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A DYNAMIC MEASURE OF CONTROLLABILITY AND OBSERVABILITY FOR THE PLACEMENT OF ACTUATORS AND SENSORS ON LARGE SPACE STRUCTURES

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Introduction

The dimensions of space structures being considered for future applications are on the order of several hundred meters to several kilometers and will require a large number of actuators and sensors for attitude and shape control. A solar power satellite, for instance, may require hundreds of control moment gyros and thrusters to damp out surface vibrations caused by periodic disturbances such as solar and gravity gradient torques. The questions which naturally arise are: (a) where the actuators and sensors should be placed, (b) what types should be used, and (c) how many should be used.

Placement represents a substantial degree of freedom available to the designer and is usually not a very straightforward question. It is even less apparent when one considers redundancy in the system to allow for failures; even if the "optimal" position of an actuator is known, it may not be so clear where a backup actuator should be placed. The answer will likely depend on, among other things, the operating strategy—such as whether or not it is intended to use all available actuators at all times.

The types of control system components to be used is normally decided early in the design process based on their utility, cost, availability, reliability and other factors. This decision will not be discussed further here although the effectiveness of different types of sensors and actuators can be evaluated using the observability and controllability measures which will be developed. The number of components to be used must reflect the trade-off of cost, weight, power, etc. vs. system performance—and the evaluation of performance should recognize the likelihood of some component failures during the lifetime of the system.

In this work we develop a methodology for measuring the performance of a system which reflects the type, number and placement of the actuators and sensors on the structure. The measures also reflect the expected loss of performance due to component failures. These performance measures are intended to be especially useful as guides to the choice of component number and placement.

Problem Definition

It would be most helpful to the control engineer to have some criterion at his disposal for placing actuators and sensors. Unfortunately, modern control theory does not provide any such measure of "controllability" and "observability." Controllability is simply a binary concept—either a system is controllable or it is not. It does not say how controllable a system is. A vibratory mode of a beam, for example, is not controllable by a force actuator placed exactly at one of the nodes, but it is controllable by an actuator placed just off the node. One would suspect that an actuator slightly farther out would have even more control capability, but one can only verify that the system will be controllable. The same conditions hold with respect to observability for a sensor.

What should a more quantitative measure of controllability take into account? First, it is necessary to define a control objective. The most likely choice is to return the system to some specified state (usually the origin) after an initial disturbance. Secondly, the criterion should include how much control effort is required to accomplish this task. Finally, one should somehow standardize the criterion by the magnitude of the initial disturbance. A larger disturbance returned to the origin with the same amount of control as a less perturbed system would likely have a more favorable degree of controllability. It will also be necessary to normalize the initial states so that one unit in each direction is equally "important," since rarely are all states expressed in the same units or of equal concern.

Many ideas for observability parallel those for controllability if the word "state" is replaced by "state estimation error" (the difference between the estimate of the state and the true state): (1) the objective of measurement is to reduce the error covariance toward zero, (2) accomplish this using the measurements optimally, and (3) standardize the criterion by the magnitude of tolerable errors.

Previous Work

Several papers have been encountered which deal with the subject of controllability and observability, but only two (Juang and Rodriguez [1] and Likins [2]) formulate measures using the types of standards just outlined. Horner [3] has considered optimum actuator placement but does it for the specific case of passive damping of a free-free beam. Skelton and Hughes [4] define measures in terms of controllability and observability "norms" which apply to the individual modes of a system rather than to the system as a whole. Their approach is also tailored to "linear mechanical systems" which have a special form of representation as a second order matrix differential equation. Although that form applies to space structure dynamics, we prefer to define measures which have a physical interpretation in terms of control or estimation error characteristics for general linear systems.

In order to get a perspective on the measures of controllability and observability in the sections which follow, it may be helpful to review the two papers which develop similar concepts. Juang and Rodriguez take an approach very similar to the linear quadratic regulator formulation. For the LTI state equation,

$$\dot{x}(t) = Ax(t) + Bu(t)$$

they define the cost function

$$J = \frac{1}{2} \int_{t_o}^{t_f} (x^T Qx + u^T Ru)\, dt$$

where Q and R are weighting matrices on the state and control, respectively. This is the same cost function as for the LQ regulator problem except that the usual additive quadratic term involving the final state is not defined because an infinite time horizon is allowed and $x(t_f)$ converges to zero. Thus the integral directly penalizes state excursion from the desired final state (the origin) as well as control effort.

Performing the minimization on J and letting $t_f - t_o \rightarrow \infty$, one obtains the optimal cost function,

$$J^o = \frac{1}{2} x^T(t_o)\ P^o x(t_o)$$

where P^o is the steady state solution of the matrix Riccati equation

$$\dot{P} = -PA - A^T P + PBR^{-1}B^T P - Q.$$

Since the control effectiveness matrix B is a function of the actuator locations $\{\epsilon_i\}$, P^o is also a function of the actuator positions ϵ_i. Thus, the optimal cost is a function of both initial state and actuator positions.

For a fixed initial state, the optimal cost with respect to actuator positions is defined as:

$$J^{o*}\ (\epsilon_b, x_o) = \min_{\epsilon} J^o(\epsilon,\ x_o)$$

where ϵ_b are the actuator locations giving the minimum cost. Now since the initial state can have several directions in state space, the expectation with respect to x_o is invoked:

$$J^{o*}(\epsilon_b) = \min_{\epsilon} E[J^o(\epsilon)]$$

or

$$J^{o*}(\epsilon_b) = \min_{\epsilon} \frac{1}{2} \mathrm{Tr}(P^o Q^o)$$

where

$$Q^o = E[x(t_o)x(t_o)^T]$$

The optimal placement of actuators is then defined to be the position vector giving the absolute minimum of the expectation of the cost function .

We found several objections to this method:

(1) The weighting of control effort versus state excursion is rather arbitrary.

(2) If there is a particular direction x_o in which the system is not very controllable, the information is largely lost when the cost is averaged over different initial states.

(3) The degree of controllability is actually an inverse measure since a higher cost function represents a lower degree of controllability and actually becomes infinite when the system is uncontrollable.

(4) While control use is penalized, no effort is made to bound it.

Likins develops a more sophisticated technique to be used in the case of bounded control effort. Using the variation of constants formula,

$$x(t) = \Phi(t,t_o)x(t_o) + \Phi(t,t_o)\int_{t_o}^{t}\Phi(t_o,\tau)Bu(\tau)d\tau$$

and choosing $t_o=0$ and $t=T$, one can define the displacement in state space δ in time T

$$\delta = x_T - x_o = [I - \Phi^{-1}(T,0)]x_T + \int_0^T \Phi(0,t)Bu(t)dt$$

Choosing $x_T=0$, δ reduces to

$$\delta = \int_0^T \Phi(0,t)Bu(t)dt = -x_o$$

where u of the original system has been normalized so that $|u_i| \leq 1$ and B redefined appropriately.

Likins then proceeds to define a "recovery region" R as the volume of initial states that can be returned to the origin in time T under bounded control $|u_i| \leq 1$; i.e.,

$$R = \left\{ x(0) \middle| \exists\, u(t),\ \ t \in [0,T],\ |u_i(t)| \leq 1 \text{ for } i=1,\ldots,m \quad x(T) = 0 \right\}$$

The measure of controllability is chosen to be the minimum distance from the origin, over all directions in initial state space, of the outer surface of this region.

$$\rho \triangleq \inf \| x(0) \| \quad \forall \; x(0) \notin R$$

The problem now reduces to finding the minimum norm of δ (or x_o) on this surface. This is a difficult problem which requires, in effect, the definition of optimum bounded control trajectories which reach the origin in the specified time from many different initial conditions. Likins expresses this problem in terms of quadratures which must, in most cases, be computed numerically. One can only compute a finite number of these and use the smallest computed δ as the controllability measure. (A parallelogram approximation to the recovery region, such as is indicated in Fig. 1, is suggested by the authors.) If a system were actually uncontrollable there is no guarantee that one would compute the trajectory for which δ is zero.

The overriding objection to this method is the complication involved in the multiple control case. An important attribute of the measure of controllability will be its easy computation. Another objection is that Likins chooses to bound control magnitude and does not attempt to perform any sort of minimization with respect to quantity of control used, citing bounded control magnitude as the more realistic situation. It is usually the case, however, that quantity of control (e.g., fuel in thruster, stored angular momentum in CMG) is the primary consideration, not saturation of the controller.

DYNAMIC MEASURE OF CONTROLLABILITY

The measure of controllability formulated here combines some of the characteristics of both of these methods. Like Juang and Rodriguez, it involves minimizing a cost function, and as Likins, the final degree of controllability involves a measurement in some "maximized" initial state space. The difference is that the cost involves only the control, where a quadratic is chosen for convenience to approximate magnitude, and the initial state is maximized with respect to integrated control utilization rather than running the control at saturation for the duration of the control period in question.

The degree of controllability is the result of a four step procedure:

(1) Find the minimum control energy strategy for driving the system from a given initial state to the origin in the prescribed time. ["Control energy" is defined as $E = \frac{1}{2} \int_0^T u^T R u \, dt$, where R is a positive definite weighting matrix.]

(2) Find the region of initial states which can be driven to the origin with constrained control energy and time using the optimal control strategy. This region is bounded by an ellipsoidal surface in state space.

(3) Scale the axes so that a unit displacement in every direction is equally important to control.

(4) The degree of controllability is a linear measure of the

weighted "volume" of the ellipsoid in this equicontrol space.

Step 1 can be stated mathematically as follows:

$$\min E = \frac{1}{2}\int_{o}^{T} u^T R u dt$$

$$\text{subject to} \quad \begin{cases} \dot{x} = Ax + Bu \\ x(0) = x_o \\ x(T) = 0 \end{cases} \tag{1}$$

The Hamiltonian for this problem is:

$$H = \frac{1}{2} u^T R u + P^T (Ax + Bu)$$

so that

$$\dot{P} = -A^T P \qquad P(o),\ P(T) \text{ free} \tag{2}$$

$$u^*(t) = -R^{-1} B^T P(t) \tag{3}$$

where $u^*(t)$ is the optimal control.

To find $P(t)$, combine the differential equations (1) and (2) into matrix form using the optimal control (3):

$$\begin{bmatrix} \dot{x} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ P \end{bmatrix} \tag{4}$$

Then denoting the state transition matrix for the augmented state vector $[x^T \quad P^T]^T$ as $\Phi(t)$, and making use of the identities $\Phi(0)=I$ and $\dot{\Phi}=\tilde{A}\Phi$, where $\tilde{A}$ is the new state matrix in (4), the costate variable is found to be:

$$P(t) = -\Phi_{pp}(t)\ \Phi_{xp}(T)^{-1}\ \Phi_{xx}(T)\ x_o \tag{5}$$

where Φ_{xx}, Φ_{xp}, and Φ_{pp} are the respective partitions of the state transition matrix $\Phi(t)$.

Step 2: In order to carry out step 2 of the procedure, we will require an expression for the optimum cost, $E^* = \frac{1}{2}\int_0^T u^{*T} Ru^* dt$, as a function of the initial state. To this end, we seek a relation of the form

$$x = VP \tag{6}$$

since P is a function of the initial state. Differentiating (6), substituting (1), and noting that the resulting equation set equal to zero must hold for arbitrary P, we find that

$$\dot{V} = AV + VA^T - BR^{-1}B^T \tag{7}$$

with the boundary condition

$$V(T) = 0 \tag{8}$$

to satisfy the requirement that $x(T)=0$ since in general $P(T)$ is not zero. We choose this boundary condition for V as a matter of convenience; any other terminal value which satisfies the requirement $V(T)\ P(T) = 0$ would produce the same result for the control energy. The reason for not using the usual relation $P=Wx$ is that in order for $P(T)$ not to be zero, $W(t)$ would have to be poorly defined at $t=T$.

Corresponding to the usual cost expression

$$J = \frac{1}{2}\, x(0)^T W(0) x(0)$$

we expect the energy cost to have the inverse form

$$E = \frac{1}{2}\, x(0)^T V(0)^{-1} x(0) \tag{9}$$

The validity of this expression can be verified as follows:

Generalize the initial time to t_o. Then

$$E = \frac{1}{2} \int_{t_o}^{T} u^T R u dt \tag{10}$$

and we would like to show

$$E = \frac{1}{2}\, x(t_o)^T\ V(t_o)^{-1}\ x(t_o) \tag{11}$$

Differentiating (10) with respect to the initial time and substituting (3) gives

$$\frac{dE}{dt_o} = -\frac{1}{2}\, P(t_o)^T B R^{-1} B^T P(t_o) \tag{12}$$

Substituting (6) into expression (11) (which is to be verified) we have

$$E = \frac{1}{2} P(t_o)^T V(t_o) P(t_o) \tag{13}$$

Differentiation of this and substitution of (2) yields the same result as equation (12) so that the derivative of the quadratic expression for E in (9) is correct.

Also, the boundary condition matches as we can see by letting $t_o \rightarrow T$. Since the optimal trajectory tends toward the constraint $x(T)=0$, the control energy $E(t_o)$ tends to 0 as $t_o \rightarrow T$ and $x(t_o) \rightarrow 0$. The property $E(t_o) \rightarrow 0$ as $t_o \rightarrow T$ is assured by the form of E given in (13) and the boundary condition on V

$$\lim_{t_o \rightarrow T} V(t_o) = V(T) = 0 \tag{14}$$

Equation (9) defines an n-dimensional ellipsoidal surface in initial state space. Any point within the ellipsoid can be returned to the origin in time T with energy E using the optimal control in eq. (3). Though the energy expression (9) is simpler than that appearing in (1), the differential equation for V in (7) remains to be solved. The solution to (7) for the case of rigid body and vibratory modes of a spacecraft is presented in the section on Applications.

Step 3 is to scale the axes so that a unit displacement in every direction is equally important. But what is meant by "important"?

It may first occur to the reader to scale each state by the magnitude of its maximum tolerable displacement, $|x_{i_{max}}|$,

$$z_o = \begin{bmatrix} \frac{1}{|x_{i_{max}}|} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{|x_{n_{max}}|} \end{bmatrix} x_o$$

so that a unit displacement in every direction is equally intolerable. But this scaling is highly inappropriate for the following reason. For a fixed amount of control energy and time, the larger the volume of initial states encompassed by the quadratic surface in eq. (9) is, the better the system can be controlled; larger initial states can be returned to the origin with the same control effort and time. Increasing the x_1 dimension of the ellipsoid, for instance, indicates a favorable control capability. But if x_1 is scaled by dividing its maximum tolerable value, $x_{1_{max}}$, we observe the following paradox: as $x_{1_{max}}$ is made smaller, meaning that smaller values of x_1 can be tolerated (or x_1 is more important in terms of system performance) then z_1, the scaled variable, becomes larger which signifies improved control capability.

It is apparent that the appropriate scaling should make a more important variable transform to a smaller value in the new space so as to emphasize the need to control that variable. The problem is that controllability should not be related to the accuracy with which a variable is ultimately controlled (which is what the above scaling does), but rather to the size of the excursion one would like to be able to achieve. Thus let $x_{i_{min}}$ be the minimum state excursions one would like to be able to return to the origin in a given time using a prescribed control energy. Then define the transformation

$$z = D\,x$$

$$\text{where} \qquad D = \begin{bmatrix} \frac{1}{|x_{1_{min}}|} & & & \bigcirc \\ & \cdot & & \\ & & \cdot & \\ \bigcirc & & & \cdot \;\; \frac{1}{|x_{n_{min}}|} \end{bmatrix} \tag{15}$$

so that unit values of z in any direction represent controllable displacements of equal importance. If controlling a given state is deemed less important (which is useful to recognize since it requires less control capability), the corresponding state in z-space is made larger.

Step 4 is to measure the controllability represented by this ellipsoid in equicontrol space (z-space). Consider a two-dimensional case in which it is as important to control an initial displacement in the x_1 direction twice as large as one in the x_2 direction. In this case the ellipsoid defined by equation (9) is an ellipse in x-space. Let the ellipse have the shape illustrated in Figure 2a. This represents the ideal allocation of control since we are able to control a maximum displacement in the x_1 direction exactly twice as large as one in the x_2 direction. Figure 2b illustrates that the ellipse becomes a circle when transformed to equicontrol space via equation (15). Thus any deviation from a circle in equicontrol space represents a less than ideal control allocation.

After considering a number of alternatives, the degree of controllability was chosen to be the following:

$$DC = \left[V_S + \frac{V_S}{V_E}\,(V_E - V_S)\right]^{1/n} \tag{16}$$

where V_E is the n-dimensional volume of the ellipsoid in equicontrol space and V_S is the volume of the largest inscribed sphere; n is the dimension of the state space. The first term on the right side of (16) is the predominant term in the controllability measure; it reflects the smallest magnitude of initial state in equicontrol space which can be driven to the origin in the specified time using the specified control energy. If the controls were ideally allocated, the initial condition surface would be a sphere and V_S would be the controllability measure. The second term in (16) adds a smaller amount to DC to recognize the larger region of state space from which the system can recover if the surface is not spherical. The additional volume, $V_E - V_S$, is scaled by $\frac{V_S}{V_E}$ so that the most this term can add, as $V_E \rightarrow \infty$, is V_S and so that DC is zero if there is any direction from which the system cannot recover at all— this is the case of traditional uncontrollability, and $V_S = 0$. The nth root of the weighted volume is taken as the controllability measure to make it proportional to the linear dimensions of the region from which the system can recover. The volume weighting scheme for a two-dimensional case (volumes are areas) is depicted in Figures 3(a-c).

Once one accepts (16) as a reasonable assessment of the controllability of the system, what remains to be shown are the mechanics of computing the n-dimensional volumes V_S and V_E. Consider the quadratic form, $x^T A\, x = d$, where x is a vector of length n, A is an nxn matrix, and d is some scalar constant. For the two dimensional case, this quadratic surface is an ellipse and the enclosed area is given by πab, where a and b are the intersections of the ellipse with its principal axes. The intersections are $\sqrt{\frac{d}{\lambda_1}}$ and $\sqrt{\frac{d}{\lambda_2}}$ where the λ's are eigenvalues of A so that the area equals $\pi d \frac{1}{\sqrt{\lambda_1}\sqrt{\lambda_2}}$. For three dimensions, the surface is an ellipsoid and the enclosed volume is

$$\frac{4}{3}\pi d^{3/2} \frac{1}{\sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3}} .$$

For n-dimensions the volume is defined by n integrations over the n axes (bounded by the intersections of the surface with the axes) and is found to be $K \cdot \frac{1}{\sqrt{\lambda_1} \cdots \sqrt{\lambda_n}}$ where K is a constant. Since volume for $n \geqslant 4$ has little absolute significance the constant K is dropped and the volume is taken to be simply

$$V = \left(\prod_{i=1}^{n} \sqrt{\lambda_i}\right)^{-1} \tag{17}$$

To apply this result to the case at hand, first substitute (15) into (9) to obtain the equation of the ellipsoidal surface in equicontrol space

$$E = \frac{1}{2} z_o^T \, (DV_oD)^{-1} \, z_o \tag{18}$$

V_E is then given by (17) where λ_i are the eigenvalues of $(DV_oD)^{-1}$. From (7) and (15) we observe that both D and V are symmetric matrices so that the product DV_cD is also symmetric. The eigenvalues of the inverse of a symmetric matrix are just the reciprocals of the eigenvalues of the original matrix. Therefore, if ν_i denote the eigenvalues of DV_oD, the ellipsoidal volume is also given by

$$V_E = \prod_{i=1}^{n} \sqrt{\nu_i} \tag{19}$$

and the spherical volume is the shortest distance to the surface, $1/\sqrt{\lambda_{max}}$, to the nth power, or alternatively,

$$V_S = \left(\sqrt{\nu_{min}}\right)^n \tag{20}$$

The degree of controllability can then be computed using (16), (19), and (20) and actually becomes zero when the system is uncontrollable; the ellipsoid collapses to zero in the uncontrollable direction so that ν_{min} is zero.

To find the least controllable direction in equicontrol space (the point closest to the origin), we note that the principal axes of the ellipsoid are in the same directions as the eigenvectors of $(DV_oD)^{-1}$, and the eigenvectors of $(DV_oD)^{-1}$ are the same as those of DV_oD. Therefore, the point of closest approach is in the direction u_{min}, where

$$DV_oDu_{min} = \nu_{min}\, u_{min} \tag{21}$$

To recover the direction in the original state space, simply multiply u_{min} by D^{-1}.

One further consideration is important in defining the Degree of Controllability of a system; that is how the measure varies with number of actuators. The Degree of Controllability has been defined in terms of a constraint on control energy with no reference to a constraint on control magnitude. But it seems appropriate to recognize the fact that a system with more actuators has greater control capability when there is a limit on control magnitude—as is always the case. The measure of controllability as defined above can be made to vary directly with the number of actuators placed at the same locations by scaling the elements of R inversely with m—the number of actuators in the system. Usually R is taken diagonal, and if the diagonal elements $R_{o_{ii}}$ are first chosen to reflect the relative cost of using the different actuators, then the final elements of R are defined to be

$$R_{ii} = R_{o_{ii}}/m \tag{22}$$

with m = total number of actuators.

Dynamic Measure of Observability

Any measure of the observability of a dynamic system should reflect as directly as possible the amount of information which can be derived about the system states from the sensor outputs in a given amount of time. The means of obtaining this information is by attaching to the system an observer whose states, x, are "estimates" of the true states of the system. The more information that is obtained about the system, the smaller the estimation error becomes.

A direct indicator of the amount of information one has about the system states is the information matrix, the inverse of the error covariance matrix. In order to maximize the amount of information, one should minimize the estimation error. The linear estimator which minimizes the state estimation error vector, $e = \hat{x} - x$, in a mean square sense, i.e., minimizes

$$S = \overline{e^T Me} \tag{23}$$

where M is some weighting matrix, is the Kalman Filter.

For the Kalman Filter, the error covariance equation is

$$\dot{P} = AP + PA^T - PC^T N^{-1} CP + Q \tag{24}$$

where P is the estimation error covariance matrix, and N and Q are the measurement and driving noise intensity matrices, respectively. Since the measurement noise is a property of the set of sensors being evaluated, we retain its inclusion in (24) in the form of N but do not include the effect of state driving noise, because that is an external influence not related to the sensor set. Thus, if we set Q=0 and call the information matrix $J(=P^{-1})$, then (24) in terms of J becomes

$$\dot{J} = - JA - A^T J + C^T N^{-1} C \tag{25}$$

Take as the standard situation the case in which there is no information about the state initially and data is collected up to a specified time T. Then $J(0) = 0$ and one is interested in $J(T)$. Having the information matrix at time T, we are interested in measuring how much information has been accumulated. One way of measuring the size of $J(T)$ is by reference to the quadratic surface

$$v^T J^{-1} v = 1 \tag{26}$$

As with equation (9) in the control case, equation (26) defines an ellipsoidal surface in v-space. If J is a diagonal matrix (one can always transform to principal coordinates), one observes that increasing an element j_{ii} will expand the ellipsoid in the direction v_i. Thus the larger J becomes, the larger the volume encompassed by the surface in (26) so that the more information obtained about the system, the larger the volume becomes.

Typically, however, some components of x will be of greater concern than others—especially considering that different units will apply to different components. Paralleling the discussion of the control case, define the transformation

$$w = Fv$$

$$F = \begin{bmatrix} |e_{1_{max}}| & & \bigcirc \\ & \ddots & \\ \bigcirc & & |e_{n_{max}}| \end{bmatrix} \qquad (27)$$

where $e_{i_{max}}$ are the maximum errors one is willing to tolerate in the direction x_i. The more error one is willing to tolerate in that direction, the greater the transformed state so the larger the volume becomes. Thus the scaling is consistent with the ideas presented in the last section. Also note that v has units of reciprocal error, so w is dimensionless as was z in the control case.

Now that the axes have been scaled so that it is equally important to obtain information in each direction, one can use the same definition for the degree of observability as was used for controllability when applied to equicontrol space.

Again, the ideal sensor distribution would produce a sphere in ν-space, so that the degree of observability involves a spherical volume plus a lesser weighted excess volume due to the nonideality of the distribution. Specifically,

$$DO = \left[V_S + \frac{V_S}{V_E}(V_E - V_S)\right]^{1/n} \tag{28}$$

with

$$V_E = \prod_{i=1}^{n} \sqrt{\nu_i}$$

$$V_S = \left(\sqrt{\nu_{min}}\right)^n$$

and the ν_i are the eigenvalues of $FJ(T)F$.

The remaining problem is to solve the differential equation (25) for J so as to write out explicitly J(T). We have

$$\dot{J} = -JA - A^T J + C^T N^{-1} C$$

$$J(0) = 0$$

This is similar to the corresponding problem in the definition of the degree of controllability. There we required V(0) with

$$\dot{V} = AV + VA^T - BR^{-1}B^T$$

$$V(T) = 0$$

Define a backward time variable, $\tau = T - t$, so that $\frac{dJ}{d\tau} = -\frac{dJ}{dt}$.

Then in terms of $\mathcal{T}$, equation (25) becomes

$$\dot{J} = JA + A^T J - C^T N^{-1} C$$
$$J(T) = 0 \tag{29}$$

This is the same as the equation and boundary condition for V with the substitutions:

V equation		J equation
A	$\Longrightarrow$	A^T
B	$\Longrightarrow$	C^T
R	$\Longrightarrow$	N

So if a subroutine is prepared to produce V(0) given A, B, and R, that same subroutine can be used to produce J(T) by use of the substitutions indicated.

It is worthy to note that the parallelism in computing the degrees of controllability and observability stems from the similarity between the quadratic forms (9) and (26), respectively. However, the concepts which drove us to those forms were quite different. Equation (9) represents an actual ellipsoid in state-space which bounds the initial states that can be returned to the origin in time T with a prescribed energy E. For the observability case, the information retrieval capability is already maximized through the use of a Kalman Filter, and one is simply trying to formulate a measure of observability based upon the size of the final information matrix. Thus equation (26) serves only as an aid to the definition of the size of J, and the space in which it is defined serves only to measure that size volumetrically.

APPLICATION TO ONE-DIMENSIONAL CASE

To demonstrate the procedure for obtaining the degree of controllability and observability, the above results were applied to the vibratory modes of a free-free beam. Start with a series expansion for the beam displacement y,

$$y(\epsilon, t) = \sum_i \phi_i(\epsilon)\psi_i(t)$$

where $\phi_i(\epsilon)$ is an orthogonal set of modal shapes and $\psi_i(t)$ are the modal amplitudes, and substitute this into the governing differential equation for a beam

$$EI \frac{\partial^4 y}{\partial \epsilon^4} + m \frac{\partial^2 y}{\partial t^2} = f(\epsilon, t)$$

where f is the forcing term and m, E, and I are the beam mass (M)/length (ℓ), modulus, and cross-section inertia, respectively. Assuming the use of m point force actuators,

$$f(\epsilon, t) = \sum_{j=1}^{m} \delta(\epsilon - \epsilon_j)\ u_j(t)$$

with ϵ_j being the actuator positions and $u_j(t)$ the control magnitudes, one obtains the relations

$$\omega_i^2 \psi_i(t) + \frac{d^2\psi_i}{dt^2} - \frac{1}{M}\sum_{j=1}^{m}\phi_i(\epsilon_j)u_j(t) = 0 \qquad (30)$$

where ω_i is the frequency of the ith mode.

The modal shapes for a free-free beam are given by

$$\phi_1(x) = 1$$

$$\phi_2(x) = \frac{\sqrt{12}}{\ell}\left(x - \frac{\ell}{2}\right) \tag{31}$$

$$\phi_i(x) = \cosh\beta_i x + \cos\beta_i x - a_i(\sinh\beta_i x + \sin\beta_i x) \quad i \geqslant 3$$

where the β_i are the solutions to

$$1 - \cosh\beta_i\ell\cos\beta_i\ell = 0$$

and

$$a_i = \frac{\sinh\beta_i\ell + \sin\beta_i\ell}{\cosh\beta_i\ell - \cos\beta_i\ell}$$

The first two modes of the beam are rigid body modes and thus have a frequency equal to zero. ψ_1 has the interpretation of the rigid body translation of the center of mass of the beam, and ψ_2 represents rotation of the beam about its center of mass.

Next, consider casting (30) into the state space form,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{32}$$

where

$$x = \left[\psi_1 \ \dot{\psi}_1 \ \psi_2 \ \dot{\psi}_2 \ \psi_3 \ \dot{\psi}_3 \cdots \psi_N \ \dot{\psi}_N\right]^T$$

$$C = \begin{bmatrix} 0 & \phi_1(\alpha_1) & 0 & \phi_2(\alpha_1) & \cdots & 0 & \phi_N(\alpha_1) \\ 0 & \phi_1(\alpha_2) & 0 & \phi_2(\alpha_2) & & 0 & \phi_N(\alpha_2) \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \phi_1(\alpha_P) & 0 & \phi_2(\alpha_P) & \cdots & 0 & \phi_N(\alpha_P) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \frac{1}{M}\phi_1(\epsilon_1) & \frac{1}{M}\phi_1(\epsilon_2) & \cdots & \frac{1}{M}\phi_1(\epsilon_M) \\ 0 & 0 & \cdots & 0 \\ \frac{1}{M}\phi_2(\epsilon_1) & \frac{1}{M}\phi_2(\epsilon_2) & \cdots & \frac{1}{M}\phi_2(\epsilon_M) \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \\ \frac{1}{M}\phi_N(\epsilon_1) & \frac{1}{M}\phi_N(\epsilon_2) & \cdots & \frac{1}{M}\phi_N(\epsilon_M) \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & & & & & & & \\ 0 & 0 & & & & & & & \mathbf{0} \\ & & 0 & 1 & & & & & \\ & & 0 & 0 & & & & & \\ & & & & 0 & 1 & & & \\ & & & & -\omega_3^2 & 0 & & & \\ & \mathbf{0} & & & & & \ddots & & \\ & & & & & & & 0 & 1 \\ & & & & & & & -\omega_N^2 & 0 \end{bmatrix}$$

where the number of modes has been truncated at N, and the use of M force actuators at positions ϵ_j and P translation rate sensors at positions α_i has been assumed. The replacement of a force actuator at ϵ_j by a torque actuator would involve replacing the corresponding elements of B by $\frac{d\phi_i(\epsilon_j)}{dx}$ for i = 1,...,N. The use of a deflection sensor at α_i would involve switching 0 and $\phi_j(\alpha_i)$ in each of the pairs $[0 \;\; \phi_j(\alpha_i)]$ in the ith row of C. To include natural damping in the model, the negative of the damping term, $2\zeta\omega_i$, would appear in each diagonal block of the system matrix of (32) multiplying the $\dot{\psi}$ term. For the present, this is considered negligible.

Equation (7) remains to be solved before the degrees of controllability and observability can be computed. The solution of this equation is facilitated by use of the following real invertible transformation:

$$T = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & \uparrow & \uparrow \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 & \underline{v}_4 & \underline{a}_3 & \underline{b}_3 & \cdots & \underline{a}_N & \underline{b}_N \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \end{bmatrix} \quad (33)$$

where the $\underline{v}_i$ are the generalized eigenvectors corresponding to the zero eigenvalues and the $\underline{a}_i$, $\underline{b}_i$ are the real and imaginary parts of the eigenvector corresponding to the complex eigenvalue $\lambda_i = \sigma_i + j\omega_i$.

If a new matrix M is defined by the relation

$$V = TMT^T \quad (34)$$

and Λ is formed from the eigenvalues,

$$\Lambda = \begin{bmatrix} 0 & 1 & & & & & & & \\ 0 & 0 & & & & & & & \\ & & 0 & 1 & & & & O & \\ & & 0 & 0 & & & & & \\ & & & & \sigma_3 & \omega_3 & & & \\ & & & & -\omega_3 & \sigma_3 & & & \\ & & & & & & \ddots & & \\ & O & & & & & & \sigma_N & \omega_N \\ & & & & & & & -\omega_N & \sigma_N \end{bmatrix} \quad (35)$$

then substitution of both of these relations into (7) yields

$$\dot{M} = \Lambda M + M\Lambda^T - T^{-1}BR^{-1}B^TT^{-T} \quad (36)$$

This equation is much simpler to solve than equation (7) for V, and the solution for M is presented in Appendix A. Conversion back to V is attained through use of (34).

A computer program was written to calculate the degree of controllability (observability) for up to four actuators (sensors) placed at various positions along a free-free or simply supported beam (FORTRAN listing appears in Appendix B). The programmer specifies the number of equally spaced positions along a half beam length to be tested (mode shapes are symmetric), and the program computes the degree of controllability for all possible arrangements of actuators. The same program is used to compute observability with the appropriate changes outlined in the last section. The present program assumes the use of force actuators or translation rate sensors but can be easily modified for torque actuators and deflection sensors.

The program accepts as input the system matrix A, the number of flexible modes to be considered (maximum 5), the number of actuators to be tested, the input weighting and control scaling matrices R and D, and the control period T. The mass, length, and modal frequencies of the beam were chosen to correspond to those of the experimental beam set up at NASA Langley Research Center (ℓ = 12 ft, m = 0.50 slugs, ω_1 = 11.47 rad/sec, ω_2 = 31.63 rad/sec.) In all trials, there was no relative weighting of actuators (R = I), and the amplitude rates were scaled by $1/\omega_i$ relative to their respective amplitudes using D (amplitudes were considered equally important).

In Figures 4 and 5, the degree of controllability (DC) is plotted for one force actuator varied along the length of a single mode beam. Figure 4 shows the expected correspondence between the DC and the first mode shape. The maximum DC is at the ends where there is maximum deflection, and the DC becomes zero at the nodes where the system is uncontrollable. The correspondence between mode shape and degree of controllability is again apparent in Fig. 5 when the second mode is considered alone.

Figures 6-8 consider the first and second modes simultaneously. In Fig. 6, a single actuator is tested along the length of the beam as in the previous two cases. The maximum DC is again at the ends but the system becomes uncontrollable at a node of either mode. The DC has an intermediate peak at the 7th test position which corresponds to an antinode of the 2nd mode.

In Fig. 7 one actuator is fixed at the middle of the beam (antinode of 1st mode) while the other is varied. There is an overall increase in controllability because of the presence of the second actuator, but the DC still goes to zero at the nodes of the second mode because the fixed actuator is at a node of the 2nd mode and thus contributes nothing to the controllability of that mode. The degree of controllability never goes to zero in Fig. 8 when the fixed actuator is at the end. The optimal placement of the other was found to be at position #7 if duplicate positioning at #1 is not allowed.

The degree of observability (DO) for two cases is illustrated in Figures 9 and 10. In Figure 9, a rate sensor was varied along the length of a single mode beam. The resultant DO is strikingly similar to the DC of Fig. 4. The first and second modes are considered in Fig. 10 where one sensor is fixed at the center of the beam and the other is varied. The DO becomes zero at three points because the second mode is unobservable at the location of the first sensor.

CONCLUSION

While it is difficult to consider the degrees of controllability and observability just developed in an absolute sense, they serve well as quick relative measures of controllability and observability. A more realistic measure of controllability, for instance, might involve the integral magnitude of control effort rather than the integral quadratic form chosen for convenience. This degree of realism has been sacrificed in favor of the analytic solution to the optimal control problem. It is also true that the "size" of the information matrix could have been defined in several other ways, e.g., tr J, in computing the degree of observability. The control period is also somewhat arbitrary, but if the modal periods are short compared to T, the measures of controllability and observability are independent of T in a relative sense.

The control measure does have several advantages over the methods in [1] and [2]: (a) it does not arbitrarily weight state excursions against control effort, (b) it calls attention to the most uncontrollable direction by primarily weighting the volume generated by that minimum distance—thus it is a worst case analysis, (c) it seeks a control law minimizing integrated control use, and (d) it is relatively simple to compute.

For the observability case, the Kalman Filter already provided the minimized least square estimate error for which the covariance matrix is P. P determined the information matrix J whose size was used to compute the degree of observability.

The choice of measuring the size of J by the weighted volume within a quadratic surface made the computation of observability analogous to controllability.

The results of the DC and DO calculations in the case of the free-free beam were entirely intuitive and could have been anticipated from knowledge of the mode shapes. But that example was taken in order that one could interpret the results easily. The purpose in defining these measures of controllability and observability is to assist the designer of a control system for a plant of realistic complexity where the best locations of sensors and actuators may not be so obvious.

Now that these tools have been developed, they will be applied to the problem of choosing the number and location of sensors and actuators in the design of a large space structure considering the likelihood of random failures among these components. It is expected that the optimum locations for components with possibility of failure will differ under certain circumstances from those with no chance of failure.

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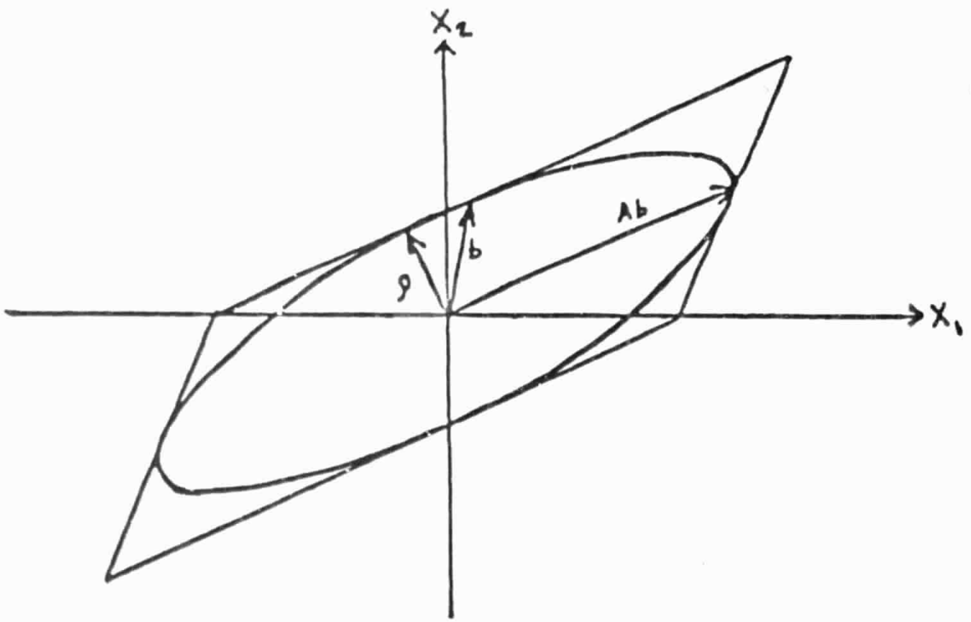


Fig. 1: Parallelogram approximation to recovery region

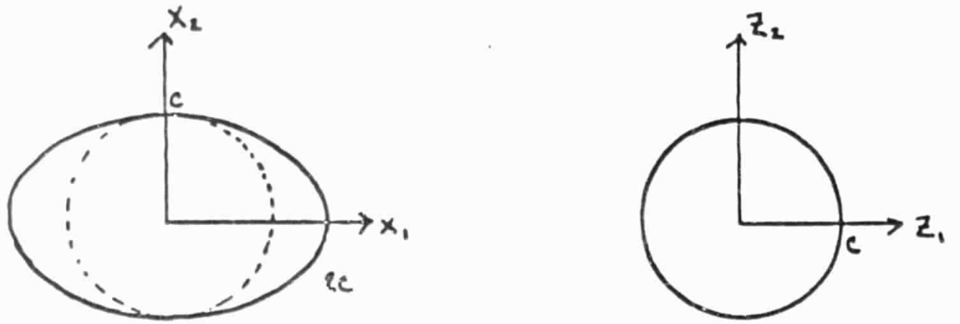
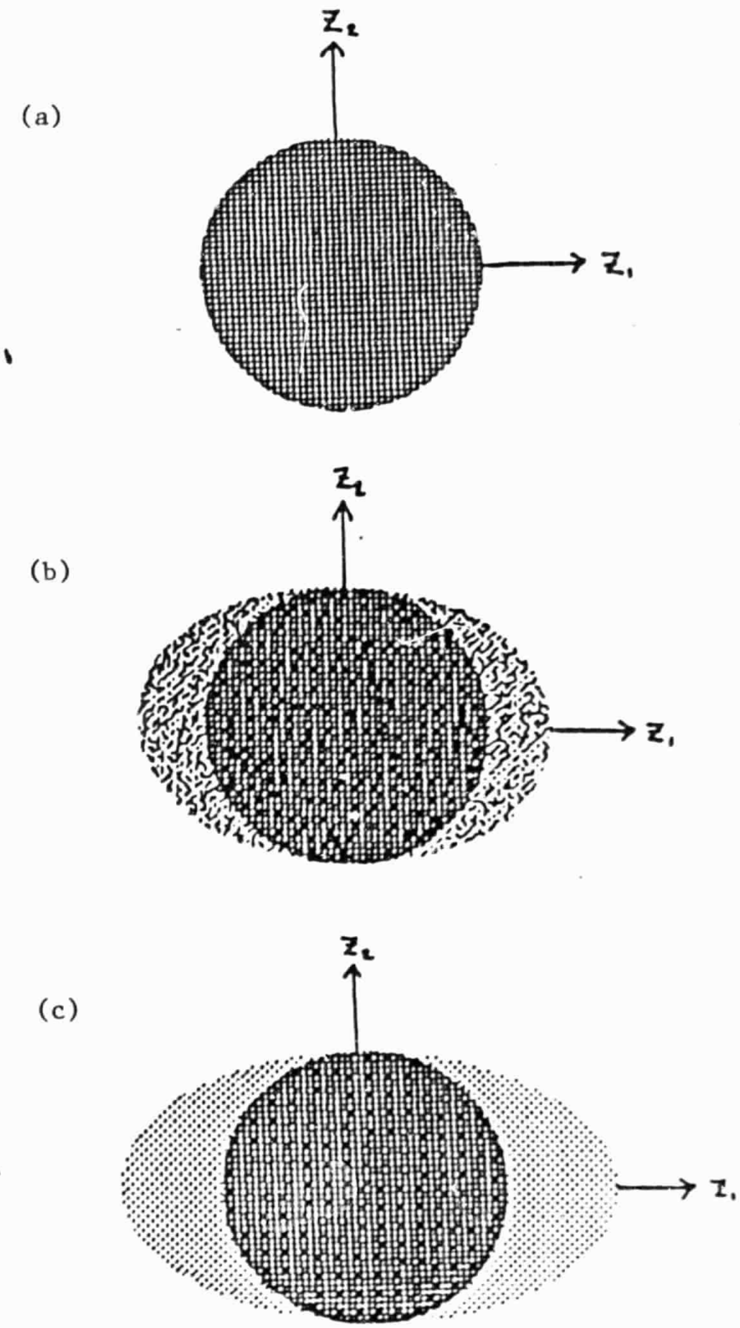


Fig. 2: Ideal control distribution transformed to equicontrol space.



Figs. 3a-c: Control weighting scheme for computing degree of controllability - (a) ideal control distribution, (b) slightly distorted distribution, (c) very distorted distribution. Shading indicates relative weighting.

Fig. 3.

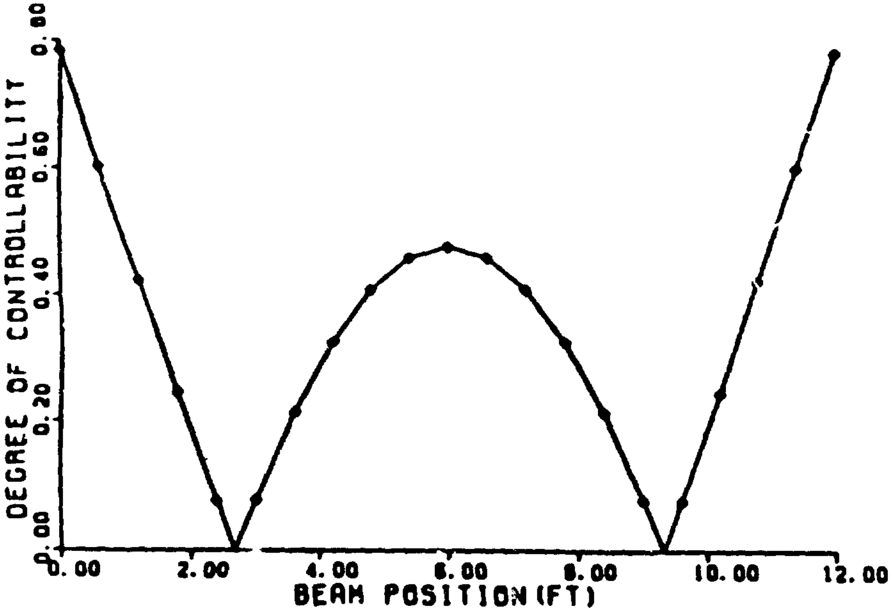


Fig. 4.

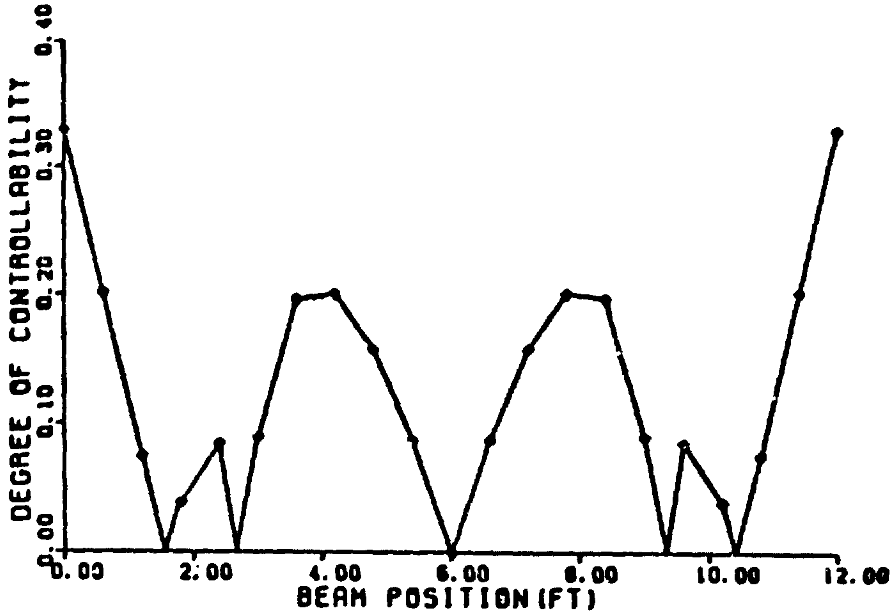


Fig. 6.

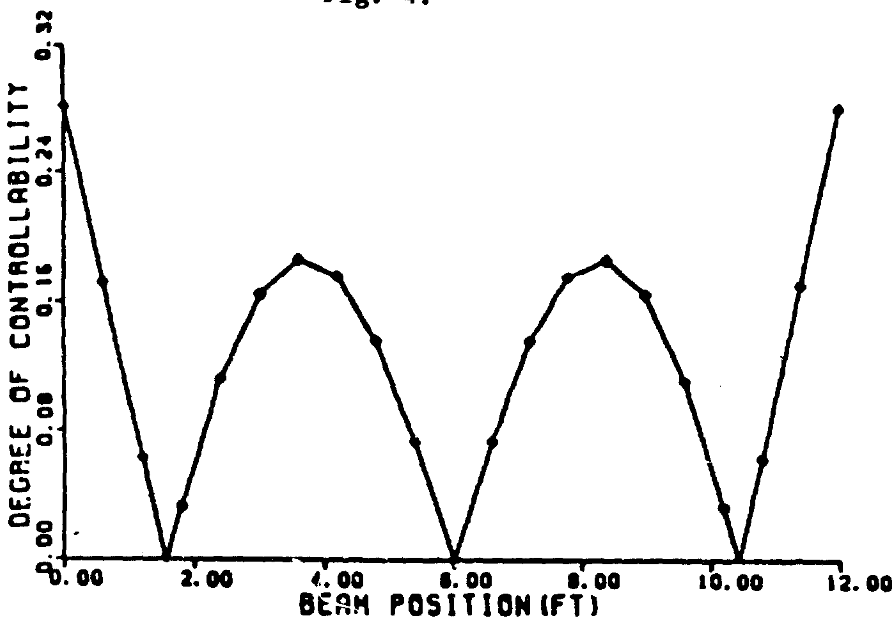


Fig. 5.

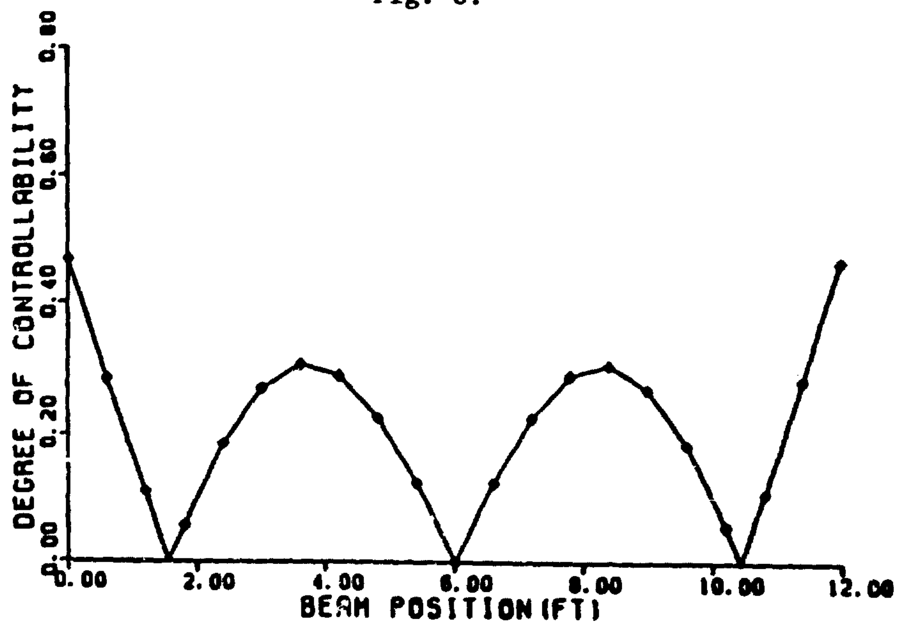


Fig. 7.

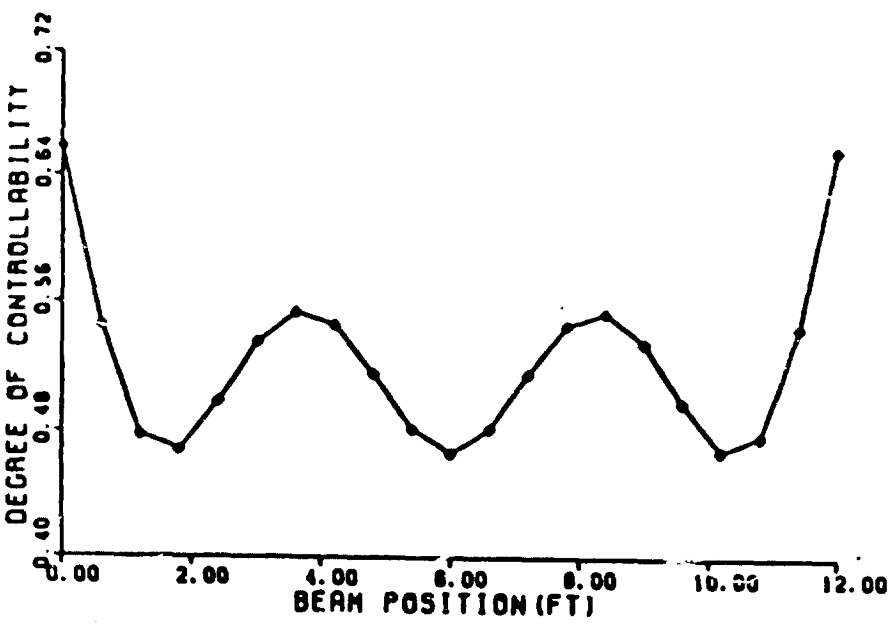


Fig. 8.

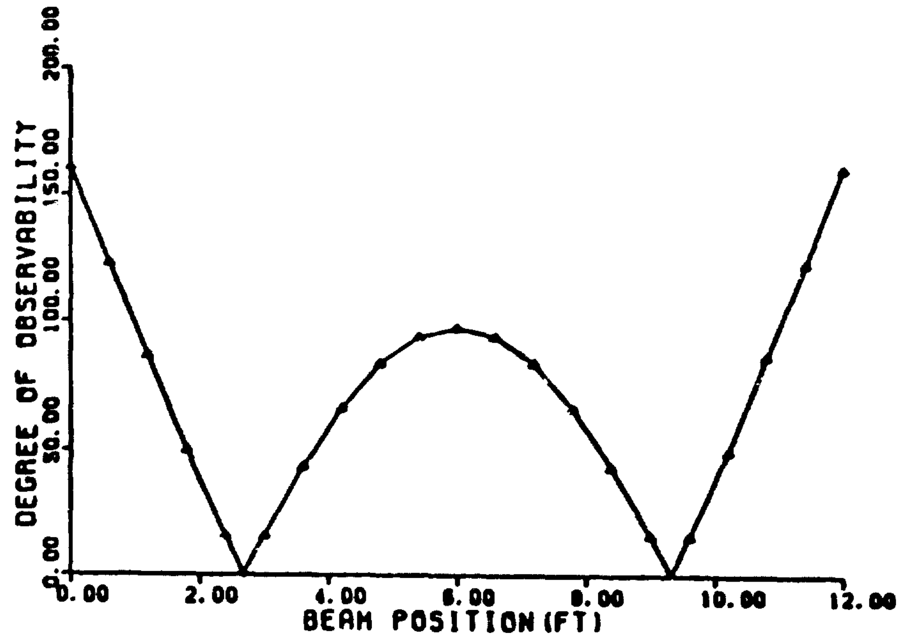


Fig. 9.

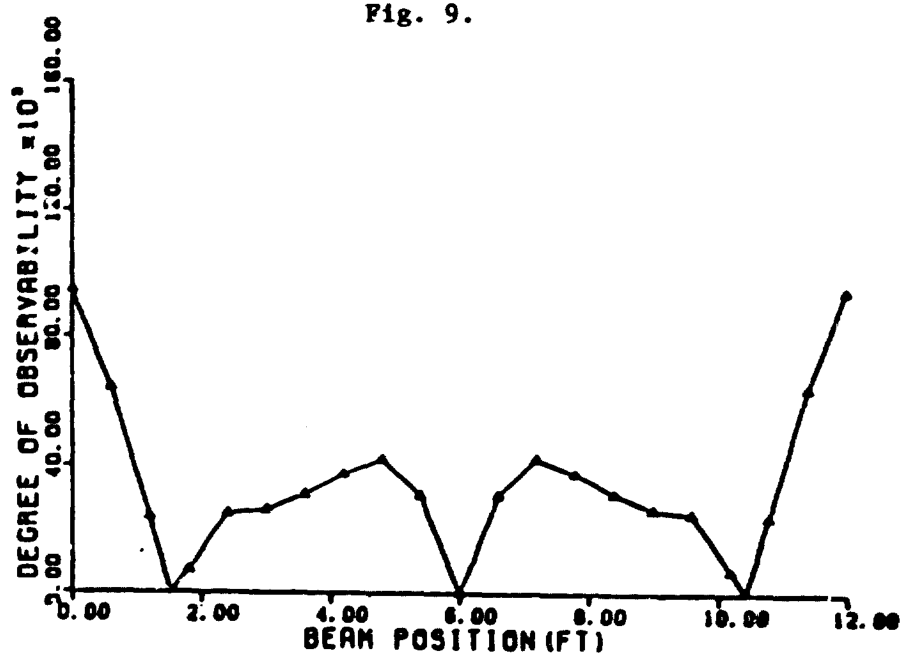


Fig. 10.

APPENDIX A

SOLUTION OF THE MATRIX DIFFERENTIAL EQUATION (36)

This Appendix presents the solution to the differential equation

$$\dot{M} = \Lambda M + M \Lambda^T - D \qquad \text{(A-1)}$$

where Λ is given by (35) and the driving matrix D is the last term in (36).

The solution matrix M(t) is symmetric and has the following form:

$$M(t) = \begin{bmatrix} I & V & II & II & II & \cdots & II \\ & I & II & II & II & \cdots & II \\ & & III & IV & IV & \cdots & IV \\ & & & III & IV & \cdots & IV \\ & & & & III & \cdots & IV \\ & & & & & & III \end{bmatrix} \qquad \text{(A-2)}$$

The Roman Numerals indicate 2x2 block solution types. If the two rigid body modes are not included in the model, the first and second row and column blocks are deleted from (A-2). The block solutions have the form

$$\begin{bmatrix} m_{ac} & m_{ad} \\ m_{bc} & m_{bd} \end{bmatrix}$$

If the solution is symmetric ($m_{bc} = m_{ad}$), only m_{ad} is given. Note that a and b are row indices, c and d are column indices.

TYPE I

$$m_{ac}(t) = \tfrac{1}{3} d_{bd}(T-t)^3 - d_{ad}(T-t)^2 + d_{ac}(T-t)$$
$$m_{ad}(t) = -\tfrac{1}{2} d_{bd}(T-t)^2 + d_{ad}(T-t)$$
$$m_{bd}(t) = d_{bd}(T-t)$$

TYPE II

$$\begin{aligned} m_{ac}(t) = {} & r\left[1-e^{\sigma_\kappa(t-T)}\cos\omega_\kappa(t-T)\right] - s e^{\sigma_\kappa(t-T)}\sin\omega_\kappa(t-T) \\ & - \frac{\sigma_\kappa d_{bc} - \omega_\kappa d_{bd}}{\omega_\kappa^2+\sigma_\kappa^2}(t-T)\, e^{\sigma_\kappa(t-T)}\cos\omega_\kappa(t-T) \\ & - \frac{\sigma_\kappa d_{bd} + \omega_\kappa d_{bc}}{\omega_\kappa^2+\sigma_\kappa^2}(t-T)\, e^{\sigma_\kappa(t-T)}\sin\omega_\kappa(t-T) \end{aligned}$$

$$\begin{aligned} m_{ad}(t) = {} & s\left[1-e^{\sigma_\kappa(t-T)}\cos\omega_\kappa(t-T)\right] + r e^{\sigma_\kappa(t-T)}\sin\omega_\kappa(t-T) \\ & - \frac{\sigma_\kappa d_{bd} + \omega_\kappa d_{bc}}{\omega_\kappa^2+\sigma_\kappa^2}(t-T)\, e^{\sigma_\kappa(t-T)}\cos\omega_\kappa(t-T) \\ & + \frac{\sigma_\kappa d_{bc} - \omega_\kappa d_{bd}}{\omega_\kappa^2+\sigma_\kappa^2}(t-T)\, e^{\sigma_\kappa(t-T)}\sin\omega_\kappa(t-T) \end{aligned}$$

$$\begin{aligned} m_{bc}(t) = {} & \frac{\sigma_\kappa d_{bc} - \omega_\kappa d_{bd}}{\omega_\kappa^2+\sigma_\kappa^2}\left[1-e^{\sigma_\kappa(t-T)}\cos\omega_\kappa(t-T)\right] \\ & - \frac{\sigma_\kappa d_{bd} + \omega_\kappa d_{bc}}{\omega_\kappa^2+\sigma_\kappa^2}\, e^{\sigma_\kappa(t-T)}\sin\omega_\kappa(t-T) \end{aligned}$$

$$\begin{aligned} m_{bd}(t) = {} & \frac{\sigma_\kappa d_{bd} + \omega_\kappa d_{bc}}{\omega_\kappa^2+\sigma_\kappa^2}\left[1-e^{\sigma_\kappa(t-T)}\cos\omega_\kappa(t-T)\right] \\ & + \frac{\sigma_\kappa d_{bc} - \omega_\kappa d_{bd}}{\omega_\kappa^2+\sigma_\kappa^2}\, e^{\sigma_\kappa(t-T)}\sin\omega_\kappa(t-T) \end{aligned}$$

where

$$r = \frac{1}{(\sigma_\kappa^2+\omega_\kappa^2)^2}\left[(\omega_\kappa^2+\sigma_\kappa^2)(\sigma_\kappa d_{ac} - \omega_\kappa d_{ad}) + (\omega_\kappa^2-\sigma_\kappa^2) d_{bc} + 2\sigma_\kappa\omega_\kappa d_{bd}\right]$$

$$s = \frac{1}{(\sigma_\kappa^2+\omega_\kappa^2)^2}\left[(\omega_\kappa^2+\sigma_\kappa^2)(\sigma_\kappa d_{ad} + \omega_\kappa d_{ac}) + (\omega_\kappa^2-\sigma_\kappa^2) d_{bd} - 2\sigma_\kappa\omega_\kappa d_{bc}\right]$$

$$K = \tfrac{1}{2}(c+1)$$

TYPE III

- for $\sigma_\kappa \neq 0$:

$$\begin{aligned} m_{ac}(t) = {} & r\left[1-e^{2\sigma_\kappa(t-T)}\right] - s e^{2\sigma_\kappa(t-T)}\sin 2\omega_\kappa(t-T) \\ & - q\left[1-e^{2\sigma_\kappa(t-T)}\cos 2\omega_\kappa(t-T)\right] \end{aligned}$$

$$\begin{aligned} m_{ad}(t) = {} & -q e^{2\sigma_\kappa(t-T)}\sin 2\omega_\kappa(t-T) \\ & + s\left[1-e^{2\sigma_\kappa(t-T)}\cos 2\omega_\kappa(t-T)\right] \end{aligned}$$

$$\begin{aligned} m_{bd}(t) = {} & r\left[1-e^{2\sigma_\kappa(t-T)}\right] + s e^{2\sigma_\kappa(t-T)}\sin 2\omega_\kappa(t-T) \\ & + q\left[1-e^{2\sigma_\kappa(t-T)}\cos 2\omega_\kappa(t-T)\right] \end{aligned}$$

where

$$r = \frac{1}{4\sigma_\kappa}(d_{ac}+d_{bd})$$

$$s = \frac{1}{\sigma_\kappa^2+\omega_\kappa^2}\left[\tfrac{1}{2}\sigma_\kappa d_{ad} - \tfrac{1}{4}\omega_\kappa(d_{bd}-d_{ac})\right]$$

$$q = \frac{1}{\sigma_\kappa^2+\omega_\kappa^2}\left[\tfrac{1}{2}\omega_\kappa d_{ad} + \tfrac{1}{4}\sigma_\kappa(d_{bd}-d_{ac})\right]$$

$$K = \tfrac{1}{2}(a+1)$$

· for $\sigma_k = 0$:

$$m_{ac}(t) = -r(t-T) + s\sin 2\omega_k(t-T) - q[1-\cos 2\omega_k(t-T)]$$
$$m_{ad}(t) = -q\sin 2\omega_k(t-T) - s[1-\cos 2\omega_k(t-T)]$$
$$m_{bd}(t) = -r(t-T) - s\sin 2\omega_k(t-T) + q[1-\cos 2\omega_k(t-T)]$$

where

$$r = \tfrac{1}{2}(d_{bd} + d_{ac})$$

$$s = \frac{1}{4\omega_k}(d_{bd} - d_{ac})$$

$$q = \frac{1}{2\omega_k} d_{ad}$$

$$k = \tfrac{1}{2}(a+1)$$

TYPE IV

$$m_{ac}(t) = s + q + e^{\sigma_T(t-T)}[r\sin(-) - s\cos(-) + p\sin(+) - q\cos(+)]$$
$$m_{ad}(t) = -r - p + e^{\sigma_T(t-T)}[s\sin(-) + r\cos(-) + q\sin(+) + p\cos(+)]$$
$$m_{bc}(t) = r - p + e^{\sigma_T(t-T)}[-s\sin(-) - r\cos(-) + q\sin(+) + p\cos(+)]$$
$$m_{bd}(t) = s - q + e^{\sigma_T(t-T)}[r\sin(-) - s\cos(-) - p\sin(+) + q\cos(+)]$$

where

$$r = -\frac{(\omega_k-\omega_j)(d_{ac}+d_{bd}) + \sigma_T(d_{ad}-d_{bc})}{2[(\omega_k-\omega_j)^2+\sigma_T^2]}$$
$$s = \frac{(\omega_k-\omega_j)(d_{bc}-d_{ad}) + \sigma_T(d_{ac}+d_{bd})}{2[(\omega_k-\omega_j)^2+\sigma_T^2]}$$
$$p = -\frac{(\omega_k+\omega_j)(d_{ac}-d_{bd}) + \sigma_T(d_{bc}+d_{ad})}{2[(\omega_k+\omega_j)^2+\sigma_T^2]}$$
$$q = -\frac{(\omega_k+\omega_j)(d_{bc}+d_{ad}) + \sigma_T(d_{bd}-d_{ac})}{2[(\omega_k+\omega_j)^2+\sigma_T^2]}$$

$$\sigma_T = \sigma_j + \sigma_k$$
$$(-) = (\omega_k-\omega_j)(t-T)$$
$$(+) = (\omega_k+\omega_j)(t-T)$$

$$j = \tfrac{1}{2}(a+1)$$
$$k = \tfrac{1}{2}(c+1)$$

TYPE V

$$m_{ac}(t) = d_{ac}(T-t) - \tfrac{1}{2}(d_{bc}+d_{ad})(T-t)^2 + \tfrac{1}{3}d_{bd}(T-t)^3$$
$$m_{ad}(t) = d_{ad}(T-t) - \tfrac{1}{2}d_{bd}(T-t)^2$$
$$m_{bc}(t) = d_{bc}(T-t) - \tfrac{1}{2}d_{bd}(T-t)^2$$
$$m_{bd}(t) = d_{bd}(T-t)$$

Appendix B

FILE: DEGCON   FORTRAN A   VM/SP CONVERSATIONAL MONITOR SYSTEM

```
C*******************************************************************************
C
C                 THIS PROGRAM COMPUTES THE DEGREE OF
C                  CONTROLLABILITY AND OBSERVABILITY
C           FOR FORCE ACTUATORS AND RATE SENSORS ON A BEAM
C
C******************************************************************************
C
C  INPUT:   N - NUMBER OF SYSTEM STATES
C           NA - NUMBER OF ACTUATORS (SENSORS)
C           IPHI - (1) FOR FREE-FREE BEAM
C                  (2) FOR SIMPLY SUPPORTED BEAM
C           ISTART - FIRST ACTUATOR TEST POSITION
C           NOPOS - NUMBER OF POSITIONS TO BE TESTED
C           IAS - (1) TO COMPUTE CONTROLLABILITY
C                 (2) TO COMPUTE OBSERVABILITY
C           IFIX - FIXED POSITION OF SECOND ACTUATOR WHEN
C                  PLOTTING CONTROLLABILITY FOR 2 ACTUATORS
C           BM - BEAM MASS
C           BL - BEAM LENGTH
C           DT - CONTROL PERIOD
C           FB - FRACTION OF BEAM LENGTH FROM END OVER
C                WHICH ACTUATORS PLACED
C           TOL - ZERO TOLERANCE FOR REAL NUMBERS
C           TAU - ACTUATOR MEAN TIME TO FAILURE
C           TOP - SYSTEM OPERATING OR MISSION PERIOD
C           OM - BEAM MODAL FREQUENCIES
C           BETA - MODAL SHAPE PARAMETERS
C           DDIAG - DIAGONAL ELEMENTS OF STATE WEIGHTING MATRIX
C           R - ACTUATOR WEIGHTING MATRIX
C           A - SYSTEM MATRIX
C
C  OUTPUT:   LOC - EIGHT DIGIT LOCATION CODE REPRESENTING
C                  POSITIONS OF 4 ACTUATORS:  RIGHTMOST
C                  PAIR REPRESENTS LOCATION OF FIRST ACTUATOR
C                  AND LEFTMOST THE FOURTH ACTUATOR (IF THE PAIR
C                  EQUALS IFCODE=NOPOS+1, THE ACTUATOR HAS FAILED)
C            LMAX - ACTUATOR LOCATIONS FOR MAXIMUM DC NOT
C                   CONSIDERING FAILURES
C            DCMAX - MAXIMUM DC NOT CONSIDERING FAILURES
C            LMAXF - ACTUATOR LOCATIONS FOR MAXIMUM AVERAGE DC
C            DCMAXF - MAXIMUM AVERAGE DC (FAILURES CONSIDERED)
C            UMIN - LEAST CONTROLLABLE DIRECTION IN ORIGINAL
C                   STATE SPACE ASSOCIATED WITH MAXIMUM DC
C            UMAX - MOST CONTROLLABLE DIRECTION IN ORIGINAL
C                   STATE SPACE ASSOCIATED WITH MAXIMUM DC
C
C*******************************************************************************
C
C
      DIMENSION A(10,10),B(10,4),R(4,4),IACT(4),OM(5),
     &          V(10,10),C1(24),WK1(55,9),AA(10,10),
     &          D(10,10),DDIAG(10),DV(10,10),DVD(10,10),EV(10),
     &          WK(200),DVDSYM(55),UMIN(10),UMAX(10),
     &          RACT(4),WKAREA(10),RINV(4,4),BRINVB(10,10),
```

```
     &          RINVBT(4,10),DC4(12,12,12,12),DC3(12,12,12),DC2(12,12),
     &          DC1(12),BETA(5),Z(10,10),T(10,10),TINV(10,10),
     &          TTRAN(10,10),TTINV(10,10),BRIBTT(10,10),
     &          DR(10,10),RW2(20),RZ2(200),AM(10,10),AMTT(10,10),
     &          DVDINV(10,10),WKAR2(55),XARRAY(23),YARRAY(23)
      COMPLEX W2(10),Z2(10,10),ZN
      EQUIVALENCE(W2(1),RW2(1)),(Z2(1,1),RZ2(1))
      DATA IN,IO,IDGT,IND,NW,IJOB,EPS/5,6,0,1,55,1,1.E-15/
      CALL PLOTS(IDUM,IDUM,9)
C
C*  READ AND ECHO INPUT
C
      READ(IN,4) N,NA,IPHI,ISTART,NOPOS,IAS,IFIX
      READ(IN,5) BM,BL,DT,FB,TOL,TAU,TOP
      READ(IN,6) (OM(I),I=1,5),(BETA(I),I=1,5),(DDIAG(I),I=1,10)
      READ(IN,7) ((R(I,J),J=1,4),I=1,4)
      READ(IN,8,END=17) ((A(I,J),J=1,[illegible],I=1,10)
    4 FORMAT(7I2)
    5 FORMAT(3F10.4/4F10.4)
    6 FORMAT(3(5F10.4/),5F10.4)
    7 FORMAT(3(4F10.4/),4F10.4)
    8 FORMAT(19(5F15.4/),5F15.4)
   17 WRITE(IO,20) N,NA,IPHI,IAS,IFIX,ISTART,NOPOS,BM,BL,DT,
     &              FB,TAU,TOP,(OM(I),I=1,5),(BETA(I),I=1,5),
     &              ((R(I,J),J=1,4),I=1,4),(DDIAG(I),I=1,10)
   20 FORMAT(1X,'N=',I2/'NA=',I2/'IPHI=',I2/'IAS=',I2/'IFIX=',I2/
     &        'ISTART=',I2/'NOPOS=',I2/
     &        'BM=',F10.4/'BL=',F10.4/'DT=',F10.4/'FB=',F10.4/
     &        'TAU=',E15.4/'TOP=',E15.4/'OM(1-5)=',5F10.4/
     &        'BETA(1-5)=',5F10.4/
     &        'R='/4(4F10.4/)//'DDIAG(1-10)=',5F10.4/12X,5F10.4////)
      IFCODE=NOPOS+1
C
C*  INITIALIZE VARIABLES
C
      DO 23 I=1,55
        DVDSYM(I)=0.
   23 CONTINUE
      DO 24 I=1,12
        DC1(I)=0.
        DO 24 J=1,12
          DC2(I,J)=0.
          DO 24 K=1,12
            DC3(I,J,K)=0.
            DO 24 L=1,12
              DC4(I,J,K,L)=0.
   24 CONTINUE
      DO 29 I=1,10
        DO 29 J=1,10
          DV(I,J)=0.
          Z(I,J)=0.
          DVD(I,J)=0.
          BRINVB(I,J)=0.
   29 CONTINUE
      DO 36 I=1,10
```

```
        D(I,I)=DDIAG(I)
        DO 36 J=1,10
          IF(I.NE.J) D(I,J)=0.
   36 CONTINUE
      NE=N*(N+1)/2
      TEND=DT
      DCMAX=0.
      DCMAXF=0.
      LMAX=0
      LMAXF=0
C
C*  FIND THE TRANSFORMATION MATRIX T USED IN COMPUTING V
C
      DO 671 I=1,N
        DO 671 J=1,N
          AA(I,J)=A(I,J)
  671 CONTINUE
      CALL EIGRF(AA,N,10,IJOB,RW2,RZ2,10,WK,IER)
      DO 672 I=1,N
        DO 672 J=1,N,2
          T(I,J)=REAL(Z2(I,J))
  672 CONTINUE
      DO 673 I=1,N
        DO 673 J=2,N,2
          T(I,J)=AIMAG(Z2(I,J-1))
  673 CONTINUE
      CALL LINV1F(T,N,10,TINV,IDGT,WK,IER)
      DO 674 I=1,N
        DO 674 J=1,N
          TTRAN(I,J)=T(J,I)
  674 CONTINUE
      CALL LINV1F(TTRAN,N,10,TTINV,IDGT,WK,IER)
C
C*  FOURTH ORDER DO-LOOP TO PERMUTE LOCATIONS OF 4 ACTUATORS
C*  (NO TWO LOCATIONS ARE ALLOWED TO BE THE SAME)
C
        DO 46 I=1,4
          IACT(I)=IFCODE
   46   CONTINUE
        IACT4=IACT(4)
        IACT3=IACT(3)
        IACT2=IACT(2)
        IACT1=IACT(1)
        IF(NA.NE.4) GO TO 49
        DO 181 IACT4=ISTART,IFCODE
          IACT(4)=IACT4
          GO TO 50
   49     IF(NA.NE.3) GO TO 51
   50     DO 171 IACT3=ISTART,IFCODE
            IACT(3)=IACT3
            GO TO 52
   51       IF(NA.NE.2) GO TO 53
   52       DO 161 IACT2=ISTART,IFCODE
              IACT(2)=IACT2
   53         DO 151 IACT1=ISTART,IFCODE
```

```
            IACT(1)=IACT1
C
C*  COMPUTE CONTROL EFFECTIVENESS MATRIX B
C
            DO 62 I=1,10
              DO 62 J=1,4
                B(I,J)=0.
   62       CONTINUE
            DO 63 I=2,N,2
              DO 63 J=1,NA
                RACT(J)=(FLOAT(IACT(J)-1)/FLOAT(NOPOS-1))*BL+FB
                IF(IPHI.EQ.2) GO TO 625
                B(I,J)=PHI(RACT(J),BETA(I/2),BL)/BM
                GO TO 627
  625           B(I,J)=PHI2(RACT(J),I/2,BM,BL)/BM
  627           IF(IAS.EQ.2) B(I,J)=B(I,J)*BM
   63       CONTINUE
C
C*  ZERO-OUT COLUMNS OF B ASSOCIATED WITH INOPERATIVE ACTUATORS
C
            IF(IACT4.NE.IFCODE) GO TO 633
            DO 632 I=2,N,2
              B(I,4)=0.
  632       CONTINUE
  633       IF(IACT3.NE.IFCODE) GO TO 635
            DO 634 I=2,N,2
              B(I,3)=0.
  634       CONTINUE
  635       IF(IACT2.NE.IFCODE) GO TO 637
            DO 636 I=2,N,2
             B(I,2)=0.
  636       CONTINUE
  637       IF(IACT1.NE.IFCODE) GO TO 65
            DO 638 I=2,N,2
              B(I,1)=0.
  638       CONTINUE
   65       NB=0
            DO 66 I=1,N
              DO 66 J=1,NA
                IF(ABS(B(I,J)).LT.TOL) NB=NB+1
   66       CONTINUE
            IF(NB.EQ.N*NA) GO TO 151
C
C*  IF ALL ACTUATORS INOPERATIVE, GO TO NEXT TEST LOCATION
C
            ITOTF=IFCODE*10**6+IFCODE*10**4+IFCODE*100+IFCODE
            LOC=IACT4*10**6+IACT3*10**4+IACT2*100+IACT1
            IF(LOC.EQ.ITOTF) GO TO 203
C
C*  ADJUST INITIAL R TO ACCOUNT FOR ACTUATOR SATURATION
C
            NOA=0
            DO 661 I=1,4
              IF(IACT(I).NE.IFCODE) NOA=NOA+1
  661       CONTINUE
```

```
                 DO 663 I=1,4
                   RINV(I,I)=FLOAT(NOA)/R(I,I)
                   DO 663 J=1,4
                     IF(I.NE.J) RINV(I,J)=0.
  663            CONTINUE
C
C*  COMPUTE DRIVING MATRIX IN D.E. FOR M
C
                 CALL VMULFP(RINV,B,NA,NA,N,4,10,RINVB,4,IER)
                 CALL VMULFF(B,RINVB,N,NA,N,10,4,BRINVB,10,IER)
                 CALL VMULFF(BRINVB,TTINV,N,N,N,10,10,BRIBTT,10,IER)
                 CALL VMULFF(TINV,BRIBTT,N,N,N,10,10,DR,10,IER)
C
C*  COMPUTE DIAGONAL BLOCKS OF M (TYPE III)
C
                 DO 675 I=1,N,2
                   SIG1=REAL(W2(I))
                   OM1=AIMAG(W2(I))
                   CALL DIAG(DT,SIG1,OM1,DR(I,I),DR(I,I+1),DR(I+1,I+1),
     &                       AM(I,I),AM(I,I+1),AM(I+1,I+1))
                   AM(I+1,I)=AM(I,I+1)
  675            CONTINUE
C
C*  COMPUTE OFF-DIAGONAL BLOCKS OF M (TYPE IV)
C
                 IF(N.LT.4) GO TO 70
                 NM3=N-3
                 NM1=N-1
                 DO 676 I=1,NM3,2
                   IP2=I+2
                   DO 676 J=IP2,NM1,2
                     SIG1=REAL(W2(I))
                     OM1=AIMAG(W2(I))
                     SIG2=REAL(W2(J))
                     OM2=AIMAG(W2(J))
                     CALL OFDIAG(DT,SIG1,SIG2,OM1,OM2,DR(I,J),
     &                           DR(I,J+1),DR(I+1,J),DR(I+1,I+1),
     &                           AM(I,J),AM(I,J+1),AM(I+1,J),
     &                           AM(I+1,J+1))
                     AM(J,I)=AM(I,J)
                     AM(J+1,I)=AM(I,J+1)
                     AM(J,I+1)=AM(I+1,J)
                     AM(J+1,I+1)=AM(I+1,J+1)
  676            CONTINUE
C
C*  TRANSFORM FROM M TO V
C
   70            CALL VMULFF(AM,TTRAN,N,N,N,10,10,AMTT,10,IER)
                 CALL VMULFF(T,AMTT,N,N,N,10,10,V,10,IER)
C
C*  TRANSFORM TO EQUICONTROL SPACE AND COMPUTE EIGENVALUES OF DVD
C
                CALL VMULFF(D,V,N,N,N,10,10,DV,10,IER)
                CALL VMULFF(DV,D,N,N,N,10,10,DVD,10,IER)
                CALL VCVTFS(DVD,N,10,DVDSYM)
```

```
           CALL EIGRS(DVDSYM,N,IJOB,EV,Z,10,WK,IER)
           IF((ABS(EV(1)).LT.TOL).OR.(EV(1).LT.O.)) GO TO 76
C
C*  COMPUTE DEGREE OF CONTROLLABILITY
C
           VS=SQRT(EV(1)**N)
           PRODEV=1.0
           DO 706 I=1,N
             PRODEV=PRODEV*EV(I)
  706      CONTINUE
           VE=SQRT(PRODEV)
           POWER=1.0/FLOAT(N)
           DEGCON=(VS+(VS/VE)*(VE-VS))**POWER
           GO TO 80
   76      DEGCON=O.
C
C*  STORE DC IN APPROPRIATE ARRAY; SEARCH FOR MAXIMUM DC
C*  AND RECORD ITS LOCATION, MAGNITUDE AND MAXIMUM AND
C*  MINIMUM CONTROLLABLE DIRECTIONS
C
   80      IF(NA.NE.4) GO TO 83
           DC4(IACT4,IACT3,IACT2,IACT1)=DEGCON
           IF(DEGCON.GT.DCMAX) GO TO 805
           GO TO 151
  805      IF((IACT1.EQ.IACT2).OR.(IACT1.EQ.IACT3).OR.
     &        (IACT1.EQ.IACT4).OR.(IACT2.EQ.IACT3).OR.
     &        (IACT2.EQ.IACT4).OR.(IACT3.EQ.IACT4)) GO TO 151
           DCMAX=DEGCON
           LMAX=LOC
           DO 807 I=1,N
             UMIN(I)=Z(I,1)/D(I,I)*SQRT(EV(1))
             UMAX(I)=Z(I,N)/D(I,I)*SQRT(EV(N))
  807      CONTINUE
           GO TO 151
   83      IF(NA.NE.3) GO TO 87
           DC3(IACT3,IACT2,IACT1)=DEGCON
           IF(DEGCON.GT.DCMAX) GO TO 835
           GO TO 151
  835      IF((IACT1.EQ.IACT2).OR.(IACT1.EQ.IACT3).OR.
     &        (IACT2.EQ.IACT3)) GO TO 151
           DCMAX=DEGCON
           LMAX=LOC
           DO 847 I=1,N
             UMIN(I)=Z(I,1)/D(I,I)*SQRT(EV(1))
             UMAX(I)=Z(I,N)/D(I,I)*SQRT(EV(N))
  847      CONTINUE
           GO TO 151
   87      IF(NA.NE.2) GO TO 89
           DC2(IACT2,IACT1)=DEGCON
           IF(DEGCON.GT.DCMAX) GO TO 875
           GO TO 151
  875      IF(IACT1.EQ.IACT2) GO TO 151
           DCMAX=DEGCON
           LMAX=LOC
           DO 877 I=1,N
```

```
                UMIN(I)=Z(I,1)/D(I,I)*SQRT(EV(1))
                UMAX(I)=Z(I,N)/D(I,I)*SQRT(EV(N))
  877         CONTINUE
              GO TO 151
   89         DC1(IACT1)=DEGCON
              IF(DEGCON.GT.DCMAX) GO TO 895
              GO TO 151
  895         DCMAX=DEGCON
              LMAX=LOC
              DO 897 I=1,N
                UMIN(I)=Z(I,1)/D(I,I)*SQRT(EV(1))
                UMAX(I)=Z(I,N)/D(I,I)*SQRT(EV(N))
  897         CONTINUE
  151       CONTINUE
            IF(NA.LT.2) GO TO 203
  161     CONTINUE
          IF(NA.LT.3) GO TO 203
  171   CONTINUE
        IF(NA.LT.4) GO TO 203
  181 CONTINUE
C
C*  COMPUTE AVERAGE DC AND SEARCH FOR MAXIMUM
C
  203 DCMAXF=0.
      IF(NA.NE.4) GO TO 300
      DO 250 I=ISTART,IFCODE
        DO 250 J=ISTART,IFCODE
          DO 250 K=ISTART,IFCODE
            DO 250 L=ISTART,IFCODE
  220         CALL PM4EXP(DC4,IFCODE,TAU,TOP,I,J,K,L,DCAVE)
              LOC=I*10**6+J*10**4+K*100+L
              WRITE(IO,225) LOC,DC4(I,J,K,L),DCAVE
  225         FORMAT('LOCATION=',I8,5X,'DC=',E11.4,5X,'DCAVE=',E11.4)
              IF(DCAVE.GT.DCMAXF) GO TO 230
              GO TO 250
  230         IF((I.EQ.J).OR.(I.EQ.K).OR.(I.EQ.L).OR.
     &           (J.EQ.K).OR.(J.EQ.L).OR.(K.EQ.L)) GO TO 250
              LMAXF=I*10**6+J*10**4+K*10**2+L
              DCMAXF=DCAVE
  250 CONTINUE
      GO TO 700
  300 IF(NA.NE.3) GO TO 400
      DO 350 I=ISTART,IFCODE
        DO 350 J=ISTART,IFCODE
          DO 350 K=ISTART,IFCODE
            CALL PM3EXP(DC3,IFCODE,TAU,TOP,I,J,K,DCAVE)
            IF(DCAVE.GT.DCMAXF) GO TO 330
            GO TO 350
  330       IF((I.EQ.J).OR.(I.EQ.K).OR.(J.EQ.K)) GO TO 350
            LMAXF=IFCODE*10**6+I*10**4+J*100+K
            DCMAXF=DCAVE
  350 CONTINUE
      GO TO 700
  400 IF(NA.NE.2) GO TO 500
      DO 450 I=ISTART,IFCODE
```

```
          WRITE(IO,422) LOC,DC2(I,J),DCAVE
  422     FORMAT('LOCATION=',I8,5X,'DC=',E11.4,5X,'DCAVE=',E11.4)
          IF(DCAVE.GT.DCMAXF) GO TO 430
          GO TO 450
  430     IF(I.EQ.J) GO TO 450
          LMAXF=IFCODE*10**6+IFCODE*10**4+I*100+J
          DCMAXF=DCAVE
  450 CONTINUE
      GO TO 700
  500 DO 550 I=ISTART,IFCODE
        LOC=IFCODE*10**6+IFCODE*10**4+IFCODE*100+I
        WRITE(IO,525) LOC,DC1(I)
  525   FORMAT('LOCATION=',I8,5X,'DC=',E11.4)
  550 CONTINUE
C
C*  OUTPUT DC'S, LOCATIONS, AND PRINCIPAL DIRECTIONS
C
  700 IF(NA.NE.4) GO TO 711
      WRITE(IO,705) DCMAX,LMAX,DCMAXF,LMAXF,
     &              (UMIN(I),I=1,10),(UMAX(I),I=1,10)
  705 FORMAT(1X,'MAX DC FOR 4 OPERATIONAL ACTUATORS IS',E11.4/
     &        'AND THE LOCATION IS ',I8//
     &        'MAX DC FOR 4 FAILING ACTUATORS IS',E11.4/
     &        'AND THE LOCATION IS ',I8//
     &        'UMIN='/5(E11.4,5X)/5(E11.4,5X)//
     &        'UMAX='/5(E11.4,5X)/5(E11.4,5X)//)
      GO TO 1000
  711 IF(NA.NE.3) GO TO 714
      WRITE(IO,715) DCMAX,LMAX,DCMAXF,LMAXF,
     &              (UMIN(I),I=1,10),(UMAX(I),I=1,10)
  715 FORMAT(1X,'MAX DC FOR 3 OPERATIONAL ACTUATORS IS',E11.4/
     &        'AND THE LOCATION IS ',I8/
     &        'MAX DC FOR 3 FAILING ACTUATORS IS',E11.4/
     &        'AND THE LOCATION IS ',I8//
     &        'UMIN='/5(E11.4,5X)/5(E11.4,5X)//
     &        'UMAX='/5(E11.4,5X)/5(E11.4,5X)//)
      GO TO 1000
  714 IF(NA.NE.2) GO TO 721
      WRITE(IO,720) LMAX,DCMAX,LMAXF,DCMAXF,
     &              (UMIN(I),I=1,10),(UMAX(I),I=1,10)
  720 FORMAT(//'LMAX=',I8,10X,'DCMAX=',E11.4/
     &        'LMAXF=',I8,10X,'DCMAXF=',E11.4//
     &        'UMIN='/5(E11.4,5X)/5(E11.4,5X)//
     &        'UMAX='/5(E11.4,5X)/5(E11.4,5X)//)
      GO TO 1000
  721 WRITE(IO,730) LMAX,DCMAX,
     &              (UMIN(I),I=1,10),(UMAX(I),I=1,10)
  730 FORMA    .MAX=',I8,10X,'DCMAX=',E11.4//
     &        'UMIN='/5(E11.4,5X)/5(E11.4,5X)//
     &        'UMAX='/5(E11.4,5X)/5(E11.4,5X)//)
C
C*  PLOT OF DC VS. ACTUATOR POSITION FOR
C*  1 FIXED AND 1 VARIABLE ACTUATOR
C
 1000 DO 1002 I=1,21
```

```
         XARRAY(I)=BL*FLOAT(I-1)/20.0
 1002 CONTINUE
      DO 1004 I=1,11
         YARRAY(I)=DC2(IFIX,I)
 1004 CONTINUE
      DO 1005 I=12,21
         YARRAY(I)=DC2(IFIX,22-I)
 1005 CONTINUE
      CALL SCALE(XARRAY,6.0,21,1)
      CALL SCALE(YARRAY,4.0,21,1)
      CALL AXIS(0.,0.,'BEAM POSITION(FT)',-17,6.0,0.0,
     &          XARRAY(22),XARRAY(23))
      CALL AXIS(0.,0.,'DEGREE OF CONTROLLABILITY',+25,4.0,90.0,
     &          YARRAY(22),YARRAY(23))
      CALL LINE(XARRAY,YARRAY,21,1,+1,5)
      CALL SYMBOL(0.5,5.0,0.21,'DEGREE OF CONTROLLABILITY',0.0,25)
      CALL SYMBOL(1.0,4.5,0.21,'FOR A FREE-FREE BEAM',0.0,20)
      CALL ENDPLT(12.0,0.0,999)
      STOP
 2000 WRITE(IO,100)
  100 FORMAT('THE MINIMUM E-VALUE IS ZERO')
      DO 2001 I=1,10
        WRITE(IO,2002) EV(I)
 2002   FORMAT('EV=',E11.4)
 2001 CONTINUE
      STOP
      END
C
C*  MODAL AMPLITUDE AT X FOR SIMPLY-SUPPORTED BEAM
C
      REAL FUNCTION PHI2(X,MODE,BM,BL)
      DATA PI/3.141592654/
      PHI2=SQRT(2.0/BM)*SIN(FLOAT(MODE)*PI*X/BL)
      RETURN
      END
C
C*  MATRIX (ARRAY) TIMES VECTOR (V)
C
      SUBROUTINE MATVEC(M,N,ARRAY,V,RET)
      DIMENSION ARRAY(M,N),V(N),RET(M)
      DO 10 I=1,M
      RET(I)=0.
      DO 10 J=1,N
   10 RET(I)=RET(I)+ARRAY(I,J)*V(J)
      RETURN
      END
C
C*  ADDS MATRIX B TO A
C
      SUBROUTINE MATADD(N,A,B,RET)
      DIMENSION A(N,N),B(N,N),RET(N,N)
      DO 10 I=1,N
      DO 10 J=1,N
   10 RET(I,J)=A(I,J)+B(I,J)
      RETURN
```

```
      END
C
C*  SUBTRACTS MATRIX B FROM A
C
      SUBROUTINE MATSUB(N,A,B,RET)
      DIMENSION A(N,N),B(N,N),RET(N,N)
      DO 10 I=1,N
      DO 10 J=1,N
   10 RET(I,J)=A(I,J)-B(I,J)
      RETURN
      END
C
C*  MODAL AMPLITUDE AT X FOR FREE-FREE BEAM
C
      REAL FUNCTION PHI(X,BETA,BL)
      ALP=BETA*BL
      SH=0.5*(EXP(ALP)-EXP(-ALP))
      CH=0.5*(EXP(ALP)+EXP(-ALP))
      A=(SH+SIN(ALP))/(CH-COS(ALP))
      PHI=0.5*(EXP(BETA*X)+EXP(-BETA*X))+COS(BETA*X)-
     &    A*(0.5*(EXP(BETA*X)-EXP(-BETA*X))+SIN(BETA*X))
      RETURN
      END
C
C*  COMPUTES AVERAGE EXPECTED PERFORMANCE MEASURE FOR
C*  4 COMPONENTS ASSUMING EACH HAS SAME EXPONENTIAL
C*  DISTRIBUTION OF TIME TO FAILURE
C
      SUBROUTINE PM4EXP(PM,IFCODE,TAU,TOP,L1,L2,L3,L4,PMAVE)
      DIMENSION PM(12,12,12,12)
      TT=TOP/TAU
      PT1=(1.0/(4.0*TT))*(1.0-EXP(-4.0*TT))
      PT25=1.0/(12.0*TT)-(1.0/(12.0*TT))*(4.0-3.0*EXP(-TT))*
     &       EXP(-3.0*TT)
      PT611=1.0/(12.0*TT)-(1.0/(12.0*TT))*(6.0-8.0*EXP(-TT)+
     &       3.0*EXP(-2.0*TT))*EXP(-2.0*TT)
      PT1215=1.0/(4.0*TT)-(1.0/(4.0*TT))*(4.0-6.0*EXP(-TT)+
     &       4.0*EXP(-2.0*TT)-EXP(-3.0*TT))*EXP(-TT)
      PT16=1.0-(1.0/(12.0*TT))*(-25.0-48.0*EXP(-TT)+
     &      36.0*EXP(-2.0*TT)-16.0*EXP(-3.0*TT)+3.0*EXP(-4.0*TT))
      PMAVE=PT1*PM(L1,L2,L3,L4)+PT25*(PM(IFCODE,L2,L3,L4)+
     &       PM(L1,IFCODE,L3,L4)+PM(L1,L2,IFCODE,L4)+
     &       PM(L1,L2,L3,IFCODE))+PT611*(PM(IFCODE,IFCODE,L3,L4)+
     &       PM(IFCODE,L2,IFCODE,L4)+PM(IFCODE,L2,L3,IFCODE)+
     &       PM(L1,IFCODE,IFCODE,L4)+PM(L1,IFCODE,L3,IFCODE)+
     &       PM(L1,L2,IFCODE,IFCODE))+PT1215*(PM(IFCODE,IFCODE,
     &          IFCODE,L4)+PM(IFCODE,IFCODE,L3,IFCODE)+
     &       PM(IFCODE,L2,IFCODE,IFCODE)+PM(L1,IFCODE,IFCODE,IFCODE))
      RETURN
      END
C
C*  SAME AS PM4EXP EXCEPT FOR 2 COMPONENTS
C
      SUBROUTINE PM2EXP(PM,IFCODE,TAU,TOP,L1,L2,PMAVE)
      DIMENSION PM(12,12)
```

```
      TT=TOP/TAU
      PT1=(1.0/(2.0*TT))*(1.0-EXP(-2.0*TT))
      PT23=(1.0/TT)*((1.0-EXP(-TT))-0.5*(1.0-EXP(-2.0*TT)))
      PMAVE=PT1*PM(L1,L2)+PT23*(PM(IFCODE,L2)+PM(L1,IFCODE))
      RETURN
      END
C
C*  SAME AS PM4EXP EXCEPT FOR 3 COMPONENTS
C
      SUBROUTINE PM3EXP(PM,IFCODE,TAU,TOP,L1,L2,L3,PMAVE)
      DIMENSION PM(12,12,12)
      TT=TOP/TAU
      PT1=(1.0/(3.0*TT))*(1.0-EXP(-3.0*TT))
      PT24=(1.0/(2.0*TT))*(1.0-EXP(-2.0*TT))-(1.0/(3.0*TT))*
     &      (1.0-EXP(-3.0*TT))
      PT57=(1.0/TT)*(1.0-EXP(-TT))-(1.0/TT)*(1.0-EXP(-2.0*TT))+
     &      (1.0/(3.0*TT))*(1.0-EXP(-3.0*TT))
      PMAVE=PT1*PM(L1,L2,L3)
     &       +PT24*(PM(IFCODE,L2,L3)+PM(L1,IFCODE,L3)+PM(L1,L2,IFCODE))
     &       +PT57*(PM(IFCODE,IFCODE,L3)+PM(IFCODE,L2,IFCODE)+
     &       PM(L1,IFCODE,IFCODE))
      RETURN
      END
C
C*  COMPUTES DIAGONAL SOLUTION BLOCKS OF M (TYPE III)
C
      SUBROUTINE DIAG(DT,SIG1,OM1,D11,D12,D22,AM11,AM12,AM22)
      DATA EPS/0.000001/
      IF(ABS(SIG1).LT.EPS) GO TO 5
      A=(D11+D22)/(4.0*SIG1)
      B=(0.5*SIG1*D12-0.25*OM1*(D22-D11))/(OM1*OM1+SIG1*SIG1)
      C=(0.5*OM1*D12+0.25*SIG1*(D22-D11))/(OM1*OM1+SIG1*SIG1)
      S=-2.0*SIG1*DT
      ARG=-2.0*OM1*DT
      AM11=A*(1.0-EXP(S))-B*EXP(S)*SIN(ARG)-C*(1.0-EXP(S)*COS(ARG))
      AM12=-C*EXP(S)*SIN(ARG)+B*(1.0-EXP(S)*COS(ARG))
      AM22=A*(1.0-EXP(S))+B*EXP(S)*SIN(ARG)+C*(1.0-EXP(S)*COS(ARG))
      GO TO 10
    5 A=0.5*(D22+D11)
      B=(D22-D11)/(4.0*OM1)
      C=D12/(2.0*OM1)
      ARG=-2.0*OM1*DT
      AM11=-A*(-DT)+B*SIN(ARG)-C*(1.0-COS(ARG))
      AM12=-C*SIN(ARG)-B*(1.0-COS(ARG))
      AM22=-A*(-DT)-B*SIN(ARG)+C*(1.0-COS(ARG))
   10 RETURN
      END
C
C*  COMPUTES OFF-DIAGONAL SOLUTION BLOCKS OF M (TYPE IV)
C
      SUBROUTINE OFDIAG(DT,SIG1,SIG2,OM1,OM2,D11,D12,
     &                  D21,D22,AM11,AM12,AM21,AM22)
      SIGT=SIG1+SIG2
      S=SIGT*(-DT)
      ARGP=(OM2+OM1)*(-DT)
```

```
      ARGM=(OM2-OM1)*(-DT)
      A=-((OM2-OM1)*(D11+D22)+SIGT*(D12-D21))/
     &   (2.0*((OM2-OM1)**2+SIGT**2))
      B=((OM2-OM1)*(D21-D12)+SIGT*(D11+D22))/
     &   (2.0*((OM2-OM1)**2+SIGT**2))
      C=-((OM2+OM1)*(D11-D22)+SIGT*(D21+D12))/
     &   (2.0*((OM2+OM1)**2+SIGT**2))
      D=-((OM2+OM1)*(D21+D12)+SIGT*(D22-D11))/
     &   (2.0*((OM2+OM1)**2+SIGT**2))
      AM11=B+D+EXP(S)*(A*SIN(ARGM)-B*COS(ARGM)+
     &                  C*SIN(ARGP)-D*COS(ARGP))
      AM12=-A-C+EXP(S)*(B*SIN(ARGM)+A*COS(ARGM)+
     &                  D*SIN(ARGP)+C*COS(ARGP))
      AM21=A-C+EXP(S)*(-B*SIN(ARGM)-A*COS(ARGM)+
     &                  D*SIN(ARGP)+C*COS(ARGP))
      AM22=B-D+EXP(S)*(A*SIN(ARGM)-B*COS(ARGM)-
     &                  C*SIN(ARGP)+D*COS(ARGP))
      RETURN
      END
```