduced, a listing of all formulae necessary for a description of an object in orbital flight is presented below.

$$(1) \quad g = \frac{\mu}{R_{eq}^2} \text{ (gravitational)}$$

The legal value of  $g$  is 32.1740486. (Note: the legal value of  $g$  is used to convert weight in lbs to mass in slugs.)

$$(2) \quad P = \frac{2 \pi a^{3/2}}{\sqrt{\mu}}$$

$$(3) \quad \eta = \frac{\sqrt{\mu}}{a^{3/2}}$$

$$(4) \quad \eta = \frac{2 \pi}{P}$$



$$(5) \quad P = \frac{2 \pi}{\eta}$$

$$(6) \quad H = r - R_{\text{eq}}$$

$$(7) \quad r = H + R_{\text{eq}}$$

$$(8) \quad M = \eta (t - t_0) + M_0$$

Computation of Semimajor axis:

$$(9) \quad a = \frac{p}{1 - e^2}$$

$$(10) \quad a = \frac{r_a}{1 + e}$$

$$(11) \quad a = \frac{r_p}{1 - e}$$

$$(12) \quad a = \frac{r_a^2}{2r_a - p}$$

$$(13) \quad a = \frac{r_p^2}{2r_p - p}$$

$$(14) \quad a = \frac{r_a + r_p}{2}$$

$$(15) \quad a = \frac{r_a + r_p}{2}$$

$$(16) \quad a = \frac{r}{2 - rV^2}$$

$$(17) \quad a = \frac{(P \sqrt{\eta})^{2/3}}{2 \pi}$$

Computation of eccentricity:

$$(18) \quad e = \frac{r_a - r_p}{r_a + r_p}$$

$$(19) \quad e = \sqrt{1 - \frac{p}{a}}$$

$$(20) \quad e = \sqrt{\frac{p}{\mu} V_A^2 + \left(\frac{p}{r} - 1\right)^2}$$

$$(21) \quad e = \frac{r_a}{a} - 1$$

$$(22) \quad e = 1 - \frac{r_p}{a}$$

$$(23) \quad e = 1 - \frac{p}{r_a}$$

$$(24) \quad e = \frac{p}{r_p} - 1$$

Computation of parameter p:

$$(25) \quad p = a(1 - e^2)$$

$$(26) \quad p = r_a \left(2 - \frac{V_A}{a}\right)$$

$$(27) \quad p = r_p \left(2 - \frac{r_p}{a}\right)$$

$$(28) \quad p = \frac{r_a r_p}{a}$$

$$(29) \quad p = r_a(1 - e)$$

$$(30) \quad p = r_p (1 + e)$$

$$(31) \quad p = \frac{2r_a r_p}{r_a + r_p}$$

$$(32) \quad p = \frac{r^2 V_H^2}{\mu} = \frac{r^2 v^2 \cos^2 \gamma}{\mu}$$

Computation of radial distance r:

$$(33) \quad r = \frac{a(1 - e^2)}{1 + e \cos \Theta}$$

$$(34) \quad r = \frac{p}{1 + e \cos \Theta}$$

$$(35) \quad r = \frac{2a}{1 + \frac{aV^2}{\mu}}$$

$$(36) \quad r = a \left( 1 \pm \sqrt{1 - \frac{(1 - e^2)}{\cos^2 \gamma}} \right)$$

$$(37) \quad r = \frac{a(1 - e^2) \tan \gamma}{e \sin \Theta}$$

$$(38) \quad r = a(1 - e \cos E)$$

$$(39) \quad r = \frac{\sqrt{\mu} a e \sin E}{V_R}$$

$$(40) \quad r = \sqrt{\mu p} V_H$$

Computation of radial distances at apogee and perigee:

$$(41) \quad r_a = a(1 + e)$$

$$(42) \quad r_p = a(1 - e)$$

$$(43) \quad r_a = \frac{ap}{r_p}$$

$$(44) \quad r_a = 2a - r_p$$

$$(45) \quad r_a = \frac{p}{1 - e}$$

$$(46) \quad r_a = r_p \left( \frac{1 + e}{1 - e} \right)$$

$$(47) \quad r_a = \frac{r_p p}{2r_p - p}$$

$$(48) \quad r_p = \frac{ap}{r_a}$$

$$(49) \quad r_p = 2a - r_a$$

$$(50) \quad r_p = \frac{p}{1 + e}$$

$$(51) \quad r_p = r_a \left( \frac{1 - e}{1 + e} \right)$$

Computation of true anomaly:

$$(52) \quad \Theta = 2 \tan^{-1} \left( \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2} \right)$$

$$(53) \quad \Theta = \sin^{-1} \left( \frac{p \tan \gamma}{er} \right)$$

$$(54) \quad \Theta = \tan^{-1} \left( \frac{p \tan \gamma}{p - r} \right)$$

$$(55) \quad \Theta = \sin^{-1} \left( \sqrt{\frac{1 - e^2 \sin E}{1 - e \cos E}} \right)$$

$$(56) \quad \Theta = \cos^{-1} \left( \frac{a}{r} (\cos E - e) \right)$$

$$(57) \quad \Theta = \cos^{-1} \left( \frac{p - r}{er} \right)$$

Computation of eccentric anomaly:

$$(58) \quad E = 2 \tan^{-1} \left( \sqrt{\frac{1-e}{1+e}} \tan \frac{\Theta}{2} \right)$$

$$(59) \quad E = \cos^{-1} \left( \frac{1 - r/a}{e} \right)$$

$$(60) \quad E = \sin^{-1} \left( \frac{r \sin \beta}{a \sqrt{1-e^2}} \right)$$

$$(61) \quad E = \cos^{-1} \left[ \frac{1}{e} \left( \frac{\frac{aV^2 - 1}{\mu}}{\frac{aV^2 + 1}{\mu}} \right) \right]$$

$$(62) \quad E = \cos^{-1} \left[ \frac{1}{e} \pm \sqrt{1 - \frac{(1-e^2)}{\cos^2 \gamma}} \right]$$

Computation of flight path angle:

$$(63) \quad \gamma = \sin^{-1} \left( \frac{V_R}{r} \right)$$

$$(64) \quad \gamma = \cos^{-1} \left( \frac{V_H}{v} \right)$$

$$(65) \quad \gamma = \tan^{-1} \left( \frac{V_R}{V_H} \right)$$

$$(66) \quad \gamma = \tan^{-1} \left( \frac{e \sin E}{\sqrt{1-e^2}} \right)$$

$$(67) \quad \gamma = \cos^{-1} \left( \frac{\sqrt{\mu p}}{rV} \right)$$

$$(68) \quad \gamma = \tan^{-1} \left[ \left(1 - \frac{r}{p}\right) \tan \Theta \right]$$

$$(69) \quad \gamma = \tan^{-1} \left[ \frac{e \sin \Theta}{1 + e \cos \Theta} \right]$$

Computation of velocity

$$(70) \quad v = \sqrt{\mu \left( \frac{2}{r} - \frac{1}{a} \right)}$$

$$(71) \quad v_{\text{circ}} = \sqrt{\frac{\mu}{a}}$$

$$(72) \quad v = \sqrt{\frac{\mu (1 + e \cos E)}{a (1 - e \cos E)}}$$

$$(73) \quad v = \sqrt{\frac{\mu p}{r \cos \gamma}}$$

$$(74) \quad v = \sqrt{v_H^2 + v_R^2}$$

Computation of radial and horizontal velocities:

$$(75) \quad v_H = v \cos \gamma$$

$$(76) \quad v_R = v \sin \gamma$$

$$(77) \quad v_R = \sqrt{\frac{\mu}{p}} (e \sin \Theta)$$

$$(78) \quad v_H = \sqrt{\frac{\mu p}{r}}$$

Escape velocity computation:

$$(79) \quad v_{\text{ESC}} = \sqrt{\frac{2\mu}{r}}$$

$$(80) \quad I_{sp} = F/W_b$$

$$(81) \quad F = W_b I_{sp}$$

$$(82) \quad W_b = F/I_{sp}$$

$$(83) \quad I_{sp} = c/g$$

$$(84) \quad W_{TB} = W_F \left( e^{\frac{\Delta v}{c}} - 1 \right)$$

$$(85) \quad \Delta v = c \ln \left( 1 + \frac{W_{TB}}{W_F} \right)$$

$$(86) \quad \Delta v_{circ} = \sqrt{\mu} a_1^{-3/2} \left( \frac{a_2 - a_1}{2} \right)$$

Kepler's equation:

$$(87) \quad M = E - e \sin E$$

$$(88) \quad E_{n+1} = E_n + \frac{M - E_n + e \sin E_n}{1 - e \cos E_n}$$

Series expansions for low eccentricities:

$$(89) \quad \begin{aligned} \Theta = & M + \left( 2e - \frac{e^3}{4} + \frac{5e^5}{96} + \frac{107e^7}{4608} + \dots \right) \sin M \\ & + \left( \frac{5e^2}{4} - \frac{11e^4}{24} + \frac{17e^6}{192} - \dots \right) \sin 2M \\ & + \left( \frac{13e^2}{12} - \frac{43e^4}{192} + \frac{95e^6}{512} - \dots \right) \sin 3M \\ & + \left( \frac{103e^4}{96} - \frac{451e^6}{480} + \dots \right) \sin 4M \end{aligned}$$

$$\begin{aligned}
& + \left( \frac{1097e^4}{960} - \frac{5957e^7}{4608} + \dots \right) \sin 5 M \\
& + \left( \frac{1223e^4}{960} - \dots \right) \sin 6 M \\
& + \left( \frac{47,273e^7}{32,256} - \dots \right) \sin 7 M \\
(90) \quad E = & M + \left( e - \frac{e^3}{8} + \frac{e^5}{192} - \frac{e^7}{9216} + \dots \right) \sin M \\
& + \left( \frac{e^2}{2} - \frac{e^4}{6} + \frac{e^6}{48} - \dots \right) \sin 2 M \\
& + \left( \frac{3e^3}{8} - \frac{27e^5}{128} + \frac{243e^7}{5120} - \dots \right) \sin 3 M \\
& + \left( \frac{e^4}{3} - \frac{4e^6}{15} + \dots \right) \sin 4 M \\
& + \left( \frac{25e^5}{384} - \frac{3125e^7}{9216} + \dots \right) \sin 5 M + \\
& \quad \left( \frac{27e^6}{80} + \dots \right) \sin 6 M
\end{aligned}$$

### C.3. TRANSFER OF ORBIT

All space vehicle trajectories, at least during some phase of the vehicle's mission, can be classified as transfer orbits. This general classification is basic to most missions, because, in almost all instances after the space vehicle has been established in an initial orbit, a velocity impulse is given to the vehicle which causes it to transfer to another desired orbit. Transfer orbits are employed because a direct launch into the final orbit is almost always physically impossible and because an intermediate orbit can be more desirable for initiating a precise transfer to a translunar or interplanetary orbit than a direct transfer.



In consideration of an orbital transfer it is necessary that the orbital characteristics of both initial and final orbits be known. With that information it can be determined whether a transfer orbit is necessary and the method by which transfer is to be accomplished.

**Exercise C.3.1.** Radar observations indicate that an orbiting satellite has an altitude at apogee of 600 n.mi and a perigee altitude of 150 n.mi

- a) Find the semimajor axis (Eq. (15) )
- b) Find the eccentricity Eq. (19) or (21) )
- c) Find the period (Eq. (2) )

**Exercise C.3.2.** The orbiter is in an orbit with a semimajor axis of 4454.91 n.mi and eccentricity of 0.2.

- a) What is the perigee altitude?  
(Eqs. (6) and (42) )
- b) What is the apogee altitude?  
(Eqs. (6) and (41) )
- c) What is the radial velocity at perigee? at apogee?  
(Eqs. (70) and (76) )
- d) What is the horizontal velocity at perigee? at apogee?  
(Eqs. (70) and (75) )
- e) What is the radial distance when the eccentric anomaly is  $90^\circ$ ? (Eq. (8) )
- f) What true anomaly coincides with this eccentric anomaly?

Transfer orbits may be classified in several ways. One of the methods of classifying transfer orbits is by the type of thrust used to either initiate or accomplish the change of trajectory. The transfer could be the result of a single velocity impulse, a sequence of velocity impulses, a low impulse thrust, a high impulse thrust, or a continuous thrust along the transfer trajectory. Engines may be fired toward the earth, adding to the velocity in a direction away from the earth, they may be fired in the opposite direction, or in any direction in between.

Velocity is really a vector quantity. Some quantities, such as mass and time,

can be represented by a single real number scaled to an appropriate unit of measure. Such quantities are referred to as scalar quantities, and the real number associated with each is a scalar. Some quantities can not be described fully by a single real number because they have both magnitude and direction. Force (thrust), and displacement are examples in addition to velocity. The mathematical object used to describe each vector quantity is called a vector.

A vector is represented geometrically by an arrow. By definition, a vector is a directed line segment, with a tail end (initial point) and a head end (terminal point), and it is understood to be directed from tail to head. The length of a vector represents its magnitude and the direction of the arrow represents the direction in which the vector acts.

On the earth, or from the earth surface reference, we make use of the up-down direction, which we designate as vertical. A direction at right angles to the vertical, perpendicular to the vertical or parallel to the earth surface, is called horizontal. Often, a vector quantity is described in terms of vertical and horizontal components as shown in Figure 1C.

If the earth surface is used as reference, a satellite in circular orbit would have a velocity that was always horizontal, perpendicular to the vertical which will be referred to as the radial direction. In an elliptical orbit, however, the velocity is completely horizontal only at apogee and perigee. Throughout all other points of the elliptical orbit the satellite has radial and horizontal components as shown in Figure 2C.

**Exercise C.3.3.** The orbiter is launched on a trajectory such that at the completion of the OMS 2 burn it is traveling horizontally with velocity of 25,495.83 feet/sec at an altitude of 607,611.55 ft.

- a) What is the semimajor axis?
- b) What is the eccentricity?
- c) What is the perigee distance?

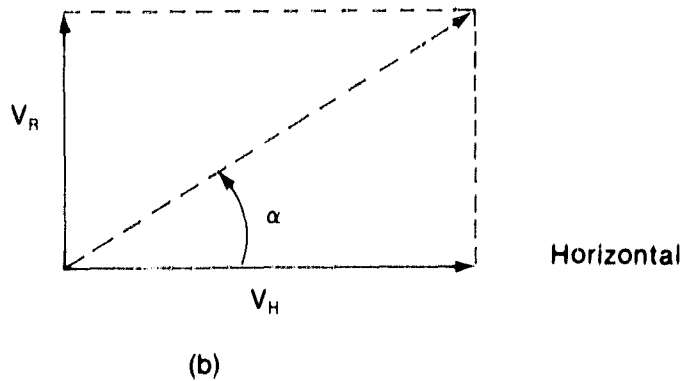
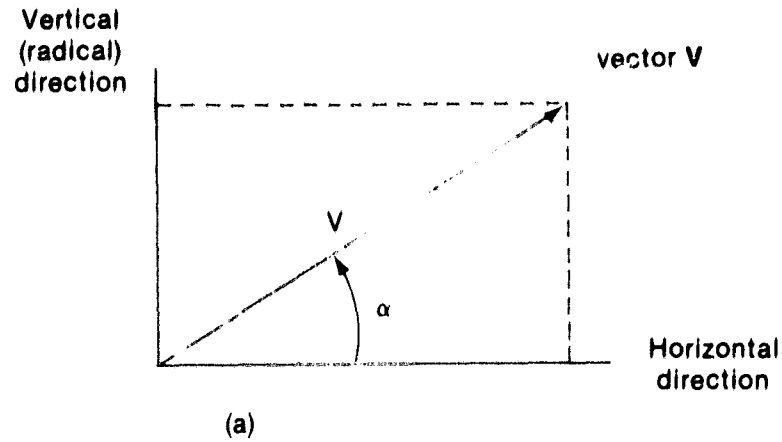
Hint: The data given is  $V_H$ ,  $V_R$  and  $H$ .

The data wanted is  $a$ ,  $e$ ,  $r_p$

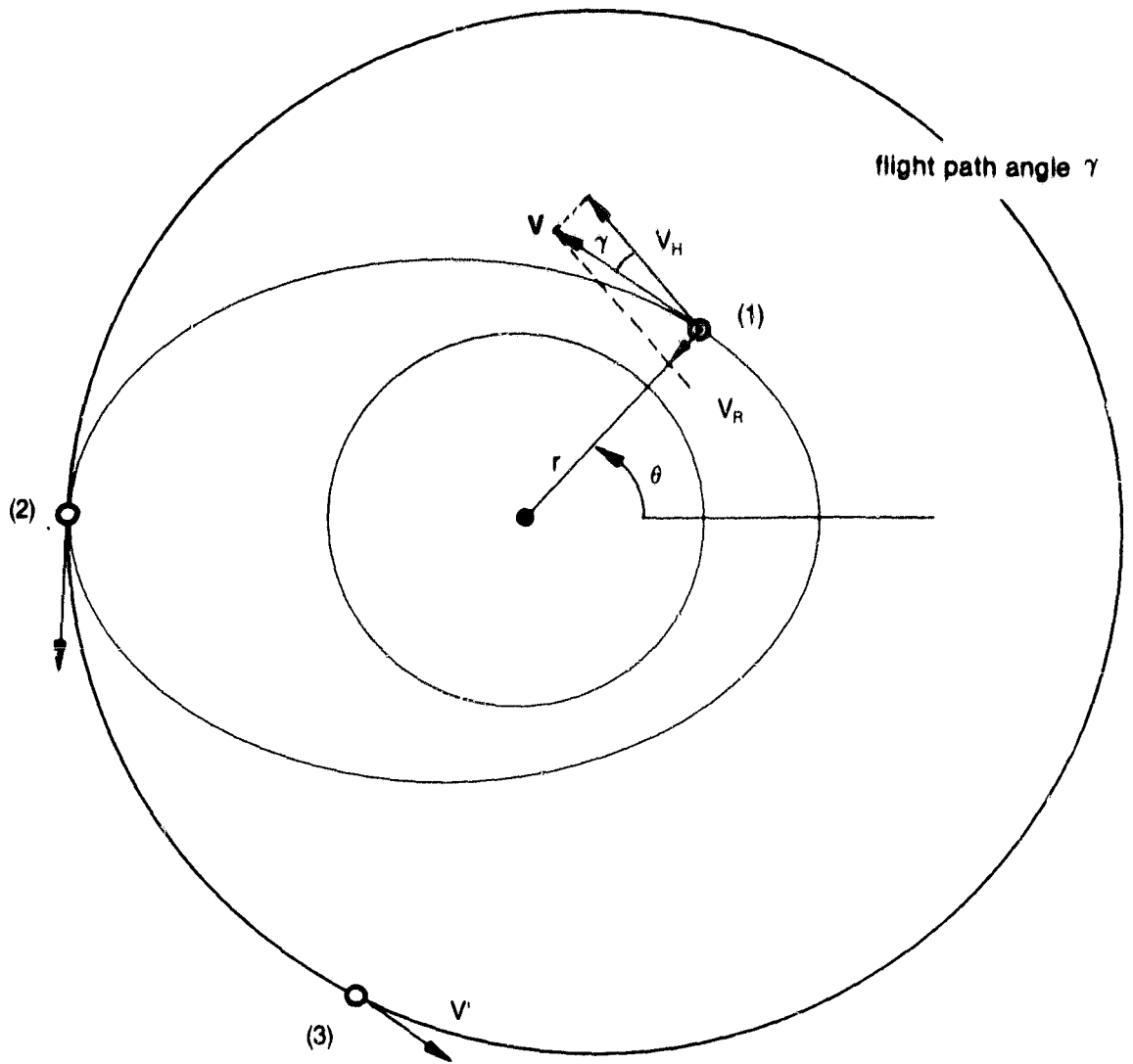
Use (74) to find  $V$  ( $V_R = 0$  from above).

Use (7) to find  $r$ .

Use (16) to find  $a$ .



**Figure 1C.** Vector components. (a) The vector labeled  $V$  is described by its magnitude,  $V$ , and direction,  $\alpha$ , from the horizontal. (b) The vector  $V$  is equivalently replaced by two vectors, a horizontal vector with magnitude  $V_H$  and a vertical vector of magnitude  $V_R$ . Magnitudes  $V_H = V \cos \alpha$  and  $V_R = V \sin \alpha$  may be called respectively the horizontal and vertical components of the vector  $V$ .



**Figure 2C.** An Orbital Transfer. For a satellite moving with velocity  $V$  in the elliptical orbit at position (1), the velocity is parallel to the orbit at that point and can be separated into vertical (radial) and horizontal components. At position (2), apogee, the vertical component of velocity is zero. If, at that point, an appropriate horizontal impulse is provided, the satellite may be transferred to a circular orbit with new velocity  $V'$  as shown in position (3). In this case the magnitude of  $V' - V = \Delta V$  is the change in velocity.  $V_H = V \cos \gamma$  and  $V_R = V \sin \gamma$ .

Use (32) to find  $p$ .

Use (19) to find  $e$ .

Use (42) to find  $r_p$ .

**Exercise C.3.4.** Calculate the velocities at the points of coincidence for each orbit in Exercise B.4.3. The difference in velocities is the  $\Delta v$  required for transfer from one orbit to the other when the burns are horizontal.

**Exercise C.3.5.** At MECO, the orbiter has a velocity of 25,731.68 ft/sec, an altitude of 55 n.mi and a flight path angle of  $1.4^\circ$  relative to the horizon.

a) Find  $V_R$  and  $V_H$ .

ANS: 628.68 ft/sec  
25,724.0 ft/sec

b) Find  $a$ ,  $e$ .

ANS: 3498.96 n.mi  
.0244

c) Find  $r_a$ , velocity at apogee.

ANS: 3548.3346 n.mi  
25,111.0847 ft/sec

The orbiter is to be put in a circular orbit with a semimajor axis equal to  $r_a$  by burning the OMS engines at apogee.

d) Find  $\Delta v$  required.

ANS: 312.07 ft/sec

e) If the orbiter weighs 246,300 lbs, how much Hydrazine will be used?

**Exercise C.3.6.** Suppose only the initial circular orbit and the final circular orbit are known. Explain how to calculate the semimajor axis and eccentricity of the transfer orbit.

Hint: The perigee of the transfer orbit must be the same as the radial distance of the lower orbit. Similarly, the apogee of the transfer orbit must be the same as the radial distance of the upper orbit.

#### C.4. RADIAL BURN TRANSFER

Transfer from a circular orbit to an elliptical orbit or from one elliptical orbit to another may be accomplished by a horizontal burn. The final orbit depends on both the magnitude of the final velocity and the flight path angle at burn.

A radial burn from a circular orbit, one directed up or down, generally causes a satellite to move to an elliptical orbit. Results of a radial burn for upward thrust is illustrated in Figure 3C. If another circular orbit is desired it could be accomplished from the elliptical orbit in the manner discussed earlier. The elliptical orbit then would be referred to as the transfer orbit. Note the velocity addition in Figure 3C. Note also the reversibility of the addition.

Had the satellite been in the elliptical orbit initially with velocity  $V_{new}$  at position (2), firing the rockets upward to create a downward thrust, or  $V_R$  downward, would have produced the resultant velocity  $V'$ , as shown in Figure 4C: The resultant final orbit would have been the circular orbit shown.

**Exercise C.4.1.** The orbiter is in a circular orbit with an altitude of 160 n.mi. A radial velocity of 120 ft/sec upward is imparted by firing the OMS engines directly toward the center of the earth.

- a) Discuss the change in the orbit.

Hint: Note that the horizontal component of velocity is not affected by a burn at right angles to it. The horizontal velocity is the same as the circular orbit velocity at 160 n. mi. altitude (Eq. (71)). The new total velocity is found from (74). The new semimajor axis is found from (16),  $p$  is found from (32), the new value of  $e$  is found from (19), and  $\theta$  is found from (57).

ANS:

$$V_{circ} = 25353.94945$$

$$V_{new} = 26354.2334$$

$$\Delta V = 0.28398 \text{ ft/sec}$$

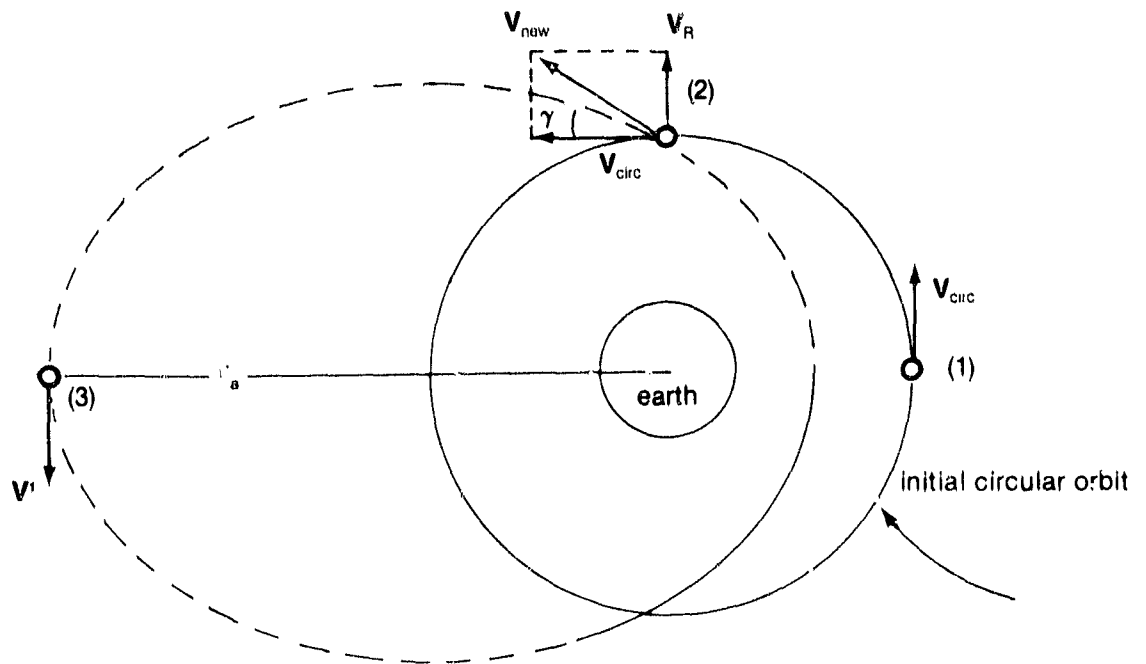
$$a_{new} = a_{old} + 490.50$$

$$e_{new} = 0.004733$$

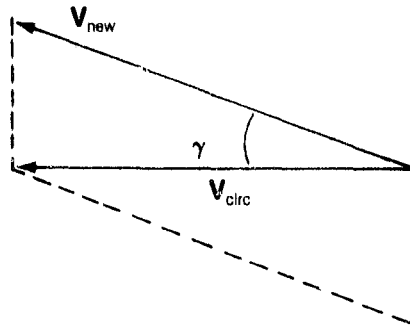
$$r_a = r_{circ} + 117.1365$$

$$r_p = r_{circ} - 17.056495 \text{ n.mi}$$

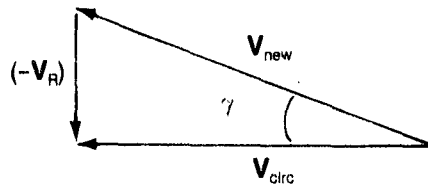
- b) How much Hydrazine is burned, if the orbiter weighs 200,000 lbs?



**Figure 3C. Radial Burn Transfer.** A satellite in circular orbit with velocity  $V_{circ}$  is shown in position (1). At position (2) engines fire creating a radial thrust giving the satellite a radial velocity  $V_R$ . Since the velocity  $V$  is horizontal and unaffected, the new velocity  $V_{new}$  is the vector sum of  $V$  and  $V_R$  with magnitude  $V_{new} = \sqrt{V_{circ}^2 + V_R^2}$ . Its direction is the flight path angle  $\gamma$ , at that point, for the new elliptical orbit to which the satellite has been transferred. When the satellite is in apogee position (3), an appropriate horizontal thrust could cause it to be transferred from the elliptical orbit to a circular one of radius  $r_a$ .



$$(a) \mathbf{V}_{\text{circ}} = \mathbf{V}_{\text{new}} + (-\mathbf{V}_R)$$



$$(b) \mathbf{V}_{\text{circ}} = \mathbf{V}_{\text{new}} + (-\mathbf{V}_R)$$

**Figure 4C.** Vector Addition. (a) parallelogram method. (b) graphic method in which vectors to be summed are constructed tail to head with the resultant vector drawn from the tail of the first vector to the head of the last. Note that  $V_{\text{circ}} = V_{\text{new}} \cos \gamma$ ,  $V_R = V_{\text{new}} \sin \gamma$ , and  $V_{\text{new}}^2 = V_{\text{circ}}^2 + V_R^2$ .



**Exercise C.4.2.** Suppose the 120 ft/sec increase in speed is imparted by a horizontal burn to the orbiter in a 160 n.mi circular orbit, instead of the radial burn as in the previous exercise.

a) Discuss the change in the orbit.

Hint: Note that the radial component is zero before and after the burn.

The new total velocity is

$$V_{\text{new}} = V_{\text{circ}} + 120 \text{ ft/sec}$$

$$\Delta V = 120 \text{ ft/sec}$$

$$a_{\text{new}} = a_{\text{old}} + 34.52 \text{ n.mi.}$$

$$e_{\text{new}} = .009444$$

$$r_a = r_{\text{circ}} + 68.881 \text{ n.mi}$$

$$r_p = r_{\text{circ}} - 956.836 \text{ ft}$$

b) How much Hydrazine is burned, if the orbiter weighs 200,000 lbs?

**Exercise C.4.3.** Discuss the relative merit, in terms of fuel used and total change in orbit, of the radial burn and a horizontal burn.

**Exercise C.4.4.** Suppose that the orbiter is in one circular orbit and must be transferred to another circular orbit at a new altitude. Is there any way to do this with only one short burn of the OMS?

**Exercise C.4.5.** Calculate the radial velocity in an orbit with the characteristics

$$a = 13,185.3858$$

$$e = .7267$$

at values of

$$\Theta = 0^\circ, 180^\circ$$

$$\Theta = 90^\circ, 270^\circ$$

$$\Theta = 7^\circ, 187^\circ$$

$$\Theta = 62^\circ, 242^\circ$$

Tabulate the velocities and compare for the cases where the true anomaly differs by  $180^\circ$ .

**Exercise C.4.6.** Calculate the radial distance, flight path angle, and radial velocity for the new orbit resulting from the radial burn in Exercise C.4.2.

- a) at the point of burn
- b)  $180^\circ$  around the orbit

**Exercise C.4.7.** What is the total  $\Delta v$  required for placing a satellite in the upper circular orbit of Exercise B.4.3. (see Exercise C.3.4.) from a 160 n.mi circular orbit? The first stage empty case and the second stage and the payload weigh 7,436 lbs. How many lbs of a fuel with an  $I_{sp} = 260$  secs is required in the first stage? If the second stage empty case and payload weigh 3800 lbs, is there enough fuel in the second stage? What is the period of the transfer orbit in hrs, minutes, seconds? (Note that all three orbits are in the same plane. In the *real problem*, we must change planes.)

Calculation of the time required for an orbiting body to go from one point in its orbit to another is important. Position and time can be accurately related. The direct time problem is to find the time associated with a given value of  $\Theta$ , where  $\Theta$  is the true anomaly. This is solved as follows. For a given  $a$ ,  $e$ , and  $\Theta$ , we may find  $E$  by Eq. (58):

$$E = 2 \tan^{-1} \left( \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2} \right)$$

The value of  $M$  then may be determined using Eq. (87),

$$M = E - e \sin E$$

Next, we find  $\eta$  from Eq (3),

$$\eta = \sqrt{\frac{\mu}{a^{3/2}}}$$

Finally  $(t - t_0)$  is obtained from the relation

$$(t - t_0) = \frac{M - M_0}{\eta}$$

where  $t_0$  is the epoch time and  $t$  the time when the body reaches the point in its orbit defined by  $\Theta$ . If the epoch is at perigee or apogee,  $M_0 = 0$ ; otherwise,  $M_0$  must be calculated.

**Exercise C.4.8.** Find the time required to travel from perigee to a radial distance,

$$r = 6800 \text{ n.mi.},$$

for the transfer orbit of Exercise C.4.7.

Hint: Find  $\Theta$  from Eq (57).

$$\text{ANS: } (t - t_0) = 1^{\text{h}} 24^{\text{m}} 44^{\text{s}}$$

where epoch is time of perigee passage

**Exercise C.4.9.** Check the answer in Exercise C.4.8. using the calculator program of Exercise B.4.1.

**Exercise C.4.10** Find the time required to travel from  $r = 6800 \text{ n.mi}$  to  $r = 18,300 \text{ n.mi}$  in the transfer orbit of Exercise C.4.7.

Hint: Using Exercise C.4.8., the epoch is  $1^{\text{h}} 24^{\text{m}} 44^{\text{s}}$  and the mean anomaly at Epoch is the mean anomaly calculated:  $M_0 = 48^{\circ} 11' 40''$ .

**Exercise C.4.11.** The ignition time, direction of burn, and magnitude of burn for the second stage is to be determined from radar tracking data acquired during the first  $1^{\text{h}} 25^{\text{m}}$  of travel after first stage ignition. Approximately three hours will be required to do calculations. Will there be enough time before  $r = 18,300 \text{ n.mi.}$ ?

**Exercise C.4.12.** A solid rocket, once ignited, cannot be throttled back or shut down. Suppose you have to use a two stage solid rocket to transfer a payload from the orbiter (160 n.mi circular) to another circular orbit at 750 n.mi, and the first stage is much too powerful for the job. Do you think you could design a trajectory that would "waste" the right amount of fuel, so that delivery can be accomplished? If not, continue.

## C.5. ENERGY MANAGEMENT: DIRECTIONAL BURNS

Since fuel economy is a vital factor in many space missions, a natural and important phase of mission design is to investigate the conditions that will lead to a minimum fuel or energy transfer between the initial and final orbits. A transfer of orbit problem may be regarded as a change of energy problem.

First determinations of minimum energy transfer orbits and calculations of mission times was done by a man named Walter Hohmann (in 1925!). Let us consider two possibilities for transfer between circular orbits. The first, shown in Figure 5C, is called a bi-elliptic transfer; three burns are required. The second, a two-stage transfer shown in Figure 6C, has been reviewed in the exercises and will be referred to as a Hohmann transfer. While the two-burn Hohmann transfer will in most cases of interest require less energy, it can be shown that the bi-elliptic transfer is most energy efficient for cases in which the radius of the final circular orbit is more than sixteen times the radius of the initial orbit. But in any case it is much more time consuming.

Suppose that the orbiter is in a 160 n.mi circular orbit, and an upper stage is deployed for transfer into a geocentric orbit. The transfer is minimum energy (Hohmann transfer) with the characteristics:

Basic 160 n.mi orbit:

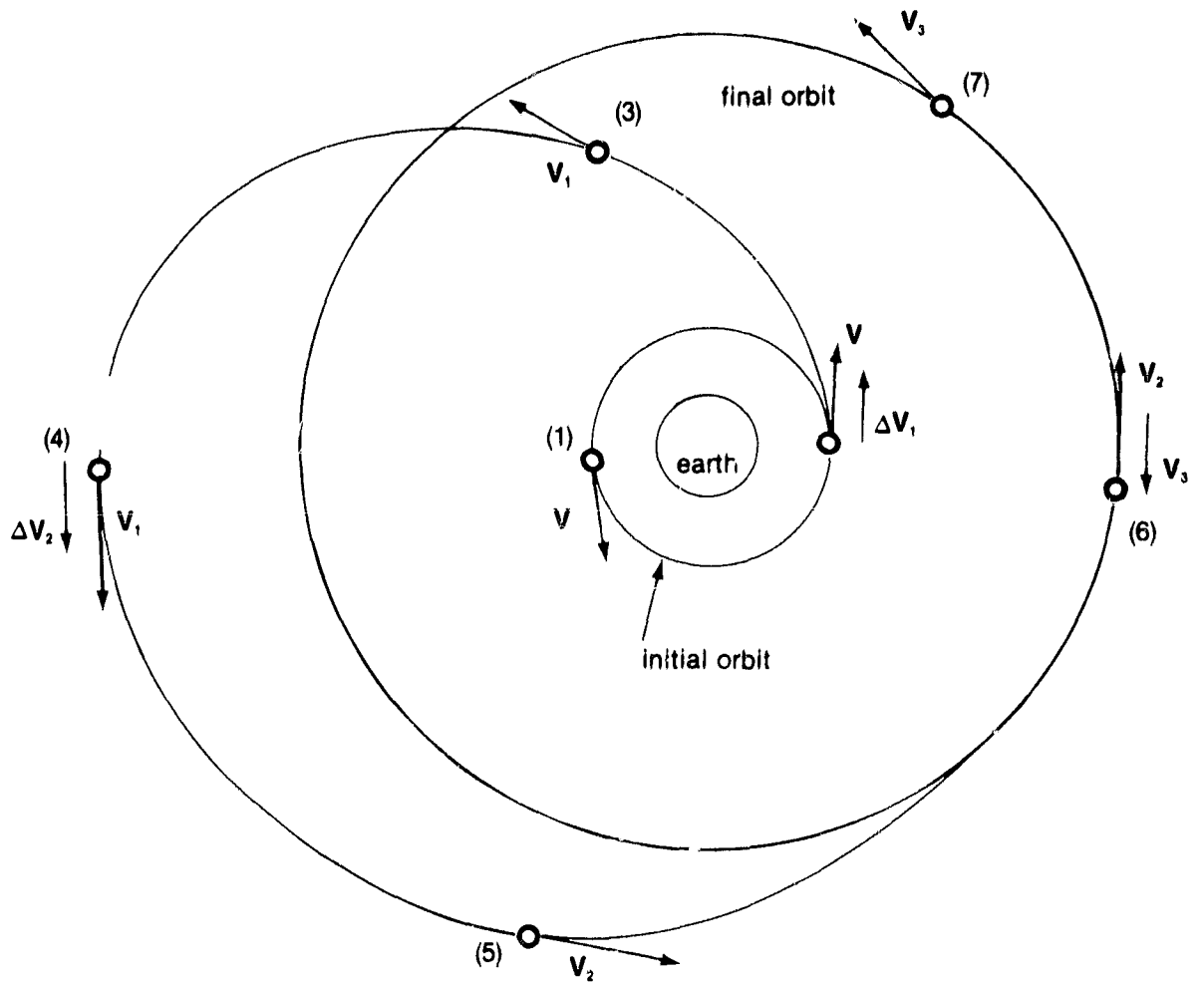
$$\begin{aligned} a &= 3603.930885 \text{ n.mi} \\ V &= 25353.9497 \text{ fps} \\ P &= 1^{\text{h}} 30^{\text{m}} 27^{\text{s}} \end{aligned}$$

Transfer orbit:

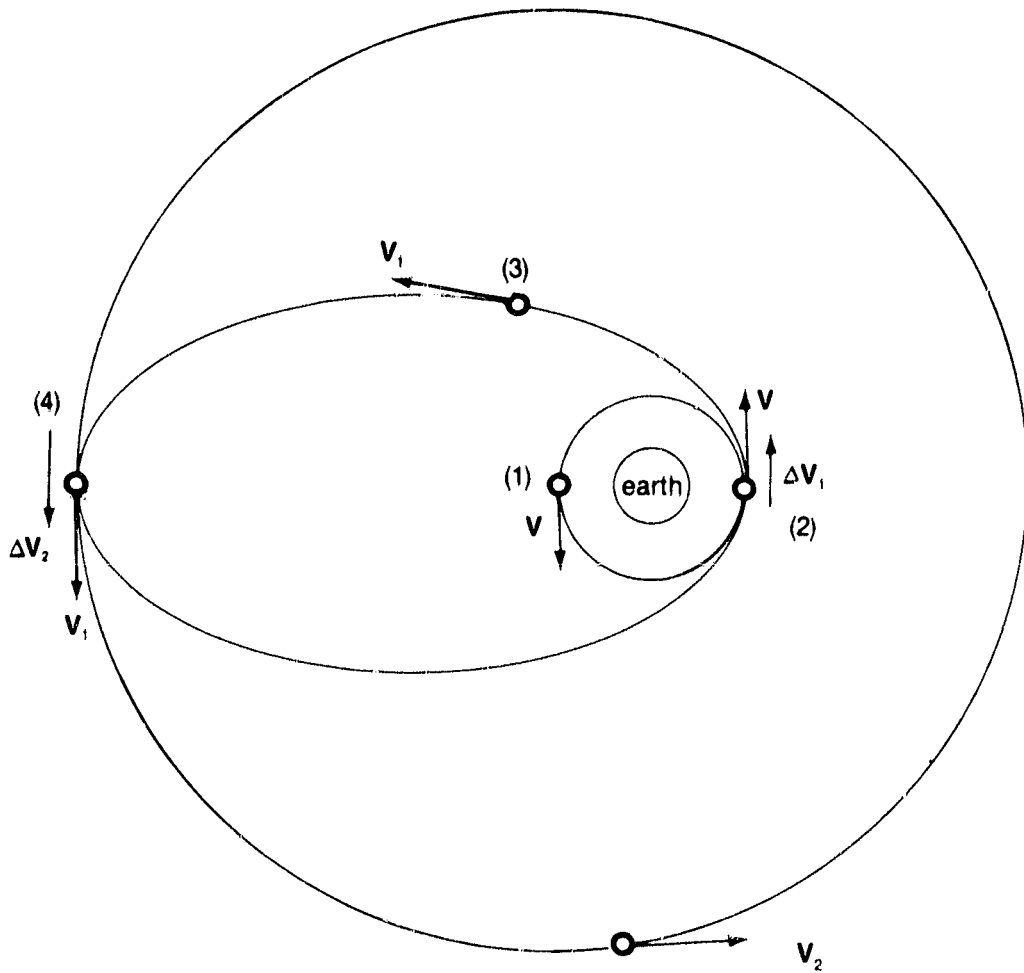
$$\begin{aligned} a &= 13,185.3858 \\ e &= .7266723 \\ V_p &= 33,315.8263 \text{ fps} \\ V_a &= 5,273.8074 \text{ fps} \\ P &= 10^{\text{h}} 32^{\text{m}} 56^{\text{s}} \end{aligned}$$

Geosynchronous orbit:

$$\begin{aligned} a &= 22,766.84089 \text{ n.mi} \\ V &= 10,087.4748 \text{ fps} \\ P &= 23^{\text{h}} 56^{\text{m}} 04^{\text{s}} \end{aligned}$$



**Figure 5C. Bi-elliptic Transfer.** The first of three horizontal burns is made at (2) resulting in velocity  $V_1$ . The second burn is at apogee (4) of the elliptical orbit, resulting in velocity  $V_2$ . The third burn, at perigee (6) in the new elliptical orbit, is made to give an opposing velocity in order to effect the final circular orbit.



**Figure 6C.** Hohmann Transfer. This is a minimum energy transfer between two circular orbits, accomplished by two horizontal burns. The first burn occurs at (2) imparting velocity  $\Delta V_1$ ; the second burn occurs at (4),  $180^\circ$  away, imparting velocity  $\Delta V_2$ .  $V_2 - V = \Delta V_1 + \Delta V_2 = \Delta V$ .

You may verify the parameters for the transfer orbit using previous exercises, and that the required  $\Delta V$  for the two upper stages is given by

$$\text{1st stage } \Delta V = 7961.88 \text{ fps}$$

$$\text{2nd stage } \Delta V = 4813.67 \text{ fps}$$

Suppose now that the upper stage has two motors, a solid rocket for the first stage and a liquid fuel rocket for the second stage. The solid rocket cannot be throttled, and has a fixed  $\Delta v$  of 8,200 fps while the liquid stage can be throttled to any desired  $\Delta V$ . The prescribed Hohmann transfer orbit cannot be made since the  $\Delta V$  is too large. But now consider the problem of choosing a direction of first stage burn such that excess  $\Delta v$  in the first stage is wasted, and of choosing a direction and value of the second stage burn to accomplish the desired circularization at geosynchronous altitude, as illustrated in Figure 7C.

The procedure for solution is as follows. Choose the burn direction for the first stage such that the horizontal component of velocity gained is the  $\Delta v$  required to go from the circular orbit to the prescribed transfer orbit, as shown in Figure 8C. Using the orbital characteristics and given information, we see that in two triangles of Figure 8C,

$$V_C = 25353.9497 \text{ ft/sec}$$

$$\Delta V_{REQ} = 7961.8766 \text{ ft/sec}$$

$$V_{circ} + \Delta V_{REQ} = 33,315.8263 \text{ ft/sec} = V_H,$$

$$\text{1st stage } \Delta V = 8,200 \text{ ft/sec.}$$

The angle  $\beta$  is given by

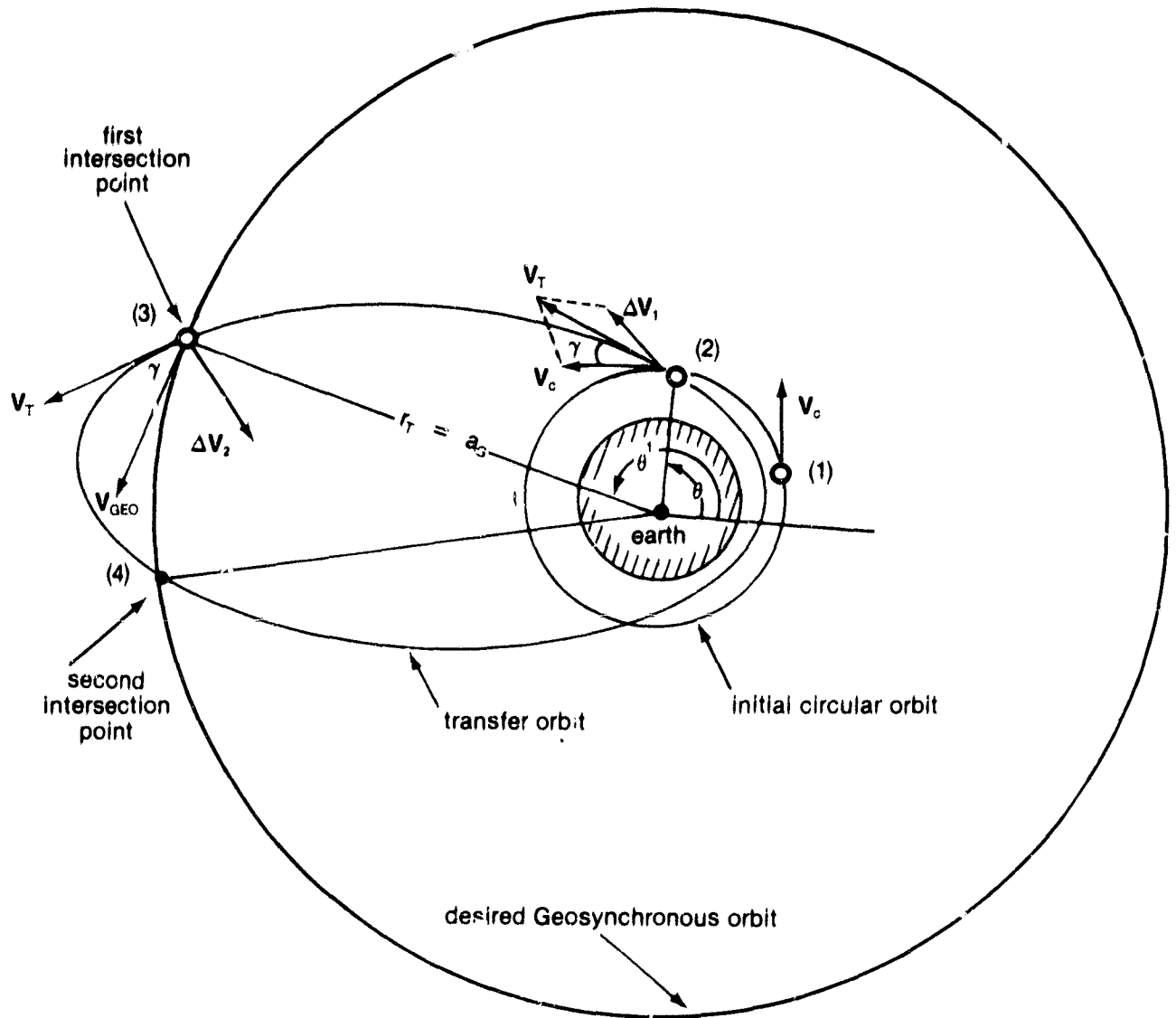
$$\beta = \cos^{-1} \left( \frac{\Delta V_{REQ}}{\text{1st stage } \Delta V} \right)$$

$$\beta = \cos^{-1} \left( \frac{7961.8766}{8200} \right) = 13.8417^\circ$$

From Figure 8C,

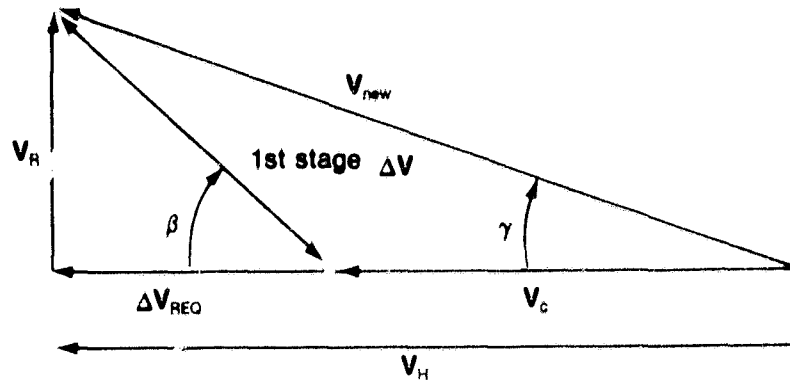
$$V_R = (\text{1st stage } \Delta V) \sin \beta .$$

$$V_R = 8200 \sin (13.8417^\circ),$$



**Figure 7C.** Orbital Transfer Away From Apogee. At (1) a satellite in circular orbit with velocity  $V_C$ . At (2) engines are fired producing a velocity  $\Delta V_1$ . The new velocity  $V_T$  is the vector sum of  $\Delta V_1$  and  $V_C$ . At (3) engines are fired in a predetermined direction producing a velocity  $\Delta V_2$ . The resultant velocity of the satellite will be the vector sum of  $\Delta V_2$  and  $V_T$  at that point. If the sum is equal to  $V_{GEO}$  and directed horizontally, the satellite will be placed in the geosynchronous orbit shown.





**Figure 8C.** Velocity Addition for Directed Burn. First stage  $\Delta v$  is the vector sum of  $\Delta V_{REQ}$  and  $V_R$ . Angle  $\beta$  is the burn angle. The resultant velocity after burn,  $V_{new}$ , is the vector sum of the initial velocity in circular orbit,  $V_C$ , and the first stage  $\Delta v$ .  $\gamma$  is the flight path angle for the transfer orbit. The horizontal velocity after burn,  $V_H$ , is the sum of  $V_C$  and  $\Delta V_{REQ}$ .

$$V_R = 1961.7648 \text{ ft/sec}$$

As a check try

$$V_R = \sqrt{(1st \text{ stage } \Delta V)^2 - (\Delta V_{REQ})^2}$$

$$V_R = \sqrt{(8200)^2 - (7961.877)^2},$$

$$V_R = 1961.7648 \text{ ft/sec}$$

The new orbital velocity is

$$V_{new} = \sqrt{V_H^2 + V_R^2}$$

$$V_{new} = \sqrt{(33,315.8263)^2 + (1961.7648)^2},$$

$$V_{new} = 33,373.53447 \text{ ft/sec.}$$

The flight path angle for the first stage is found from Eq. (65).

$$\gamma = \tan^{-1} \frac{V_R}{V_H}$$

$$\gamma = \tan^{-1} \left( \frac{1961.7648}{35,315.8263} \right),$$

$$\gamma = 3.36990^\circ.$$

The new transfer orbit has a set of characteristics different from that given. The parameter  $p$  is obtained from Eq. (32).

$$p = \frac{r^2 V_B^2}{\mu}$$

$$p = \left( \frac{3603.93088}{62750.2781} \right)^2 \left( \frac{33,315.8263}{6076.1155} \right)^2,$$

$$p = 6222.8079 \text{ n.mi.}$$

The new semimajor axis is given by Eq. (16).

$$a = \frac{r}{2 - r \frac{V_{\text{new}}^2}{\mu}}$$

$$a = 13480.6653 \text{ n.mi.}$$

The new eccentricity is given by Eq. (19).

$$e = \sqrt{1 - \frac{p}{a}}$$

$$e = 0.73375074$$

The line of apsides of the new transfer orbit has been rotated by the radial burn. The new true anomaly is given by Eq. (54).

$$\Theta = \tan^{-1} \left( \frac{p \tan \nu}{p - r} \right)$$

$$\Theta = 7.9649^\circ$$

We have now completed the calculation of

- (i) the required burn direction,
- (ii) the characteristics of the new transfer orbit.

The direction and magnitude of the 2nd stage burn is determined by the intersection of the transfer orbit and the required geosynchronous orbit. To determine where this intersection occurs (see Figure 7C) we determine the true anomaly for  $r$  equal to the radius of the geosynchronous orbit. Using Eq. (57),

$$\theta' = \cos^{-1} \left( \frac{p - r_{\text{GEO}}}{er_{\text{GEO}}} \right)$$

$$\theta' = \cos^{-1} \left( \frac{6222.8079 - 22,766.8409}{(0.73373) 22,766.8409} \right)$$

$$\theta' = 172.0355^\circ.$$

Note that the angle of intersection,  $\theta'$ , when added to the angle at burn, is  $180^\circ$ . Under the circumstances of burn we have chosen, orbital intersection will occur at angles  $\theta' = 180^\circ - \theta$  and at  $\theta' = 180^\circ + \theta$ .

Assume that we have chosen for orbital change the second intersection point,  $\theta' = 180^\circ + \theta$ . The radial velocity at this point of intersection is given by Eq. (77).

$$V_R = \sqrt{\frac{\mu}{p}} e \sin(180 + \Theta),$$

where  $\Theta$  is the true anomaly of the 1st burn. Inserting the data,

$$V_R = 1961.7661 \text{ ft/sec.}$$

Note that this differs from the radial velocity of the first burn by: (i) it is in opposite direction; (ii) it differs in magnitude by less than 0.002 ft/sec, which we can safely attribute to rounding error in the computations. See Exercise C.4.5. The orbital velocity is given by Eq. (70).

$$V = \sqrt{\mu \left( \frac{2}{r} - \frac{1}{a} \right)}$$

or,

$$V = 5,626.861 \text{ ft/sec.}$$

The flight path angle is given by Eq. (63).

$$\gamma = \sin^{-1} \frac{V_R}{V}$$

yielding

$$\gamma = 20.4043^\circ.$$

The horizontal velocity is given by

$$V_H = V \cos \gamma,$$

or

$$V_H = 5,273.507 \text{ ft/sec}$$

As a check on the calculated horizontal velocity, use

$$V_H = \sqrt{\frac{\mu p}{r}}$$

to obtain

$$V_H = 5,273.8068 \text{ ft/sec.}$$

Knowing the velocity required to circularize at geosynchronous altitude to be

$$V_{GEO} = 10,087.4748 \text{ ft/sec,}$$

The  $\Delta V$  required is

$$V_{GEO} - V_H = 4,813.668 \text{ ft/sec.}$$

The second stage must be burned in a direction with a magnitude that (i) cancels the radial component in the transfer orbit, and (ii) provides the  $\Delta v$  necessary in the horizontal direction. From Figure 9C, the magnitude of the 2nd stage  $\Delta v$  required is

$$\begin{aligned} \text{2nd Stage } \Delta v &= \sqrt{(\Delta V_{REQ})^2 + V_R^2} \\ &= \sqrt{(4813.688)^2 + (1961.7661)^2} \end{aligned}$$

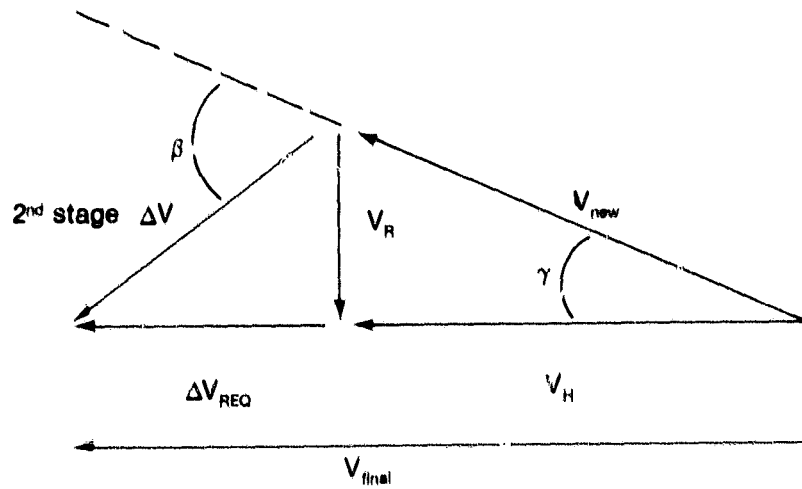
$$\text{2nd Stage } \Delta v = 5,198.069 \text{ ft/sec}$$

The angle at which the 2nd stage must be burned is given by

$$\beta = 180 - (90 - |\gamma|) - \cos^{-1} \left( \frac{V_R}{\text{2nd stage } \Delta v} \right)$$

$$\beta = 90 + |\gamma| - \cos^{-1} \left( \frac{1961.7661}{5198.069} \right)$$

where we have used the facts that the sum of the angles in a triangle is  $180^\circ$  and that the total on a straight line is  $180^\circ$ . Then



**Figure 9C. Second Stage Velocity Addition.** The resultant of  $V_{\text{new}}$  and the 2nd stage  $\Delta v$  is  $V_{\text{final}}$ , the final velocity for the desired circular orbit. This velocity must be horizontal and equal to  $V_{\text{GEO}}$ . The radial component of  $\Delta v$  must be identical to and cancel the radial component of  $V_{\text{new}}$ . The flight path angle  $\gamma$  is identical to the flight path angle caused by the first burn.

$$\beta = 42.5772^\circ$$

and the burn direction has been found, completing the calculations. Note again that in the real world, a change of plane, which we have ignored, is also necessary.

**Exercise C.5.1.** Since an intersection with the transfer orbit just determined occurs at a true anomaly of

$$180 - \Theta$$

in the new transfer orbit, derive the magnitude and direction of the 2nd stage burn required to circularize into a geosynchronous orbit at this first intersection.  $\Theta$  is the true anomaly at 1st stage burn ( $\Theta = 7.9649^\circ$ ).

**Exercise C.5.2.** Find the mean anomaly at epoch and the epoch after time of perigee passage for the new transfer orbit of the last exercise at  $\Theta = 7.9649^\circ$ .

$$t_0 = 0^h 01^m 31^s$$

$$M_0 = 0.8333^\circ$$

Verify these values with the calculator program of Exercise B.4.1.

**Exercise C.5.3.** Find the times required to intersect the geosynchronous orbit altitude in the new transfer orbit, after the time of first burn.

ANS:

$$\text{1st opportunity } M_1 = 145^\circ 22' 19''$$

$$(t - t_0)_1 = 4^h 21^m 11^s$$

$$\text{2nd opportunity } M_2 = 214^\circ 32' 31''$$

$$(t - t_0)_2 = 6^h 28^m 34^s$$

These times are lapsed times after the first stage burn.

**Exercise C.5.4.** Calculate the  $\Delta v$  required for escape from earth for a 160 n.mi altitude orbit (Eq. (79)). Compare this with the total  $\Delta v$  required to place a payload in a geosynchronous orbit from 160 n.mi. Compare the two.

**Exercise C.5.5.** The orbiter is in an orbit with the characteristics

perigee altitude = 120 n.mi

apogee altitude = 260 n.mi

An upper stage is to be deployed from perigee for transfer into geosynchronous orbit. The first stage is a solid rocket with a  $\Delta v$  of 8,200 ft/sec. The second stage is a liquid fuel stage.

Design the upper stage burn directions, magnitudes, and burn locations. There are two opportunities for upper stage ignition during each revolution in the transfer orbit. Find the times of each opportunity

## **C.6. EVALUATION**

### **C.6.1. Learning Objectives**

The principal objectives of Section C were fourfold: first, to show how sets of parameters characterize an orbit and to develop methods for calculating the position and velocity of a satellite in the plane of its orbit; second, to show how particular orbits may be attained through orbital transfers; thirdly, to show how orbital maneuvers may be influenced by upper stage rocket characteristics and fuel efficiency requirements; and lastly, to raise the level of analysis from single to multistage and multivariable.

Having completed your study of this section and related references, you should have an understanding of

- the elements required for describing orbit characteristics and satellite position in the plane
- the problems relating to transfer orbits and fuel efficiency
- the use of burn directions, thrust magnitudes, and orbital position in effecting orbital transfers



## **C.6.2. Outcome Measures**

**You should now be able to**

1. Use the appropriate basic transfer equations in solving problems.
2. Use the appropriate formulas and specific data to calculate (a) the position of an orbiter, (b) the velocity of an orbiter at a particular point in its orbit, and (c) the time required for an orbiter to travel from one point to another.
3. Describe orbital transfers.
4. Add and resolve vectors information components;
5. Design upper stage burn directions, magnitudes, and burn locations for designated orbital transfers.
6. Define fuel requirements for orbital maneuvers, given appropriate information.

## D. TIME AND RELATIVE ORBITAL POSITION

### D.0. INTRODUCTION

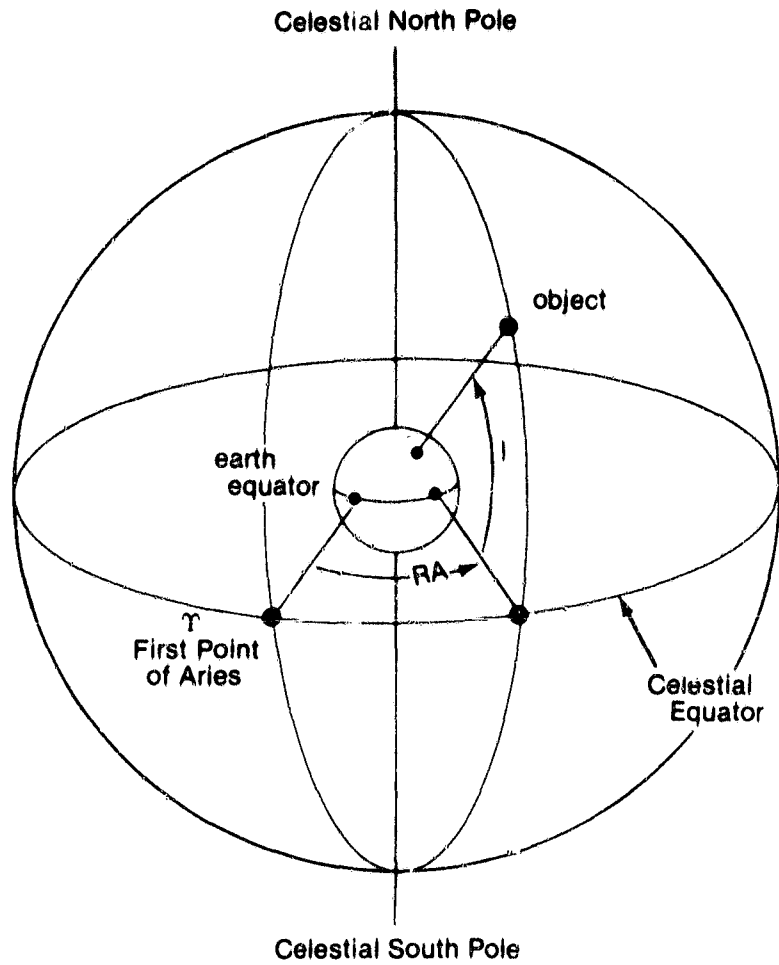
We have discussed the position and velocity of a satellite, orbiter, or other spacecraft relative to the apogee and perigee points of its orbit, and orbital transfers within the orbital plane: now we turn our attention to a more complete description of the position of an orbiting object, one relative to its base — the earth — and its energy source — the sun. The earth is a ball, not quite spherical, traveling at high speed in its orbit about the sun and spinning like a top about its axis. When we calculate the velocity of a satellite, it is usually the velocity relative to an earth surface, and we want to be able to specify its position relative to some position on that surface.

### D.1. CELESTIAL COORDINATES

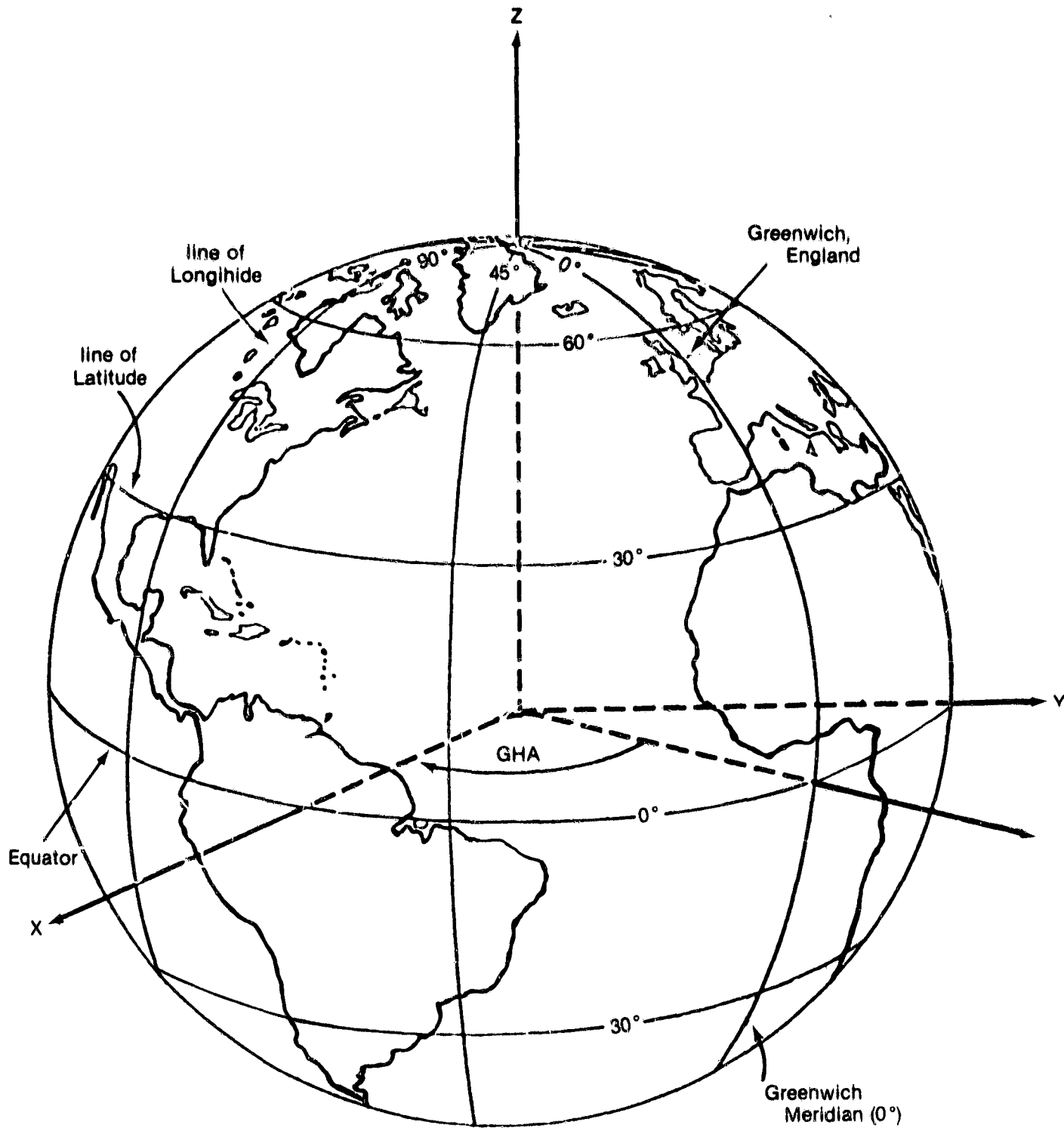
The position of stars and other celestial or space objects may be specified in the same way that a position is defined on the earth's surface. The *location* of any spot on earth is given by latitudinal and longitudinal meridians. *Latitude* is measured north or south of the *equator*, which is assigned a value of  $0^\circ$ , as shown in Figure 1D. The *poles* are  $90^\circ$  to the north and south. *Longitude* is measured east or west of the *Greenwich Meridian*, which is the  $0^\circ$  reference. One nautical mile at the equator was originally the equivalent of one minute of arc.

To define a set of Celestial coordinates, we construct a *Celestial Sphere* with the earth at center as shown in Figure 2D. The intersection of the projected plane of the earth's equator with the Celestial Sphere defines the *Celestial Equator*. A line from the Celestial North Pole to the Celestial South Pole passing through the point in the sky, the *First Point of Aries* or *Vernal Equinox*, where the sun crosses the equator in the spring, provides the longitude type reference for the sky. The symbol,  $\gamma$ , sign of the Ram, is used to denote the Vernal Equinox and may be recognized by those of you born under it. Angles are measured to the east along the Celestial Equator from the First Point of Aries.

**Exercise D.1.1.** Using the value of  $1 \text{ n.mil} = 1 \text{ min of arc}$ , calculate the equatorial radius in nautical miles. Compare this with the value in Appendix A. (See, for example, the entry *nautical mile* in Webster's Unabridged Dictionary for an indication of the reason for the discrepancy.)



**Figure 2D.** Celestial Coordinates. The line from the Celestial North Pole through the First Point of Aries to the South Pole provides a Celestial "longitude" reference. An object positioned as shown is located through the angle RA, called the right ascension (defining "longitude") and the angle I, the declination (defining "latitude").



**Figure 1D. Earth Meridians.** The Greenwich meridian at 0° is longitudinal reference for the earth. The equator is the latitudinal reference at 0°. The angle shown in the equatorial plane is called the Greenwich Hour Angle (GHA).  $GHA = GHA_{epoch} + \omega_e(t)$  is the rotation rate of the earth and  $t$  is the elapsed time from epoch.

## D.2. TIME MEASUREMENT

The clock time interval between successive meridian passages of the sun is called a *true solar day*. That is to say, if a time check is made when the sun is directly overhead on one day and again when the sun is directly overhead the following day, that time interval is the true solar day, as illustrated in Figure 3D. The length of the true solar day varies throughout the year, since it is dependent upon the speed of the earth which varies (according to Kepler's second law) in its orbit about the sun. The average clock time length of the solar day over the year is called the *mean solar day*. The mean solar day is divided into 24 hours or 86,400 mean solar seconds.

We know the motion to be relative. It is accepted fact that the earth spins on its axis and moves in orbit about a focal point at the center of the sun. But such motion can be viewed from different perspectives (different coordinate or reference systems). Choosing to regard the earth as stationary in its orbit we obtain a different view: the sun would appear to move in orbit about the earth with a period of one year. Choosing to regard the earth as stationary in its rotational motion, the sun would appear to orbit about the earth with a period of one day: this is our normal "illusion," with the solar day defined in terms of successive earth meridian passages of the sun in its orbit, shown in Figure 4D. We use the view or system that is most convenient for our problem description.

The time divisions of the clock for a mean solar day define the quantity called *Universal Time (UT)*. They are also the same for *Greenwich Mean Time (GMT)*, 12 hours GMT (noon) defined to be the instant of passage of the mean sun over the Greenwich Meridian ( $0^\circ$  longitude). The *Local Mean Time (LMT)* at any longitude on earth can be calculated by

$$\text{LMT} = \text{GMT} + (\text{East longitude})/15$$

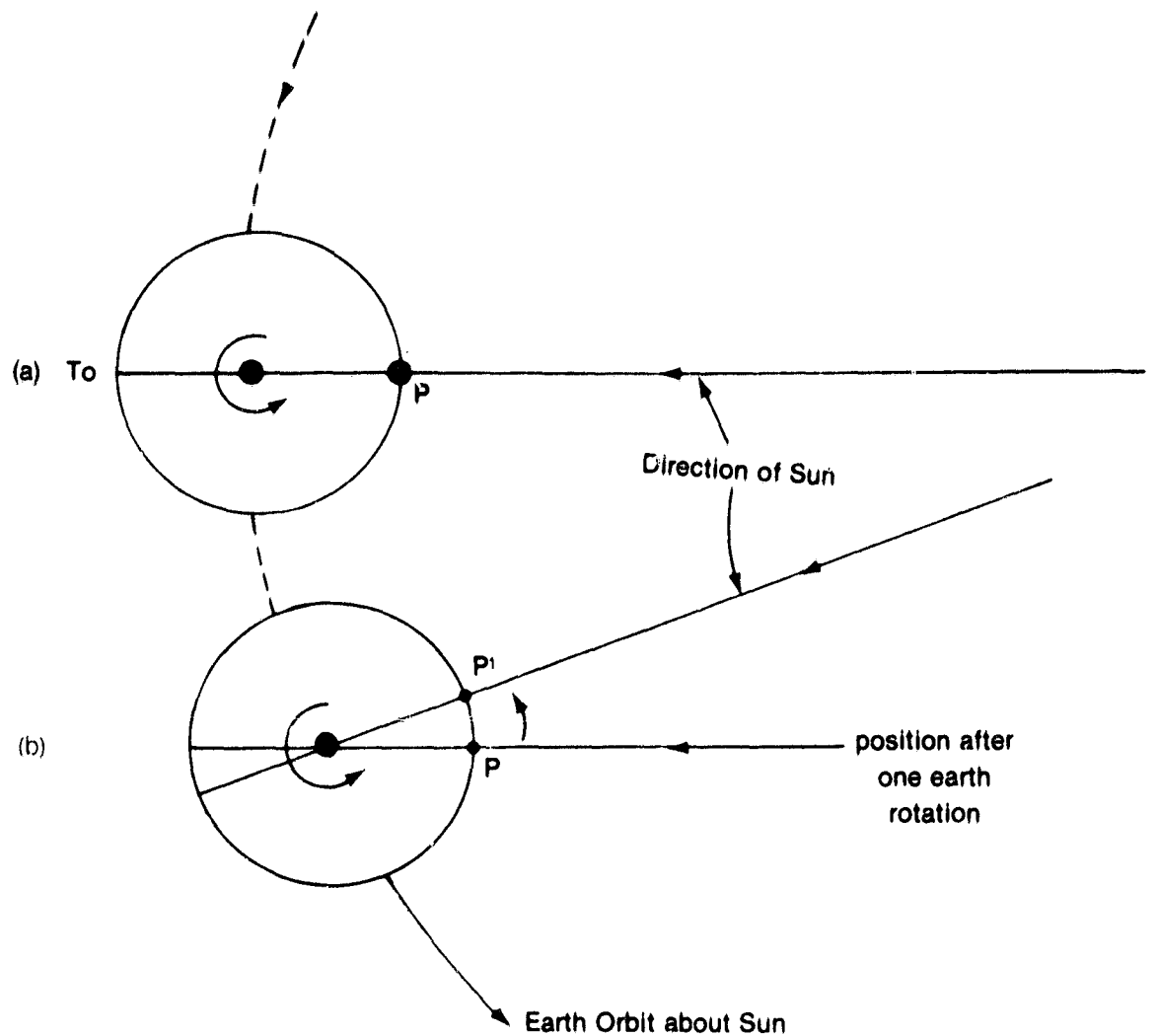
or

$$\text{LMT} = \text{GMT} - (\text{West longitude})/15$$

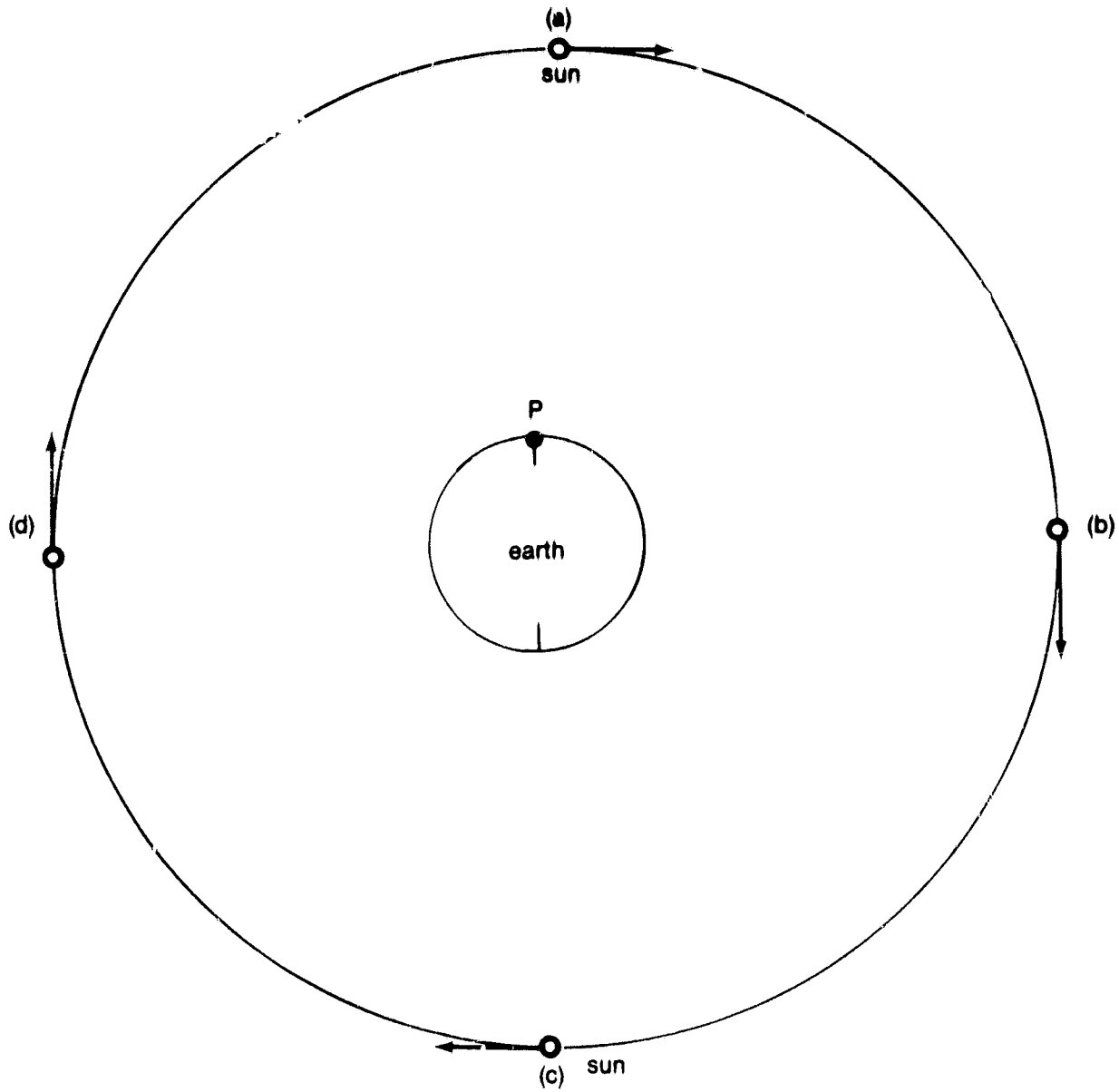
where the longitude is given in degrees and fractions of degrees.

The time interval between successive meridian passages of the vernal equinox is called a *sidereal day*. It is roughly the time for one earth rotation, as was shown in Figure 3D. But because of the "wobble" of the earth's poles, called precession, the sidereal day varies slightly in length. As with the solar day, we may define a mean sidereal day, *Greenwich Mean Sidereal Time (GST)*, and *Local Sidereal Time (LST)*. A sidereal time interval is somewhat shorter than a solar time interval:

$$\text{sidereal time interval} = 0.9972695664 \\ (\text{solar time interval})$$



**Figure 3D. The Solar and Sidereal Day.** With the earth considered in its motions, the time  $T_0$  at P is noon with the sun directly overhead in (a). When the earth makes one complete rotation, the point P will be located as shown in (b). Due to the movement of the earth in its orbit, the sun is now directly overhead at the point P'. The time required for the earth to rotate from P (a) to P (b) is one sidereal day. The time required for the earth to rotate from P (a) to P' (b) is one solar day. If P is on the Greenwich Meridian, the time at P' (a) is 12<sup>h</sup> Greenwich Mean Time (GMT).



**Figure 4D.** The Solar Day. From a point P on the surface of the earth it “appears” that the sun moves about the earth as shown. With the sun in position (a) it is noon. Sunset occurs at (b), midnight at (c), and sunrise at (d). The time required to return to position (a) is one solar day.

**Exercise D.2.1.** Find the length of a sidereal day in terms of the mean solar day.

**Exercise D.2.2.** A point near Houston, Texas, has the longitude  $95^{\circ} 00' 28''$ W. Find the Local Mean Time at  $12^{\text{h}}$  Greenwich Mean Time.

**Exercise D.2.3.** Find the length of a mean solar day in sidereal time.

### D.3. ANGULAR TIME MEASURE

The time, in angle measure, since the last passage of the first point of Aries over the Greenwich Meridian is called the *Greenwich Hour Angle* (GHA) (See Figure 1D). The GHA expressed in time measure is identical to the Greenwich Mean Sidereal Time (GST). Thus, conversion of time measure to angle measure is based on exactly 15 seconds of arc per sidereal second, or

$$15.041069 \text{ sec of arc} = 1 \text{ mean solar second}$$

**Exercise D.3.1.** Using the number of revolutions per day from Exercise A.7.1., calculate the number of seconds of arc per mean solar day, and compare your answer with the number above. Calculate the number of degrees per solar hour.

The GHA of Aries can be computed at any time from the equation

$$\text{GHA}(t) = \text{GHA}(t_0) + \frac{15.041069}{3600} (t - t_0)$$

where time is in mean solar seconds and GHA (t) is in degrees and fractions of degrees.

**Exercise D.3.2.** Compute the GHA of Aries at  $17^{\text{h}}$  GMT on July 2, 1979, using the epoch value,  $\text{GHA}(t_0) = 278^{\circ} 27' 12''$ , on July 1, 1979. Compare your answer with the A.E. Nautical Almanac entry.

### D.4. TIME CORRECTIONS

Time is given in the *American Ephemeris* and *Nautical Almanac* in what is called *Ephemeris Time* (ET). This is a uniform time measurement based in prin-



cial on observations of the planets and moon. There is no theoretical connection between Ephemeris time and Universal time, or GMT, which fluctuates as a result of changing earth motion. The difference is observed over several years and projected forward one year at a time. For example, the predicted difference between GMT and Ephemeris time in 1979 is given as:

Jan. 1, 1979 + 49.9 seconds

Apr. 1, 1979 + 50.1 seconds

July 1, 1979 + 50.4 seconds

Oct. 1, 1979 + 50.7 seconds

The difference is quite small, but correction to Universal Time should be made if high accuracy is required.

For example, the right ascension of the sun at 0<sup>h</sup>, July 1, 1979, is given in the Almanac as 6<sup>h</sup> 37<sup>m</sup> 23.45<sup>s</sup>. The correction for GMT is

$$50.4^s \times \frac{248.26}{86400} \times 15.0 = 2.16''$$

Therefore, the corrected right ascension for 0<sup>h</sup> GMT on the given date is 99° 20' 54.16".

**Exercise D.4.1.** Use the American Ephemeris and Nautical Almanac to find the Universal Time of the meridian passage of the equinox at Greenwich on July 1, 1979.

## D.5. THE RIGHT ASCENSION AND LONGITUDE

The angle measured to the east along the Celestial Equator from the first point of Aries to the celestial meridian of an object in the sky is called the *right ascension* (RA) of the object. This has been illustrated in Figure 2D. It is the difference between the Greenwich Hour Angle of Aries and the GHA of the object.

The *Local Hour Angle* (LHA) is the time since the local meridian passage of the first point of Aries. It is identical to the Local Sidereal Time when expressed in time measure.

$$\text{LHA} (\gamma) = \text{GHA} (\gamma) \begin{cases} + \text{ East longitude } (\lambda \text{ E}) \\ - \text{ West Longitude } (\lambda \text{ W}) \end{cases}$$

If the right ascension of a celestial object is equal to the Local Hour Angle, i.e.,

$$\text{when RA} = \text{LHA} (\Upsilon)$$

the object is directly over the local meridian. Therefore, the *earth fixed longitude*,  $\lambda$ , of a subsatellite point can be found if the right ascension of the satellite can be found. The earth longitude is given by

$$\lambda_{\epsilon} = \text{RA} - \text{GHA} (\Upsilon)$$

where measurement is in the eastern direction (see Figure 5D).

**Exercise D.5.1.** Using the Nautical Almanac, find the right ascension of the sun at 0<sup>h</sup> GMT on July 1, 1981.

**Exercise D.5.2.** At 0<sup>h</sup> GMT July 1, 1981, calculate where on earth it is exactly noon. Note that it is the "subsatellite" longitude of the sun requested. Compare your answer with the GHA (sun) from the Almanac for that date.

## D.6. THE RIGHT ASCENSION OF THE MEAN SUN

In our work a rapid means of calculating the right ascension of the mean sun, RA (sun), is an advantage over the use of tables, for reasons of convenience and efficiency. The orbital elements of the sun are used; again, we pretend that the sun is in orbit about the earth when doing these calculations. The coordinate system used is the true equator and true equinox of date, as shown in Figure 6D.

Consider the epoch 0<sup>h</sup> GMT, January 1, 1979 (JD 2443874.5). The orbital elements of the sun will be

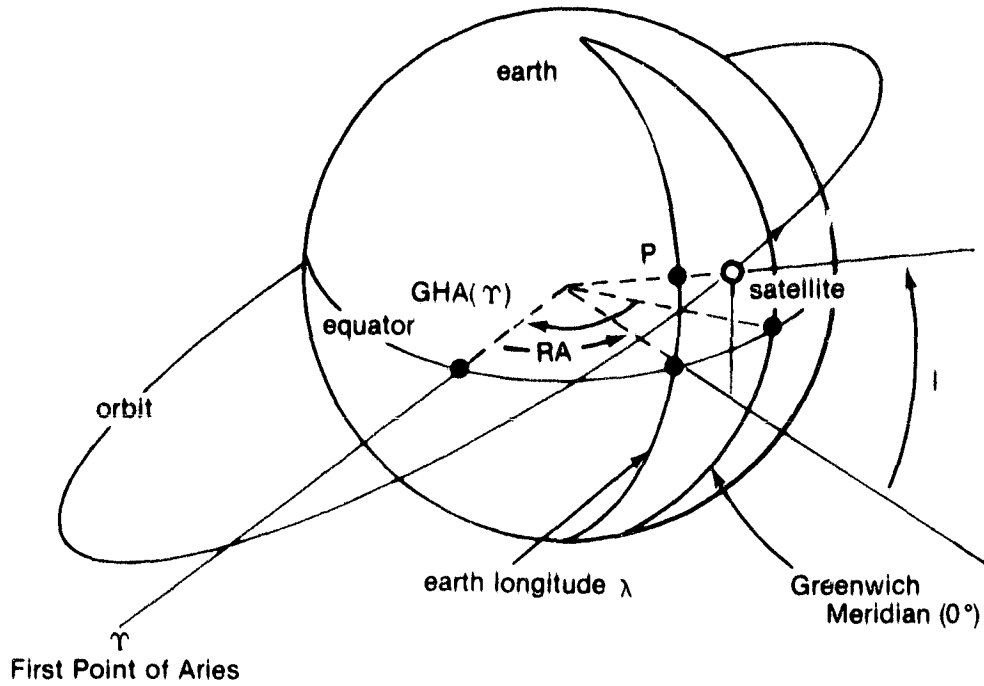
$$g = 282.5711432^{\circ} + (0.0000470684^{\circ}) d,$$

$$M = 357.47863 + (0.985600267) d,$$

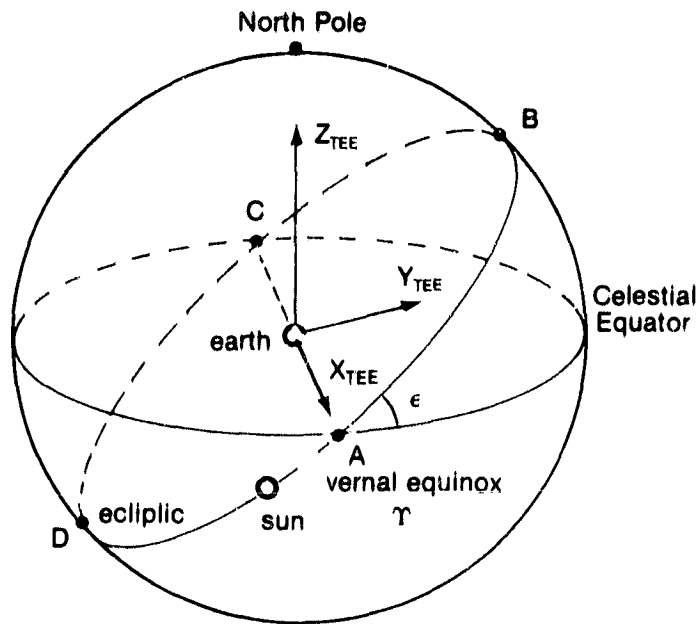
$$e = 0.016736825,$$

$$\epsilon = 23.44207294^{\circ}.$$

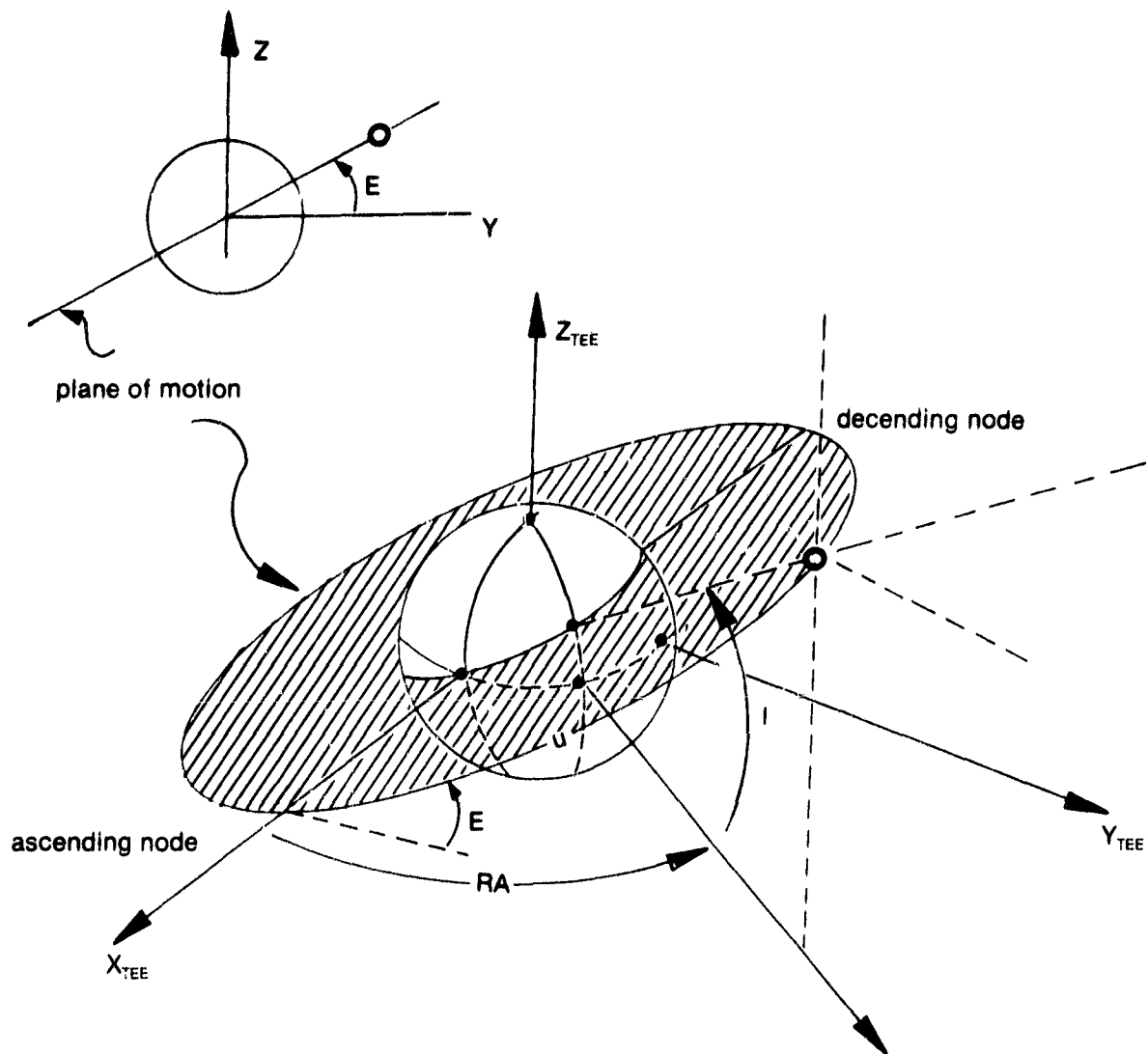
$g$  is called the *argument of perigee*, the angle measured in the plane of the *ecliptic* (sun's orbital plane illustrated in Figure 8D) from the *true equinox of date* (First Point of Aries) to the perigee of the sun's orbit, in degrees.



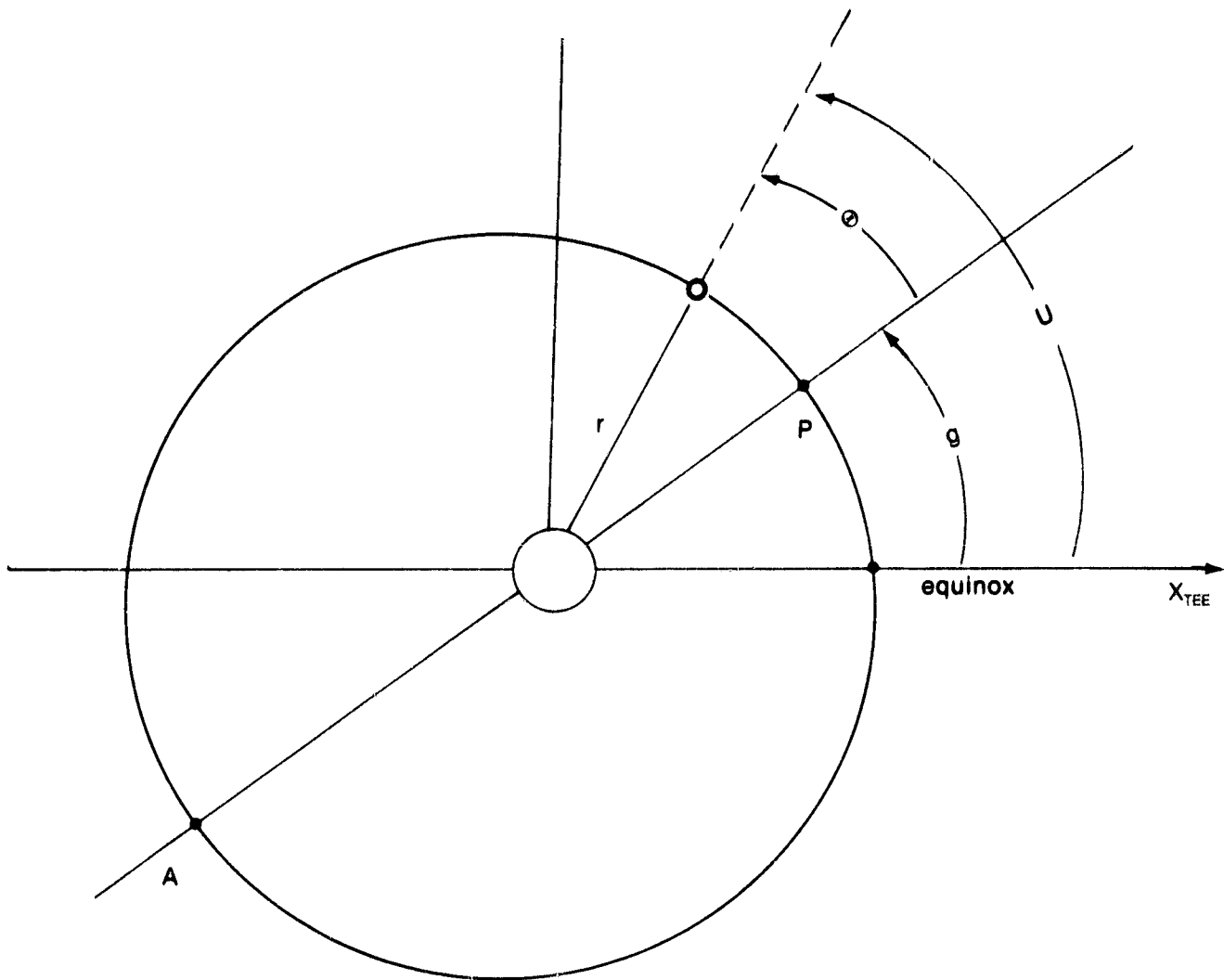
**Figure 5D.** Longitude of Subsatellite Point.  $GHA(\gamma)$  is the angle measured from the Greenwich Meridian to a line pointing to the First Point of Aries or the Vernal Equinox.  $RA$  is the right ascension of the satellite, measured in the equatorial plane from the Vernal Equinox as shown. The satellite is directly over the point  $P$  which lies on the earth longitude  $\lambda$ . With  $GHA(\gamma)$  greater than  $RA$ , the longitude is west with  $\lambda_w = GHA(\gamma) - RA$ .



**Figure 6D.** The Ecliptic. The yearly path that the sun appears to follow among the stars observed from earth is called the elliptic. It makes an angle  $\epsilon = 23^{\circ} 27^m$  with the celestial equator. Point A is the Vernal Equinox or First point of Aries, the ascending node; with the sun in that position, on March 21, the length of day and night are equal. Point C, the descending node, occurs on September 21. The coordinate system is the True Equator and Equinox of Epoch (TEE) reference axis system. The epoch is defined to be  $0^h 0^m 0^s$  GMT on the user specified base date.



**Figure 7D.** Location of Orbiting Body. A body in orbit is specified by its right ascension, RA, and declination, measured in and from the equatorial plane. The angle  $u$  is measured in the orbital plane.



**Figure 8D.** Angles in the Plane of Motion. P refers to the orbit perigee and A to apogee. The angle  $g$  is the argument of perigee,  $\theta$  is the true anomaly, and  $U$ , measured from the ascending node to the object, is the argument of latitude.

d is the number of Julian days elapsed after 0<sup>h</sup> GMT, January 1, 1979, in days and decimal fractions of days. (Note: The correction term added to g represents the perturbation of the earth's orbit by the moon and planets. It amounts to 0.1694 secs of arc per day rotation of the line of apsides of the orbit.)

M is the mean anomaly of the sun in its orbit. The term involving the number of Julian days since epoch reflects the period of the sun in its orbit:

$$P = \frac{360 \text{ deg}}{0.985600267} \text{ deg/deg} = 365.259641 \text{ days}$$

e is the eccentricity. It changes due to perturbations of the moon and planets about 41.8 parts per million per century (decreasing during our geological era). We will ignore this change.

$\epsilon$  is the *obliquity of the ecliptic* (the inclination of the plane of the sun's orbit) in degrees. The obliquity of the ecliptic is also changing at the rate of 47 secs of arc per century. We will ignore this change.

The procedure for calculating the right ascension is as follows.

- (a) Find the Julian date of the new date.
- (b) Subtract the Julian date of epoch 0<sup>h</sup> GMT January 1, 1979 (JD 2443874.5) to find d,

$$d = \text{JD} - 2443874.5$$

Note that the Julian date may include a decimal fraction of a day. For example, at 17<sup>h</sup> 32<sup>m</sup> 03.3<sup>s</sup> GMT on 30 June 1979, the Julian date is

$$2444054.5 + \frac{17.53425}{24} = 2444055.23$$

- (c) Find the new elements of the orbit at the new date.
- (d) Find the true anomaly (see programs written in Exercises B.2.1 and B.4.1.),

$$\Theta = M + 2 e \sin M + \frac{5}{4} e^2 \sin 2 M$$

$$+ \frac{e^3}{12} (13 \sin 3M - 3 \sin M)$$

- (e) Find the argument of latitude,  $u$ , defined as the angle in the plane of motion from the *ascending node*, i.e., the true equinox of date to the position of the object),

$$u = g + \Theta$$

- (f) Calculate

$$x = \cos u,$$

$$y = \cos \epsilon \sin u,$$

$$\text{RA (Sun)} = \tan^{-1} \left( \frac{y}{x} \right)$$

- (g) Do a quadrant check.

**Exercise D.6.1.** Repeat the calculation for the Julian date for  $17^{\text{h}} 32^{\text{m}} 03.3^{\text{s}}$  GMT on 30 June 1979. Rearrange the computations as necessary to carry the number of decimal fractions of a day to 0.1 second of time. How many decimal places must be retained?

**Exercise D.6.2.** Find the right ascension of the sun at  $17^{\text{h}} 32^{\text{m}} 03.3^{\text{s}}$  GMT on June 30, 1979. In your solution you should find:

$$d = 180.73059 \text{ days},$$

$$g = 282.57965^{\circ},$$

$$M = 175.60675,$$

$$u = 98.18640 + 57.29578 \frac{\text{deg}}{\text{rad}} (0.0025118),$$

$$w = 98.18640 + 0.143911,$$

$$x = -0.144880,$$

$$y = 0.907783,$$

$$\text{RA(sun)} = -80.93222^{\circ} + 180^{\circ},$$

$$\text{RA(sun)} = 99^{\circ} 04' 04''.$$

**Exercise D.6.3.** Check the result of the last exercise against the tabular entry in the American Ephemeris and Nautical Almanac. What is the discrepancy?



ANS: at 0<sup>h</sup>ET, RA = 6<sup>h</sup> 33<sup>m</sup> 14.94<sup>s</sup>, corresponding to 98.3125°

at + 17<sup>h</sup> 32<sup>m</sup> 03.3<sup>s</sup> ET, RA = 99° 04' 08"

$$(0.130590 \times 248.51 = 181,5589 = 2723.385 \text{ sec arc} = 0.756496^\circ)$$

ET to GMT correction, RA = 99° 04' 10.16"

$$(50.1 \times \frac{248.51}{86400} \times 15 = 2.16 \text{ sec of arc})$$

**Exercise D.6.4.** (a) What clock error corresponds to 6.2 secs of arc error in RA (Sun)?

ANS: 143.7 sec

(b) How many decimal places in the Julian Day number does this represent?

ANS: 1E - 06

**Exercise D.6.5.** What is the maximum error,  $\delta$ , in RA (Sun) for an observer on the equator if he uses the tabular entry in the Almanac? See Figure 9D.

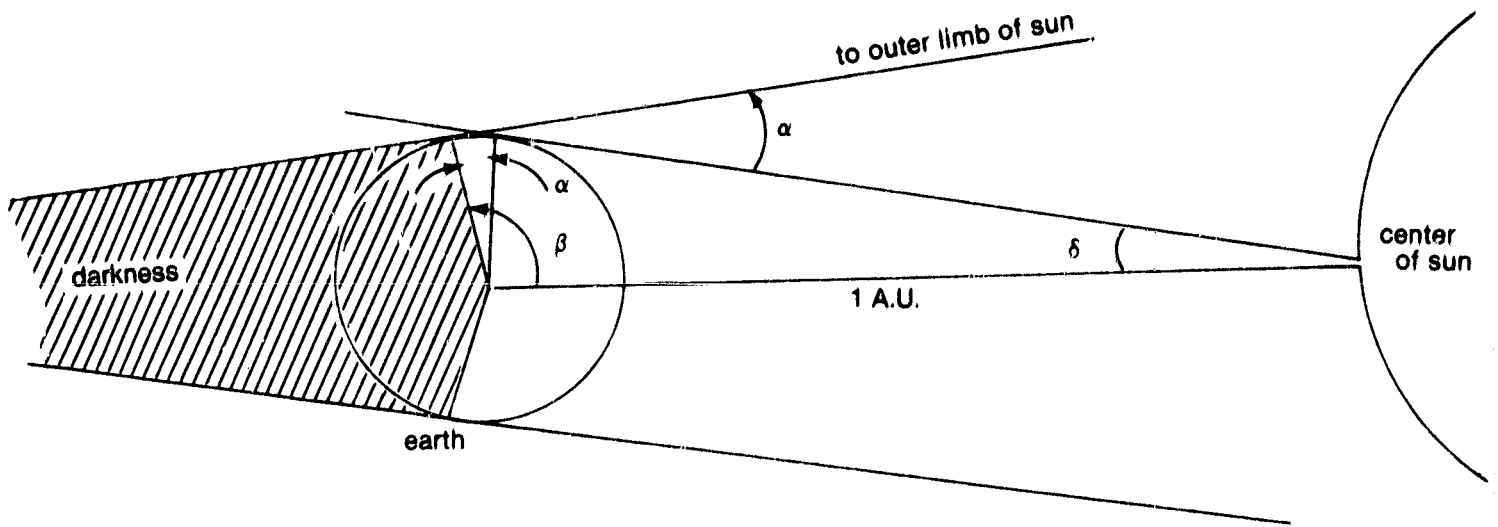
ANS: 9 secs of arc

**Exercise D.6.6.** What is the maximum error in RA (Sun) for the orbiter in a 160 n. mi orbit using the tabular entries in the Almanac?

**Exercise D.6.7.** For an observer at earth's distance, what is the maximum error in RA (Sun) if one limb is observed (see  $\alpha$  in Figure 9D) instead of the center of the sun? Note that the sun has a diameter of 752,827.06 n.mi, and find its distance on June 30, 1979 from the Almanac.

ANS: 15' 45" arc

**Exercise D.6.8.** Check this last answer against the apparent semi-diameter of the sun given in the Almanac for 0<sup>h</sup> ET June 30, 1979.



**Figure 9D.** Horizontal Parallax. The angle  $\delta$  is defined through  $\tan \delta = (R_{eq}/1 \text{ A.U.})$  and  $\alpha$  through  $\tan \alpha = (R_{sun}/1 \text{ A.U.})$ . The angle  $\beta$ ,  $90^\circ + \alpha$  is the maximum central angle of daylight.

**Exercise D.6.9.** Write a calculator program for calculating the RA (Sun) and GHA (Sun) given the time, GMT, and number of Julian days elapsed since epoch. Choose the epoch to be 0<sup>h</sup> GMT January 1, 1979.

## D.7. THE DECLINATION OF THE SUN

The *declination* of a celestial object is the angle from the celestial equator to the object, measured positive to the north, negative to the south. Check Figures 2 and 8D. The declination of the sun is given by

$$e = \sin^{-1}(\sin \epsilon \sin u),$$

where

$e$  is the declination,

$\epsilon$  is the obliquity of the ecliptic,

$u$  is the argument of latitude, calculated as in the previous section.

**Exercise D.7.1.** Find the declination of the sun at 17<sup>h</sup> 32<sup>m</sup> 03.3<sup>s</sup> GMT on June 30, 1979. Compare your answer with the Almanac tabulated value.

ANS: 23° 10' 49"; 23° 10' 36" tabulated

**Exercise D.7.2.** Add the computation of the declination to the calculator program of Exercise D.6.9.

## D.8. SUNRISE AND SUNSET

Daylight and darkness onset times are calculated in terms of the maximum central angle of daylight. This was shown in Figure 9D.  $\beta$  would be 90° if the sun were a point (infinitely far away), if the earth were a perfect sphere, and if the atmosphere did not bend the rays of the sun. The correction for the fact that the sun has an observable diameter gives

$$\beta = 90^\circ 15' 45''$$

The longitudes of sunrise and sunset are found from

$$\lambda_{\text{sunrise}}^{\text{sunset}} = \text{GHA}(\text{sun}) \pm \cos^{-1} \left( \frac{\cos \beta - \sin l \sin \text{LAT}}{\cos l \cos \text{LAT}} \right)$$

where

$l$  is the declination of the sun at date,

LAT is the given latitude,

$\beta$  is the central angle ( $\beta = 90^\circ 15' 45''$ ).

$$\text{GHA}(\text{sun}) = \text{GHA}(\Upsilon) - \text{RA}(\text{sun})$$

gives the longitude on the earth where it is exactly noon.

**Exercise D.8.1.** Find the longitude of sunset at  $17^{\text{h}} 32^{\text{m}} 03^{\text{s}}$  GMT June 30, 1979 at latitude  $40^\circ \text{N}$ .

Let us look at the solution for the above exercise. From the Nautical Almanac, the GHA ( $\Upsilon$ ) at  $0^{\text{h}}$  GMT June 30, 1979 is  $277^\circ 28.0'$ . At  $17^{\text{h}} 32^{\text{m}} 03.3^{\text{s}}$ ,

$$\text{GHA}(\Upsilon) = 277.46667^\circ + 15.041069 (17.53425),$$

$$\text{GHA}(\Upsilon) = 181^\circ.200534.$$

From Exercise D.6.2,  $\text{RA}(\text{sun}) = 99^\circ 04' 04''$ . Therefore,

$$\text{GHA}(\text{sun}) = -82^\circ 07^{\text{m}} 58^{\text{s}}.$$

From Exercise D.7.1, the declination is

$$l = 23^\circ 10' 49''.$$

Using the given relation,

$$\begin{aligned} \lambda_{\text{sunset}} &= -82^\circ 07' 58'' + \cos^{-1} \left( \frac{-0.004408688 - .25301}{0.704202} \right) \\ &= -82^\circ 07' 58'' + 111^\circ 27' 25'' \\ &= 29^\circ 19' 27'' \end{aligned}$$

**Exercise D.8.2.** (a) Find the time of sunset for the previous exercise in GMT.

$$\text{ANS: } 17^{\text{h}} 32^{\text{m}} 03.3^{\text{s}}$$

(b) Find time of sunset in local time.

We will note that the solution for (b) above is

$$12^{\text{h}} + \frac{111^{\circ} 27' 25''}{15} = 19^{\text{h}} 25^{\text{m}} 50^{\text{s}}$$

That solution follows because

- (a) the longitude of high noon corresponds to  $12^{\text{h}}$ , and
- (b) the longitude to the east converts to time at the rate of  $15^{\circ}$  per hour, *later*.

Check the local time against the entry in the Nautical Almanac for the June 30 date.

When the second term in the equation for longitude of sunrise/sunset is  $180^{\circ}$  or undefined, the sun never rises (or sets) at these latitudes. Thus, the condition for a continuous daylight (or darkness) is

$$\frac{\cos \beta - \sin I \sin \text{LAT}}{\cos I \cos \text{LAT}} \geq 1$$

This occurs when the absolute value of the latitude is greater than  $\beta$  minus the absolute value of the declination,

$$|\text{LAT}| \geq \beta = |I|$$

Thus, for a declination of  $23^{\circ} 10' 49''$  ( $17^{\text{h}} 32^{\text{m}} 03.3^{\text{s}}$  GMT on June 30, 1979) continual daylight occurs above latitude  $67^{\circ} 04' 56''\text{N}$  and continual darkness exists below  $67^{\circ} 04' 56''\text{S}$ . These latitudes must be modified to account for the oblate figure of the earth. Check this result ( $67^{\circ} 04' 56''$ ) against the Sunrise/Sunset table in the Nautical Almanac for this date.

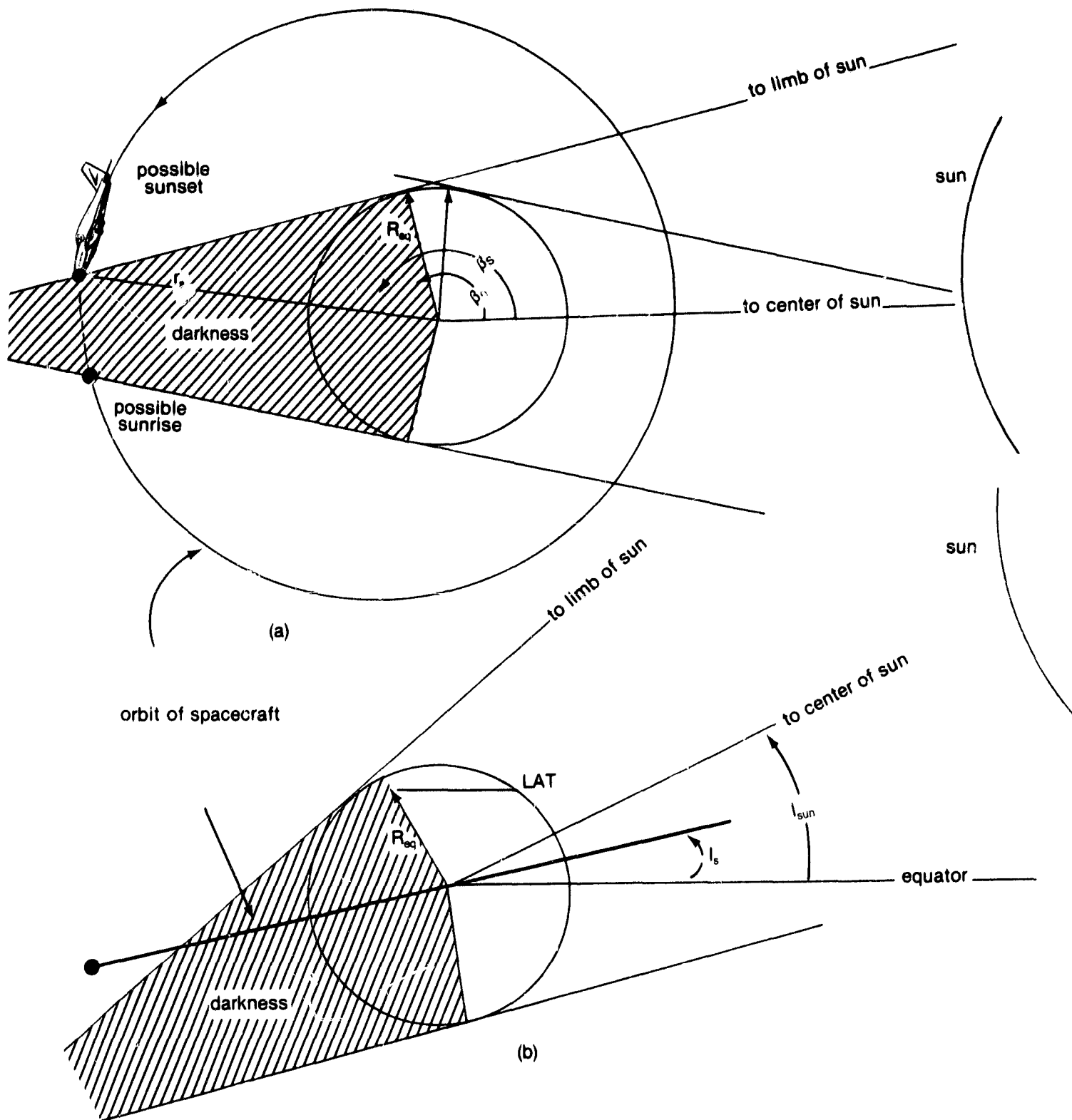
#### D.9. DAYLIGHT/DARKNESS FOR A SPACECRAFT

For a spacecraft above the earth, the central angle  $\beta$  must be modified as shown in Figure 10D. The central angle is given by

$$\beta = \cos^{-1} \left( \frac{R_{\text{eq}}}{r_s} \right) + 90^{\circ} 15' 45''$$

**Exercise D.9.1.** Find the central angle for the orbiter in a 160 n.mi circular orbit.

ANS:  $107^{\circ} 23' 57''$



**Figure 10D.** (a) shows a view perpendicular to orbital plane.  $\beta_0$  is the central angle for the spacecraft at a radial distance  $r_s$ , with  $\beta_0$  equal to  $90^\circ 15' 45''$ . (b) shows a view parallel to equatorial plane with orbital plane given parallel.  $\lambda_s$  is the declination of the orbiting spacecraft and  $\lambda_{\text{sun}}$  the declination of the sun. In (a) is shown possible sunrise and sunset points, but (b) shows that the spacecraft is always in sunlight due to the declinations of sun and orbital planes. Latitudes above  $\text{LAT}_0$  on earth can be seen to be always in sunlight.

The coordinates of the spacecraft and the sun are obtained by projections against the celestial sphere, so that the equation for sunrise/sunset becomes

$$RA(s) = RA(\text{sun}) \pm \cos^{-1} \left( \frac{\cos \beta - \sin i_s \sin i_{\text{sun}}}{\cos i_s \cos i_{\text{sun}}} \right)$$

where

$i_{\text{sun}}$  is the declination of the sun, in degrees,

$i_s$  is the declination of the spacecraft, in degrees.

For the *special case* of a spacecraft in an *equatorial orbit*,  $i_s = 0^\circ$ , the relation reduces to

$$RA(s) = RA(\text{sun}) \mp \cos^{-1} \left( \frac{\cos \beta}{\cos i_{\text{sun}}} \right)$$

**Exercise D.9.2.** For a geosynchronous satellite in a circular equatorial orbit, find the amount of time it spends in darkness on July 1, 1979.

ANS:

$$\beta = 171^\circ 33' 43''$$

$$RA(\text{Sun}) = 99^\circ 20' 54''$$

$$i_{\text{sun}} = 23^\circ 09' 40''$$

Inverse cosine undefined—

Spacecraft always in sunlight

**Exercise D.9.3.** Find the amount of time the geosynchronous satellite spends in darkness on September 23, 1979.

For a solution to Exercise D.9.3, we use the program of Exercises D.6.9, and D.7.2. On 0<sup>h</sup> GMT, September 23, 1979, RA (sun) is found to be given by

$$RA(\text{Sun}) = 179^\circ 25' 40''$$

$$RA(\text{Sun}) = 40^\circ 15' 01''$$

Then

$$RA(s) \begin{array}{l} \text{sunrise} \\ \text{sunset} \end{array} = 179^\circ 25' 40'' \pm 171^\circ 33' 56'',$$

or

$$RA(s) = 350^\circ 59' 36'' \text{ Sunset}$$

$$RA(s) = 7^\circ 51' 44'' \text{ Sunrise}$$

It follows that

$$\text{angle in Darkness} = 16^{\circ} 52' 08''$$

$$\text{total period in hours} = 23.93446$$

$$\text{time in darkness} = \frac{16^{\circ} 34' 30''}{3600} \times 23.93446374,$$

$$t = 1^{\text{h}} 07^{\text{m}} 17^{\text{s}}$$

The ratio of time spent can be found this way only because it is a circular orbit and the motion is uniform throughout the orbit (i.e., constant velocity).

#### D.10. EARTH FIXED LONGITUDES

The earth fixed longitude is calculated by the equation given in D.5 (see Figure 5D).

$$\lambda_{\text{E}} = \text{RA (object)} - \text{GHA} (\Upsilon)$$

Similarly, if the longitude of an observation and the time of an observation is known,

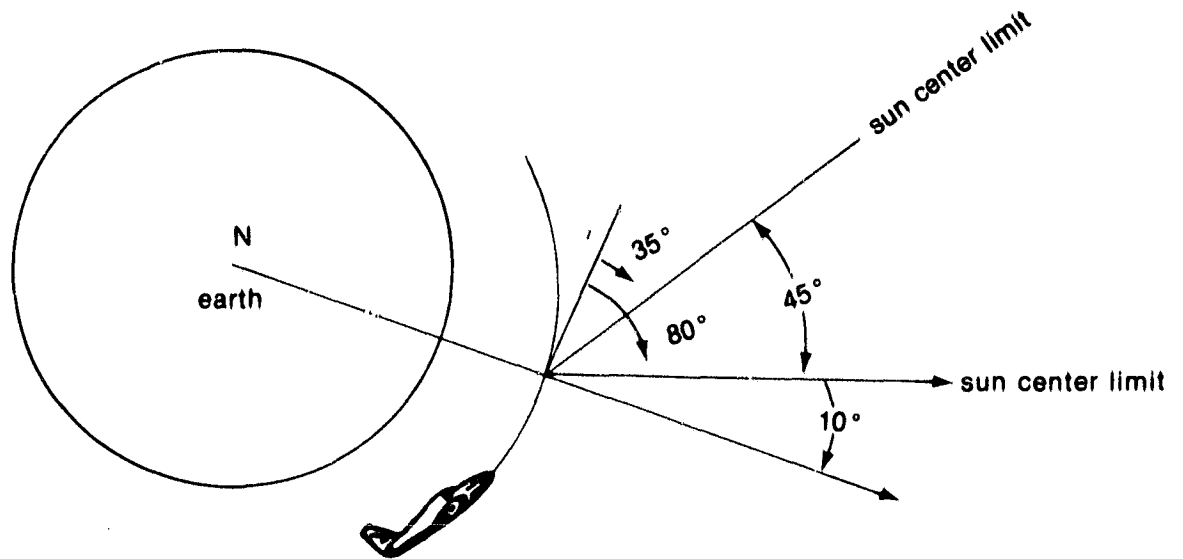
$$\text{RA (object)} = \lambda_{\text{E}} + \text{GHA} (\Upsilon)$$

**Exercise D.10.1.** A satellite passes directly overhead at Houston ( $95^{\circ} 28'$  W) at  $14^{\text{h}} 23^{\text{m}} 16^{\text{s}}$  GMT on July 2, 1979. Find the right ascension of the satellite.

ANS:  $40^{\circ} 22' 45''$

**Exercise D.10.2.** The orbiter is to deploy an upper stage in full sunlight under the condition that the angle between the sun and the flight path angle of the upper stage shall be at least  $35^{\circ}$  and no greater than  $80^{\circ}$ . The deployment is also to be over a tracking site at  $91^{\circ} 32'$  W. Launch is to be between  $10^{\text{h}}$  GMT and  $11^{\text{h}}$  GMT on July 15, 1981. How long after the launch will deployment be possible? We will assist in the solution starting as you should, with a sketch.





The RA (Sun) must be between the values  
 $RA(\text{Sun}) = 91^\circ 32' 0'' + 10^\circ + GHA(\gamma)$   
 and

$$RA(\text{Sun}) = 91^\circ 32' 0'' + 55^\circ + GHA(\gamma)$$

The required GHA ( $\gamma$ ) at deploy is between

$$GHA(\gamma) = RA(\text{Sun}) + 91^\circ 32' 0'' - 10^\circ$$

and

$$GHA(\gamma) = RA(\text{Sun}) + 91^\circ 32' 0'' - 55^\circ$$

Use Programs of Exercise D.6.9.

- Find RA (Sun) at 10<sup>h</sup> GMT on July 15, 1981

- Find GHA ( $\gamma$ ) at 10<sup>h</sup> GMT on July 15, 1981

- GHA ( $\gamma$ ) at 11<sup>h</sup> GMT on July 15, 1981

(Note RA (sun) does not change rapidly.)

- Find differences between GHA ( $\gamma$ ) at  $10^h$  and  $11^h$  and required GHA ( $\gamma$ )
- Convert to time
- Correct for change in RA (sun) at new times

ANS: • The *deploy window* opens at  $14^h 32^m 0^s$  GMT

- Deploy window closes at  $17^h 32^m 01^s$
- For  $10^h$  GMT launch, the times prior to deploy are  $4^h 32^m 0^s$  and  $7^h 32^m 01^s$
- For  $11^h$  GMT launch, the times are reduced by 1 hour.
- The above includes a correction of  $0^h 0^m 45^s$  window opening.
- $0^h 1^m 16^s$  window closing for change in RA (sun) between above times and  $10^h$  GMT.
- Does *not* include a correction for the semi-diameter of the sun.

**Exercise D.10.3.** For an orbiter in a 160 n.mi circular orbit, assuming no latitude requirements for deploy (e.g., no equatorial crossing display requirement), how many opportunities, *at most*, will the orbiter have to deploy the payload in the above window?

ANS: 2 opportunities

Note: If an equatorial crossing deploy requirement exists (e.g., plane change requirement), *no* opportunity may exist in this window for this launch time.

**Exercise D.10.4.** In the above window, how much time will the crew have to deploy the payload?

ANS:  $0^h 11^m 18^s$

**Exercise D.10.5.** Suppose something precludes the deployment. Where is the next window?

ANS: window opens on July 16, 1981 14<sup>h</sup> 32<sup>m</sup> 05<sup>s</sup> GMT  
closes 17<sup>h</sup> 32<sup>m</sup> 04<sup>s</sup>

**Exercise D.10.6.** Suppose the satellite in the previous example is to be placed in geosynchronous orbit by a direct Hohmann transfer. What is the Earth-fixed longitude of placement? (Assume it is deployed from a 160 n.mi circular orbit.)

ANS: 9° 08'E

**Exercise D.10.7.** If a geosynchronous satellite is to remain over the Houston, Texas, meridian (95° 28' W), and it has been placed in geosynchronous orbit by a Hohmann transfer from a 160 n.mi orbit, what is the Earth-fixed longitude of deployment? (This is the longitude of radar placement if the satellite is to be tracked from the ground during deployment.)

ANS: 163° 51' 31" E

## D.11. SUMMARY FOR EARTH-SUN CALCULATIONS

### D.11.1. Symbols

- LMT = Local Mean Time
- GMT = Greenwich Mean Time
- LST = Local Sidereal Time
- GST = Greenwich Sidereal Time
- GHA = Greenwich Hour Angle
- RA = Right Ascension
- g = argument of perigee
- M = mean anomaly
- e = eccentricity
- ε = obliquity of plane of motion
- Θ = true anomaly

- u** = argument of latitude
- l** = declination of object
- d** = number of days since epoch
- λ** = earth fixed longitude, positive east
- λ<sub>E</sub>** = earth fixed longitude east
- λ<sub>W</sub>** = earth fixed longitude west
- LAT** = earth fixed latitude
- β** = central angle
- t** = time
- t<sub>0</sub>** = time of epoch
- JD** = Julian Day number

#### D.11.2. Relations

$$LMT = GMT + \lambda_E$$

$$LMT = GMT - \lambda_W$$

$$LST = GST + \lambda_E$$

$$LST = GST - \lambda_W$$

$$GHA = GST$$

$$u = g + \theta$$

$$l = \sin^{-1}(\sin \epsilon \sin u)$$

$$RA_{(sun)} = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = \cos u$$

$$y = \cos \epsilon \sin u$$

#### Solar/Sidereal Conversion

$$\frac{\text{solar time interval}}{\text{sidereal time interval}} = 1.0027379093$$

$$\frac{\text{sidereal time interval}}{\text{solar time interval}} = 0.9972695664$$

#### Arc/Time Conversion - earth rotation relative to stars

1 hour mean solar time = 15.014069 degrees of arc

Arc/Time Conversion - earth rotation relative to sun

1 hour mean solar time = 15.00 degrees of arc

Longitude of sunrise/sunset on earth

$$\lambda_{\text{sunrise/sunset}} - \text{GHA (sun)} \pm \cos^{-1} \left( \frac{\cos \beta - \sin I_{\text{sun}} \sin \text{LAT}}{\cos I_{\text{sun}} \cos \text{LAT}} \right)$$

$$\text{GHA (sun)} = \text{GHA} (\Upsilon) - \text{RA (sun)}$$

Sunrise/sunset for spacecraft (s)

$$\text{RA (s)}_{\text{sunrise/sunset}} = \text{RA (sun)} \pm \cos^{-1} \left( \frac{\cos \beta - \sin I_s \sin I_{\text{sun}}}{\cos I_s \cos I_{\text{sun}}} \right)$$

Earth Fixed Longitude for subsatellite point

$$\lambda_E = \text{RA (s)} - \text{GHA} (\Upsilon)$$

## D.12. EVALUATION

### D.12.1. Learning Objectives

The principal objectives of Section D were threefold: first, to introduce the measurement of relative position and time; second, to increase your problem solving capability in the conceptual range; and thirdly, to teach the methods for evaluating time related problems, such as launch window determination

Having completed this section and related references, you should have an understanding of:

- the relationship between various time measurements
- the use of reference points, lines, and angles in the location of objects in space
- the use of reference tables
- the importance of earth-sun relations

### **D.12.2. Outcome Measures**

You should now be able to:

1. Convert time from one measurement set to another by use of formula or table.
2. Convert time measurements to angular measure.
3. Find the right ascensions, declinations, or other parameters of the sun and earth orbiters at specified times given appropriate information.
4. Find spacecraft longitude and longitudes of sunrise and sunset for earth and spacecraft.
5. Calculate deploy windows.
6. Consider advancing to changes of orbital plane and perturbations or going through Section D again, or asking lots of questions.

# APPENDIX A

Selected data from

JSC-14262

## SPACE SHUTTLE ASTRODYNAMICAL CONSTANTS

By B. F. Cockrell and Bruce Williamson  
Mathematical Physics Branch

### 1.0 SUMMARY

This document provides basic space Shuttle astrodynamical constants for use in mission planning and construction of ground and onboard software input loads. The data included here are provided to facilitate the use of consistent numerical values throughout the project. The document supersedes reference 1.

### 2.0 INTRODUCTION

The astrodynamical constants presented in this document are taken primarily from reference 2, "Natural Environmental and Physical Standards for Apollo and AAP". Those data have been expanded, and other data have been included when there was an indication that additional information would be useful. The values of the constants in reference 3 have been informally adopted by the Interplanetary Trajectory Committee.

### 3.0 ASTRODYNAMIC CONSTANTS AND PARAMETERS

#### 3.1 Introduction

This section presents the values and associated uncertainties of the constants and models used in trajectory prediction. A list of conversion factors and a description of the gravitational potential equation are also included.

Uncertainties ( $1\sigma$ ) are presented if available. For consistent conversion between units, more decimal digits are given for some quantities than are justified by the uncertainties. The values presented are in agreement with those adopted by NASA Headquarters (ref. 2).

#### 3.2 Constants And Parameters

##### 3.2.1. *Angular Velocity of the Earth's Rotation With Respect to the Vernal Equinox*

The Earth's angular rotational velocity with respect to a precessing equinox ( $\omega_p$ ) and inertial equinox ( $\omega_i$ ) for OFT (calendar year 1979) based on references 6 and 16 is

$$\omega_p = .7292115854918357 - 004 \text{ radians per second}$$

$$\omega_l = .7292115146459210 - 004 \text{ radians per second}$$

### 3.2.2. Speed of Sound Constant

This is a derived constant used in the calculation of the speed of sound. The speed of sound is defined by:

$$C_s = \left( \frac{\gamma R^* T_m}{M_0} \right)^{1/2}$$

where  $\gamma = 1.4$  and is the ratio of the specific heat of air at constant pressure to that at a constant volume (dimensionless).

Because  $\gamma$ ,  $R^*$ , and  $M_0$  are constants, they are combined for computational ease and the equation is written:

$$C_s = K \sqrt{T_m}$$

where K is the speed of sound constant

$$K = 65.77035 \text{ ft/sec/}\sqrt{K^\circ}$$

### 3.2.3. Gravitational Potential Function

The classical expression for the gravitational potential V exerted at a point in space located at a distance r from the center of the attracting body of radius  $R_E$  and gravitational parameter  $\mu_E$  is given in equation as

$$V(r, \phi, \lambda) = \sum_{n=0}^{\infty} \frac{\mu_E}{r} \left( \frac{R_E}{r} \right)^n \sum_{m=0}^n P_{nm}(\sin \phi) C_{nm} \cos(m \lambda) + S_{nm} \sin(m \lambda)$$

$\phi$  is latitude (geodetic)

$\lambda$  is longitude

where  $c_{nm}$  and  $S_{nm}$  are the harmonic coefficients of the potential function and  $P_{nm}(\sin \phi)$  represents the associated Legendre functions of the first kind, of degree n and order m. Because  $\sin \phi = z/r = u$ , where u is a direction cosine, the associated Legendre functions may be expressed as:



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DEPARTMENT OF PHYSICS

November 7, 1981

Mr. Robert Brown, Technical Monitor  
Lyndon B. Johnson Space Center  
National Aeronautics and Space Administration  
Houston, Texas 77058

RE: NASA Grant NAG 9-6

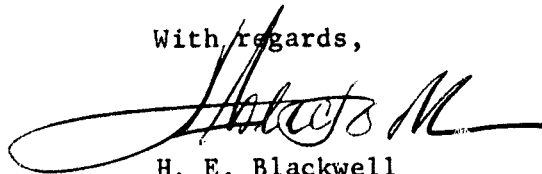
Dear Mr. Brown:

Please find enclosed the communication prepared for you dated August 31, 1981, my anticipated date for transmittal of the final grant report. I did not transmit it then and thought that my conversation with you and Mr. Scarlett concerning the matter would suffice for the delay. Having offered to send the report minus one portion of typed material, you both thought that it should be sent complete and that a delay was acceptable. But it was an error on my part not to have had the fiscal report sent, since it is normally sent separately and was probably of greater immediate significance.

The small delay stretched itself and we now believe the report too long past due. In the enclosed material, therefore, is one part in a more or less complete but unfinished form. If it is acceptable to and agreeable with you, finished copies will be mailed to you and to the STIF as soon as I receive it in appropriate form (November 30 projected).

It has been a pleasure working with you, and I wish to extend from Texas Southern University our hopes for continued cooperative efforts.

With regards,



H. E. Blackwell

# TEXAS SOUTHERN UNIVERSITY

HOUSTON, TEXAS 77004

DEPARTMENT OF PHYSICS

August 31, 1981

TO: Robert H. Brown, Technical Monitor  
National Aeronautics and Space Administration  
Lyndon B. Johnson Space Center

FROM: H. E. Blackwell  
Texas Southern University

RE: NASA Grant NAG 9-6

National Aeronautics and Space Administration (NASA) grant NAG 9-6 was awarded to Texas Southern University for work related to Space Shuttle flight and operation. The initial phase of the work was projected for one year beginning January 1, 1980. The project was divided into two areas: (1) a review, criticism, testing and preparation of course materials for training of space flight design paraprofessionals; and (2) analysis and development of the kinematics and approximate flexibility models of the Remote Manipulator System (RMS) of the Space Shuttle.

Course development proceeded from initial materials provided by the Johnson Space Center (JSC). This work utilized principally the teaching and course material development expertise of TSU faculty in the design and documentation of a system for the training of individuals in the mechanics of space flight design.

A program of research was planned to delve into the subtler aspects of remote manipulator systems, systems similar to those that will be required for future applications in space. The initial investigations used the configuration planned for the Shuttle Transportation System. This approach was used to allow Texas Southern to build upon the substantial work already accomplished on this system, thereby gaining a significant jump toward addressing the problems of more sophisticated systems. A significant aspect of this area of work was the transfer of computer programs used to investigate remote manipulator systems from JSC to TSU. These programs permit the rapid display and study of the control and response characteristics of System manipulator and represent many years of analysis, coding, and testing. NASA was to have provided engineering data, algorithms used in remote control, software documentation, and other information to aid TSU in building a remote manipulator capability.

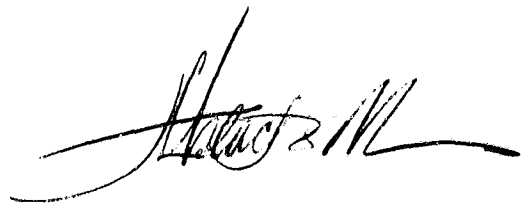
The first phase or initial grant period was listed as January 1 through December 31, 1980, although the award was made several weeks after that beginning date. Funds for equipment (computer) came later in the year by way of a grant supplement. As a result of this and the incapacitation of the project director during the latter part of the year, a no-cost extension of the grant period through June 1981 was obtained. Although significantly new developments in the RMS were not expected as a product of

this initial phase, the extension allowed completion of computer program transfer from JSC, along with other tasks, and was thought necessary to provide a more complete preparation for the second phase of this work. In May 1981, we were informed that funding and work force in this area at JSC had been reduced, and that funds for continued research at TSU along the lines of the initial grant could not be supported.

A large portion of the initial work was designed to provide project faculty with appropriate background, primarily the configuration and basic operation of the Shuttle RMS along with the tools and skills necessary for a description of that system. The RMS is a rather complex system having at that time a number of engineering problems remaining after years of work by a number of engineering groups. Our initial work consisted of learning tasks, moving toward more specific problems of the RMS system and more creative approaches in developing such general systems which were to be accomplished in the second phase.

Because most of the RMS work consisted of investigative tasks that were not new but oriented toward continuance, we will not detail that work. We are including, however, a report describing the RMS computer simulation program developed at JSC which we transferred to and worked with in our system: this includes work by E. L. Copeland of the Lockheed Engineering and Management Company whose assistance we acknowledge. One new approach on which we worked dealt with errors, in which a general analysis of error generation in the complete operation of the remote arm system was projected. Another considered the spatial transformations necessary to describe end effector position and orientation: this approach used the Alpine concept in a move toward better control of time-response characteristics of the simulated system. Our principal reference for RMS study was documentation by SPAR Aerospace Products of Toronto, Canada. Study of Vought's Teleoperator Maneuvering System was anticipated as general development proceeded.

Addressing the first area of the grant, course materials for training of space flight design paraprofessionals, we have included material labeled Introduction to Flight Planning (I).

A handwritten signature in black ink, appearing to be "A. L. Copeland" followed by a flourish.

$$P_{nm}(u) = \frac{\cos^m \phi}{2^n n!} \frac{d^{n+m}}{du^{n+m}} (u^2 - 1)^n$$

The constants are for real time operations are:

$$\begin{aligned} C_{2,0} &= -1082.7 \times 10^{-6} \\ C_{3,0} &= 2.56 \times 10^{-6} \\ C_{4,0} &= 1.58 \times 10^{-6} \\ C_{22} &= 1.57 \times 10^{-6} \\ S_{22} &= -.897 \times 10^{-6} \end{aligned}$$

all others are zero.

(Higher order models are sometimes used for analysis and postmission trajectory reconstruction.)

### 3.2.3.1. Equatorial Earth Radius (Gravitational)

$$\begin{aligned} 1 \text{ E.r.} &= .6378160000000000 + 007 \pm .50000 + 001 \text{ m (ref. 2).} \\ 1 \text{ E.r.} &= .2092572178477690 + 008 \pm .16404 + 002 \text{ int. ft.} \\ 1 \text{ E.r.} &= .3443930885519158 + 004 \pm .26998 - 002 \text{ n. mi.} \\ 1 \text{ E.r.} &= .9999992160754699 + 000 \pm .78392 - 006 \text{ E.r. (MCC).} \end{aligned}$$

### 3.2.3.2. Gravitational Parameter ( $GM_e = \mu_e = \mu$ Earth)

$$\begin{aligned} \mu_e &= .39860,2000000000 + 015 \pm .40000 + 009 \text{ m}^3/\text{sec}^2 \text{ (ref. 2).} \\ \mu_e &= .1407646853278542 + 017 \pm .14126 + 001 \text{ (int. ft.)}^3/\text{sec}^2. \\ \mu_e &= .6275027808522208 + 005 \pm .62970 - 001 \text{ (n. mi.)}^3/\text{sec}^2. \\ \mu_e &= .1990931661816326 + 002 \pm .19979 - 004 \text{ (E.r.)}^3/\text{hr}^2 \text{ (MCC).} \end{aligned}$$

### 3.2.3.3. Mass of the Earth

$$\begin{aligned} M_e &= .5973343323842350 + 025 \text{ KG} \\ M_e &= .1316896781981220 + 026 \text{ LBM} \end{aligned}$$

### 3.2.4. Lunar Constants

#### 3.2.4.1. Earth-Moon Mass Ratio (ref. 2)

$$M_e/M_m = 81.3010 (\pm 0.0010)$$

#### 3.2.4.2. Mean Lunar Radius

$$\begin{aligned} R_m &= .1738090000000000 + 007 \pm .70000 + 002 \text{ m (ref. 2).} \\ R_m &= .5702395013123360 + 007 \pm .22966 + 003 \text{ int. ft.} \\ R_m &= .9384935205183585 + 003 \pm .37797 - 001 \text{ n. mi.} \\ R_m &= .2725062772756741 + 000 \pm .10975 - 004 \text{ E.r. (MCC).} \end{aligned}$$

#### 3.2.4.3. Principal Axes

$$\begin{aligned} a &= .1738570000000000 + 007 \pm .70000 + 002 \text{ m (ref. 2).} \\ a &= .5703969816272966 + 007 \pm .22966 + 003 \text{ int. ft.} \\ a &= .9387526997840173 + 003 \pm .37797 - 001 \text{ n. mi.} \\ a &= .2725815340305558 + 000 \pm .10975 - 004 \text{ E.r. (MCC).} \end{aligned}$$

$$\begin{aligned}
b &= .1738210000000000 + 007 \pm .70000 + 002 \text{ m (ref. 2).} \\
b &= .5702788713910761 + 007 \pm .22966 + 003 \text{ int. ft.} \\
b &= .9385583153347732 + 003 \pm .37797 - 001 \text{ n. mi.} \\
b &= .2725250914643945 + 000 \pm .10975 - 004 \text{ E.r. (MCC).} \\
c &= .1737490000000000 + 007 \pm .70000 + 002 \text{ m (ref. 2).} \\
c &= .5700426509186352 + 007 \pm .22966 + 003 \text{ int. ft.} \\
c &= .9381695464362851 + 003 \pm .37797 - 001 \text{ n. mi.} \\
c &= .2724122063320720 + 000 \pm .10975 - 004 \text{ E.r. (MCC).}
\end{aligned}$$

where  $a$  is directed toward the center of the Earth,  $c$  is coincident with the Moon's rotational axis, and  $b$  is perpendicular to  $a$  and  $c$ .

#### 3.2.4.4. Gravitational parameter for the Moon ( $GM_m = \mu_m = \mu$ Moon)

$$\begin{aligned}
\mu \text{ Moon} &= .4902780000000000 + 013 \pm .60000 + 008 \text{ m}^3/\text{sec}^2 \text{ (ref. 2).} \\
\mu \text{ Moon} &= .1731400417087798 + 015 \pm .21189 + 010 \text{ (int. ft.)}^3/\text{sec}^2. \\
\mu \text{ Moon} &= .7718260968373028 + 003 \pm .94456 - 002 \text{ (n. mi.)}^3/\text{sec}^2. \\
\mu \text{ Moon} &= .2448838571715250 + 000 \pm .29969 - 005 \text{ E.r.}^3/\text{hr}^2 \text{ (MCC).}
\end{aligned}$$

### 3.2.5. General Constants

#### 3.2.5.1. Astronomical Unit

$$\begin{aligned}
\text{AU} &= .1495978930000000 + 012 \pm .50000 + 004 \text{ m (ref. 2).} \\
\text{AU} &= .4908067355643045 + 012 \pm .16404 + 005 \text{ int. ft.} \\
\text{AU} &= .8077640010799136 + 008 \pm .26996 + 001 \text{ n. mi.} \\
\text{AU} &= .2345469159233102 + 005 \pm .78793 - 003 \text{ E.r. (MCC).}
\end{aligned}$$

#### 3.2.5.2. Velocity of Light in a Vacuum

$$\begin{aligned}
c &= .2997925000000000 + 009 \pm .30000 + 003 \text{ m/sec (ref. 2).} \\
c &= .9835711942257218 + 009 \pm .98425 + 003 \text{ int. ft./sec.} \\
c &= .1618750000000000 + 006 \pm .16199 + 000 \text{ n. mi./sec.} \\
c &= .1692105801590269 + 006 \pm .16933 + 000 \text{ E.r./hr (MCC).}
\end{aligned}$$

#### 3.2.5.3. Gravitational Parameters for the Sun

$$\begin{aligned}
\mu \text{ Sun} &= .1327124990000000 + 021 \pm .15000 + 014 \text{ m}^3/\text{sec}^2 \text{ (ref. 2).} \\
\mu \text{ Sun} &= .4686697671960388 + 022 \pm .52972 + 015 \text{ (int. ft.)}^3/\text{sec}^2. \\
\mu \text{ Sun} &= .2089242635906454 + 011 \pm .23614 + 004 \text{ (n. mi.)}^3/\text{sec}^2. \\
\mu \text{ Sun} &= .6628718533157138 + 007 \pm .74922 + 000 \text{ E.r.}^3/\text{hr}^2 \text{ (MCC).}
\end{aligned}$$

## 3.3. EPHEMERIS TAPE SYSTEMS

### 3.3.1. DE19 Tape

The ephemeris tape system to be used for all missions is provided by JPL and is called the JPL Development Ephemeris Number 19 (DE19). For additional information, see references 7 and 8.

### 3.3.2. Tape System Conversion Factors

The following values are to be used to convert DE19 units to kilometers (refs. 7,8, and 11).

- AU = 149 597 893 km (scale factor for planetary ephemerides)
- $R_{em}$  = 6378.1492 km (scale factor for lunar ephemerides)
- $\mu^{-1}$  = 81.301 (ratio of Earth mass to Moon mass)
- GE = 398 601.2 km<sup>3</sup>/sec<sup>2</sup> (gravitational parameter of the Earth)

### 3.4. FISCHER EARTH MODEL

The following constants describe the Fischer Earth model (1960), which is used for the location of radar stations and other Earth surface features (ref. 2)

#### 3.4.1. Equatorial Earth Radius

- a = .637816600000000 + 007 m.
- a = .2092574146981627 + 008 int. ft.
- a = .3443934125269978 + 004 n. mi.
- a = .1000000156784906 + 001 E.r. (MCC).

#### 3.4.2. Flattening

- f = flattening = 1 - b/a
- f = 1/298.30 = 0.3352329869259135 x 10<sup>-2</sup>

#### 3.4.3. Polar Earth Radius

- b = .6356784283607107 + 007 m.
- b = .2085559148165061 + 008 int. ft.
- b = .3432388922034075 + 004 n. mi.
- b = .9966478263900521 + 000 E.r. (MCC).

#### 3.4.4. Eccentricity of Ellipsoid

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

- e = .8181333401693114 - 001
- e<sup>2</sup> = 2f - f<sup>2</sup>
- e<sup>2</sup> = .6693421622965943 - 002

### 4.0 EQUIVALENTS AND CONVERSION FACTORS

- 1 int. ft = .304800000000000 + 000 m (exact) (ref. 2)
- 1 n. mi. = .185200000000000 + 001 km (exact) (ref. 2)
- 1 E.r. = 637816500000000 = 004 km (exact for scaling) for MCC internal use
- 1 lbm = .453592370000000 + 000 kg (exact) (ref. 2)

1 hr = 3600.000000000000 + 000 sec (exact)  
 1 rad =  $180/\pi$  deg  
 1 deg = 3600 arc sec  
 1 km = .5399568034557235 + 000 n. mi.  
 1 m = .3280839895013123 + 001 int. ft  
 1 n. mi. = .6076115485564304 + 004 int. ft.  
 1 rad = .5729577951308233 + 002 deg  
 1 deg = .1745329251994329 - 001 rad  
 1 kg = .2204622621848776 + 001 lbm  
 1 int. stat. mi. = 5280 ft (exact)  
 1 lbf = 32.174048556 (int. ft/sec<sup>2</sup>) lbm  
 $\pi$  = .3141592653589793 + 001 (ref. 2)

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