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Twelve Month Technical Progress Report

to the

National Aeronautics and Space Administration

on

NASA Grant NSG-3048*

ALTERNATIVES FOR JET ENGINE CONTROL

October 1, 1980 - September 30, 1981

*Under the Direction of

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ABSTRACT

This report deals with progress made on the Grant NSG-3048 during the twelve month period beginning October 1, 1980 and ending September 30, 1981. The NASA Technical Officer for this period was Dr. Kurt Seldner of Lewis Research Center. The director of the research at the University of Notre Dame was Dr. Michael K. Sain, who has been assisted by Mr. Stephen Yurkovich, Mr. Joe P. Hill, and Mr. Thomas A. Klingler, research assistants, in the Department of Electrical Engineering. Mr. Yurkovich received the degree of Master of Science during this period, for his January 1981 thesis entitled "Application of Tensor Ideas to Nonlinear Modeling and Control". Mr. Hill and Mr. Klingler expect to complete research investigations for the Master of Science degree within the next calendar year. Mr. Yurkovich may complete requirements for the degree of Doctor of Philosophy in 1982.

Researches during the preceding calendar year have centered on basic topics in the modeling and feedback control of nonlinear dynamical systems. Of special interest have been the following topics: (1) the role of series descriptions, especially insofar as they relate to questions of scheduling, in the control of gas turbine engines; (2) the use of algebraic tensor theory as a technique for parameterizing such descriptions; (3) the relationship between tensor methodology and other parts of the nonlinear literature; (4) the improvement of interactive methods for parameter selection within a tensor viewpoint; and (5) study of feedback gain representation as a counterpart to these modeling and parameterization ideas.

Progress has been made in all five of the areas just described. Of

special interest, we believe, are the natural design ties which exist between scheduling and series representations and the natural mathematical ties which exist between symmetric tensor representations and formal series. In the light of rapidly evolving capabilities of microcomputers and minicomputers, in view of the qualitative tensor model possibilities established by Mr. Yurkovich in M.S. studies, and taking into account both the state of the art and prospects for further advance in tensor techniques for feedback from such models, we believe that significant opportunities for research progress are occurring in this area.

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ACKNOWLEDGMENTS

The funded research on this grant has been aided by the voluntary, unfunded efforts of a number of individuals. We would like to thank Mr. Robert Spann and Mr. John Monaghan for helpful discussions relative to College computing systems. Mr. Spann and Mr. Monaghan completed studies for the Bachelor of Science degree in May 1981. We also appreciate the interest of Mr. Joseph O'Sullivan, who participated in the readings and discussions pertaining to portions of this work.

Special thanks are due to Dr. R. Michael Schafer, who has been most helpful in regard to issues concerning the PDP-11 computer.

We also acknowledge encouragement and support extended to the project by the Department of Electrical Engineering.

Finally, we are pleased to thank Mrs. T. Youngs, who has prepared the typescripts.

I. BACKGROUND

1.1 INTRODUCTION

In this report, we discuss progress which has been made on NASA Grant NSG-3048, entitled "Alternatives for Jet Engine Control", during the twelve month period beginning on October 1, 1980 and ending on September 30, 1981.

This section contains, in subsections 1.2 and 1.3, some mathematical background material, which may be useful for reference during an examination of later sections.

Section II reports on the results of an extensive literature examination carried out by Mr. Stephen Yurkovich. This material explains many of the relationships between the theoretical machinery in use under this grant and various other methodologies involved in other theoretical studies. Insofar as we can determine, this group remains the pioneer in assessing practical utility of such methods for use in realistic application simulations. This means that our viewpoint and evaluation may be weighted in a manner different from that of the pure theoretical investigator.

Section III treats in an introductory way the polynomic and formal series implications of controller scheduling. The practical process of scheduling linear multivariable controllers leads naturally to families such as those which we have under investigation.

Section IV gives an update on the group's progress in developing parameter selection methods for choosing coefficients in tensor representations. This work has been carried out by Mr. Thomas Klingler. As will be apparent in the comparison of present capabilities with those of last year, a number of very positive steps have been taken. We expect these steps to be of considerable assistance in subsequent work.

Section V deals with the use of feedback on nonlinear tensor dynamical models. This study is in the formative stages and is being carried out by Mr. Joe Hill.

1.2 ABSTRACT DIFFERENTIATION

As will be indicated in Section 3.2, the idea of polynomic scheduling, of gains or time constants, suggests a state description in terms of series. Because we wish to use operator theoretic methods to some extent, it is convenient here to ask a few introductory questions about derivatives in such a context.

Let V and W be normed real vector spaces, with Z open in V. A function $f : Z \to W$ is differentiable at a point p in Z if there exists a continuous linear map $F : V \to W$ such that, for $(p+h)$ in Z and h in V,

$$\lim_{||h|| \to 0} \frac{||f(p+h) - f(p) - Fh||}{||h||} = 0 .$$

If F exists, then it is unique and is called the derivative of f at p, and is denoted by

$$(Df)(p) : V \to W .$$

In case f is differentiable on Z, then we have a construction

$$Df : Z \to L(V,W) ,$$

where $L(V,W)$ denotes the real vector space of $\mathbb{R}$-linear maps $V \to W$. Higher order derivatives are defined in a recursive fashion,

$$(D^r f)(p) = (D(D^{r-1} f))(p) ,$$

with r a positive integer, provided that the indicated limit exists.

An important connection exists between the calculus on normed vector spaces and the tensor algebra. Indeed,

$$D^2 f(p) \in L(V,L(V,W)) ,$$

$$D^3 f(p) \in L(V,L(V,L(V,W)))$$

$$\vdots \qquad \qquad \vdots$$

whenever the limits exist. Let us denote by

$$L(V_1, V_2, \ldots, V_n, W)$$

the real vector space of n-linear functions

$$V_1 \times V_2 \times \ldots \times V_n \to W ,$$

an n-linear function being one which is linear in its remaining argument whenever $(n-1)$ of its arguments are fixed. It can be shown that there exist isomorphisms

$$\begin{array}{ccc} L(V_1, V_2, W) & \to & L(V_1, L(V_2, W)) , \\ L(V_1, V_2, V_3, W) & \to & L(V_1, L(V_2, L(V_3, W))) , \\ \vdots & & \vdots \end{array}$$

so that $(D^r f)(p)$ can be regarded as an r-linear map $V^r \to W$, up to isomorphism. We suppress this isomorphism and think of $(D^r f)(p)$ as just such a map.

It is now straightforward to establish a connection with the tensor algebra, and we do so in the section following. The importance of the connection lies in its parametric possibilities: Every r-linear map can be composed from a linear map and a universal r-linear construction called tensor product. In a sense, the linear map embodies the parameters which are available for scheduling; and we pursue this view in a later section.

1.3 TENSOR ALGEBRA

In this section, we develop some of the structures with which we can subsequently discuss scheduling questions in Section 3.3. Let V be a real vector space. For each integer r which is two or greater, let

$$(\otimes^r V, \otimes^r)$$

be a tensor product for r copies of V. The notion extends to 1 and 0 by the definitions

$$\otimes^1 V = V \quad , \quad \otimes^0 V = \mathbb{R} \ .$$

The sequence $\otimes^r V$, $r = 0,1,2,\ldots,$ can be developed into a biproduct, and the images of $\otimes^r V$ under insertion can be given the same notation. Then the tensorial powers $\otimes^n V$ can be developed into an associative algebra by defining the internal direct sum

$$\otimes V = \sum_{n=0}^{\infty} \otimes^n V \ ,$$

and by equipping $\otimes V$ with the bilinear mapping $(\alpha,\beta) \to \alpha\beta$ for α, β, $\alpha\beta \in \otimes V$ whose result is defined by

$$\alpha\beta = \sum_{n,m} \alpha_n \otimes \beta_m \ ,$$

where $\alpha = \sum_n \alpha_n$, $\beta = \sum_m \beta_m$ for $\alpha_n \in \otimes^n V$ and $\beta_m \in \otimes^m V$. With this multiplication, $\otimes V$ becomes the graded tensor algebra over V with elements $(\alpha_0, \alpha_1, \ldots)$, which are sequences of the tensors $\alpha_i \in \otimes^i V$, $i = 0,1,\ldots,$ and with unit element $(1,0,\ldots)$. We emphasize the fact that multiplication in the tensor algebra is not a tensor product.

Now let $\otimes V$ and $\otimes W$ be tensor algebras as defined above, over V and W respectively. For every pair n, $m \geq 1$, let $\otimes^n_m(V,W)$ be a tensor product of $\otimes^n V$ and $\otimes^m W$, that is,

$$\otimes^n_m(V,W) = (\otimes^n V) \otimes (\otimes^m W) .$$

We set $\otimes^n_0(V,W) = \otimes^n V$ and $\otimes^0_m(V,W) = \otimes^m W$. In a manner similar to that preceding,

$$\otimes^n_m(V,W) \quad , \quad n = 0,1,2,\ldots, \ m = 0,1,2,\ldots$$

can also be developed into a biproduct; and the images of each of these spaces under natural insertion into the biproduct can again be given the same symbolic representation. Again, then, we construct the internal direct sum

$$\otimes(V,W) = \sum_{n,m\geq 0}^{\infty} \otimes^n_m(V,W)$$

with

$$\otimes(V,W) = \sum_{k=0}^{\infty} [\sum_{n+m=k} \otimes^n_m(V,W)]$$

functioning as the induced gradation on $\otimes(V,W)$.

Now consider two spaces $\otimes^n_m(V,W)$ and $\otimes^r_s(V,W)$. There exists a unique bilinear mapping

$$\mu : \otimes^n_m(V,W) \times \otimes^r_s(V,W) \to \otimes^{n+r}_{m+s}(V,W)$$

with action

$$\mu(\alpha_n \otimes \beta_m, \alpha_r \otimes \beta_s) = (\alpha_n \otimes \alpha_r) \otimes (\beta_m \otimes \beta_s) ,$$

where $\alpha_n \in \otimes^n V$, $\alpha_r \in \otimes^r V$, $\beta_m \in \otimes^m W$, $\beta_s \in \otimes^s W$. The pair $(\otimes^{n+r}_{m+s}(V,W),\mu)$ is a tensor product, or

$$\otimes^{n+r}_{m+s}(V,W) = \otimes^n_m(V,W) \otimes \otimes^r_s(V,W)$$

and

$$(\alpha_n \otimes \alpha_r) \otimes_1 (\beta_m \otimes \beta_s) = (\alpha_n \otimes \beta_m) \otimes_2 (\alpha_r \otimes \beta_s) .$$

We have subscripted the product symbol $\otimes$ in this equation in order to emphasize the fact that the defining product $\otimes_1$ on the left side is between an (n+r)-tensor and an (m+s)-tensor, while the defined product $\otimes_2$ on the right is between an (n+m)-tensor and an (r+s)-tensor.

An algebra structure may be placed on $\otimes(V,W)$ by defining a multiplication operation. To this end, let $\alpha_m^n \in \otimes_m^n(V,W)$ and $\beta_s^r \in \otimes_s^r(V,W)$ so that the tensors

$$\alpha = \sum_{n,m} \alpha_m^n \quad , \quad \beta = \sum_{r,s} \beta_s^r$$

are elements of $\otimes(V,W)$. Then the product of two such tensors is given by

$$\alpha\beta = \sum_{\substack{n,m\\ r,s}} (\alpha_m^n \otimes \beta_s^r) \ ,$$

where the symbol $\otimes$ is the same as $\otimes_2$ above. Notice that the multiplication rule implies

$$\begin{aligned}(\alpha_n \otimes \beta_m)(\alpha_r \otimes \beta_s) &= (\alpha_n \otimes \beta_m) \otimes_2 (\alpha_r \otimes \beta_s)\\ &= (\alpha_n \otimes \alpha_r) \otimes_1 (\beta_m \otimes \beta_s)\\ &= (\alpha_n\alpha_r) \otimes (\beta_m\beta_s) \ .\end{aligned}$$

This relation shows that the algebra $\otimes(V,W)$ is the canonical tensor product of the subalgebras $\otimes V$ and $\otimes W$, or

$$\otimes(V,W) = (\otimes V) \otimes (\otimes W) \ .$$

Our motivation is, of course, the expansion of functions $f : X \times U \to X$, for X a real vector space of states and U a real vector space of controls.

In concluding this section, which goes into considerable detail, we remark that there is more than one way in which to develop a tensor algebra.

A portion of the difficulty in applications studies is to determine how to develop the sequence of tensor vector spaces into an algebra. Various choices on multiplying tensors may be made. The foregoing choice fits well with preceding grant studies and is suitable for use in later sections of this report.

II. PROGRESS IN MODELING THEORY*

2.1 INTRODUCTION

The study of nonlinear systems has become increasingly more active with efforts focused on overcoming the well known analytical difficulties that accompany them. A vast collection of literature exists relative to this activity, particularly noticeable in the last 15 years. It is this body of literature, then, that this section considers, focusing primarily on the topics dealing in system approximation, bilinear systems, and algebraic structures. While these areas themselves represent a large body of the literature, only those papers deemed directly relevant to the present research aims are reported on here.

By approximate systems we mean that branch of study which attempts to model complex nonlinear systems, such as

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= g(x,u), \\ x(0) &= x_o, \end{aligned} \tag{2.1.1}$$

for $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^k$, by simpler, workable forms which possess the desirable properties of stability, causality, controllability, and so on. First order linearization schemes form a subset of this class of systems and, as we will point out, the problem has been well studied. Polynomic systems, which we also consider as representing a subclass of approximate systems, are equally important and are thus reviewed here in the subsection to follow. Topics in analysis, treated thoroughly in the classic works of Dieudonne [1] and Apostle [2], are crucial in all of these studies.

*Contributed by Stephen Yurkovich. See Section 2.5 for references [A..], [B..], [C..].

Bilinear systems may be considered as a specialization of (2.1.1) when we add the assumption of linearity in the control or in the state; that is, bilinear systems are linear separately with respect to the state x and the control u, but not jointly. We characterize them by the following dynamical equation:

$$\dot{x} = Ax + Bu + \sum_{i=1}^{m} N_i u_i x ,$$
$$y = Cx \tag{2.1.2}$$

for the matrices A, B, N_i, and C of appropriate dimensions (time invariant case), where u_i is the ith component of u. In a more concise form, the system (2.1.2) is illustrated in Figure 1. There are several practical and theoretical motivations for the study of such systems, as seen, for example, in [3].

Algebraic system theory is the main vehicle toward the goals of the present research. The works of Wonham [4] and Sain [5] in the area of multivariable systems offer a necessary springboard for studies in this field. In [7,8,9,10] the motivations put forth in [6] are extended toward real modeling problems, utilizing the symmetric tensor algebra. Unfortunately, the literature is rather sparse on this topic relative to nonlinear system theory. However, the papers we will cite in the area of algebraic tensors are examples of the uses of similar ideas in the literature. The intent is not to expound the details of the theory; this may be found in such works as [11] and [12]. Similarly, the theory of Lie algebras apparently plays a pertinent role in the research. Several leading works will be cited, while a more complete exposition of the theory is given in [13] and [14].

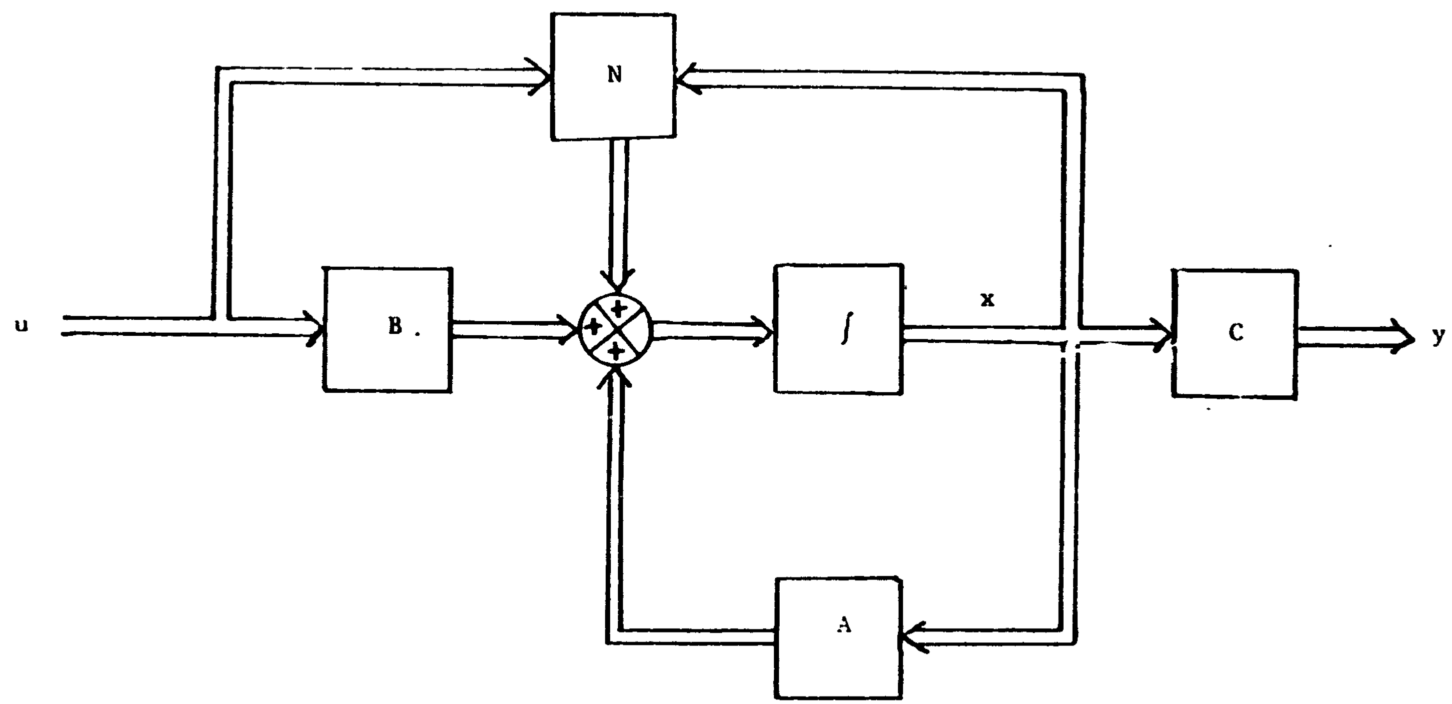

Figure 1 A Typical Bilinear System

2.2 APPROXIMATE SYSTEMS

We begin this subsection by discussing an important linearization technique which a handful of authors have utilized in recent years. It will be seen to be useful not only in linearization of nonlinear systems but in bilinearizations as well. This approach appears to have been introduced first by Carleman [A1] in 1932.

Let us initially consider the following scalar nonlinear differential equation,

$$\dot{x} = f(x) \ , \tag{2.2.1}$$

where $f(x)$ may be required to have certain analytic properties. The Carleman Linearization Process (CLP) is based on the fact that any homogeneous nonlinear differential equation (2.2.1) can be converted into a linear differential equation of infinite order by defining new variables

$$x_k = x^k \ . \tag{2.2.2}$$

By cutting off this infinite system at a finite stage a closed set of equations which model (2.2.1) may be obtained.

One of the earliest (1963) applications of this linearization approach appears in [A2], where the basic idea is to employ the CLP in rewriting nonlinear equations as an infinite sequence of coupled linear equations. This sequence is then truncated by a linear closure approximation involving a mean-square error minimization. The multidimensional case is treated in the following manner. Consider the set of nonlinear differential equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{N} a_{ij} \, x_j + \sum_{j,k=1}^{N} a_{ijk} \, x_j \, x_k \ ,$$

$$x_i(0) = c \ . \tag{2.2.3}$$

Taking $x_j x_k$ as new variables $y_{jk} = x_j x_k$, and using vector notation, (2.2.3) becomes

$$\begin{aligned} \dot{x} &= Ax + By \ , \\ \dot{y} &= A_2 y + \phi(x) \ , \end{aligned} \tag{2.2.4}$$

where x is an n-vector so that y and $\phi(x)$ are of dimension n^2, A is nxn, A_2 is $n^2 \times n^2$, and B is $2n \times n$. Similarly, we can form the $n^3 \times n^3$ system satisfied by functions $x_i x_j x_k$. While the notion of tensor products involving x or y (to form the monomial terms) is not used, it is pointed out that the A_i are the iterated Kronecker sums (denoted by $\oplus_K$) of A,

$$\begin{aligned} A_2 &= A \oplus_K A \ , \\ A_3 &= A \oplus_K A_2 \\ &= A \oplus_K A \oplus_K A \ , \end{aligned} \tag{2.2.5}$$

and so on. (A discussion of the Kronecker sum may be found in Bellman [15]; this and the Kronecker product are major topics in 2.4 of this review.) Stability of (2.2.4) is related to the characteristic values of A.

Alternate linearizations are also considered in [A2] in which any continuous function (2.2.1) (not necessarily analytic) can be expanded in an orthogonal series. For example, assuming that x varies only over $-1 \leq x \leq 1$, Legendre polynomials offer the best linear approximation in the mean square sense. A similar statement can be made for Chebychev polynomials.

A form of the CLP is used in [A7] in estimating the "domain of attraction" (stability results) for a class of nonlinear systems. A statement of the CLP is presented with a detailed proof in which an algorithm is developed for an iterative procedure for estimating the domain of attraction. The theorem involves an error bound in terms of Euclidean norms.

The work of Sira-Ramirez in [A9,A10] follows the work in [A7], particularly in [A10] where the main theorem used by Loparo, et.al., forms the basis of the paper. In [A10], the use of the CLP is proposed for feasible set (set of all possible solutions of the systems of differential equations) computation on a class of nonlinear analytic feedback systems. The error-bound result of [A7] is used, then, to approximate arbitrarily close the feasible region of a nonlinear system (whose initial state is bounded by a compact generalized polyhedron). Moreover, the higher dimensional linear system obtained from the CLP has parameters which are shown to be computable in terms of the Volterra series expansions of the nonlinear map.

Let us now introduce some notation which is generally accepted and typically attributed to Brockett (see [B2] and [C12]) for use of the CLP. Given the n-vector x with components x_i, denote by $x^{[p]}$ the $\binom{n+p-1}{p}$ dimensional vector with elements of the form

$$\alpha \prod_{i=1}^{n} x_i^{p_i} , \tag{2.2.6}$$

with $\sum_{i=1}^{n} p_i = p$, $p_i \geq 0$, and α a constant scalar. For example, we can represent typical terms as

$$x^{[0]} = 1 ,$$

$$x^{[1]} = x ,$$

$$x^{[2]} = (x_1^2, x_1x_2, \ldots, x_1x_n, x_2^2, x_2x_3, \ldots, x_2x_n, \ldots, x_n^2)' , \tag{2.2.7}$$

and so on, where $'$ denotes transposition. The elements of $x^{[p]}$ are ordered lexicographically[1], in the manner (2 2.7), which becomes important when any actual calculations are done. In [9] is given a general algorithm which accomplishes this for use on a digital computer. There it is pointed out that such objects (2.2.7) are actually elements of a tensor product between p vectors.

With this notation defined, we consider now the results of Krener in [A5] and a specific class of nonlinear control systems. We restrict ourselves here to the case of scalar u, entering linearly, yielding the differential system

$$\begin{aligned} \dot{x} &= f_o(x) + uf_1(x) \\ y &= g_o(x) + ug_1(x) \\ |u| &\leq 1, \ x(0) = 0 \ . \end{aligned} \tag{2.2.8}$$

With the assumption that f_i and g_i are as smooth as needed, (2.2.8) in general gives rise to an infinite dimensional bilinear system of the form

$$\begin{aligned} \dot{x}^{[p]} &= \sum_{i=p-1}^{\infty} A_i^p x^{[i]} + u \, B_i^p x^{[i]} \ , \\ y &= \sum_{i=0}^{\infty} C_i x^{[i]} + u \, D_i x^{[i]} \ , \end{aligned} \tag{2.2.9}$$

for matrices A_i^p, B_i^p, C_i, and D_i of appropriate dimensions. Now (2.2.9) may be truncated by setting $x^{[p]} = 0$ for $p \geq q$, and by defining a new

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[1]There has appeared in the literature at least three different words for this same connotation: lexicographically, lexigraphically, and lexographically. Interestingly enough, the third of these apparently is not an accepted word (according to Merriam-Webster) but is seen most often, probably due to its use by Brockett [B2]. We will adopt the first of these, lexicographically, an accepted term from formal language theory.

state vector as

$$x = (x^{[0]}, x^{[1]}, \ldots, x^{[q-1]})' , \tag{2.2.10}$$

the result is a finite dimensional bilinear system

$$\dot{x} = Ax + uBx$$

$$y = Cx + uDx$$

$$|u| \leq 1, \; x(0) = (1,0,\ldots,0) . \tag{2.2.11}$$

We state now the main theorem for bilinearization about a point.

<u>Theorem</u> [A5] Consider the nonlinear control system (2.2.8). For any integer $q \geq 0$ there exists a bilinear control system (2.2.11) such that for some constants M, $T > 0$ for any admissable inputs the outputs $y(t)$ and $\bar{y}(t)$ of the nonlinear and bilinear system, respectively, satisfy

$$|y(t) - \bar{y}(t)| \leq Mt^{q}$$

for all $t \in [0,T]$. Also, if x is the state of (2.2.8) and $\bar{x}$ consists of x_1 to x_n of the state of (2.2.11), then

$$|x(t) - \bar{x}(t)| \leq Mt^{q+1}$$

for all $t \in [0,T]$.

An equivalent result is proved for bilinearization about a reference trajectory.

An earlier work by Krener, [A4], studies the problem of when two control systems (where the control enters linearly) are equivalent, i.e., that there exist a local diffeomorphism which takes the solution of one system for each control into the solution of the other for the same control. Necessary and sufficient conditions are derived. As a corollary, necessary and sufficient conditions are derived for a nonlinear system to be locally diffeomorphic to a linear system. These equivalence and linearization results hinge on the theories of manifolds and Lie brackets.

A method of formal linearization is presented in [A11] in which the state of a nonlinear system is au mented with linearly independent functions (lif's) of the state variables. The result is a system, where the dynamical equation of the augmented state is expanded in a series of lif's, which is li ar in the function space spanned by the lif's. This of course amounts to a form of the CLP and, in fact, a result using Taylor's Theorem (with remainder term) is given. Moreover, a numerical example is reported on.

In [A13] Crouch offers a rigorous development in which he considers nonlinear systems described by finite Volterra series, with certain analyticity and linear-in-the-control requirements. The natural properties of Lie algebra of the system lead to the formulation of the state space as a homogeneous space of nilpotent Lie groups. This leads to showing that the state space is homeomorphic to a Cartesian space. Thus, when these systems are set in natural coordinate systems it is seen that the state space admits a natural vector space structure. A finer structure is also identified which shows that these systems are cascades of linear systems with polynomial link maps.

Further results dealing with Volterra series expansions are discussed in [A6]. There, a general methodology is developed for obtaining fundamental expansions consisting of multilinear integral operators. Validity conditions for the expansions are obtained, as are results concerning the approximation errors for appropriately defined normed spaces. Several such error bounds have been mentioned thus far, and are crucial in any approximate system results. Another method, which defines a dynamical error system, is described in [A3].

As a lead-in to the discussion of polynomic systems, we point out two papers by Porter. In [A8] polynomial operators (one example is a usual Volterra series) are used in the approximation of nonlinear systems. The classic Weierstrass result is used in which the function to be approximated need not be differentiable; rather, emphasis is placed on approximating the function by polynomials over a compact set. The Bernstein system (employing Bernstein polynomials) is one constructive realization of the Weierstrass approach. A comparison is given for this methodology versus power series expansions. In [A12], Porter utilizes a Hilbert space setting and considers two distinct problems, interpolating and approximating (for a "black box" phenomenon). The basic theorem here shows that interpolators which can be realized linearly on a vectorized space have a specific approximation property.

A rather complete overview of the theory of polynomic systems is given by Porter (1976) in [A14], containing 75 references on the topic. In this framework, a function is said to be <u>polynomic</u> if it is a finite sum of multipower maps (defined also in [A15]) and said to be <u>analytic</u> if it is an infinite sum of multipower maps with an appropriate convergence. Thus, polynomic operators are a subset of analytic operators. In [A15] it is shown that a Weierstrass-type approximation result does not hold between the finite memoryless polynomic functions and the memoryless continuous function. A symmetric multilinear operator W from H^n to H (where H is a Hilbert space) is said to generate a <u>multipower</u> function $\hat{W} : H \to H$ by the rule

$$\hat{W}(x) = W[x,x,\ldots,x] \;. \tag{2.2.12}$$

For causality studies, orthoprojectors are introduced.

The problem treated in [A18] can be constructed as an identification problem, the representation of a black box phenomenon by polynomic or multilinear models. From a collection of observed input-output pairs a polynomic operator is constructed.

Conditions are derived in [A16] which guarantee that a feedback system modeled by a "general quadratic and cubic" plant and controller will be of the type

$$\dot{x} = Ax + kBx + Du + N(x,x) + M(x,x,x)$$

$$y = Cx \ , \tag{2.2.13}$$

for vectors x, u, and y and the feedback factor $k \in \mathbb{R}$, where N is a bilinear form in x and M is a trilinear form in x. These systems do not, however, contain forms which are trilinear in x_i and u_i mixed.

The concept of span reachability is considered in [A17] where discrete polynomial state-affine systems are treated. The class of systems studied are said to be span-reachable if the set or state vectors which are reachable from the origin span the entire state space.

The use of tensor products has recently emerged in the literature relative to polynomic system theory. For example, in [A19] a multivalued switching function f is said to be realized by "polylogic" (over an index set) if there exists a polynomic function which computes f (on the domain of f). The implication is that a polynomic realization $\phi(x)$ of a given switching function $f : A^n \to A$ exists if and only if a linear realization exists,

$$f(x) = \phi(x) = \Gamma\bar{x} \tag{2.2.14}$$

for $x \in A^n$, where

$$\bar{x}_j = \otimes^j(x), \; j = 1,\ldots,n \; . \tag{2.2.15}$$

To illustrate, if the index set is $\{0,1,2,3\}(n=4)$, then

$$\bar{x} = (1,\; x,\; x \otimes x,\; x \otimes x \otimes x) \tag{2.2.16}$$

since $\bar{x}_o$ is defined to be 1. Computation of such Φ nd T in (2.2.14) is discussed.

A further example of the use of tensor products is given in [A20] where the topic is state representations of polynomic maps. Briefly, if H is a Hilbert space then for $x \in H$, the quadratic operator

$$\gamma(x) = (1,\; x,\; x \otimes x) \tag{2.2.17}$$

is defined in order to create the new Hilbert space

$$\hat{H} = \text{closed span } \{\gamma(x) : x \in H\} \tag{2.2.18}$$

with inner product induced by that of H. Moreover, $\hat{H}$ is shown to be a Hilbert resolution space. With this, causality properties of non-epic polynomic maps such as $\gamma : H \to \hat{H}$ are discussed, and the treatment of higher order polynomic operators is alluded to. These concepts are then employed for state decompositions.

2.3 BILINEAR SYSTEMS

A brief introduction to bilinear systems was given in Section 2.1 where (2.1.2) and Figure 1 served to depict such systems in mathematical and block diagram forms. The importance of bilinear systems to the present research is evident by the fact that any bilinear function may be represented in terms of the universal tensor product function. In fact, bilinear (or "2-linear") functions are merely a subset of the class of multilinear (or r-linear) functions which in turn, with appropriate operations defined, can be identified up to an isomorphism with a space of algebraic tensors. Furthermore, inner product spaces have inherent relations to such ideas since over the real numbers any inner product is a bilinear form. So the area of bilinear system theory, while in itself a large and growing field, contributes in many ways to ongoing research in multilinear (and thus nonlinear) dynamical systems.

An introduction to bilinear systems and the accompanying body of literature can be found in the survey papers [B5] and [B6] in 1974, and [B17] in 1980. Bruni, Dipillo, and Koch in [B5] (an often cited work) outline some basic definitions of bilinear systems. To summarize, let us rewrite (2.1.2) here for convenience, in a slightly different form:

$$\dot{x} = A(t)x + B(t)u + N(t)xu$$
$$y = C(t)x \qquad (2.3.1)$$

where the input u is assumed a priori to be of a specific class. The matrix $A(t)$ belongs to $\mathbb{R}^{nxm}$, $B(t)$ to $\mathbb{R}^{n}$ and $N(t)$ is a bilinear form in x and u which can be rewritten in the manner

$$N(t)\ xu = \sum_{i=1}^{p} N_i(t)\ xu_i\ , \qquad (2.3.2)$$

for $N_i(t)$ in $\mathbb{R}^{nxm}$. This definition (2.3.1) of a time varying bilinear system (time invariant if A, B, C, and N are not time dependent) can be further specialized under additional hypothesis. Bilinear systems are defined to be homogeneous in the state if $B = 0$, homogeneous in the input if $A = 0$, and strictly bilinear if $A = B = 0$. Along with these and further definitions, the authors stress the fact that there have been no effective contributions to the application of bilinear system theory to the solution of practical modeling problems . While there has been some recent contributions, the general identification problem remains unsolved today, and only a few results for the special cases seem to be available. The topics of stability and distributed parameter systems are also listed as trends for future research.

In 1974 Mohler [B6] published another such survey-type paper in which he discusses the evolution of bilinear systems, with emphasis on their application to population models, biological systems, nuclear fission processes, and socioeconomics. It is pointed out that in these various instances bilinear mathematical models arise in a natural manner, while in others they represent another degree of approximation beyond that of linear models. This paper may be overshadowed now by a more recent work (1980) by Mohler and Kolodziez [B17]. Here, feedback combinations of bilinear systems are discussed, and the following point is made. In many systems feedback combinations result in multilinear models which may be decomposed into open loop bilinear systems for certain analyses. In this manner multilinear models and bilinear systems may be used to approximate more highly nonlinear systems. An approximation theorem due to H.J. Sussman is quoted where it is stated that arbitrary functions satisfying certain causality and continuity conditions can be approximated arbitrarily close by maps which arise from bilinear systems for measurable and bounded inputs. The authors note, however, that this does not give a method for constructing the approximate bilinear system for a given nonlinear system and that the basic assumptions may in fact be too restrictive.

Possibly the most often cited paper in the bilinear system literature is [B2]. Brockett considers the algebraic structure of bilinear systems and sketches the general procedures for constructing a theory parallel to that in linear systems for parallel and series interconnections, canonical forms, controllability, observability, and equivalent realizations. The starting point uses the fact that (see also [C11]) any input-output map which can be realized by

$$\dot{x}(t) = [A + \sum_{i=1}^{m} u_i(t)\, B_i]\, x(t) + [\sum_{i=1}^{m} u_i(t)\, b_i]$$

$$y(t) = Cx(t)\ , \tag{2.3.3}$$

for appropriate matrices A, B_i and C and vectors b_i, can be realized by

$$\dot{z}(t) = [F + \sum_{i=1}^{m} u_i(t)\, G_i]\, z(t)$$

$$y(t) = H\, z(t)\ . \tag{2.3.4}$$

A more involved result says that any input-output map realized by

$$\dot{x}(t) = [A + \sum_{i=1}^{m} u_i(t)\, B_i]\, x(t)$$

$$y(t) = \sum_{p=1}^{q} \psi_p(x(t), x(t), \ldots, x(t)), \tag{2.3.5}$$

where ψ_p is a p-linear map in $x(t)$, can also be realized by the form

(2.3.4). One such construction uses a form of the Carleman linearization process. These results rest on the fact that if x satisfies a homogeneous in the state bilinear system, then so does $x^{[m]}$ (the lexicographically ordered vector defined in the preceding section). That is, if z is given by $x^{[2]}$, then there exist matrices $A^{[2]}$ and $B_i^{[2]}$ such that

$$\dot{z}(t) = (A^{[2]} + \sum_{i=1}^{m} u_i(t)\, B_i^{[2]})\, z(t) \;. \tag{2.3.6}$$

At this point Brockett alludes to the use of Kronecker product relationships (for iterative construction of the $A^{[i]}$ and $B^{[i]}$) and the theory of symmetric tensors, citing reference [13]; again, however, tensors are not used in the development.

Several other general points of interest are made in [B2]. With respect to interconnections, if the parallel connection of two bilinear realizations is defined, the resulting system will have a bilinear realization. The same is not true for series connections; that is, bilinear systems are not closed under series connections. However, if the series connection of a system having a bilinear realization followed by a system having a linear realization is defined, then the resulting system has a bilinear realization. As a final point, Brockett notes that in classifying systems and in determining equivalent realizations, the results available in the study of Lie algebras are of fundamental importance.

A global bilinearization result is given by Lo in [B12], summarized in the following. Consider the nonlinear differential system

$$\begin{aligned} \dot{x}(t) &= f(x) + [G(x)]\, u \\ z(t) &= h(x) + [Q(x)]\, v \;, \end{aligned} \tag{2.3.7}$$

for $x \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, $u \in \mathbb{R}^m$, and $v \in \mathbb{R}^p$. The nonlinear system (2.3.7) is dynamically equivalent to the bilinear system

$$\dot{y} = (A + \sum_{i=1}^{m} B_i u_i)\, y(t)$$

$$z(t) = (C + \sum_{i=1}^{p} D_i v_i)\, y(t) \tag{2.3.8}$$

for some $M_i > 0$, $i = 0,1,\ldots,p$, such that

$$\begin{aligned} \text{rank}\ [C', A'C', \ldots, (A')^{M_0-1} C', D_1', A'D_1', \ldots, (A')^{M_1-1} D_1', \\ \ldots D_p', A'D_p', \ldots, (A')^{M_p-1} D_p'] \\ = \dim A \end{aligned} \tag{2.3.9}$$

if and only if (2.3.7) has a finite-dimensional sensor orbit. Briefly, if $L(g(x)) = g_x(x)f(x)$ where $g_x(x)$ is the gradient of g, and if $h \in C^\infty$, the set of functions

$$\{h(x), L(h(x)), L^2(h(x)), \ldots\}$$
$$\cup [\cup_{i=1}^{p} \{Q_i(x), LQ_i(x), L^2Q_i(x), \ldots\}] , \tag{2.3.10}$$

where Q_i denotes the i-th column of Q, is called the <u>sensor orbit</u> of (2.3.7) at time t for any input. In a final note, Lo points out that a method of constructing (2.3.8), attributed to Brockett [B2], is achieved by letting $y = (x^{[1]}, x^{[2]}, \ldots, x^{[\max(k,p)]})'$.

Stochastic bilinear systems are treated in [B7] in which systems with multiplicative noise processes (thus, bilinear) are considered. Brockett's "moment equations" ([C13]) are used to compute the expected value of $x^{[p]}$ for zero mean white noise Gaussian processes. The condition that $E\{x^{[p]}\}$ have a closed form solution is that the Lie algebra be solvable (see [14]). If the Lie algebra is not solvable, an approximation method is used by truncation of cumulants (coefficients of the Taylor Series expansion of the logarithm of the characteristic function).

In [B16] the optimal control of a class of single-input discrete bilinear systems is considered. Through dynamic programming solutions are obtained for the deterministic and stochastic problem, where the performance index is the usual quadratic cost function in discrete time.

Controllability of bilinear systems has been treated by many authors during the 1970's. One of the original works (1968) on the subject is that of Rink and Mohler [B1]. There, two sufficient conditions are given for a bilinear system such as (2.3.1) to be completely controllable. Several examples are given, and in an appendix the set of equilibrium points for bilinear systems is described. This work is extended in [B13] where the solution of the parameterized equation

$$\dot{x} = [A(t) + \sum_{i=1}^{m} B_i(t)\, v_i]\, x(t) + N(t)\, u(t) \;, \qquad (2.3.11)$$

with $x(t_o) = x_o$ and v an element of a Banach space of continuous $\mathbb{R}^m$-valued functions on a finite interval, is given by

$$x(t) = \Phi(t, t_o; v)\, x_o + \int_{t_o}^{t} \Phi(t, s; v)\, N(s)\, u(s)\, ds \;. \qquad (2.3.12)$$

Here, $\Phi(t, t_o; v)$ is the state transition matrix associated with the matrix which premultiplies $x(t)$ in (2.3.11). From this, then, a nonnegative symmetric controllability matrix is defined which is used to obtain global and totally controllable results by bounding $\Phi(t, t_o; v)$.

In [B3] is given a description of the "least linear subspace" that contains all the states of the system (2.3.1) (for scalar u) reachable from the origin. For this purpose a canonical decomposition of the state space into a direct sum of four subspaces is considered. Sufficient conditions for the reachable set of a bilinear system at a fixed time to be convex are given in [B9]. Under the hypothesis for reachability (convexity), the minimum time control for transferring x_o to any other reachable point is discussed under the guise of bang-bang control. A rigorous treatment of reachability (and observability) concepts is found in [B14]. This paper shows that any two "quasi-reachable" and observable realizations of bilinear systems are isomorphic. This leads to the construction of canonical forms utilizing the Kronecker product of matrices.

As mentioned previously, Lie algebras play a vital role in bilinear system theory, particularly in the study of controllability. In fact, in the opinion of Elliot [B8] in 1974, the most important criterion for controllability and accessability of a homogeneous (in the state) bilinear system is the transitivity of the associated Lie algebra. We briefly state such a controllability (necessary) condition concerning the bilinear system

$$\dot{x} = (A + uB)\,x\ . \tag{2.3.13}$$

Let L be the smallest real linear subspace of matrices A and B closed under the Lie product, and let $\{c_1,\ldots,c_m\}$ be a basis for this Lie algebra L. If system (2.3.13) is controllable then L is transitive (see below also), that is,

$$\operatorname{rank}\,(c_1x,\ldots,c_mx) = n \tag{2.3.14}$$

for all $x \in \mathbb{R}^n_o = \mathbb{R}^n - \{0\}$. Since the origin is an isolated equilibrium point for a bilinear system such as (2.3.13), $\mathbb{R}^n_o$ is the usual state space considered.

The work of Cheng, Tarn, and Elliot [B10] offers a brief survey of works concerning controllability of bilinear systems. Moreover, a discussion of Lie algebras and Lie subalgebras is given. A definition of transitivity is given, stated in the following. We say that a set M of matrices is transitive on $\mathbb{R}^n_o$ if for every $x, y \in \mathbb{R}^n_o$ there exists an X in M such that $Xx = y$. In this paper both discrete and continuous time controllability is discussed.

Observability is considered in [B18] for homogeneous in the state bilinear systems. It is noted that an observable (in the usual sense) bilinear system may be unobservable for some inputs. The primary concern, then, is the design of inputs u^* which are as close to the given input u (in the L_2 sense) as required so that the bilinear system will be observable relative to u^*; an algorithm for choosing such inputs is developed. Other methods for achieving this are discussed, such as optimization of observability matrix eigenvalues. Here, however, appropriate inputs are achieved by slightly perturbing given inputs.

Identification of bilinear systems is discussed in [B4]. A deterministic approach using Newton's method is employed, then statistical hypotheses are allowed and Maximum Likelihood Estimation is carried out forming a differential bilinear model. Similar aims are pursued in [B11] where Isidori and Ruberti consider time varying bilinear systems such as (2.3.1) in finding internal descriptions. The state transition matrix associated with $\dot{x} = A(t)x$ is used to express the response in terms of Volterra kernels. This leads to necessary and sufficient conditions for realizability by a finite dimensional bilinear internal description. This paper follows closely along the lines of [B22]. In [B15] is given necessary and sufficient conditions for the existence of a nonsingular matrix with real entries which transforms the given multi-input, multi-output bilinear system into a triangular canonical form, which amounts to a coordinate transformation within the state space. Conditions on the internal description are outlined, and on the external description (external data) conditions are specified via a realizable formal power series.

The review of the literature for bilinear systems up to this point suggests an adequate foundation on which the study of realization theory can be undertaken. A detailed discussion of this broad topic and numerous theories involved is of course beyond the scope of this review. However, because of its overall importance, several papers on the subject of bilinear system realization are listed and will be briefly discussed.

An early (1969) work by Arbib [B19], following the work of Kalman, obtains a decomposition for multilinear discrete-time constant systems in terms of linear subsystems and multipliers. For instance, it is shown that a bilinear system may be characterized by two layers of linear systems. While most of the paper concerns automaton minimization, an appendix includes a summary of the theory and use of the tensor product to achieve some of the results for bilinear systems concerning the construction of canonical forms. This decomposition idea is further developed in [B20] where explicit conditions for minimal realizations of time-varying multilinear maps are obtained. The Nerode realization theory is applied with algebraic concepts such as quotient spaces. Again, as suggested in [B19], the bilinear map $\otimes$ is used in the canonical factorizations Another work concerning multilinear maps and their realizations is [B29] which further extends these ideas, studying also observability and quasi-reachability of the multilinear systems.

Similar are discussed in [B27] by the same author, with Denham, for bilinear systems.

In [B26] a technique for bilinear system identification is developed which uses a finite orthonormal expansion to approximate input-output functions. The basis of the expansion is Walsh functions which form a complete orthonormal set. Two useful properties of Walsh functions are that with the proper multiplication defined they form a commutative group and that the integral of a Walsh function can be represented in terms of Walsh functions. The technique is illustrated with four computational examples.

Tarn and Nonoyama [B24] obtain algorithms for the construction of discrete-time internally bilinear state space models. The notions of the tensor product and the less known affine tensor product are used to describe such systems.

Minimal realizations are studied in [B21] by introducing a "generalized Hankel matrix", analogeous to linear system theory, formed from input-output map parameters. In [B22] the realization theory for bilinear systems is developed in terms of Volterra series expansions of the zero state response, while [B25] uses functional series expansions, building on previously cited works.

Based on the theorem for global bilinearization given by Lo (as discussed in [B12]) in [B28] is developed an approximation theorem of linear-in-the-control bilinear systems. Use of Taylor's Theorem is discussed and construction for the bilinear approximations is given in the proof of the theorem. In [B23] Krener developes a result similar to that of [A5], where again it is shown that every nonlinear realization can be approximated by a bilinear realization with an error that grows like an arbitrary power of t. In [B23], however, Lie algebraic concepts are employed, making the development somewhat more rigorous than that of [A5].

2.4 ALGEBRAIC STRUCTURES

Several references have already been cited for a general introduction to the algebraic topics with which the current research is concerned. Since the work of [7-10] hinges primarily on the usage of algebraic tensors and spaces of multilinear functions, our main emphasis here is on that of the tensor algebra. While [11] and [12] offer a formal treatment for the necessary background, we mention several works in the literature to add to these sources.

We begin with two tutorial-type introductory papers on these topics. In [C2] the theory of multilinear forms is reviewed and the main discussion centers on the notion of the linear operator contraction. A technical difference between the contraction of tensors (which exists independent of its expression in particular bases) and the contraction of multilinear forms (which in general is basis dependent) is outlined. Beginning with vector spaces and their duals, several types of contractions are discussed and are shown to coincide with the "usual" engineering definition of contraction. The second tutorial paper, [C3], discusses the properties of bilinear forms (or "second-order tensors"). For V an F-vector space (F a field), the set of all bilinear forms on $V \times V$ can be constituted as a linear vector space itself with dimension equal to $(\dim V)^2$ by the definitions

$$(a_1 + a_2)(\cdot) = a_1(\cdot) + a_2(\cdot)$$
$$(ka_1)(\cdot) = ka_1(\cdot), \tag{2.4.1}$$

where a_1 and a_2 are arbitrary bilinear forms on $V \times V$, $k \in F$. Many other elementary topics are introduced and extended, including a discussion of the inner product as a real positive definite symmetric bilinear form $a : V \times V \to \mathbb{R}$.

We have already witnessed various uses of the Kronecker product and of Kronecker sums of matrices (or, abstractly, linear transformations) in the literature. The Kronecker product is of course itself a tensor product. Bellman ([15], Chapter 12) has supplied a solid foundation for the properties of the Kronecker product, and has shown [A2] how they may be employed in the Carleman linearization process. The utility of the concept for computational aspects has also been explored for use in such topics as solution of linear equations and algorithms for Fast Fourier transforms[2]. Because of its versatility, then, we will discuss some works concerning the usefulness of the Kronecker product which relate to the topics explored thus far.

Brewer [C6][3] gives a general overview of the algebra related to the Kronecker product, surveying the literature and quoting many useful theorems, definitions and properties. Furthermore, the calculus of matrix valued functions is reviewed. The main emphasis of the paper is the development of a parameter identification method, based on Newton Raphson Iteration, for linear time invariant systems using the matrix calculus and the Kronecker algebra. In an earlier work (1973) of Barnett [C1] matrix calculus ideas are explored and a solution to the matrix differential equation

$$y^{(r)} + a_1 K y^{(r-1)} + \dots + a_r K^r y = 0 \qquad (2.4.2)$$

is developed. The interesting point here is that y is a vector given by stacking the rows of an $m \times n$ matrix $X = [x_{ij}]$, denoted by

$$v_r(X) = [x_{11}, x_{12}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{mn}]' \qquad (2.4.3)$$

[2] We mention this application for the interested reader, particularly in reference to a paper by H. Sloate in IEEE Trans. Cir. Sys., Jan. 1974, p. 109.

[3] Brewer adds corrections to this paper the following year, IEEE Trans. Cir. Sys., May 1979, p. 360.

and $K = A \oplus_K B = A \otimes I_n + I_m \otimes B'$ is the Kronecker sum of some matrices A and B. Results involving v_r are given, such as

$$v_r(CXD') = (C \otimes D)\, v_r(X), \tag{2.4.4}$$

for the $p \times m$ matrix C and the $q \times n$ matrix D. The Kronecker product does not in general commute, but it is shown that, for C and D as defined above,

$$D \otimes C = P(C \otimes D)Q \tag{2.4.5}$$

where P (depends only on p and q) and Q (depends only on m and n) are permutation matrices.

Similar results have been derived by Kuo [C4], where it is shown that the nonhomogeneous product system

$$(A_1 \otimes A_2)y = b_1 \otimes b_2 \tag{2.4.6}$$

is solvable if and only if

$$A_1x_1 = b_1 \text{ and } A_2x_2 = b_2 \;, \tag{2.4.7}$$

for some x_1 and x_2. And, in fact, if y is a solution to (2.4.6), then $y = x_1 \otimes x_2$. This result is used in accordance with the column stacking operation (analogous to that for rows in (2.4.3)) to develope tensor factor equations (2.4.7) for a system such as (2.4.4). In [C7] the notion of the "extended" Kronecker product and its accompanying properties is given, denoted by

$$A \odot B = (A_1 \otimes B_1 | \ldots | A_r \otimes B_r) \;, \tag{2.4.8}$$

where $A = (A_1 | \ldots | A_r)$ and $B = (B_1 | \ldots | B_r)$ are two partitioned real matrices (in [C6] a similar notion is described and called the "Khatri-Rao" product). Results similar to those found in [C4] are given involving the product (2.4.8). The use of tensor products in linear programming is also discussed.

Another use of these ideas is given in [C10] where state transition matrices are utilized. Consider the system

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2(t) \quad . \tag{2.4.9}$$

This can be rewritten in the form (2.4.2),

$$\frac{d}{dt}[\dot{v}_r(X)] = A(t)v_r(X) \quad , \tag{2.4.10}$$

where $A(t) = A_1 \oplus_K A_2$, with state transition matrix $\Phi_1(t,t_o) \otimes \Phi_2(t,t_o)$ for Φ_i the state transition matrix associated with A_i. It should be noted that in a follow-up comment[4] on this paper, Barnett presents an alternate derivation for this result.

As an introduction to a segment of the literature involving computational aspects of tensor products, we cite [C9]. This numerical work describes the tensor factorization algorithm for tensor spline[5] approximation and how it applies to least-squares fitting. Also used is the singular value decomposition and matrix condition number. Here, "tensor" again refers to the Kronecker product of matrices.

A technique for identification of nonlinear systems using tensor ideas is developed in [C5]. Systems which admit a finite Volterra series representation are considered, where each multidimensional system transform is a product of single variable transforms. It is shown that this type of system

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[4] IEEE Trans. Autom. Contr., April 1981, p. 603.

[5] A spline function is one which approximates a, say, continuous and differentiable function on an interval in a piecewise fashion using low degree interpolating polynomials.

can be modeled, from an input-output point of view, as a cascade of a linear system, a homogeneous nonlinearity, and another linear system, as shown in Figure 2.

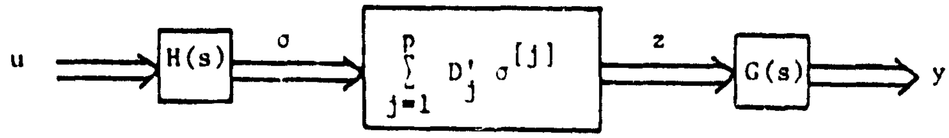

Figure 2. Input-Output Model.

A (minimal) realization of the system in Figure 2 is given by

$$\dot{x} = Ax + Bu,\ \sigma = Cx$$
$$z = \sum_{j=1}^{p} D'_j \sigma^{[j]}$$
$$\dot{v} = Lv + Mz,\ y = Nv. \tag{2.4.11}$$

Steady state sinusoidal analysis is used in the identification. As an alternative to an association of variables method, techniques are used to identify the system transforms leading to a response analysis (where the input consists of a finite sum of sinusoids and/or exponentials) based on tensored transfer functions. In short, for $H_a(s) = H(s)u(s)$, we have

$$y(s) = [\sum_{j=1}^{p} D'_j H_a^{[j]}(s)]\ G(s)\ , \tag{2.4.12}$$

so that the key is to compute the tensored transfer function $H_a^{[k]}(s)$, given $H_a(s)$. It is important to note that $F^{[k]}(s) \neq (F(s))^{[k]}$ for a transfer function $F(s)$. The authors point out the fact that because of the recursive nature of the approach, the practical question of error propagation is under investigation.

In [C8] Buric treats the problem of optimal state feedback regulation of polynomial nonlinear systems[6]. Tensor algebraic operators are the main vehicle towards this end, and the symmetric tensor algebra forms the foundation for the development. Both time-varying and time invariant systems are treated over finite and infinite regulation intervals.

The final body of literature to be considered in this section---Geometry and Lie aglebra---represents rich mathematical notions which contribute to a wealth of useful concepts. A thorough understanding of the ideas developed in these papers would contribute immensely to the understanding of all previous citations in this review and their importance cannot be overemphasized. Due to their complex and rigorous nature, we shall move quickly through most of the discussion and outline only the general concepts encountered.

We begin by centering our attention on the work of Brockett in these areas, citing seven papers from 1972 through 1976. Five of these seven form a foundation on which much of the literature builds; two others, [C15] and [C17], represent significant contributions in general nonlinear and linear systems theory, respectively. In [C11] the system

$$\dot{X}(t) = (A + \sum_{i=1}^{m} u_i(t)B_i)\, X(t) \qquad (2.4.13)$$

$$y(t) = \psi X(t)$$

is studied, under the hypothesis that X belongs to a matrix group Ω and where A and B_i belong to the Lie aglebra associated with Ω. The notation $\psi X(t)$ is to be interpreted as being a coset in Ω for the matrix

[6]This paper summarizes the authors Ph.D thesis at the University of Minnesota, 1978.

group ψ. The primary interest in the class of systems (2.4.13) is controllability in so far as it contributes to a framework for studying other system theoretic questions such as observability and realization theory. In this way, the objective is to reduce all questions about the system to questions about Lie algebras and group manifolds.

The study of Lie aglebras in control theory was motivated mainly by the confrontation of some physical problems which proved linear system theory to be inadequate, and by work on Lie algebraic methods in differential equations. This latter topic is treated in [C12] where the expression

$$\frac{\partial f}{\partial x}g(x) - \frac{\partial g}{\partial x}f(x) \tag{2.4.14}$$

arises naturally for smooth functions f and g from $\mathbb{R}^n$ to $\mathbb{R}^n$. The quantity (2.4.14), usually written as $[f,g]$, is called the Lie Bracket of f and g. An extension of the Carleman linearization process is described, summarized in the following. If $N = \binom{n+p-1}{p}$ and $x \in \mathbb{R}^n$, then, as discussed previously, associated with each map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a sequence of maps, the p-th one mapping $\mathbb{R}^N$ into $\mathbb{R}^N$. A convenient basis choice contains elements (2.2.6), or

$$\sqrt{\binom{p}{p_1}\binom{p-p_1}{p_2} \cdots \binom{p-p_1-\cdots-p_{p-1}}{p_p}}\; x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}\,, \tag{2.4.15}$$

with $\sum_{i=1}^{n} p_i = p,\ p_i \geq 0$. The constants multiplying the monomials in (2.4.15) are chosen such that

$$||x^{[p]}|| = ||x||^p \tag{2.4.16}$$

for $||x|| = (\langle x,x\rangle)^{1/2}$, where $\langle\cdot,\cdot\rangle$ is the standard inner product. More generally,

$$<x,y>^p = <x^{[p]},y^{[p]}> . \qquad (2.4.17)$$

Finally, denote by $A^{[p]}$ the map (matrix) which satisfies

$$y = Ax \Rightarrow y^{[p]} = A^{[p]}x^{[p]} . \qquad (2.4.18)$$

Another construction, $A^{(p)}$, defined exclusively in terms of matrices, is the compound of the matrix A (see also [16]), and a theory analogous to that for $A^{[p]}$ is outlined. These two constructions are shown to be specializations of the tensor product. Many other topics are treated in [C12], including controllability and observability, optimal control, stochastic differential equations, and stability theory. These ideas are expanded upon in [C13] where Brockett constructs a theory for control problems defined on spheres in which results from Lie theory again play a natural role. Results analogous to those for linear systems are developed for systems of the type

$$\dot{x}(t) = (A + \sum_{i=1}^{m} u_i(t)B_i)\, x(t)$$

$$y = Cx(t) , \qquad (2.4.19)$$

where A and B_i are skew symmetric matrices and (2.4.19) can be thought of as evolving on the sphere $||x(t)|| = ||x(0)||$.

Differential geometric methods are used in the treatment of singular optimal control problems in [C16]. Volterra series expansions and function space Taylor series expansions are the main tool in the studies, as the $x^{[p]}$ notation is utilized for the expansion of the kernels. Here, however, an expression such as $u^{[i]}(\sigma_1,\ldots,\sigma_i)$ is represented as a tensor product $u(\sigma_1) \otimes \ldots \otimes u(\sigma_i)$. Volterra series and geometric control theory are expounded in the often-cited [C18]. Again the Volterra kernals are computed in terms of the power series expansions of the functions defining the controlled differential equation. Some applications are considered, including singular control and multilinear realization theory.

In [C15] Brockett surveys some of the main results available then (1976) on the use of differential geometry in nonlinear system theory. To this end, background information on manifold theory is supplied in the form of an appendix. Some geometric aspects of linear system theory are studied in [C17], where single input-single output systems are considered (in the frequency domain).

The duality between controllability and observability for nonlinear systems is investigated in [C19]. Instead of constructing a "dual" system (as might be done in linear system study but is a much harder problem for the nonlinear counterpart) the duality between "vector fields" and "differential forms" on manifolds is exploited, along with the use of Lie algebraic concepts. In [C23] the topic of nilpotent Lie algebras is considered for the derivation of an optimal bilinear filter.

We cite two papers by Baillieul in which optimal control is dicusssed. In [C14] classical optimization techniques (the calculus of variations) are used in the context of Lie groups. Multilinear optimal control is treated in [C20]. There, the nonlinear differential equation

$$\dot{x} = A(t)x^{[p]}, \; x(0) = x_o , \tag{2.4.20}$$

where $x^{[p]}$ has elements as in (2.4.15), is solved by a particular series of successive approximations involving the terms $x^{[p]}$. The condition for solution is the convergence of such a series. Using operator norms and the differential equation for the "k-fold Kronecker product",

$$\frac{d}{dt}(x \otimes \dots \otimes x) = \sum_{i=1}^{k} x \otimes \dots \otimes A(t)x^{[p]} \otimes \dots \otimes x , \qquad (2.4.21)$$

where the term $A(t)\, x^{[p]}$ is in the i-th position, the condition for the uniform convergence of the series is derived.

Another paper by Baillieul [C24] offers methods alternative to the usual Lie theoretic approach to the study of nonlinear systems. These methods are based on topics of algebraic geometry and manifold theory, the knowledge of which is assumed of the reader. Systems of equations of the form

$$\dot{x} = Ax^{[p]} + \sum_{i=1}^{n} u_i B_i x^{[q_i]} ; \; x(0) = x_o \qquad (2.4.22)$$

are treated, as are systems of multilinear differential equations such as (2.4.20).

Necessary and sufficient conditions for the invertibility of a class of nonlinear systems are derived in [C21]. Included in this class are matrix bilinear systems for which Lie algebraic invertibility criteria are obtained. In [C22] an abstract realization theory for finite dimensional discrete time internally biaffine systems is presented. The affine tensor product is introduced in terms of the ordinary tensor product, then used in describing biaffine systems.

2.5 BIBLIOGRAPHICAL APPENDIX

This appendix contains an organized list of the references cited in the literature review section. There are three major divisions, each with two subdivisions. The papers are listed in chronological order within each subdivision, numbered consecutively within each major division. So, for example, citation [B19] refers to reference 19 under group B, Bilinear Systems.

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III. MOTIVATION: SCHEDULING

3.1 INTRODUCTION

In the applications, one common way to design a control system for a nonlinear plant is to localize its behavior along lines of operation specified by the plant manufacturer, to develop linear multivariable controls for these localizations, and to schedule those controls with key plant variables which vary smoothly along operating lines. An important part of practical design lore, the art of controller scheduling has received little modern attention from the conceptual point of view.

Bristol [17,18] has likened the process of control design to the use of idioms in a language. At least three types of idioms can be identified. First, there are idioms which have been with mankind for such a length of time that they seem universal to the human psyche. In some sense, feedback itself is an example of such an idiom, inasmuch as it may be traced at least back to ancient Arabian water clocks. Second, there are idioms which are the characteristic of certain authors. Several classic examples are the Nichols chart, the Bode plot, the Evans loci, and the Nyquist plot. And third, there are idioms which are typical of certain types of control applications. An example is that of gas turbine control systems [19].

Because of the idioms of type three, any application of control design has idiomatic features. In a sense, the task of the control designer is to blend the idioms of the application with universal idioms, with idioms of classical and modern authors, and with his or her own idioms, so as to produce a melodious and effective composition.

It goes without saying that some idioms do not play well together. In some areas of application, this may account for the famous theory/application gap.

One universal idiom is to attack the overall system design by breaking it down into manageable pieces. An important case of this type of thinking arises in the design of certain classes of nonlinear systems. Examples in point may be found in the area of gas turbine control. In brief, the nonlinear engine is linearized locally along lines of operation agreed upon by the manufacturer and the control contractor. These linear multivariable localizations are used to develop a family of local controllers, which are then sewn together by scheduling control gains and dynamics with some engine variable, as for example speed, which varies smoothly along operating lines.

As pointed out by Bristol [17], the idioms have to blend together. In the case of scheduling, the methods used for design of the local, linear multivariable controllers have to be amenable to a common thread of smooth scheduling, else a global whole is not obtained, but only a sum of parts.

The goal of this section is to examine in an introductory way certain of the conceptual questions associated with scheduling. What follows should be regarded as exploratory in nature.

3.2 SIMPLE EXAMPLE

Consider the elementary dynamical system

$$\dot{x} = -ax + bu \ . \tag{3.2.1}$$

The transfer function associated with (3.2.1) is of course

$$\frac{b}{s+a} \ . \tag{3.2.2}$$

Rewritten in terms of gain and time constant, (3.2.2) becomes

$$\frac{k}{\tau s+1} \ , \tag{3.2.3}$$

where

$$k = b/a \quad , \quad \tau = 1/a \ . \tag{3.2.4}$$

Suppose that we wanted to schedule the gain k as a function of the input u, say

$$k(u) = \alpha_1 + \beta_1 u + \gamma_1 u^2 \ . \tag{3.2.5}$$

Then the scheduled system would look like

$$\dot{x} = -ax + a\alpha_1 u + a\beta_1 u^2 + a\gamma_1 u^3 \ . \tag{3.2.6}$$

Alternatively, we might schedule the time constant τ as such a function, for example

$$\tau(u) = \alpha_2 + \beta_2 u + \gamma_2 u^2 \ , \tag{3.2.7}$$

in which case we would have

$$\begin{aligned} a &= 1/(\alpha_2 + \beta_2 u + \gamma_2 u^2) \\ &= \alpha_2^{-1} - \beta_2 \alpha_2^{-2} u + \ldots \end{aligned} \tag{3.2.8}$$

so that

$$\dot{x} = -\alpha_2^{-1} x + \beta_2 \alpha_2^{-2} ux + bu + \ldots \tag{3.2.9}$$

on out to a denumerably infinite number of terms. Next suppose that we wanted to schedule the gain k or the time constant τ as a function not of u but of x, in the manner

$$k(x) = \alpha_3 + \beta_3 x + \gamma_3 x^2 , \qquad (3.2.10)$$

$$\tau(x) = \alpha_4 + \beta_4 x + \gamma_4 x^2 . \qquad (3.2.11)$$

Then the scheduled systems would be

$$\dot{x} = -ax + a\alpha_3 u + a\beta_3 xu + a\gamma_3 x^2 u , \qquad (3.2.12)$$

$$\dot{x} = -\alpha_4^{-1} x + \beta_4 \alpha_4^{-2} x^2 + bu + \dots , \qquad (3.2.13)$$

again with a denumerably infinite number of terms.

Generally speaking, the polynomic scheduling concept tends to convert the system (3.2.1) into a system of the form

$$\dot{x} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} r_{ij} x^i u^j . \qquad (3.2.14)$$

Indeed, if the original system were of the more general form

$$\dot{x} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_{km} x^k u^m , \qquad (3.2.15)$$

and if the parameters were scheduled in an analogous way, such as

$$a_{km} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{kmpq} x^p u^q , \qquad (3.2.16)$$

then (3.2.15) becomes

$$\dot{x} = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{kmpq} x^{k+p} u^{m+q} , \qquad (3.2.17)$$

which can be formally rearranged in the same form as (3.2.15). In broad terms, then, (3.2.15) is closed under formal power series scheduling.

Because of this closure feature, we find interest in systems of this type.

3.3 ABSTRACT SERIES

As intimated in the section preceding, the formal series is a natural candidate in studies of scheduling. Various approaches can be made to the description of such series. Based upon the background of Sections 1.2 and 1.3, we wish to indicate briefly here the viewpoint toward which we are tending at the time of this report.

Consider a nonlinear state description of the general form

$$\dot{x} = f(x,u)$$

for

$$f : X \times U \to X$$

with X and U real vector spaces, equipped with norm. Let $(\bar{x},\bar{u})$ be a point in $X \times U$, and suppose that

$$D^r f : Z \to L(X \times U,\ldots,X \times U,X)$$

is available for $r = 0,1,2,\ldots,$ with Z open in $X \times U$ and $(\bar{x},\bar{u})$ in Z. Then, formally,

$$f(\bar{x} + \Delta x,\ \bar{u} + \Delta u) = \sum_{k=0}^{\infty} \frac{1}{k!} (D^k f)(\bar{x},\bar{u})(\Delta x,\Delta u)^{(k)} ,$$

where $(\Delta x,\Delta u)^{(k)} = ((\Delta x,\Delta u),(\Delta x,\Delta u),\ldots,(\Delta x,\Delta u))$, the right member having $(\Delta x,\Delta u)$ k times. It should be recognized that this series could be replaced by a finite number of terms together with a remainder. However, the above representation is adequate for brief illustrative purposes. Space does not permit a discussion of whether, or how, the series acceptably describes the function. Along the same lines, we pass over the related question of how it affects the vector field associated with the differential equation, and therefore its solutions. Instead, we remind the reader that $(D^k f)(\bar{x},\bar{u})$ is a k-linear function on $(X \times U)^k$ to X; and this suggests that we can use tensor algebra to parameterize it. Indeed, denote by $(\Delta x,\Delta u)^k$ this k-fold tensor product of $(\Delta x,\Delta u)$ with itself. Then the k-linear function $(D^k f)(\bar{x},\bar{u})$ can be factored uniquely in the manner

$$L_k(\bar{x},\bar{u}) \circ \otimes^k ,$$

where

$$(\otimes^k(X \times U), \otimes^k)$$

is a tensor product for k copies of $X \times U$, or what is sometimes called a kth tensorial power for $X \times U$. In this case, the kth parameter map operates in the manner

$$L_k(\bar{x},\bar{u}) : \otimes^k(X \times U) \to X .$$

We have, therefore, that

$$f(\bar{x} + \Delta x,\ \bar{u} + \Delta u) = \sum_{k=0}^{\infty} \frac{1}{k!} L_k(\bar{x},\bar{u}) \circ \otimes^k(\Delta x,\Delta u)^{(k)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} L_k(\bar{x},\bar{u})(\Delta x,\Delta u)^k .$$

Next consider the rearrangement of a term of type

$$L_k(\bar{x},\bar{u})(\Delta x,\Delta u)^k .$$

Consider, for example, the case $k = 2$, namely

$$(\Delta x,\Delta u)^2 = (\Delta x,\Delta u) \otimes (\Delta x,\Delta u) .$$

Such a form does not relate directly to the structure of the section preceding, which would involve terms of type $(\Delta x)^j \otimes (\Delta u)^m$. However, there is a natural way to convert to that structure. Define projections

$$\pi_u : X \times U \to U \ ; \ \pi_x : X \times U \to X ;$$

and injections

$$i_{uu} : U \otimes U \to S \quad ; \quad i_{ux} : U \otimes X \to S \; ;$$
$$i_{xu} : X \otimes U \to S \quad ; \quad i_{xx} : X \otimes X \to S \; ;$$

for

$$S = (U \otimes U) \times (U \otimes X) \times (X \otimes U) \times (X \otimes X) \; .$$

Then we can write

$$\begin{aligned}(\Delta x,\Delta u) \otimes (\Delta x,\Delta u) = \; & i_{xx}(\pi_x(\Delta x,\Delta u) \otimes \pi_x(\Delta x,\Delta u)) \\ & + i_{xu}(\pi_x(\Delta x,\Delta u) \otimes \pi_u(\Delta x,\Delta u)) \\ & + i_{ux}(\pi_u(\Delta x,\Delta u) \otimes \pi_x(\Delta x,\Delta u)) \\ & + i_{uu}(\pi_u(\Delta x,\Delta u) \otimes \pi_u(\Delta x,\Delta u)) \; .\end{aligned}$$

If we identify images of the injections with their domains, as for example

$$i_{uu}(U \otimes U) = U \otimes U \; ,$$

then we can write

$$(\Delta x,\Delta u) \otimes (\Delta x,\Delta u) = \Delta x \otimes \Delta x + \Delta x \otimes \Delta u + \Delta u \otimes \Delta x + \Delta u \otimes \Delta u \; .$$

According to the conventions of $\otimes(X,U)$, however, discussed in the section preceding, we agree to write

$$\Delta u \otimes \Delta x = T_{ux,xu} \; \Delta x \otimes \Delta u$$

for an appropriate isomorphism $T_{ux,xu}$. In that way, we can proceed to

$$\begin{aligned}L_2(\bar{x},\bar{u})(\Delta x,\Delta u)^2 = \; & L_2(\bar{x},\bar{u})(\Delta x)^2 \\ & + L_2(\bar{x},\bar{u}) \; \Delta x \otimes \Delta u \\ & + L_2(\bar{x},\bar{u}) T_{ux,xu} \; \Delta x \otimes \Delta u \\ & + L_2(\bar{x},\bar{u}) \; (\Delta u)^2 \; ,\end{aligned}$$

which we re-notate to (with factorials included)

$$L_{20}(\bar{x},\bar{u})(\Delta x)^2 + L_{11}(\bar{x},\bar{u}) \; \Delta x \otimes \Delta u + L_{02}(\bar{x},\bar{u})(\Delta u)^2 \; .$$

In this way, the formal expansion becomes

$$f(\bar{x} + \Delta x,\ \bar{u} + \Delta u) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} L_{ij}(\bar{x},\bar{u})\ (\Delta x)^{i} \bullet (\Delta u)^{j}$$

from which point we can examine the scheduling questions previously raised.

The clear distinctions established by these notations are expected to make possible a deeper investigation of the issues of controller scheduling.

IV. PROGRESS IN PARAMETER SELECTION*

4.1 INTRODUCTION

The purpose of Section IV is to provide some visual indication of progress which is being made on the interactive approach for nonlinear tensor model identification, simulation, and validation.

To begin this process, we wish to recall the situation for previous computer studies of this type. Probably the quickest and most efficient way to do this is to excerpt an example from the previous grant report, which was for the period from March 1, 1979 to September 30, 1980. This excerpt is included in the following pages. It is primarily a reference in subsection 4.3.

*Contributed by Thomas Klingler.

Excerpt

Pages 115 - 128

Technical Report

on

NASA Grant NSG-3048

March 1, 1979 - September 30, 1980

The intent of this chapter is to illustrate the notions discussed in Chapter V with two representative case studies. The first example is a system of two nonpolynomic, nonlinear differential equations with two states and two inputs. A degree-2 approximation is used in constructing a model of the system; following this, a degree-3 approximation is discussed. The second example consists of a system of three polynomic differential equations of three states and three inputs. A degree-2 approximation is used to generate the third-order model. The equations of this example are chosen as sums of monomials from the tensor products to illustrate the manner in which the identification scheme weights the appropriate parameters of the linear operators in the model.

Simulation and verification of each model make up the bulk of the chapter. Plots illustrating comparison of the simulated, linear, and true solutions are given, and extensive use of the error analysis described in Section 5.5 is made. For each model, an operating region of validity about the origin is established.

6.2 SECOND ORDER SYSTEM

In this first example, let the state x be given by the 2-vector (x_1,x_2) and the input u by (u_1,u_2). Consider the system

$$\begin{aligned} f_1(x,u) &= \frac{dx_1}{dt} \\ &= u_2\cosh(x_1x_2) - e^{2u_1}\sinh(2x_1) - 3\sinh(x_2)\ , \\ f_2(x,u) &= \frac{dx_2}{dt} \\ &= e^{u_1u_2}\sinh(x_1) - e^{u_1}u_1\cosh(x_1^2) + \sinh(x_2)\ . \end{aligned}$$

The input forcing functions are cosinusoids and are each a function of two parameters, amplitude and frequency. Notice that

$$f(0,0) = 0 ,$$

so that the origin is an equilibrium point and will thus be the point of expansion in the series truncation approximation.

The linear operators which form the standard linear approximation are calculated according to

$$L_{10} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\substack{x = (0,0) \\ u = (0,0)}}$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} ,$$

and

$$L_{01} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}\right|_{\substack{x = (0,0) \\ u = (0,0)}}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Observe that L_{01} is full rank; each function has linear terms in the inputs. Moreover, local stability of the system is ascertained by the fact that L_{10} has eigenvalues with negative real parts. Thus, the origin is a stable equilibrium point.

Consider a truncation approximation up to second degree tensor product terms only. As discussed earlier, identification of an accurate model requires that the system be perturbed a small distance from the point of expansion by choice of the initial condition vector x_0. To this end, the system is integrated with

$$x_0 = (0.005, -0.005) ,$$

while the input amplitude vector α is taken to be

$$\alpha = (0.05, 0.05) ,$$

and the vector of frequencies is

$$\phi = (0.75, 1.0) ,$$

in hertz. The solutions to the coupled differential equations are then sampled at 200 time points, evenly spaced at intervals of 0.02 seconds. In order to ensure accurate derivative estimates, the integration stepsize is taken to be 0.005 seconds. A degree-2 approximation results in a model comprised of five linear operators, as the matrix equation

$$\dot{X} = [\tilde{L}_{10}\ \tilde{L}_{01}\ \tilde{L}_{20}\ \tilde{L}_{11}\ \tilde{L}_{02}]\ X_T .$$

is formulated for the least squares minimization identification scheme.

The linear operators for the above-mentioned formulation are given in the following:

$$\tilde{L}_{10} = \begin{bmatrix} -2.001 & -3.009 \\ 1.006 & 1.011 \end{bmatrix} , \quad L_{01} = \begin{bmatrix} 0.002 & 0.997 \\ -1.000 & 0.000 \end{bmatrix} ,$$

$$\tilde{L}_{20} = \begin{bmatrix} 0.239 & -0.145 & -0.720 \\ -0.323 & -0.128 & 0.359 \end{bmatrix},$$

$$\tilde{L}_{11} = \begin{bmatrix} -4.150 & -0.074 & -0.048 & -0.176 \\ -0.007 & 0.083 & 0.008 & 0.102 \end{bmatrix},$$

$$\tilde{L}_{02} = \begin{bmatrix} -0.105 & 0.027 & 0.012 \\ -0.982 & 0.015 & -0.013 \end{bmatrix}.$$

Note that $\tilde{L}_{10}$ and $\tilde{L}_{01}$ closely approximate the analytical expressions given by the Jacobian matrices of first partial derivatives. The task that remains, then, is the model verification, presented in the following.

Verification tests involve numerous simulations of the model for various combinations of the parameters x_0, α, ϕ. Two tests will be given here, the first of which consists of twelve different choices for x_0, with nine choices for α and one pair (ϕ_1,ϕ_2); this gives a total of 108 simulations. Results of the test are tabulated in Table 6.1, where $\phi =$ (0.75, 1.0) for all simulations. The two columns on the right of the table give the values ε_1 and ε_2 for x_1 and x_2, respectively, as the maximum relative error between the model simulation solution and the linear approximation. As discussed in Section 5.5, a negative value for the ε_i indicates that the model has outperformed the linear approximation of L_{10} and L_{01}. Observe that ε_1 and ε_2 are negative for all individual simulations in this test, for single precision calculations. While these results show that the model has outperformed the linear approximation in a region about the expansion point, comparison plots of these solutions against the true solution offer a final indication of the validity of the

Table 6.1: Error Analyses for Degree-2 Model: $\phi = (0.75, 1.0)$

$(\times 10^{-2})$		$(\times 10^{-1})$		$(\times 10^{-1})$	
x_{10}	x_{20}	α_1	α_2	ϵ_1	ϵ_2
0.1	0.1	0.0	0.0	0.000	0.000
0.1	0.1	0.50	0.50	-0.044	-0.027
0.1	0.1	0.50	-0.50	-0.042	-0.025
0.1	0.1	-0.50	-0.50	-0.045	-0.027
0.1	0.1	-0.50	0.50	-0.042	-0.025
0.1	0.1	1.50	1.50	-0.404	-0.255
0.1	0.1	1.50	-1.50	-0.419	-0.235
0.1	0.1	-1.50	-1.50	-0.447	-0.257
0.1	0.1	-1.50	1.50	-0.407	-0.240
0.1	-0.1	0.0	0.0	0.000	0.000
0.1	-0.1	0.50	0.50	-0.044	-0.027
0.1	-0.1	0.50	-0.50	-0.042	-0.025
0.1	-0.1	-0.50	-0.50	-0.046	-0.027
0.1	-0.1	-0.50	0.50	-0.042	-0.026
0.1	-0.1	1.50	1.50	-0.406	-0.257
0.1	-0.1	1.50	-1.50	-0.420	-0.235
0.1	-0.1	-1.50	-1.50	-0.452	-0.257
0.1	-0.1	-1.50	1.50	-0.407	-0.240
-0.1	-0.1	0.0	0.0	0.000	0.000
-0.1	-0.1	0.50	0.50	-0.043	-0.027
-0.1	-0.1	0.50	-0.50	-0.043	-0.025
-0.1	-0.1	-0.50	-0.50	-0.047	-0.027
-0.1	-0.1	-0.50	0.50	-0.042	-0.026
-0.1	-0.1	1.50	1.50	-0.407	-0.257
-0.1	-0.1	1.50	-1.50	-0.421	-0.235
-0.1	-0.1	-1.50	-1.50	-0.434	-0.258
-0.1	-0.1	-1.50	1.50	-0.407	-0.239
-0.1	0.1	0.0	0.0	0.000	0.000
-0.1	0.1	0.50	0.50	-0.044	-0.027
-0.1	0.1	0.50	-0.50	-0.042	-0.025
-0.1	0.1	-0.50	-0.50	-0.046	-0.027
-0.1	0.1	-0.50	0.50	-0.042	-0.025
-0.1	0.1	1.50	1.50	-0.405	-0.256
-0.1	0.1	1.50	-1.50	-0.420	-0.235
-0.1	0.1	-1.50	-1.50	-0.449	-0.256
-0.1	0.1	-1.50	1.50	-0.408	-0.240
0.5	0.5	0.0	0.0	0.000	0.000
0.5	0.5	0.50	0.50	-0.045	-0.026
0.5	0.5	0.50	-0.50	-0.041	-0.024
0.5	0.5	-0.50	-0.50	-0.042	-0.026
0.5	0.5	-0.50	0.50	-0.041	-0.025
0.5	0.5	1.50	1.50	-0.395	-0.251
0.5	0.5	1.50	-1.50	-0.415	-0.235
0.5	0.5	-1.50	-1.50	-0.433	-0.252
0.5	0.5	-1.50	1.50	-0.407	-0.241

Table 6.1, continued

0.5	-0.5	0.0	0.0	0.000	0.000
0.5	-0.5	0.50	0.50	-0.044	-0.028
0.5	-0.5	0.50	-0.50	-0.042	-0.025
0.5	-0.5	-0.50	-0.50	-0.048	-0.028
0.5	-0.5	-0.50	0.50	-0.041	-0.026
0.5	-0.5	1.50	1.50	-0.410	-0.259
0.5	-0.5	1.50	-1.50	-0.420	-0.235
0.5	-0.5	-1.50	-1.50	-0.457	-0.258
0.5	-0.5	-1.50	1.50	-0.406	-0.239
-0.5	-0.5	0.0	0.0	0.000	0.000
-0.5	-0.5	0.50	0.50	-0.043	-0.028
-0.5	-0.5	0.50	-0.50	-0.043	-0.026
-0.5	-0.5	-0.50	-0.50	-0.050	-0.028
-0.5	-0.5	-0.50	0.50	-0.041	-0.026
-0.5	-0.5	1.50	1.50	-0.415	-0.259
-0.5	-0.5	1.50	-1.50	-0.425	-0.235
-0.5	-0.5	-1.50	-1.50	-0.467	-0.260
-0.5	-0.5	-1.50	1.50	-0.404	-0.238
-0.5	0.5	0.0	0.0	0.000	0.000
-0.5	0.5	0.50	0.50	-0.043	-0.027
-0.5	0.5	0.50	-0.50	-0.042	-0.025
-0.5	0.5	-0.50	-0.50	-0.045	-0.026
-0.5	0.5	-0.50	0.50	-0.042	-0.025
-0.5	0.5	1.50	1.50	-0.401	-0.253
-0.5	0.5	1.50	-1.50	-0.420	-0.235
-0.5	0.5	-1.50	-1.50	-0.443	-0.255
-0.5	0.5	-1.50	1.50	-0.408	-0.240
1.0	1.0	0.0	0.0	0.000	0.000
1.0	1.0	0.50	0.50	-0.047	-0.027
1.0	1.0	0.50	-0.50	-0.040	-0.022
1.0	1.0	-0.50	-0.50	-0.043	-0.025
1.0	1.0	-0.50	0.50	-0.039	-0.024
1.0	1.0	1.50	1.50	-0.385	-0.245
1.0	1.0	1.50	-1.50	-0.410	-0.233
1.0	1.0	-1.50	-1.50	-0.415	-0.243
1.0	1.0	-1.50	1.50	-0.401	-0.243
1.0	-1.0	0.0	0.0	0.000	0.000
1.0	-1.0	0.50	0.50	-0.043	-0.028
1.0	-1.0	0.50	-0.50	-0.042	-0.026
1.0	-1.0	-0.50	-0.50	-0.049	-0.029
1.0	-1.0	-0.50	0.50	-0.042	-0.026
1.0	-1.0	1.50	1.50	-0.413	-0.258
1.0	-1.0	1.50	-1.50	-0.421	-0.235
1.0	-1.0	-1.50	-1.50	-0.464	-0.259
1.0	-1.0	-1.50	1.50	-0.404	-0.239
-1.0	-1.0	0.0	0.0	0.000	0.000
-1.0	-1.0	0.50	0.50	-0.045	-0.028
-1.0	-1.0	0.50	-0.50	-0.043	-0.026
-1.0	-1.0	-0.50	-0.50	-0.053	-0.030
-1.0	-1.0	-0.50	0.50	-0.041	-0.026
-1.0	-1.0	1.50	1.50	-0.424	-0.257
-1.0	-1.0	1.50	-1.50	-0.428	[illegible]34
-1.0	-1.0	-1.50	-1.50	-0.482	-0.262
-1.0	-1.0	-1.50	1.50	-0.401	-0.237
-1.0	1.0	0.0	0.0	0.000	0.000
-1.0	1.0	0.50	0.50	-0.043	-0.026
-1.0	1.0	0.50	-0.50	-0.042	-0.025
-1.0	1.0	-0.50	-0.50	-0.043	-0.025
-1.0	1.0	-0.50	0.50	-0.041	-0.025
-1.0	1.0	1.50	1.50	-0.397	-0.249
-1.0	1.0	1.50	-1.50	-0.420	-0.235
-1.0	1.0	-1.50	-1.50	-0.436	-0.251
-1.0	1.0	-1.50	1.50	-0.406	-0.240

model. To best illustrate the tracking ability, consider a simulation of the model with the initial condition set at (-0.01, 0.01) but with

$$\alpha = (0.25, -0.25)$$

as the input amplitude pair, at the same frequency pair (0.75, 1.0). Simulation of the model for these conditions is depicted in Figure 6.1a for variable x_1 and Figure 6.1b for variable x_2. Clearly, the model solution, curve C, tracks the true solution, curve A, well throughout the integration interval.

An interesting feature of this example concerns the sensitive behavior of $f(x,u)$ for low frequency inputs; in the D.C. case, input amplitude steps of over 0.1 in magnitude cause instability in the system. The second test here, then, is for low frequency inputs with small amplitudes. Four choices of ϕ, four of α, and two of x_0 are utilized, a total of 32 simulations. Table 6.2 illustrates the results by way of the comparative error analyses where it is seen that the model again outperforms the linear approximation for the various conditions tested. The next two figures depict simulations of the model for two of these tests verifying its ability to track the true solution. In Figure 6.2 is given the response of the system for

$$x_0 = (0.01, -0.01) ,$$

$$\alpha = (-0.075, -0.075) ,$$

$$\phi = (0.05, 0.05) .$$

A step response (that is, for $\phi = (0,0)$ in the test) is given in Figure 6.3 at the same value for x_0 and α. In both instances the model simulation matches the true solution well while the linear approximation is poor.

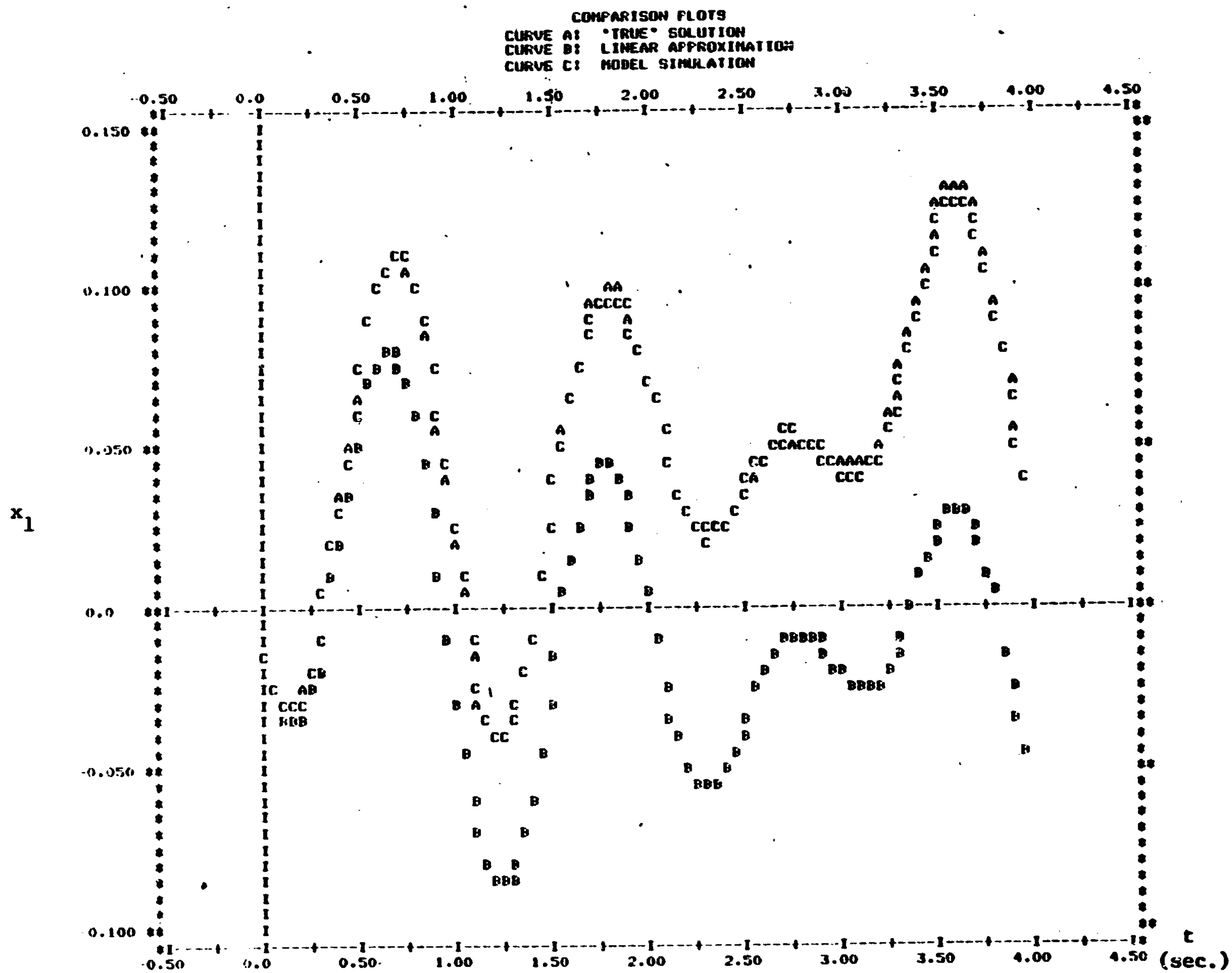
COMPARISON PLOTS
CURVE A: 'TRUE' SOLUTION
CURVE B: LINEAR APPROXIMATION
CURVE C: MODEL SIMULATION
0.150
0.100
0.050
0.0
-0.050
0.100
x_1
-0.50
0.0
0.50
1.00
1.50
2.00
2.50
3.00
3.50
4.00
4.50
t (sec.)

Figure 6.1a

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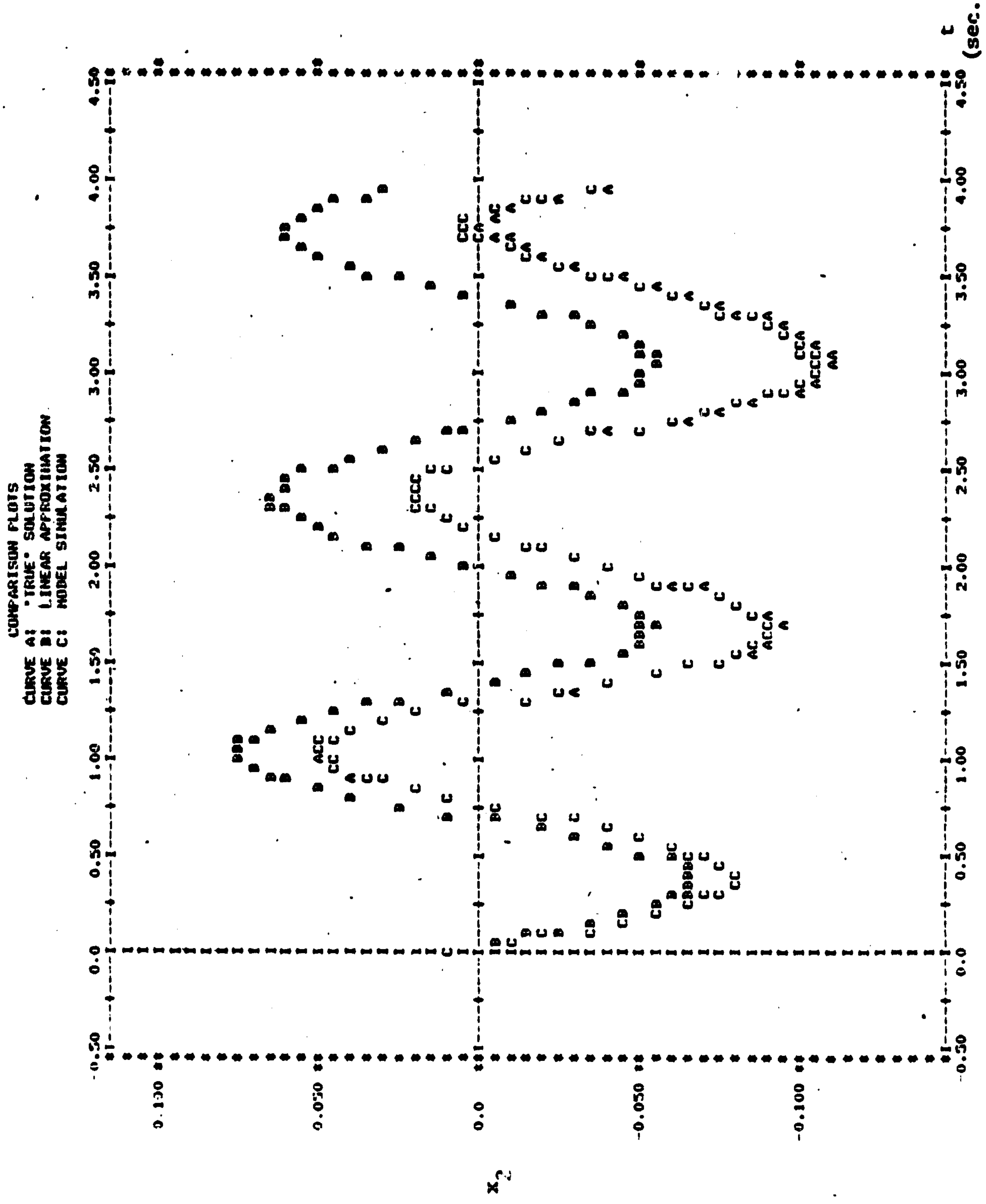

Figure 6.1b

Table 6.2: Low Frequency Error Analyses for Degree-2 Model.

$(\times 10^{-2})$		$(\times 10^{-1})$		$(\times 10^{-[illegible]})$		$(\times 10^{-1})$	
x_{10}	x_{20}	α_1	α_2	ϕ_1	ϕ_2	ε_1	ε_2
0.1	-0.1	0.10	-0.75	0.0	0.0	-0.082	-0.081
0.1	-0.1	0.10	-0.75	0.10	0.10	-0.082	-0.076
0.1	-0.1	0.10	-0.75	0.25	0.25	-0.061	-0.061
0.1	-0.1	0.10	-0.75	0.50	0.50	-0.062	-0.061
0.1	-0.1	-0.50	0.50	0.0	0.0	-0.293	-0.263
0.1	-0.1	-0.50	0.50	0.10	0.10	-0.492	-0.512
0.1	-0.1	-0.50	0.50	0.25	0.25	-0.724	-0.673
0.1	-0.1	-0.50	0.50	0.50	0.50	-0.745	-0.629
0.1	-0.1	-0.75	0.10	0.0	0.0	-0.589	-0.520
0.1	-0.1	-0.75	0.10	0.10	0.10	-1.010	-1.040
0.1	-0.1	-0.75	0.10	0.25	0.25	-1.490	-1.370
0.1	-0.1	-0.75	0.10	0.50	0.50	-1.430	-1.210
0.1	-0.1	-0.75	-0.75	0.0	0.0	-0.532	-0.454
0.1	-0.1	-0.75	-0.75	0.10	0.10	-0.704	-0.737
0.1	-0.1	-0.75	-0.75	0.25	0.25	-1.080	-0.960
0.1	-0.1	-0.75	-0.75	0.50	0.50	-1.010	-0.876
1.0	-1.0	0.10	-0.75	0.0	0.0	-0.079	-0.079
1.0	-1.0	0.10	-0.75	0.10	0.10	-0.080	-0.074
1.0	-1.0	0.10	-0.75	0.25	0.25	-0.061	-0.061
1.0	-1.0	0.10	-0.75	0.50	0.50	-0.062	-0.061
1.0	-1.0	-0.50	0.50	0.0	0.0	-0.297	-0.266
1.0	-1.0	-0.50	0.50	0.10	0.10	-0.492	-0.512
1.0	-1.0	-0.50	0.50	0.25	0.25	-0.724	-0.673
1.0	-1.0	-0.50	0.50	0.50	0.50	-0.745	-0.629
1.0	-1.0	-0.75	0.10	0.0	0.0	-0.595	-0.525
1.0	-1.0	-0.75	0.10	0.10	0.10	-1.010	-1.040
1.0	-1.0	-0.75	0.10	0.25	0.25	-1.490	-1.370
1.0	-1.0	-0.75	0.10	0.50	0.50	-1.430	-1.210
1.0	-1.0	-0.75	-0.75	0.0	0.0	-0.528	-0.454
1.0	-1.0	-0.75	-0.75	0.10	0.10	-0.704	-0.737
1.0	-1.0	-0.75	-0.75	0.25	0.25	-1.080	-0.960
1.0	-1.0	-0.75	-0.75	0.50	0.50	-1.010	-0.876

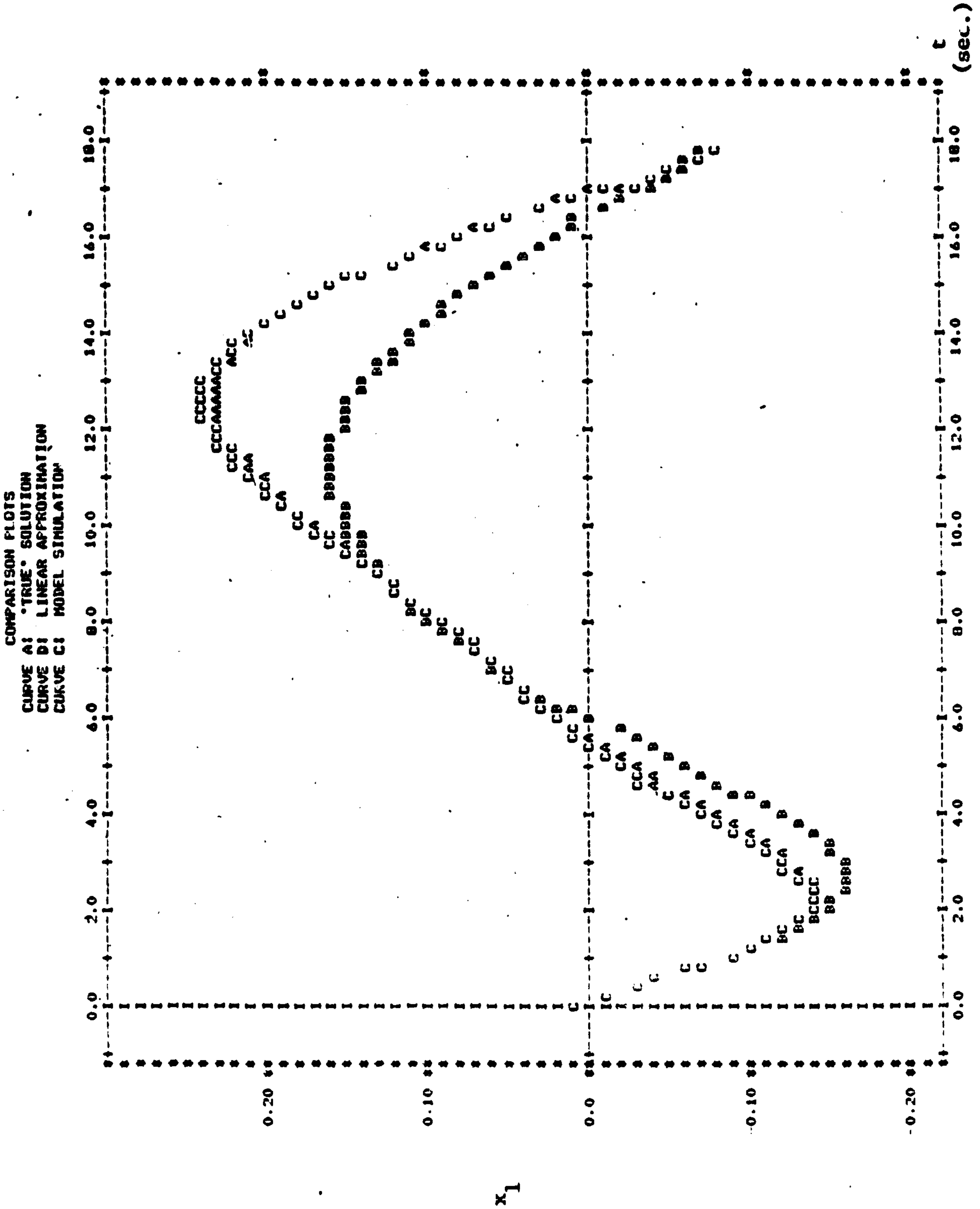
COMPARISON PLOTS
CURVE A: 'TRUE' SOLUTION
CURVE B: LINEAR APPROXIMATION
CURVE C: MODEL SIMULATION
0.0
2.0
4.0
6.0
8.0
10.0
12.0
14.0
16.0
18.0
t (sec.)
0.20
0.10
0.0
-0.10
-0.20
x1

Figure 6.2a

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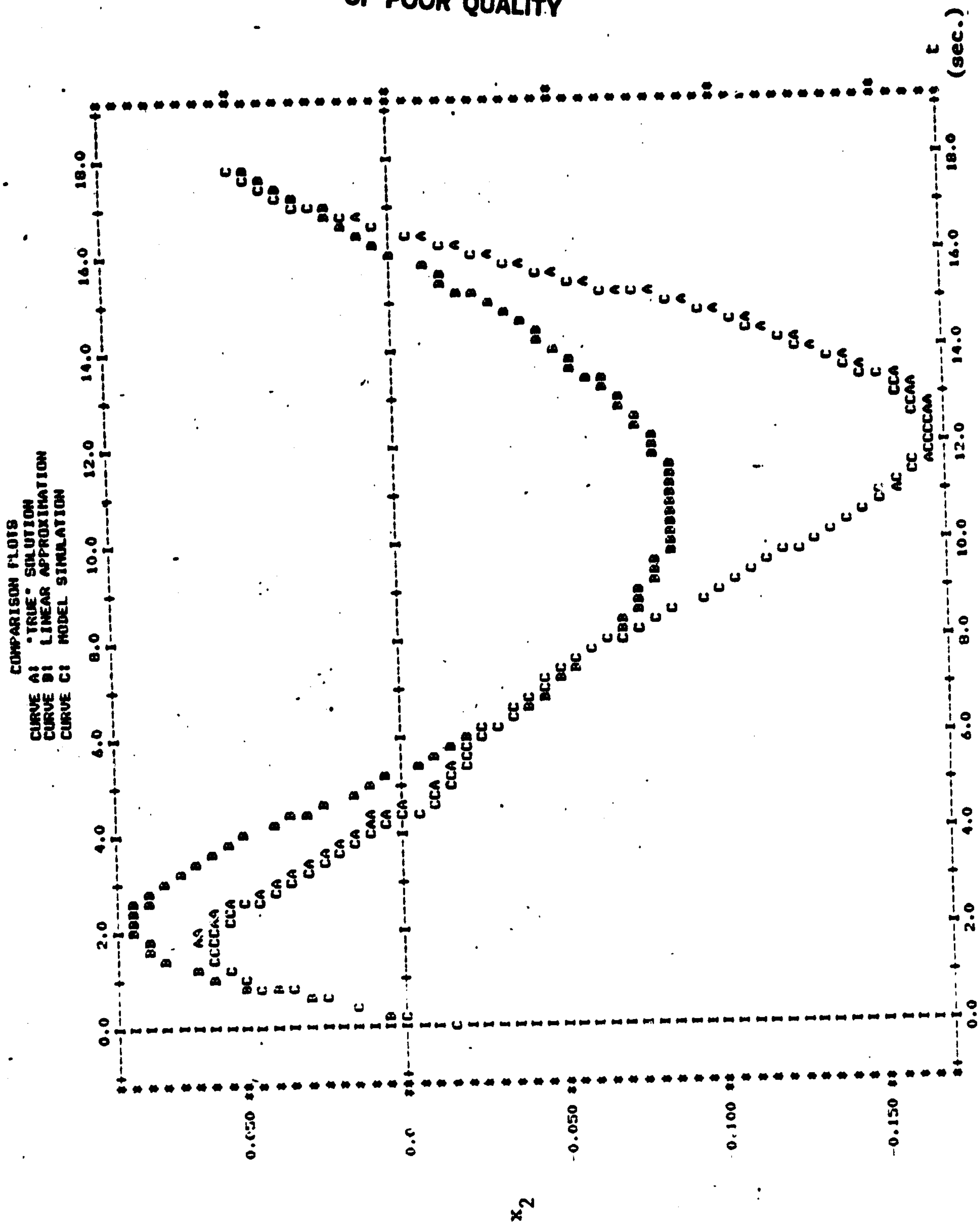

Figure 6.2b

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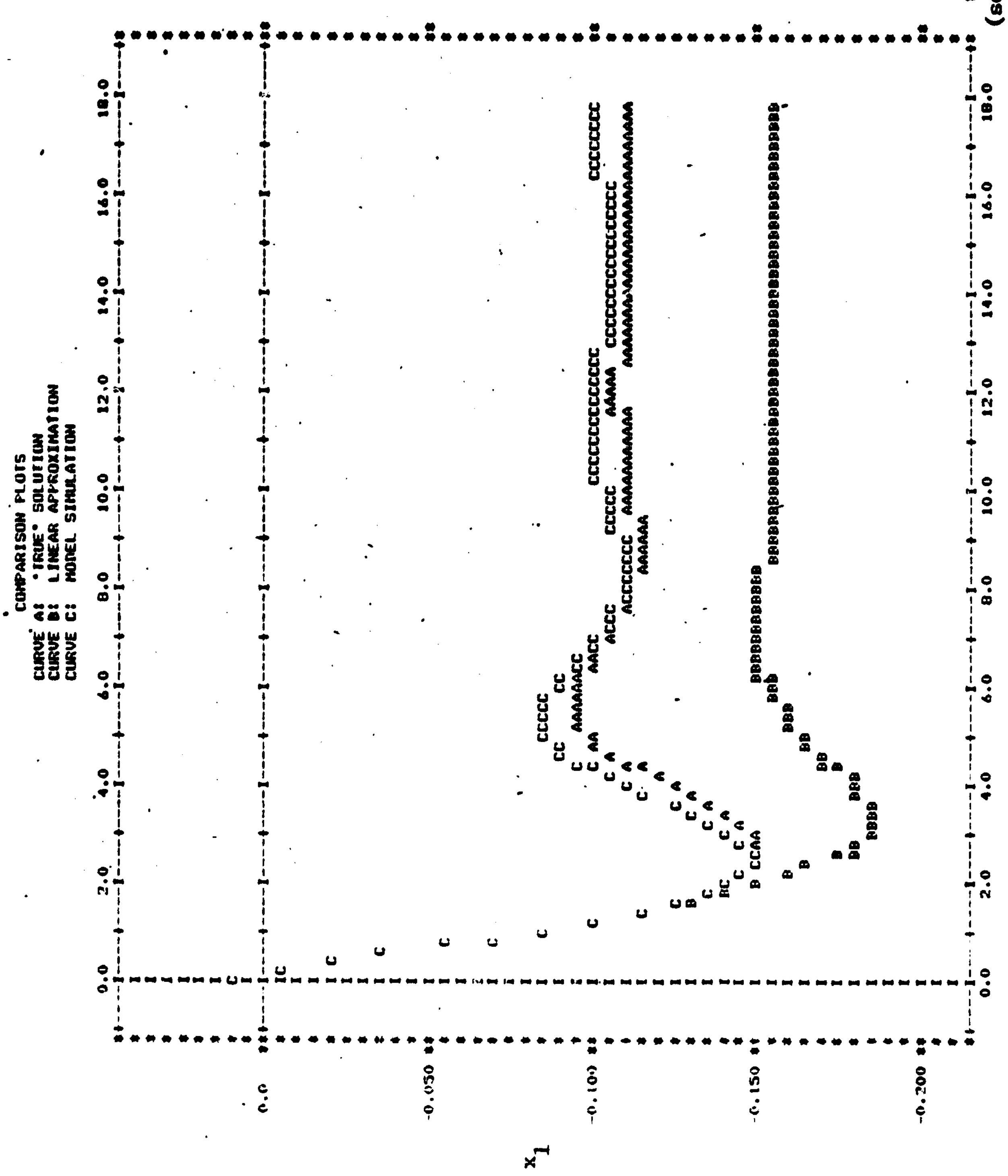

Figure 6.3a

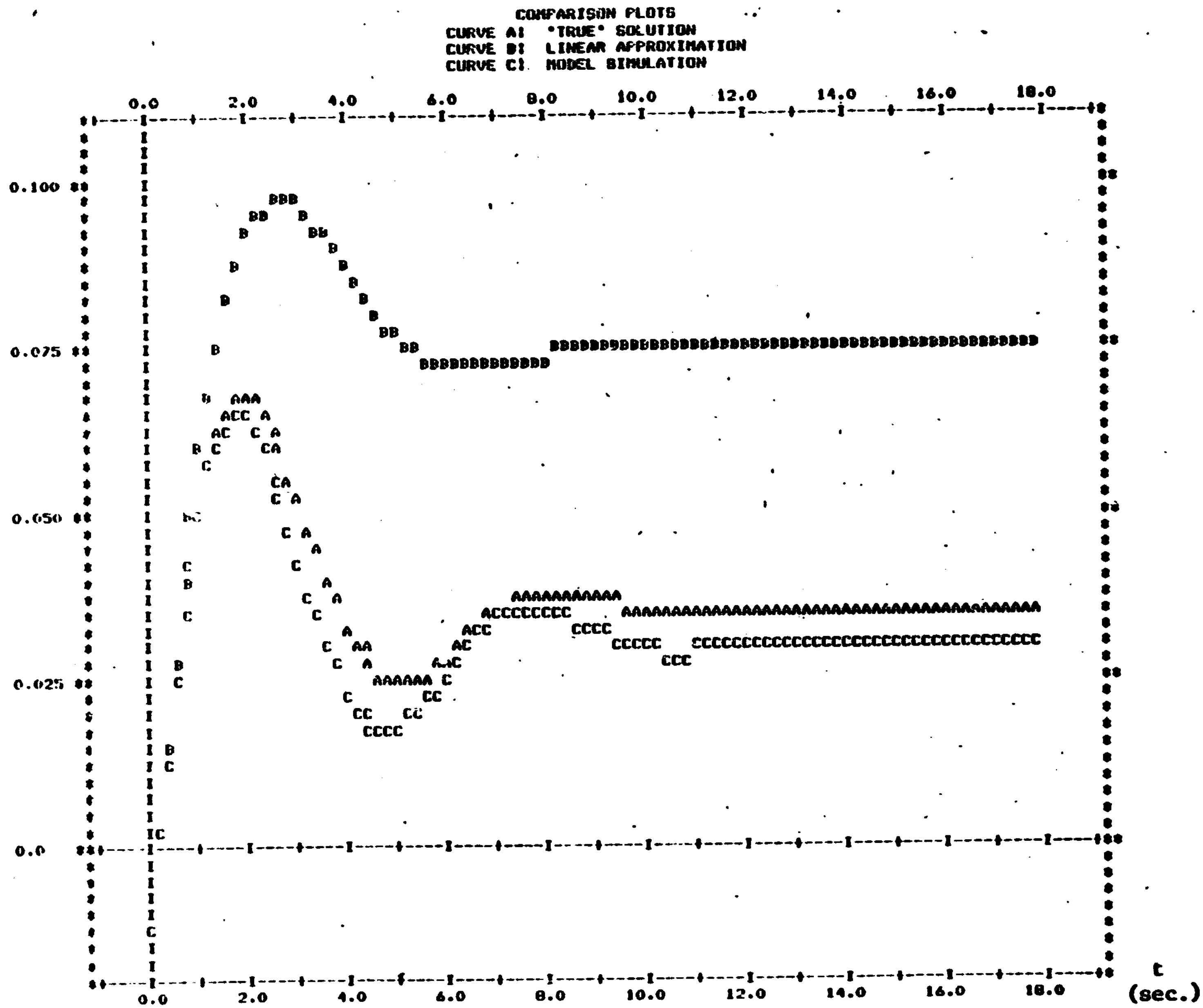

Figure 6.3b

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4.2 SOFTWARE GOALS

The intent of this phase of the research has been to devise an algorithmic procedure and implement it in the testing portion of the overall modeling scheme.

According to Figure 3, the overall modeling scheme is broken up into three main divisions - LOADER, IDENTIFY, and SIMULATE. A brief description of each follows:

LOADER - generates a Model Parameter File containing the number of states, number of controls, length of tensor term vector, degree of approximation, and number of sample points. This routine also samples the states and derivatives of the system and stores them in the Temporary Data File.

IDENTIFY-uses the data in the Temporary Data and Model Parameter Files and generates a model using the SIMEQUAT routine in the SPEAKEZY package. The model is then stored in the Model File; and the Temporary Data File is deleted.

SIMULATE-uses the data in the Model and Model Parameter Files and performs a comparative simulation between the true, linear, and nonlinear solutions. An error analysis procedure is also contained in the routine.

The entire modeling scheme has previously been implemented on the University's IBM 370-168 computer system. Results from the use of this software have been very acceptable; however, the use of the system itself has become increasingly difficult due to the immense number of users bidding for time. Consequently, it was advantageous to develop a modified version of the software and to implement it on the Department of Electrical Engineering's DEC PDP11/60 computer system.

Figure 4 illustrates the peripheral units available on the PDP11/60 system. Two of these units are of particular interest in the modeling

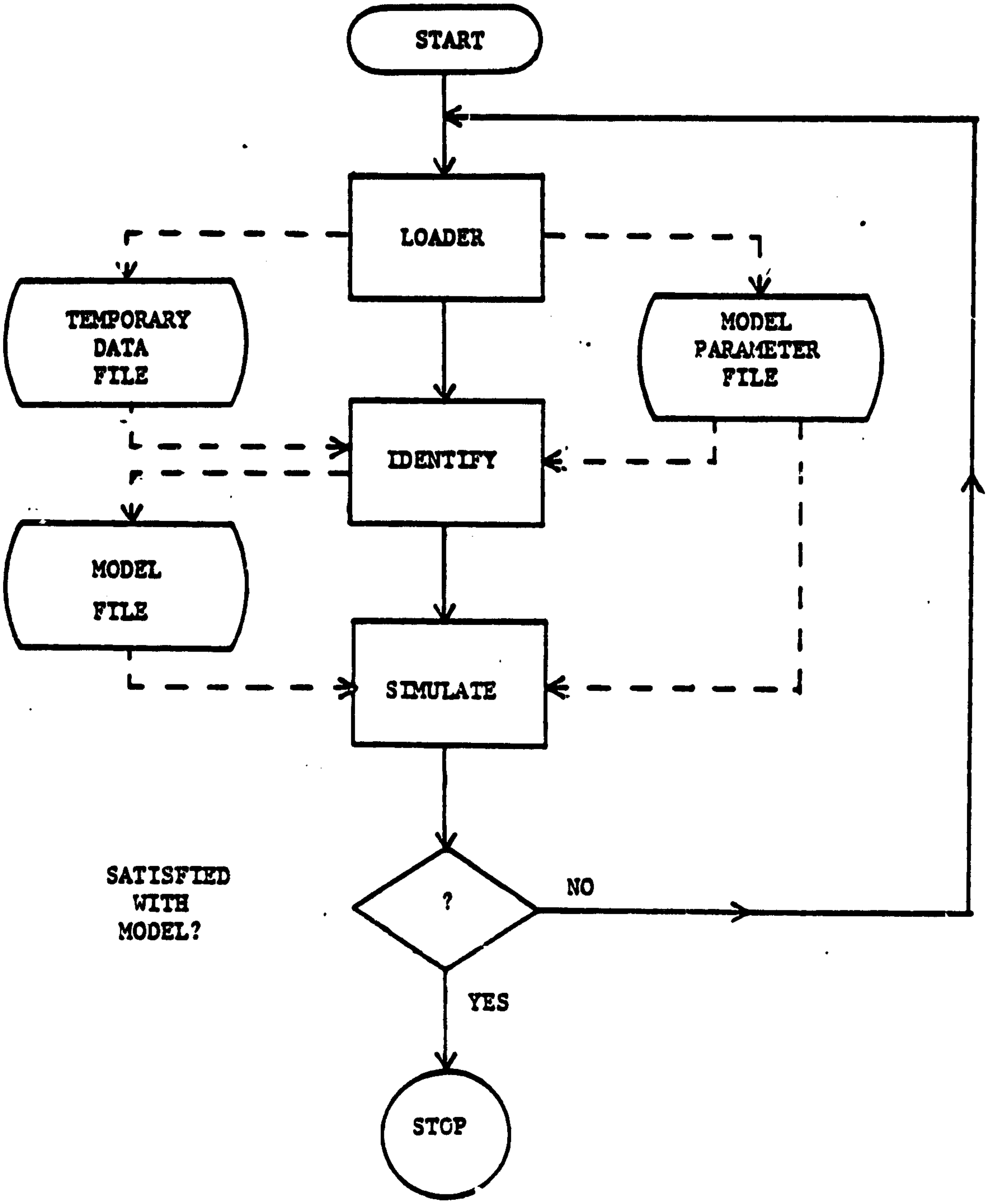


Figure 3: Flowchart of Overall Modeling Scheme

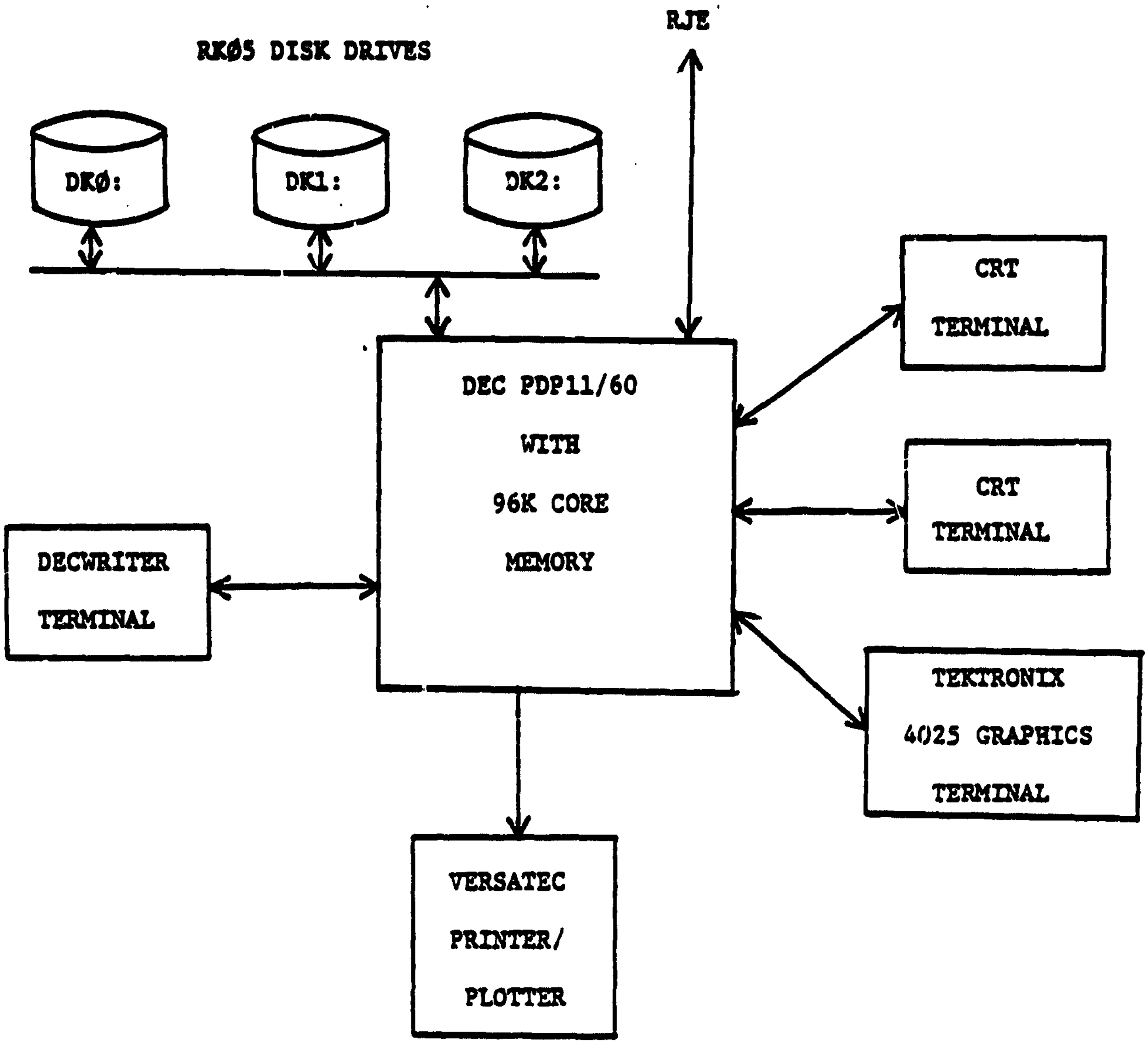


Figure 4: Block Diagram of PDP11/60 Peripherals

scheme. The first is the Tektronix 4025 video graphics terminal, and the second is the Versatec electrostatic printer/plotter. Use of both these peripherals is a valuable plus in the simulation phase, for the trajectory curves can be quickly and easily displayed on the Tektronix tube, and upon request can be spooled to the Versatec plotter. This definitely enhances the routine and improves the interactive ability of the modeling scheme.

Another view indicates that two drawbacks currently exist with the implementation of the modeling scheme on the PDP11/60. First, the 96K of core memory is divided into thirds, with 32K being allocated to each terminal. Unfortunately 32K of memory is not a sufficient amount to perform the identification phase of the scheme. Secondly, the PDP11/60 does not currently support floating point hardware. In other words, all floating point operations are presently performed in software which greatly increases the execution time of routines which contain a large number of computations, such as LOADER and IDENTIFY.

With this in mind, work has been underway to institute an interactive nonlinear model identification and testing scheme whereby the PDP11/60 computer will be linked via a data link to a Remote Job Entry (RJE) port on the IBM 370-168*. In this configuration, a user could sit at the Tektronix terminal and have both the IBM 370-168 and PDP11/60 facilities at his or her fingertips. Consequently, the memory dependent and highly computational routines LOADER and IDENTIFY could be executed on the IBM 370-168, and the Model and Model Parameter Files could be transferred to

*This link is not yet complete.

the PDP11/60 where they could be used by the SIMULATE routine. In this fashion, the SIMULATE routine could utilize both the graphics capabilities of the Tektronix, and the plotting capabilities of the Versatec.

Figures 5 and 6 contain flowcharts which describe the simulation and testing phase of the modeling scheme. Specifically Figure 5 illustrates the order in which various program functions are performed. The systems (true, linear, and nonlinear) are integrated, using a unique set of initial conditions and control parameters, and the error analysis is displayed. From an interpretation of this analysis, the user has the option to: 1) print the simulated solution at the Versatec; 2) display comparative solution curves on the Tektronix; 3) store the comparative solutions in a plot file; or 4) resolve the systems using a different set of input parameters. This portion of the routine is highly interactive and allows the user to test out a given model over a specified region.

At the termination of the routine, if a Solution Plot File exists, the user can execute the hardcopy plot procedure shown in Figure 6. This procedure uses the Solution Plot File and creates binary parameter and data files which are used by the system routine RAS to perform the actual plotting at the Versatec.

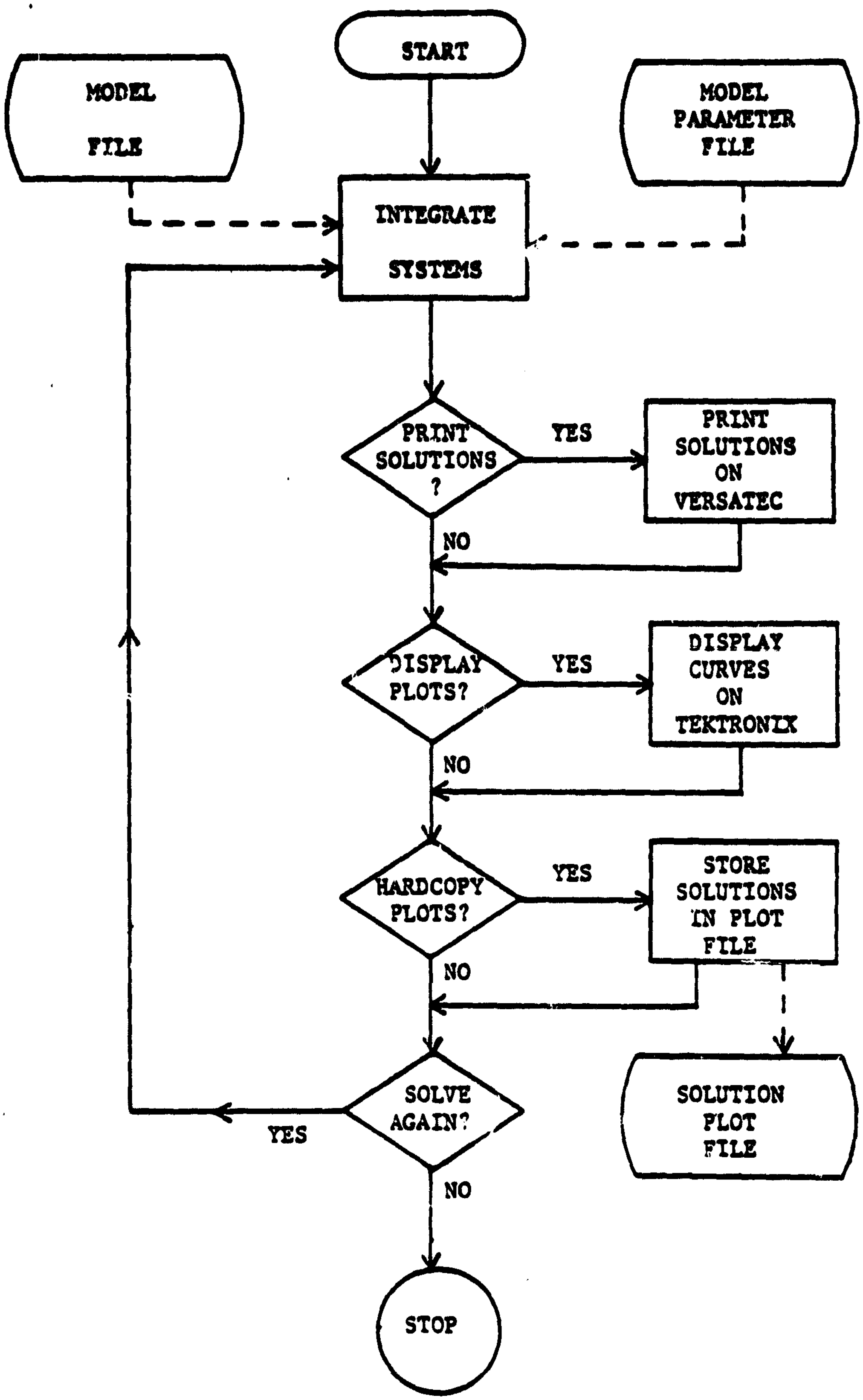


Figure 5: Flowchart of Simulation Procedure

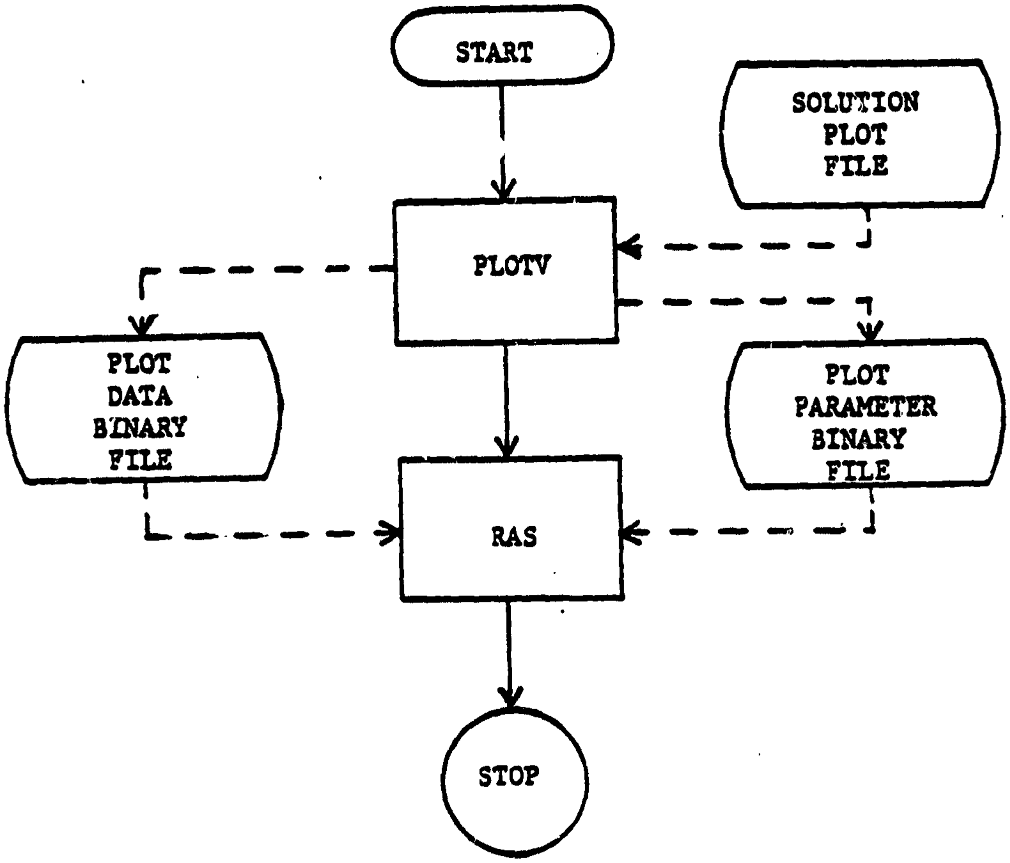


Figure 6: Flowchart of Hardcopy Plot Procedure

4.3 DEVELOPING DISPLAYS

The following pages contain examples of the displays associated with discussions in subsection 4.2. These are self-explanatory and may be compared with the excerpt from the preceding technical report, which has been included in subsection 4.1. In addition to comparison curves for Figures 6.1, 6.2, and 6.3 in that excerpt, some additional curves have been selected from Tables 6.1 and 6.2 to illustrate the capability.

PLOT SET # 1

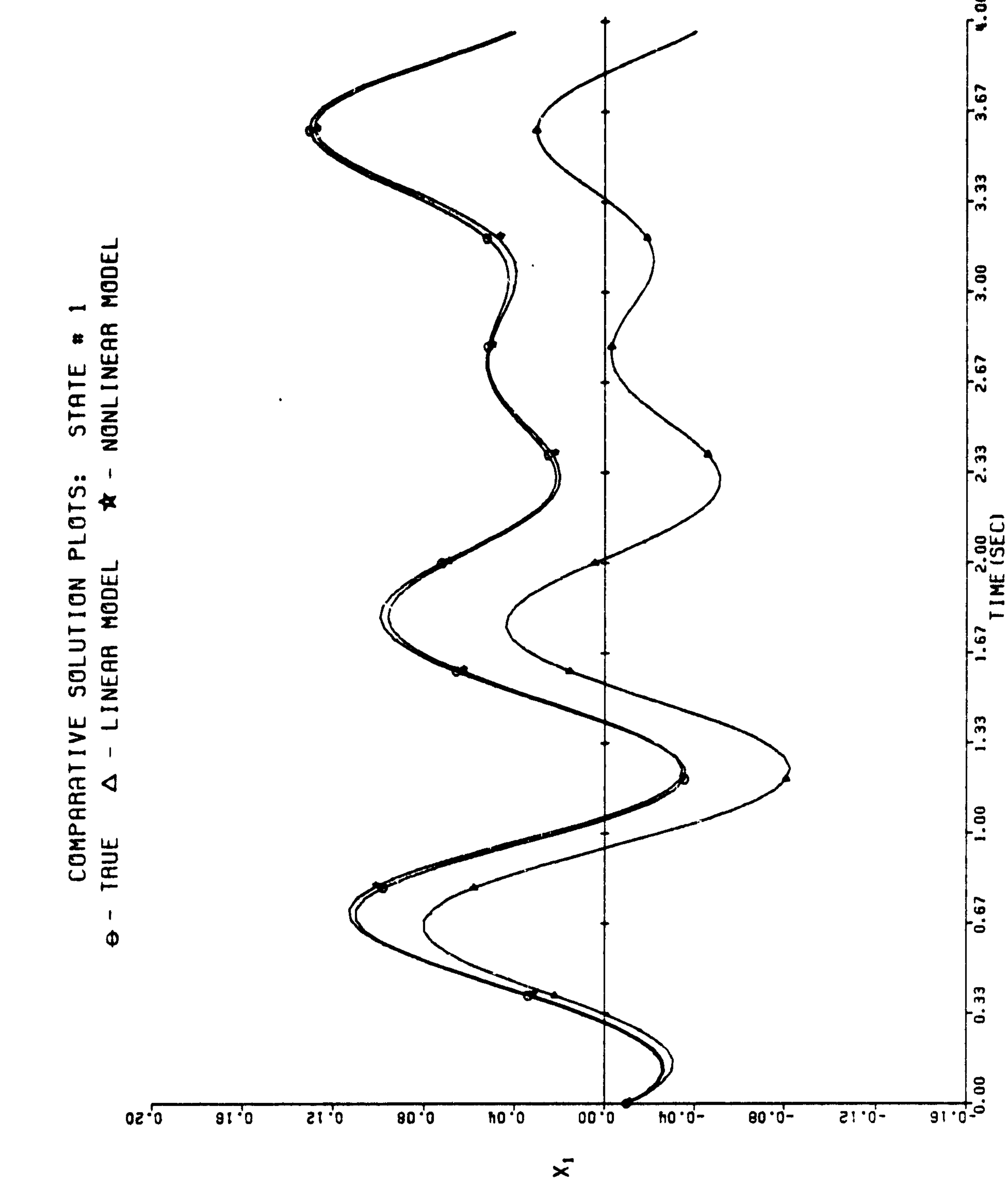
```
*************** PROBLEM SUMMARY ***************
    CONFIGURATION:  TRUE, LINEAR & NONLINEAR
    NUMBER OF STATES:  2
    NUMBER OF CONTROLS:  2
    LENGTH OF TENSOR TERM VECTOR:  14
    DEGREE OF APPROXIMATION:  2
***********************************************
```

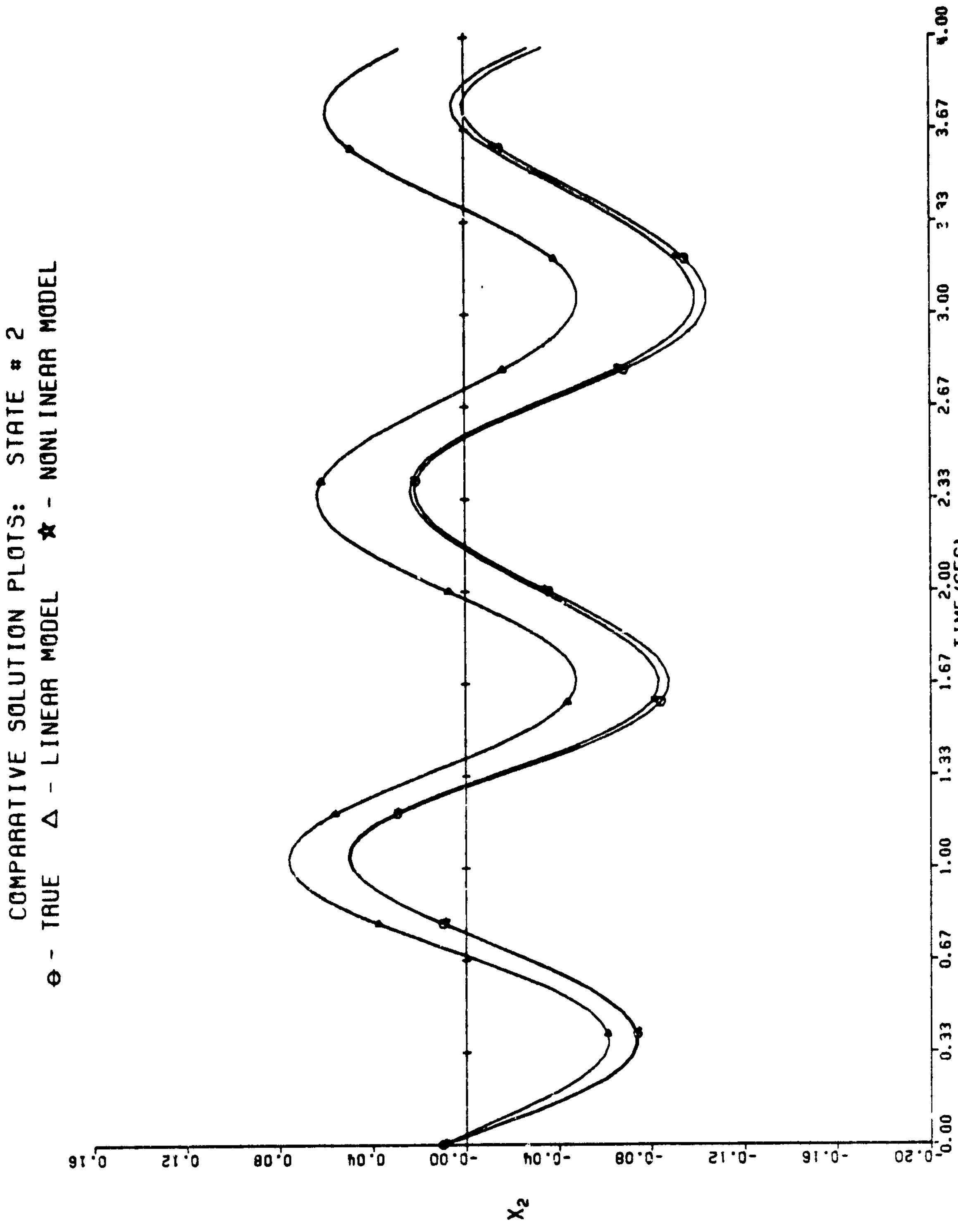
SOLUTION PARAMETERS:

```
INTEGRATION STEPSIZE:  Ø.Ø1Ø
UPPER TIME LIMIT OF INTEGRATION:  4.ØØØ
NUMBER OF PLOT POINTS: 1ØØ
SPACING BETWEEN PLOT POINTS:  Ø.Ø4Ø
```

STATE NUMBER	INITIAL CONDITION	MAXIMUM ERROR
1	-Ø.Ø1Ø	-Ø.987E-Ø1
2	Ø.Ø1Ø	-Ø.612E-Ø1

CONTROL NUMBER	AMPLITUDE	FREQUENCY (CYCLES/SEC)
1	Ø.25Ø	Ø.75Ø
2	-Ø.25Ø	1.ØØØ


COMPARATIVE SOLUTION PLOTS: STATE # 1
Φ - TRUE Δ - LINEAR MODEL ☆ - NONLINEAR MODEL
TIME (SEC)
X_1
0.00 0.33 0.67 1.00 1.33 1.67 2.00 2.33 2.67 3.00 3.33 3.67 4.00
-0.16 -0.12 -0.08 -0.04 0.00 0.04 0.08 0.12 0.16 0.20



COMPARATIVE SOLUTION PLOTS: STATE # 2
Φ - TRUE Δ - LINEAR MODEL ✫ - NONLINEAR MODEL
0.00
0.33
0.67
1.00
1.33
1.67
2.00
2.33
2.67
3.00
3.67
4.00
TIME (SEC)
0.16
0.12
0.08
0.04
-0.00
-0.04
-0.08
-0.12
-0.16
-0.20
X_2

PLOT SET # 2

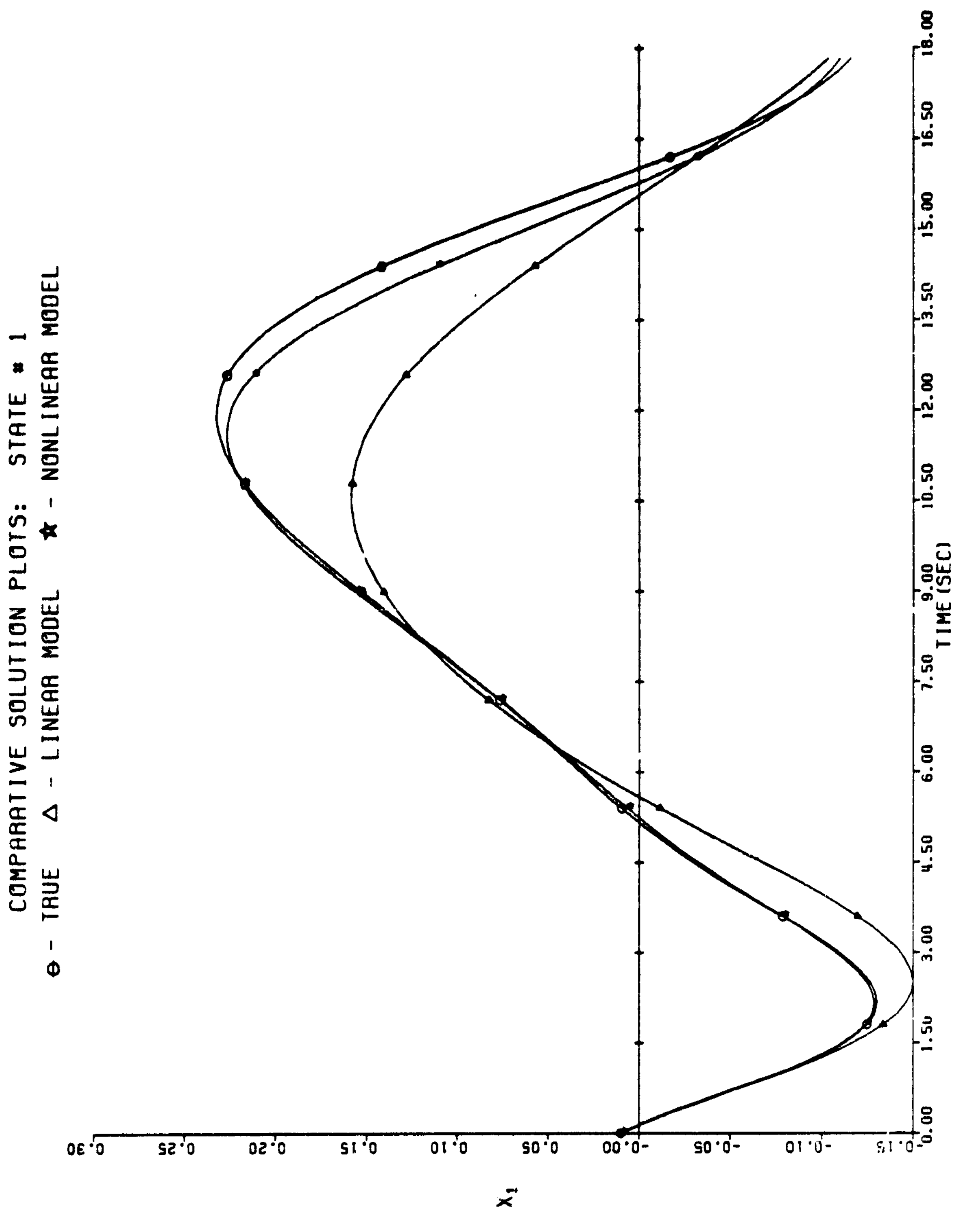
```
**************** PROBLEM SUMMARY ***************
    CONFIGURATION:  TRUE, LINEAR & NONLINEAR
    NUMBER OF STATES:  2
    NUMBER OF CONTROLS:  2
    LENGTH OF TENSOR TERM VECTOR:  14
    DEGREE OF APPROXIMATION:  2
************************************************
```

SOLUTION PARAMETERS:

```
INTEGRATION STEPSIZE:  0.010
UPPER TIME LIMIT OF INTEGRATION: 18.000
NUMBER OF PLOT POINTS: 100
SPACING BETWEEN PLOT POINTS:  0.180
```

STATE NUMBER	INITIAL CONDITION	MAXIMUM ERROR
1	0.010	-0.844E-01
2	-0.010	-0.717E-01

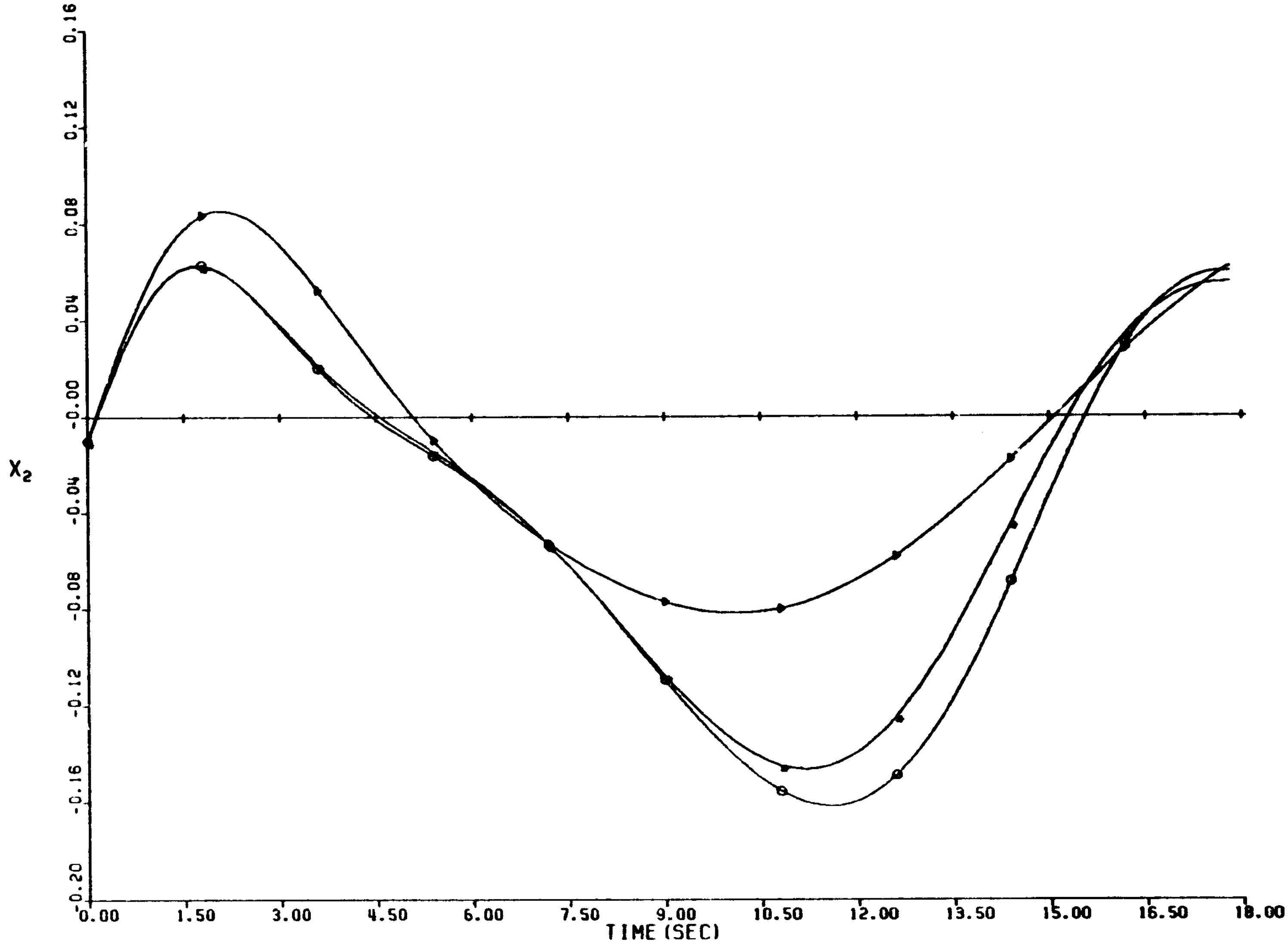
CONTROL NUMBER	AMPLITUDE	FREQUENCY (CYCLES/SEC)
1	-0.075	0.050
2	-0.075	0.050



COMPARATIVE SOLUTION PLOTS: STATE # 1
Θ - TRUE Δ - LINEAR MODEL ☆ - NONLINEAR MODEL
TIME (SEC)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
10.50
12.00
13.50
15.00
16.50
18.00
X_1
-0.15
-0.10
-0.05
-0.00
0.05
0.10
0.15
0.20
0.25
0.30

COMPARATIVE SOLUTION PLOTS: STATE # 2

⊖ - TRUE Δ - LINEAR MODEL ★ - NONLINEAR MODEL



PLOT SET # 3

```
*************** PROBLEM SUMMARY ***************
    CONFIGURATION:  TRUE, LINEAR & NONLINEAR
    NUMBER OF STATES:  2
    NUMBER OF CONTROLS:  2
    LENGTH OF TENSOR TERM VECTOR:  14
    DEGREE OF APPROXIMATION:  2
***********************************************
```

SOLUTION PARAMETERS:
****

INTEGRATION STEPSIZE: 0.010
UPPER TIME LIMIT OF INTEGRATION: 18.000
NUMBER OF PLOT POINTS: 100
SPACING BETWEEN PLOT POINTS: 0.180

STATE NUMBER	INITIAL CONDITION	MAXIMUM ERROR
1	0.010	-0.634E-01
2	-0.010	-0.528E-01

CONTROL NUMBER	AMPLITUDE	FREQUENCY (CYCLES/SEC)
1	-0.075	0.000
2	-0.075	0.000

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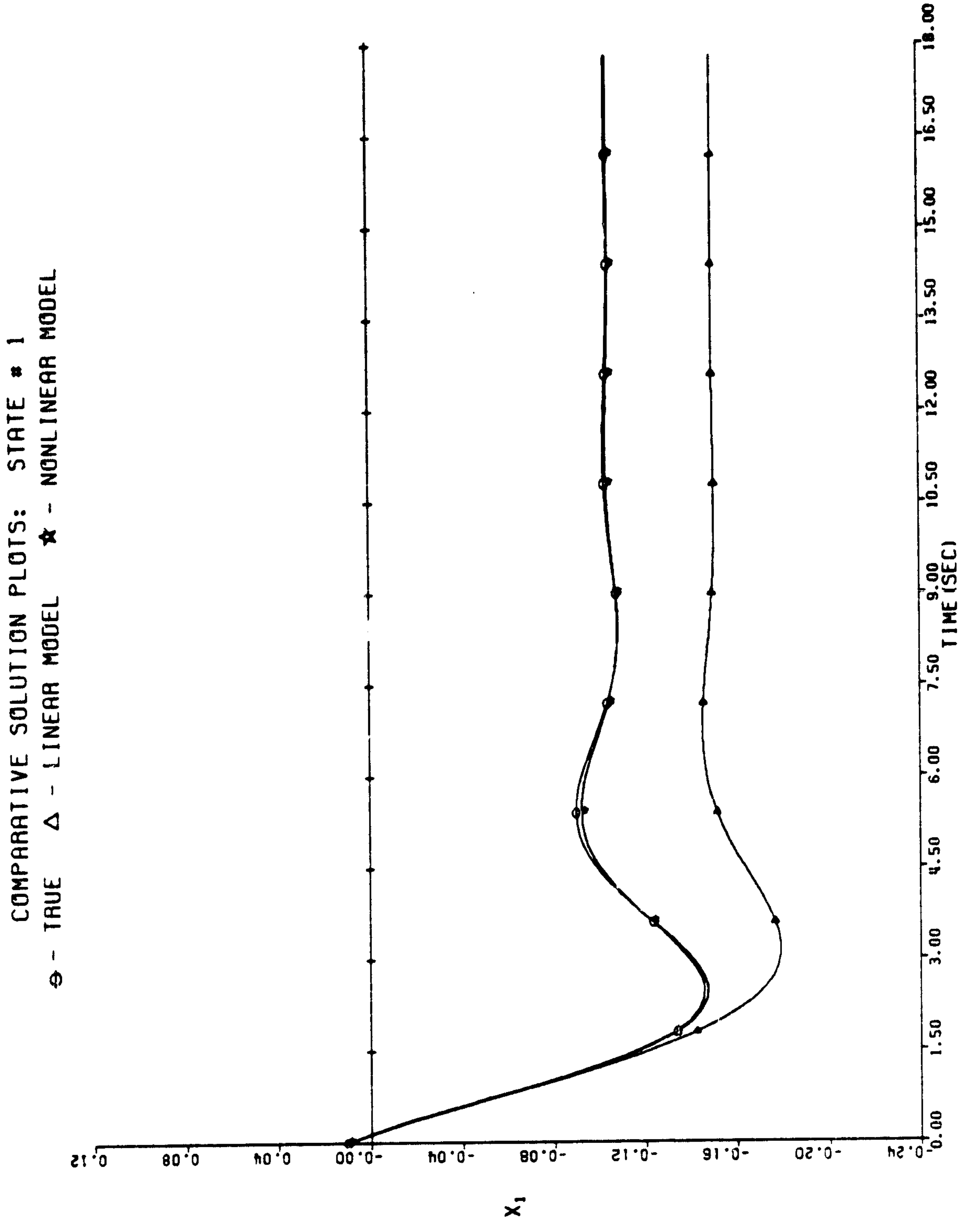
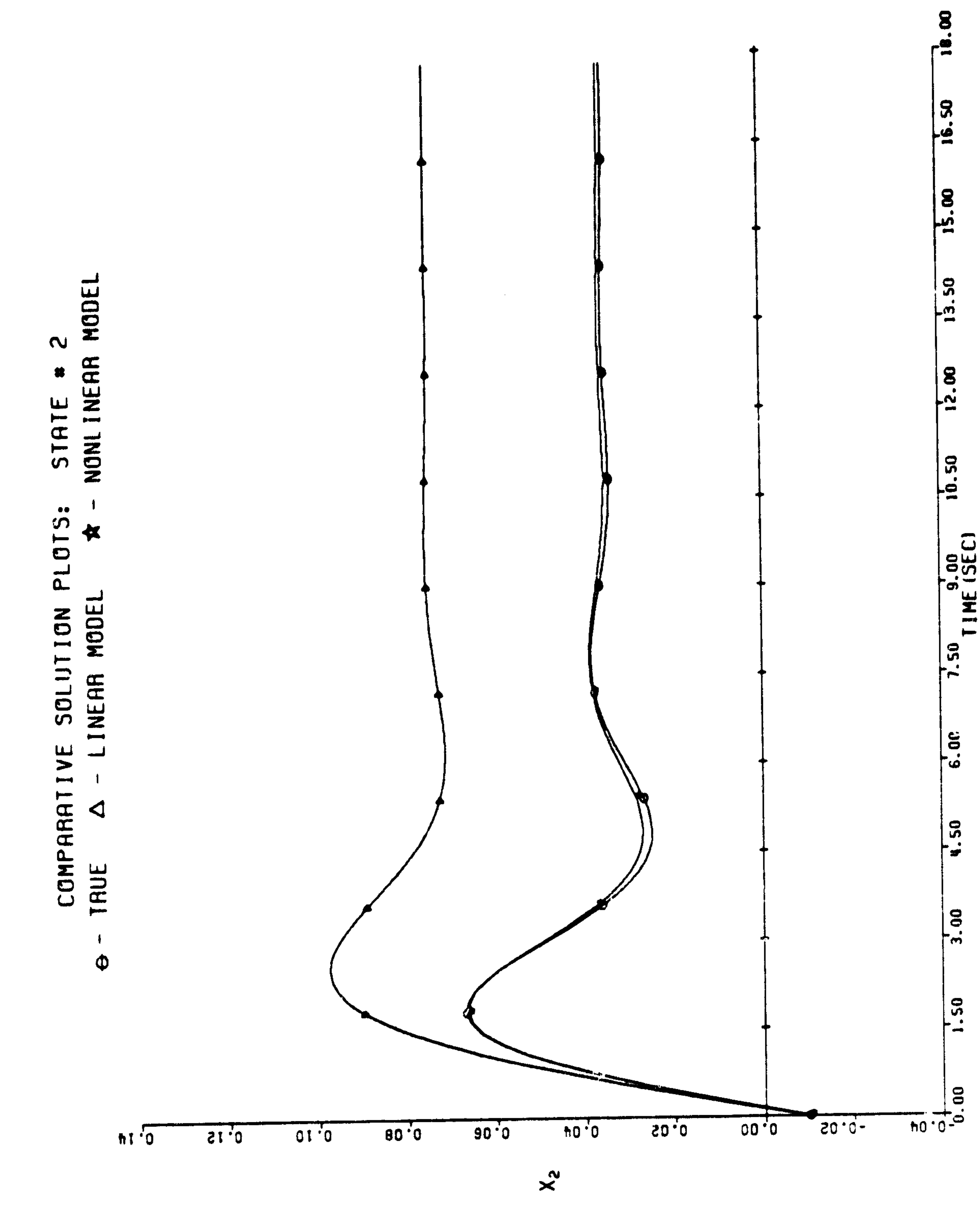


COMPARATIVE SOLUTION PLOTS: STATE # 1
⊕ - TRUE
Δ - LINEAR MODEL
✲ - NONLINEAR MODEL
TIME (SEC)
0.00
1.50
3.00
4.50
6.00
7.50
9.00
10.50
12.00
13.50
15.00
16.50
18.00
X_1
0.12
0.08
0.04
-0.00
-0.04
-0.08
-0.12
-0.16
-0.20
-0.24



COMPARATIVE SOLUTION PLOTS: STATE # 2
ϴ - TRUE Δ - LINEAR MODEL ✯ - NONLINEAR MODEL
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
-0.02
-0.04
X_2
0.00
1.50
3.00
4.50
6.00
7.50
9.00
10.50
12.00
13.50
15.00
16.50
18.00
TIME (SEC)

PLOT SET # 4

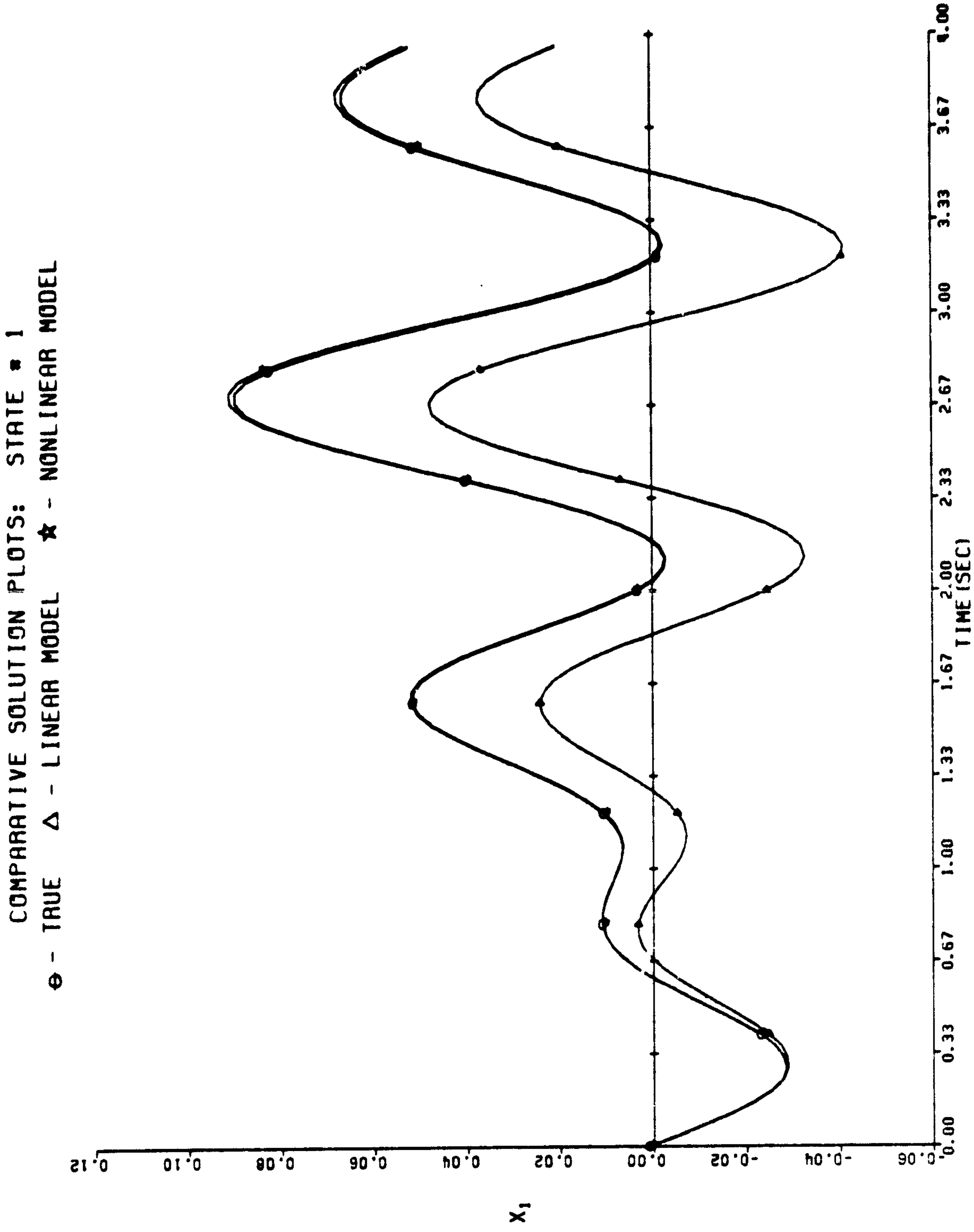
```
*************** PROBLEM SUMMARY ***************
    CONFIGURATION:  TRUE, LINEAR & NONLINEAR
    NUMBER OF STATES:  2
    NUMBER OF CONTROLS:  2
    LENGTH OF TENSOR TERM VECTOR:  14
    DEGREE OF APPROXIMATION:  2
***********************************************
```

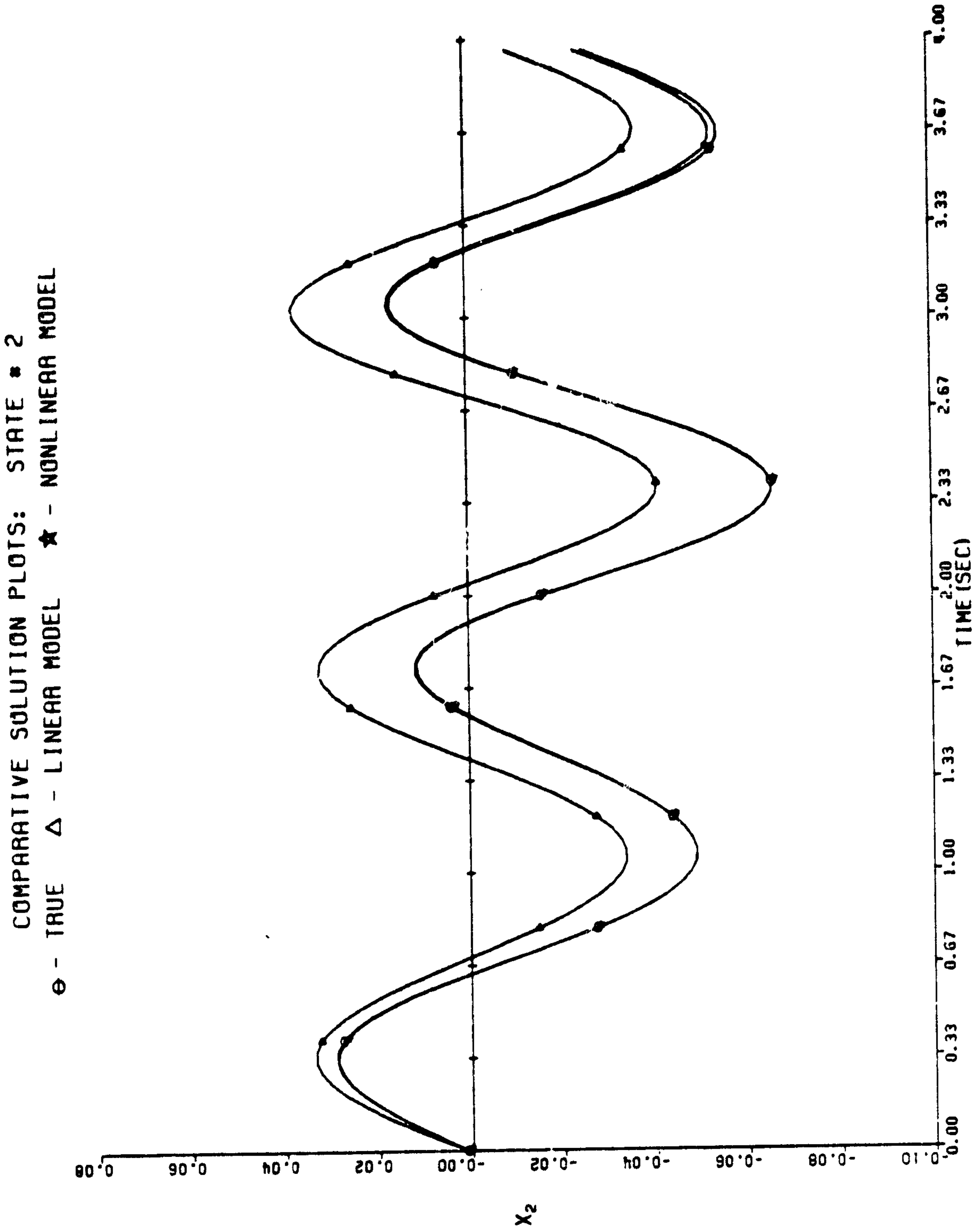
SOLUTION PARAMETERS:

```
INTEGRATION STEPSIZE:  0.010
UPPER TIME LIMIT OF INTEGRATION:  4.000
NUMBER OF PLOT POINTS: 100
SPACING BETWEEN PLOT POINTS:  0.040
```

STATE NUMBER	INITIAL CONDITION	MAXIMUM ERROR
1	0.001	-0.445E-01
2	0.001	-0.257E-01

CONTROL NUMBER	AMPLITUDE	FREQUENCY (CYCLES/SEC)
1	-0.150	0.750
2	-0.150	1.000


COMPARATIVE SOLUTION PLOTS: STATE = 1
⊕ - TRUE Δ - LINEAR MODEL ☆ - NONLINEAR MODEL
X1
0.12
0.10
0.08
0.06
0.04
0.02
0.00
-0.02
-0.04
-0.06
0.00
0.33
0.67
1.00
1.33
1.67
2.00
2.33
2.67
3.00
3.33
3.67
4.00
TIME (SEC)


COMPARATIVE SOLUTION PLOTS: STATE = 2
⊖ - TRUE Δ - LINEAR MODEL ★ - NONLINEAR MODEL
0.00
0.33
0.67
1.00
1.33
1.67
2.00
2.33
2.67
3.00
3.33
3.67
4.00
TIME (SEC)
0.08
0.06
0.04
0.02
-0.00
-0.02
-0.04
-0.06
-0.08
-0.10
X_2

PLOT SET # 5

```
*************** PROBLEM SUMMARY ***************
    CONFIGURATION:  TRUE, LINEAR & NONLINEAR
    NUMBER OF STATES:  2
    NUMBER OF CONTROLS:  2
    LENGTH OF TENSOR TERM VECTOR:  14
    DEGREE OF APPROXIMATION:  2
***********************************************
```

SOLUTION PARAMETERS:

```
INTEGRATION STEPSIZE:  0.010
UPPER TIME LIMIT OF INTEGRATION:  4.000
NUMBER OF PLOT POINTS: 100
SPACING BETWEEN PLOT POINTS:  0.040
```

STATE NUMBER	INITIAL CONDITION	MAXIMUM ERROR
1	0.005	-0.475E-02
2	-0.005	-0.276E-02

CONTROL NUMBER	AMPLITUDE	FREQUENCY (CYCLES/SEC)
1	-0.050	0.750
2	-0.050	1.000

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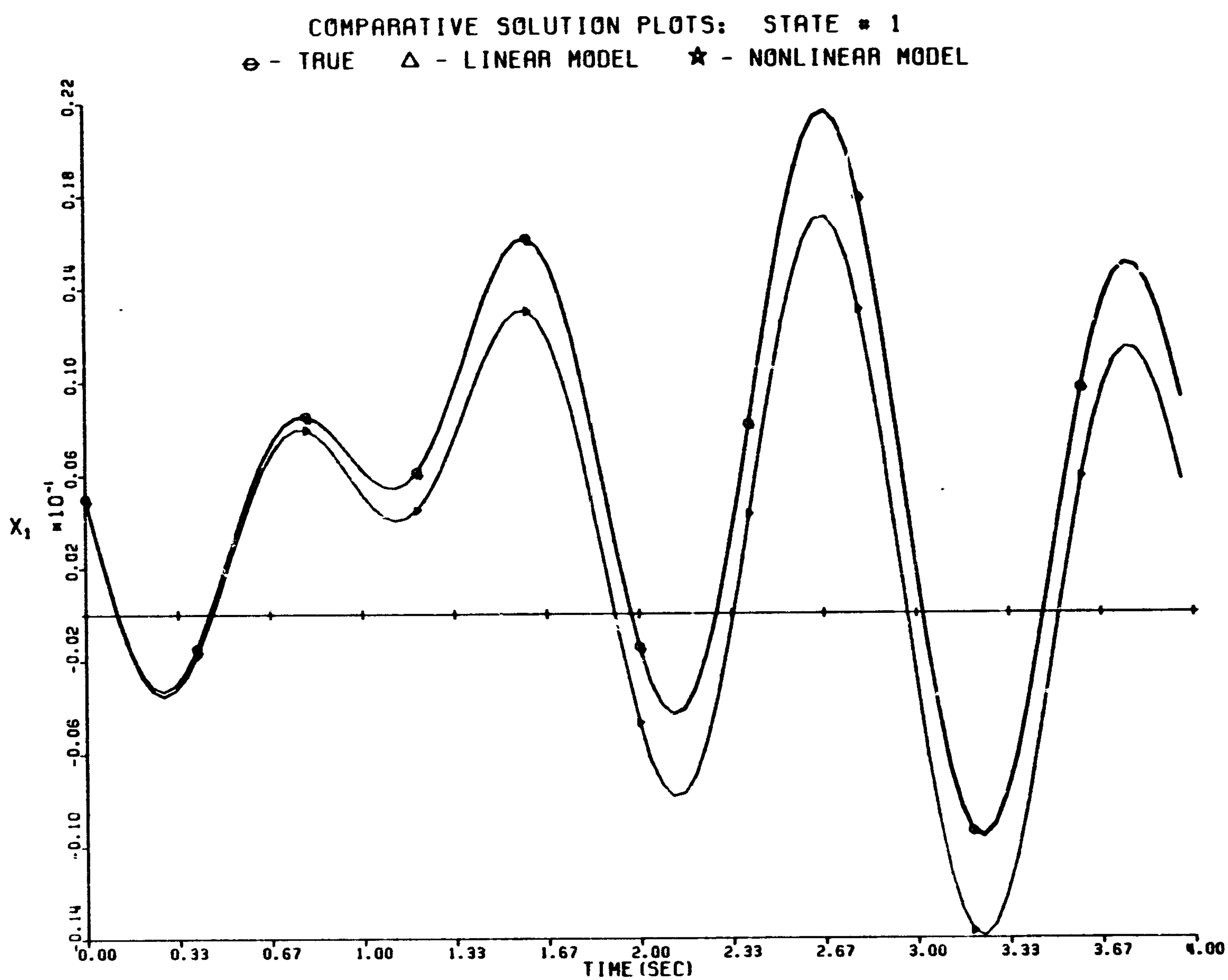
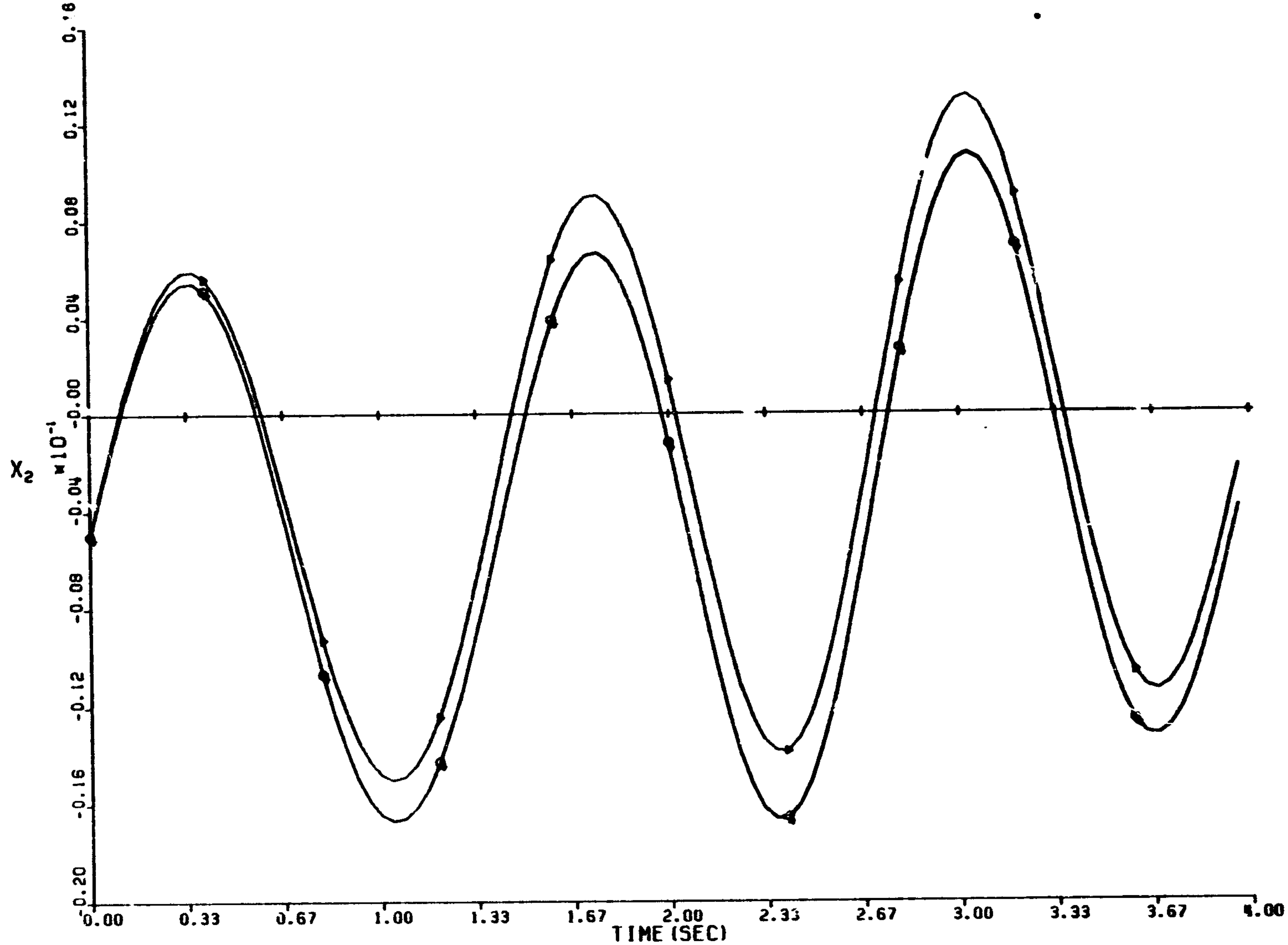


COMPARATIVE SOLUTION PLOTS: STATE * 1
⊖ - TRUE Δ - LINEAR MODEL ★ - NONLINEAR MODEL
0.22
0.18
0.14
0.10
0.06
0.02
-0.02
-0.06
-0.10
-0.14
X_1 $*10^{-1}$
0.00
0.33
0.67
1.00
1.33
1.67
2.00
2.33
2.67
3.00
3.33
3.67
4.00
TIME (SEC)

COMPARATIVE SOLUTION PLOTS: STATE # 2

⊖ - TRUE Δ - LINEAR MODEL ★ - NONLINEAR MODEL



PLOT SET # 6

**************** PROBLEM SUMMARY ****************

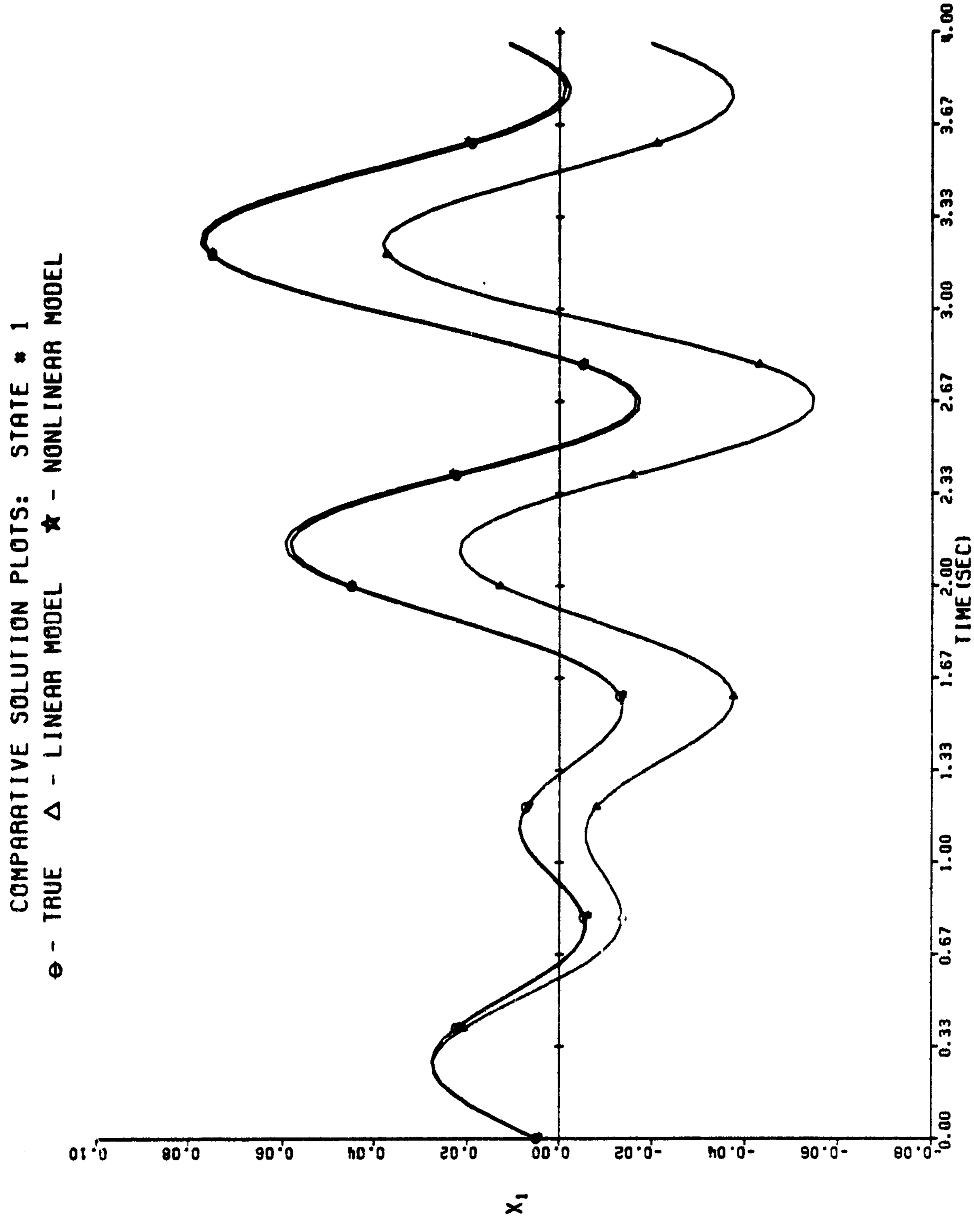
CONFIGURATION: TRUE, LINEAR & NONLINEAR
NUMBER OF STATES: 2
NUMBER OF CONTROLS: 2
LENGTH OF TENSOR TERM VECTOR: 14
DEGREE OF APPROXIMATION: 2

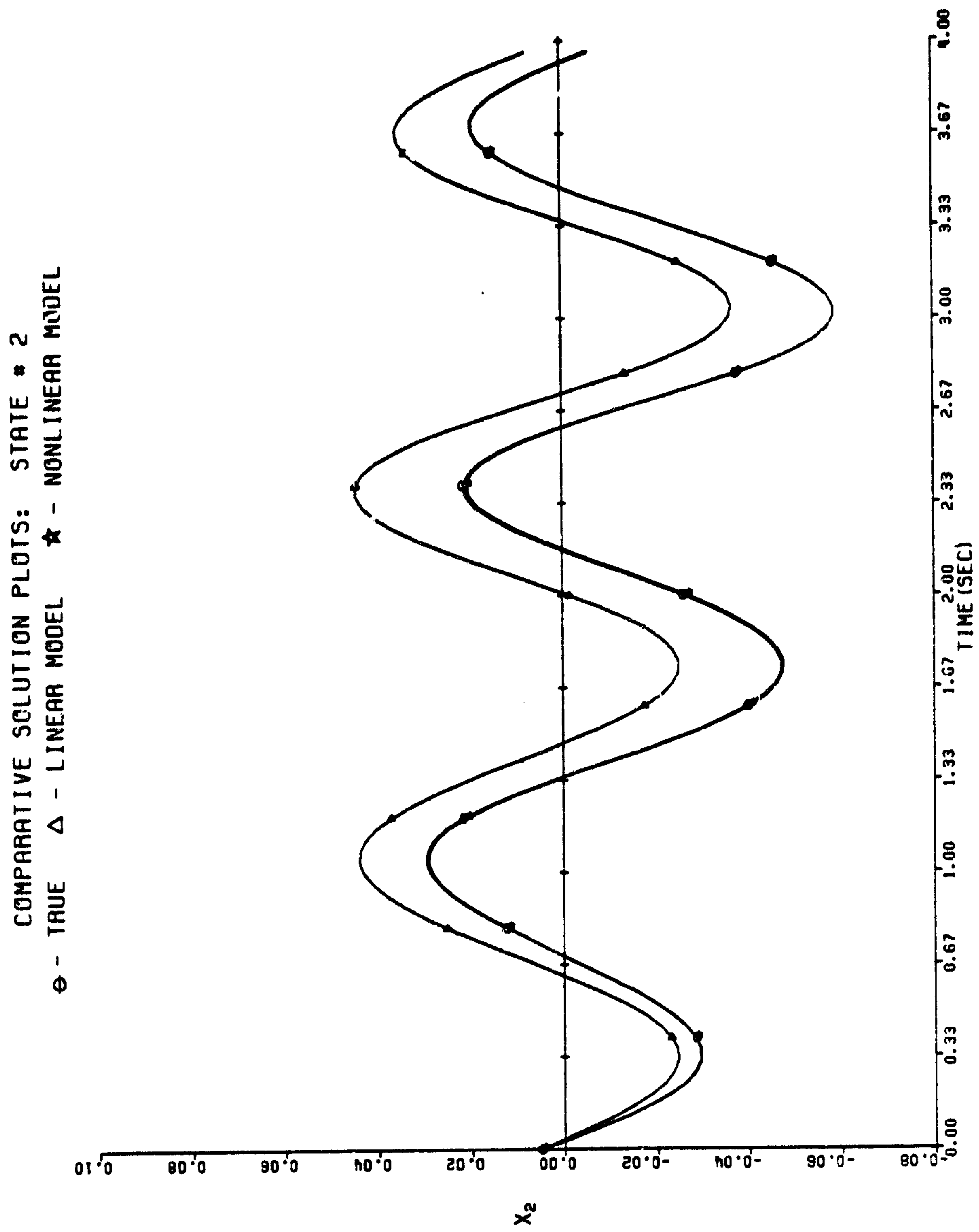
SOLUTION PARAMETERS:

INTEGRATION STEPSIZE: 0.010
UPPER TIME LIMIT OF INTEGRATION: 4.000
NUMBER OF PLOT POINTS: 100
SPACING BETWEEN PLOT POINTS: 0.040

STATE NUMBER	INITIAL CONDITION	MAXIMUM ERROR
1	0.005	-0.400E-01
2	0.005	-0.250E-01

CONTROL NUMBER	AMPLITUDE	FREQUENCY (CYCLES/SEC)
1	0.150	0.750
2	0.150	1.000


COMPARATIVE SOLUTION PLOTS: STATE # 1
⊕ - TRUE Δ - LINEAR MODEL ★ - NONLINEAR MODEL
0.00
0.33
0.67
1.00
1.33
1.67
2.00
2.33
2.67
3.00
3.33
3.67
4.00
TIME (SEC)
0.10
0.08
0.06
0.04
0.02
0.00
-0.02
-0.04
-0.06
-0.08
X1


COMPARATIVE SOLUTION PLOTS: STATE # 2
Φ - TRUE Δ - LINEAR MODEL ☆ - NONLINEAR MODEL
X_2
TIME (SEC)
0.00
0.33
0.67
1.00
1.33
1.67
2.00
2.33
2.67
3.00
3.33
3.67
4.00
-0.08
-0.06
-0.04
-0.02
0.00
0.02
0.04
0.06
0.08
0.10

PLOT SET # 7

```
**************** PROBLEM SUMMARY ***************
    CONFIGURATION:  TRUE, LINEAR & NONLINEAR
    NUMBER OF STATES:  2
    NUMBER OF CONTROLS:  2
    LENGTH OF TENSOR TERM VECTOR:  14
    DEGREE OF APPROXIMATION:  2
************************************************
```

SOLUTION PARAMETERS:
****

INTEGRATION STEPSIZE: 0.010
UPPER TIME LIMIT OF INTEGRATION: 10.000
NUMBER OF PLOT POINTS: 100
SPACING BETWEEN PLOT POINTS: 0.100

STATE NUMBER	INITIAL CONDITION	MAXIMUM ERROR
1	0.001	-0.566E-01
2	-0.001	-0.553E-01

CONTROL NUMBER	AMPLITUDE	FREQUENCY (CYCLES/SEC)
1	-0.050	0.050
2	0.350	0.050

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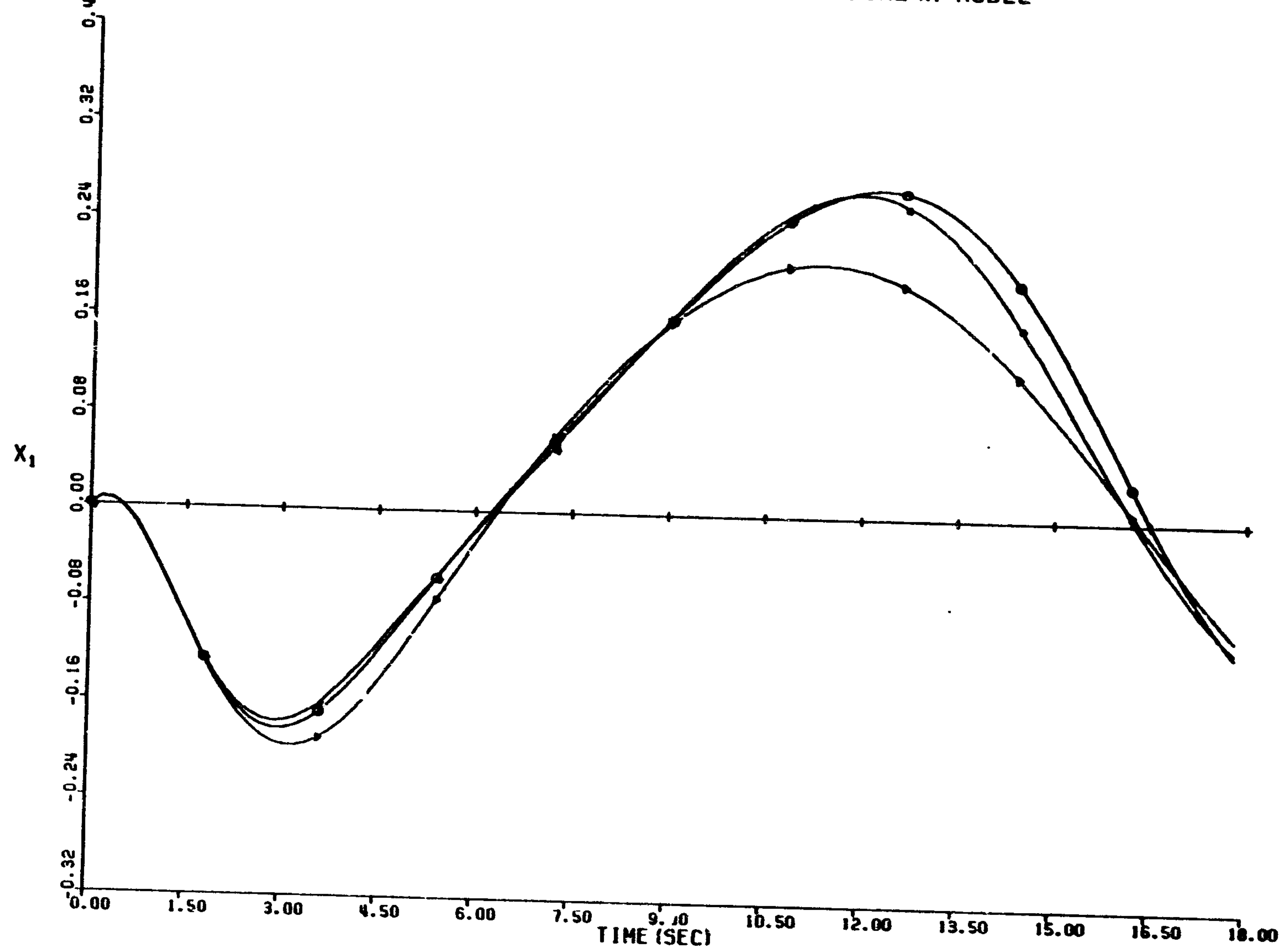
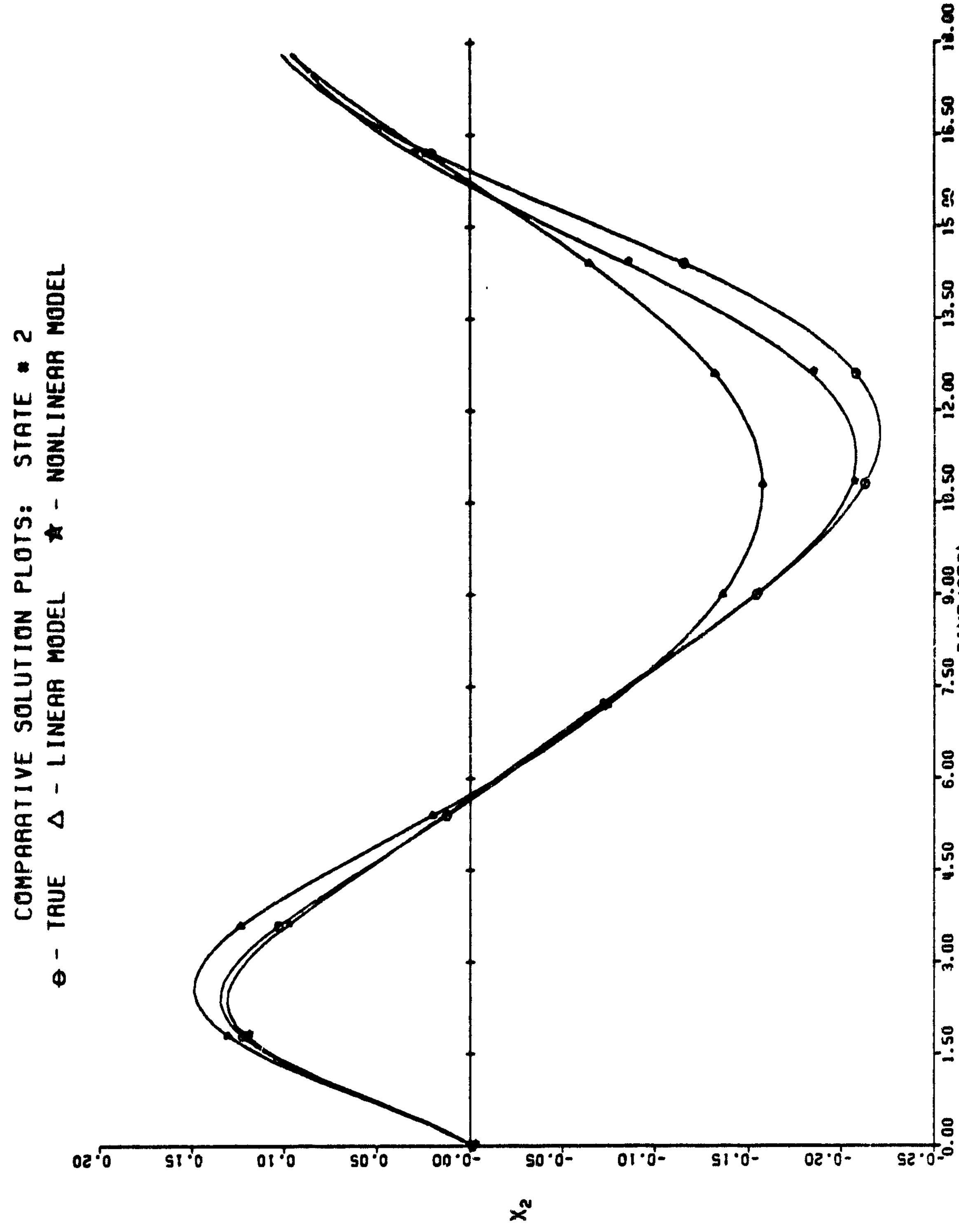
COMPARATIVE SOLUTION PLOTS: STATE * 1

⊖ - TRUE   Δ - LINEAR MODEL   ★ - NONLINEAR MODEL





COMPARATIVE SOLUTION PLOTS: STATE # 2
⊕ - TRUE Δ - LINEAR MODEL ✪ - NONLINEAR MODEL
TIME (SEC)
X_2
0.00
1.50
3.00
4.50
6.00
7.50
9.00
10.50
12.00
13.50
15.00
16.50
18.00
-0.25
-0.20
-0.15
-0.10
-0.05
-0.00
0.05
0.10
0.15
0.20

PLOT SET # 8

```
*************** PROBLEM SUMMARY ***************
    CONFIGURATION:  TRUE, LINEAR & NONLINEAR
    NUMBER OF STATES:  2
    NUMBER OF CONTROLS:  2
    LENGTH OF TENSOR TERM VECTOR:  14
    DEGREE OF APPROXIMATION:  2
***********************************************
```

SOLUTION PARAMETERS:

INTEGRATION STEPSIZE: 0.010
UPPER TIME LIMIT OF INTEGRATION: 10.000
NUMBER OF PLOT POINTS: 100
SPACING BETWEEN PLOT POINTS: 0.100

STATE NUMBER	INITIAL CONDITION	MAXIMUM ERROR
1	0.001	-0.732E-01
2	-0.001	-0.645E-01

CONTROL NUMBER	AMPLITUDE	FREQUENCY (CYCLES/SEC)
1	-0.075	0.000
2	-0.010	0.000

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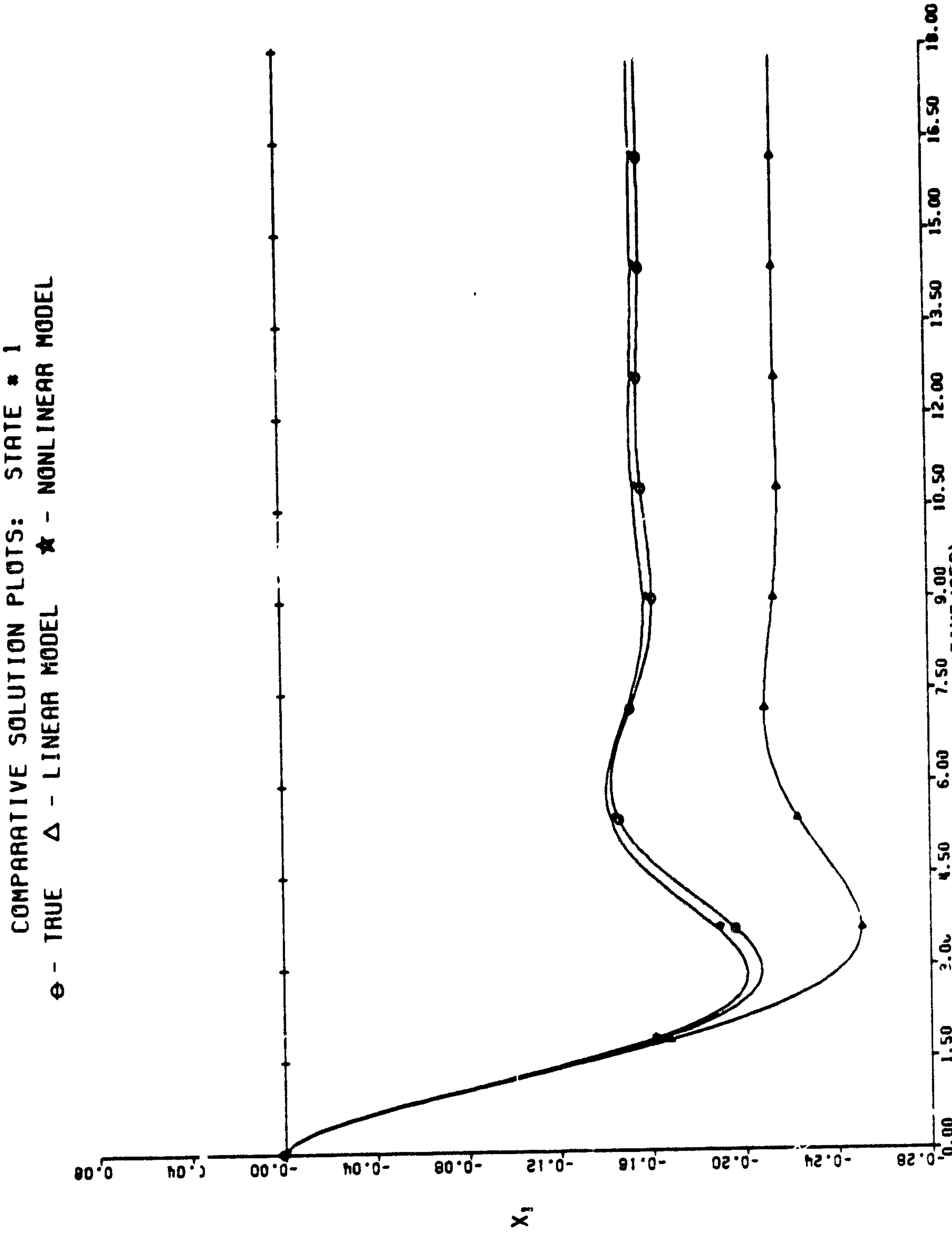
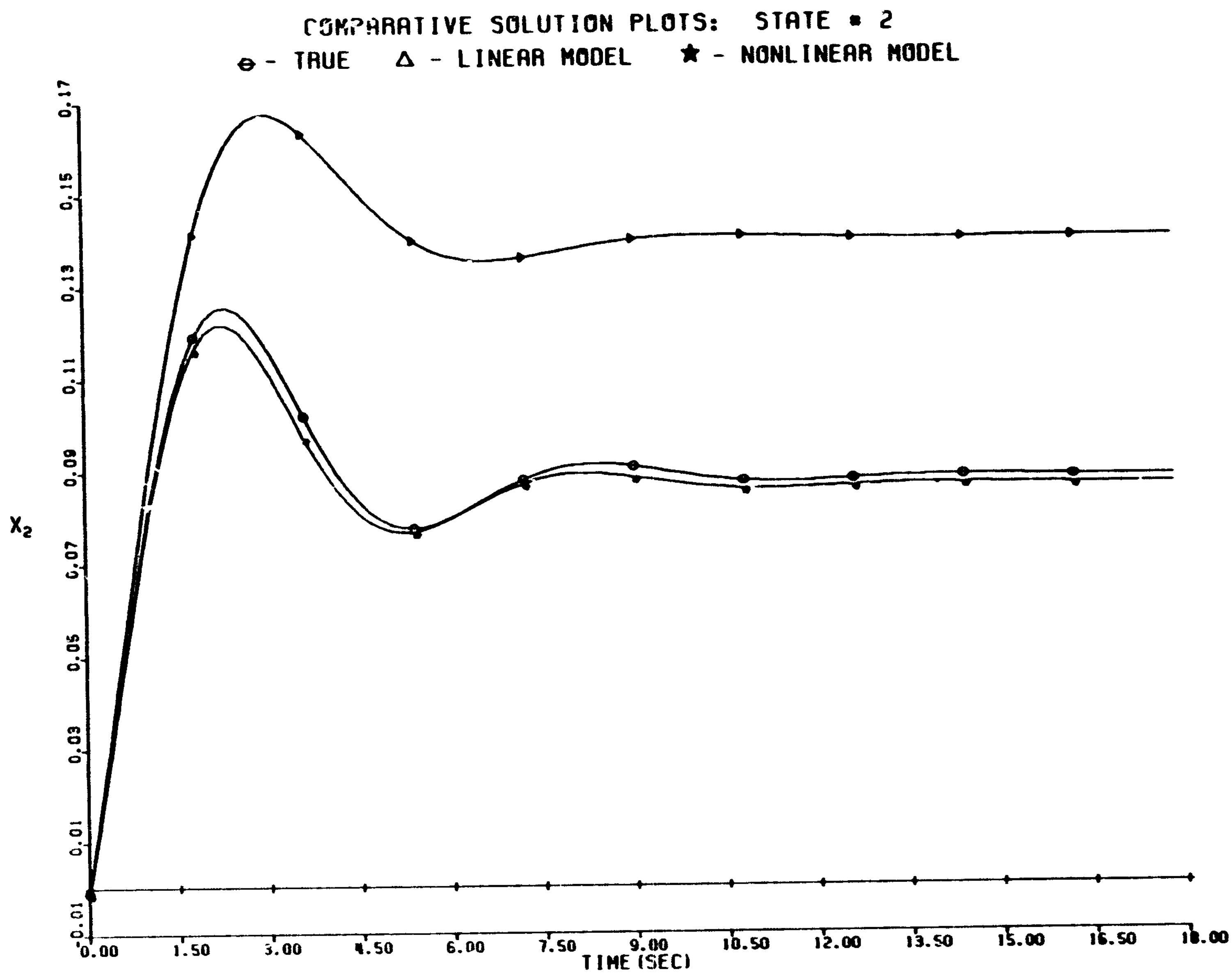

COMPARATIVE SOLUTION PLOTS: STATE # 1
Φ - TRUE
Δ - LINEAR MODEL
★ - NONLINEAR MODEL
X_1
TIME (SEC)
0.08
-0.00
-0.04
-0.08
-0.12
-0.16
-0.20
-0.24
-0.28
0.00
1.50
4.50
6.00
7.50
9.00
10.50
12.00
13.50
15.00
16.50
18.00


COMPARATIVE SOLUTION PLOTS: STATE = 2
⊖ - TRUE Δ - LINEAR MODEL ★ - NONLINEAR MODEL
X2
0.17
0.15
0.13
0.11
0.09
0.07
0.05
0.03
0.01
-0.01
0.00
1.50
3.00
4.50
6.00
7.50
9.00
10.50
12.00
13.50
15.00
16.50
18.00
TIME (SEC)

V. PROGRESS ON NONLINEAR FEEDBACK FROM TENSOR MODELS

Work on this aspect of the research has been underway only a few months. The goal is to assess the practical issues involved in an implementation of the nonlinear feedback scheme proposed by Buric [20] for use on tensor models.

In particular, it is desired to determine exactly what is involved in calculating the feedback gains, to study whether the theory must be applied without modification or whether it may be possible to begin with certain simplification of method, and to carry out the software steps needed to execute a nontrivial example.

At the time of this report, the group is nearing completion of the first of the three steps above. The principal issues involved for the second of the three steps appear to be the following: (1) How intrinsic is the use of duality, which necessitates an indirect approach to vector-valued tensors and applies the less-than-intuitive method of contractions? (2) Should the initial example employ the symmetric tensor algebra, with its additional learning overhead but with computational advantages, or should it employ the easier and more intuitive parent algebra embodying both symmetry and skew symmetry?

Decisions on these issues are expected to be made in the near future.

VI. CONCLUSIONS

This report has described progress on NASA Grant NSG-3048, entitled "Alternatives for Jet Engine Control", during the twelve month period beginning on October 1, 1980 and ending on September 30, 1981. Included have been reports on modeling theory, controller scheduling, interactive parameter selection, and nonlinear feedback from tensor models.

In light of rapidly evolving capabilities of microcomputers and minicomputers, in view of the qualitative tensor model possibilities established earlier by Mr. Stephen Yurkovich, and taking into account both the state of the art and prospects for further advance in tensor techniques for feedback from such models, we believe that current progress continues to point out significant new opportunities for productive research in this area.

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APPENDIX A

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APPENDIX B

Reprint

"Sensitivity Issues in Decoupled Control System Design"

M.K. Sain, A. Ma, D. Perkins

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SENSITIVITY ISSUES IN DECOUPLED CONTROL SYSTEM DESIGN*

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Abstract

Recent discussions in the literature have pointed out the existence of decoupled control system examples with high classical margins in the individual loops but little loop stability tolerance to gain variation. This paper points out the applicability of the Cruz-Perkins sensitivity matrix to design problems involving such plants, which may be foreseen with graphically interactive methods developed by Schafer and Sain.

Introduction

Consider the system of Figure 1. Here r represents a vector of requests, u a vector of control actions, and y a vector of plant responses. Assume that

$$y = P\,u\,, \tag{1}$$

$$u = M\,r\,, \tag{2}$$

$$y = T\,r\,, \tag{3}$$

for appropriate linear operators P, M, and T. Combine these three equations so that

$$Tr = PMr, \tag{4}$$

and require that (4) hold for all requests r. Then

$$T = PM, \tag{5}$$

or in matrix form

$$[P \mid -I]\begin{bmatrix} M \\ \hline T \end{bmatrix} = [0]. \tag{6}$$

Any possible control action-plant response pair (u,y) can thus be represented as an element in

$$\ker\,[P \mid -I], \tag{7}$$

where the kernel can be conceived either in vector space or module theoretic terms.

A very broad type of feedback system within the class indicated by Figure 1 has been studied by Bengtsson [1], for the case in which the plant matrix $[P(s)]$ arises from a controllable and observable triple (A,B,C) with B monic. In particular, there exists an internally stable feedback realization of Bengtsson type for matrices $[M(s)]$ and $[T(s)]$ satisfying (6) if and only if $[M(s)]$ is proper and both $[M(s)]$ and $[T(s)]$ are stable. For an extension to the case $[P(s)]$ proper, see [2].

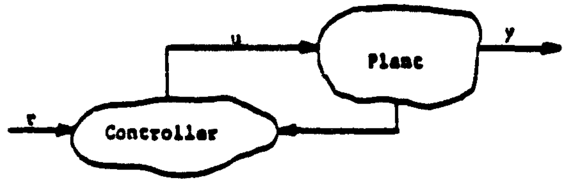

Figure 1.

Case in Point

The introductory discussion above leads rapidly to some very practical guidelines when $[T(s)]$ is diagonal and nonsingular and when $[P(s)]$ is square. This is the case of desired decoupling. From (5), $[P(s)]$ will have to be invertible if decoupling is to be attained.

For discussion in this paper, we wish to make use of the example

$$[P(s)] = \begin{bmatrix} \frac{9}{s+1} & -\frac{10}{s+1} \\ -\frac{8}{s+2} & \frac{9}{s+2} \end{bmatrix} \tag{8}$$

studied by Rekasius [3], who in turn attributes it to J.C. Doyle. A straightforward calculation then gives

$$[P(s)]^{-1} = \begin{bmatrix} 9(s+1) & 10(s+2) \\ 8(s+1) & 9(s+2) \end{bmatrix}. \tag{9}$$

For decoupling,

$$T(s) = \begin{bmatrix} t_{11}(s) & 0 \\ 0 & t_{22}(s) \end{bmatrix}, \tag{10}$$

*This work was supported in part by the Office of Naval Research under Contract Number N 00014-79-C-0475 and in part by the National Aeronautics and Space Administration under Grant NSG 3048.

and (5) implies

$$[H(s)] = [P(s)]^{-1}[T(s)] = \begin{bmatrix} 9(s+1)t_{11} & 10(s+2)t_{22} \\ 8(s+1)t_{11} & 9(s+2)t_{22} \end{bmatrix} \quad (11)$$

which for internally stable feedback realizability must be proper and stable by the Bengtsson conditions. The choice

$$T(s) = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} \quad (12)$$

meets the conditions. In fact, this choice can be realized by an output error feedback configuration of the type shown in Figure 2, while maintaining internal stability, with the choice

$$G(s) = \begin{bmatrix} 9 & 10 \\ 8 & 9 \end{bmatrix}. \quad (13)$$

The reasons for the interest attracted by this example can be explained as follows. The forward path gain

$$[Q(s)] = [P(s)][G(s)] \quad (14)$$

in Figure 2 is

$$\begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}. \quad (15)$$

Ideally, then, the two-input, two-output problem has been reduced to two one-input, one-output problems. Moreover, each of the one-input, one-output problems has <u>infinite gain margin</u> in the usual classical sense. Unfortunately, the infinite gain margin is but an illusion, as shown in the following section.

Stability Analysis

The key to a stability margin determination, for an operator gain with matrix

$$[K] = \begin{bmatrix} 1+k_1 & 0 \\ 0 & 1+k_2 \end{bmatrix} \quad (16)$$

inserted between $[P(s)]$ and $[G(s)]$, is the relation

$$CLCP(s) = |I+[P(s)][K][G(s)]| \; OLCP(s) \quad (17)$$

between the closed loop and open loop characteristic polynomials CLCP(s) and OLCP(s) respectively. For this case, (17) becomes

$$CLCP(s) = s^2 + (5 + k_1 + k_2)s + (6 + 83k_1 - 78k_2 + k_1k_2 . \quad (18)$$

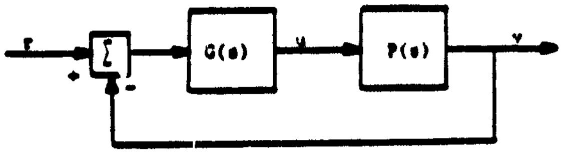

Figure 2.

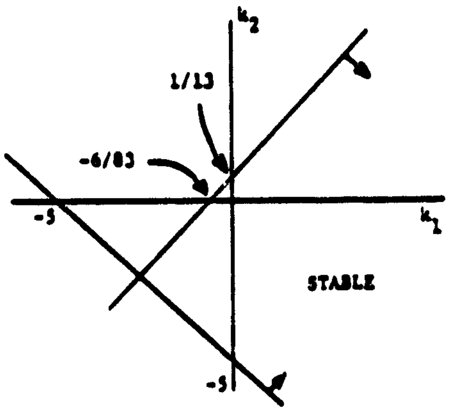

Figure 3. (Not to scale).

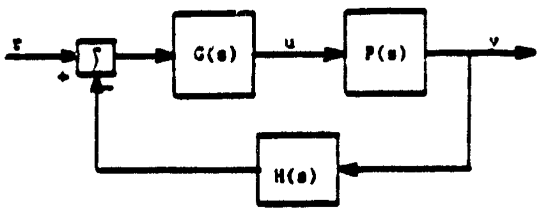

Figure 4.

which generates a stability region

$$5 + k_1 + k_2 > 0 \quad (19)$$

$$6 + 83k_1 - 78k_2 + k_1k_2 > 0 \quad (20)$$

in the (k_1,k_2) plane. This region is sketched in Figure 3, which is not, however, drawn to scale. For this sketch, the boundary of (20) was written

$$k_2 = (6 + 83k_1)/(78 - k_1), \quad (21)$$

and the assumption $|k_1| \ll 78$ gives a close approximation to a straight line. Figure 3 makes clear that very small changes in the length of control space basis vectors can destabilize the loop.

Sensitivity Matrix Design

The stability margin observations of the two sections preceding are instructive, in that they point out the potential fallacies which can be associated with filter design of control loops, when sensitivity aspects have not been explicitly considered.

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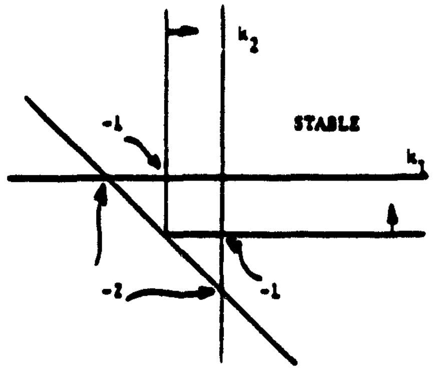

Figure 5.

To incorporate this aspect, we make use of the comparison sensitivity matrix [4] of Cruz and Perkins, within the feedback structure of Figure 4. For the design of $G(s)$ and $H(s)$, we make use of the equations

$$G = P_0^{-1} S_0^{-1} T_0, \tag{22}$$

$$H = T_0^{-1} (I - S_0), \tag{23}$$

where the subscript (0) denotes nominal representations of the plant P, the response operator T, and the Cruz-Perkins operator S, the last satisfying a well known equation

$$S_0 = (I + P_0 GH)^{-1}. \tag{24}$$

Regard (22) and (23) as design equations in terms of a given plant and of specifications on filter response T_0 and comparison sensitivity S_0. For T_0, choose (12); then we maintain the same nominal filter behavior obtained in Figure 2. For S_0, choose

$$[S_0(s)] = \frac{.0159s}{.0159s + 1} I , \tag{25}$$

which represents an improvement both in gain and bandwidth over the comparison sensitivity matrix

$$\begin{bmatrix} \frac{s+1}{s+2} & 0 \\ 0 & \frac{s+2}{s+3} \end{bmatrix} \tag{26}$$

which occurs in Figure 2 when $[H(s)]$ is I. It is straightforward to calculate

$$[G(s)] = \begin{bmatrix} \frac{9(s+1)(.0159s+1)}{.0159s\ (s+2)} & \frac{10(s+2)(.0159s+1)}{.0159s\ (s+3)} \\ \frac{8(s+1)(.0159s+1)}{.0159s\ (s+2)} & \frac{9(s+2)(.0159s+1)}{.0159s\ (s+3)} \end{bmatrix}, \tag{27}$$

$$[H(s)] = \begin{bmatrix} \frac{s+2}{.0159s+1} & 0 \\ 0 & \frac{s+3}{.0159s+1} \end{bmatrix} \tag{28}$$

for which it follows that

$$\begin{aligned} CLCP(s) = (.0159^2 s^2 + .0159s\ (2 + k_1 + k_2) \\ + (1 + k_1 + k_2 + k_1 k_2))\ (.0159s + 1)^2 \\ \cdot (s+1)\ (s+2)^2\ (s+3), \end{aligned} \tag{29}$$

and that

$$2 + k_1 + k_2 > 0 \tag{30}$$

$$1 + k_1 + k_2 + k_1 k_2 > 0 \tag{31}$$

are counterparts of (19) and (20). The new stability region is sketched in Figure 5.

Clearly, the shape and character of the stability region shown in this figure represents a substantial improvement over that of Figure 3. In effect, the boundaries shown in Figure 5 are very practical, because $[K]$ in (16) becomes singular at those points. As a result, $[P(s)][K]$, considered as a new plant matrix $[\hat{P}(s)]$, would be singular and could not produce a decoupled system.

Plant Character

The special character of the Rekasius-Doyle example could have been foreseen before a decoupling design was completed. To see this, notice that a decoupled $T(s)$ in configuration of the Figure 2 will always generate a diagonal forward path operation $Q(s)$, as in (14). As a consequence, $G(s)$ may be regarded as a pre-compensator chosen to achieve column dominance of $Q(s)$. A graphically interactive procedure for assessing such questions has been developed by Schafer and Sain [5,6]. Precompensation is taken to be of the form

$$[\hat{G}(s)] = \begin{bmatrix} 1 & r_{12}(s)+ji_{12}(s) \\ r_{21}(s)+ji_{21}(s) & 1 \end{bmatrix}. \tag{32}$$

For s an element of a Nyquist contour, pairs $(r_{mn}(s), i_{mn}(s))$ which achieve column dominance of $Q(s)$ are visualized as interiors of solid circles or exteriors of dashed circles in the (r_{mn}, i_{mn}) plane. As s follows the Nyquist contour, these circles generate a CARDIAD (Complex Acceptability Region for DIAgonal Dominance) plot. Figures 6 and 7 give the CARDIAD plots for the original plant (8). Now write (13) in the form

$$[G(s)] = \begin{bmatrix} 1 & 10/9 \\ 8/9 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \tag{33}$$

$$= [\hat{G}(s)]\ [9I] . \tag{34}$$

In Figure 6, observe that $(8/9 = .888...)$ is just slightly to the left of all the dashed circles; in Figure 7, $(10/9 = 1.111...)$ is within all the solid circles. Thus the CARDIAD plot predicts column dominance of $\hat{Q}(s)$. However, the situation with regard to this dominance condition is precarious, inasmuch as $(8/9)$ remains quite close to the dashed circles while $(10/9)$ must be delicately placed to remain inside all the solid circles. Figures 8 and 9, and Figures 10 and 11, present the CARDIAD plots of $[\hat{P}(s)]$ for (k_1, k_2) equal to $(0,-.1)$ and $(0,.1)$ respectively. The former is a stable condition, the latter unstable, as seen in

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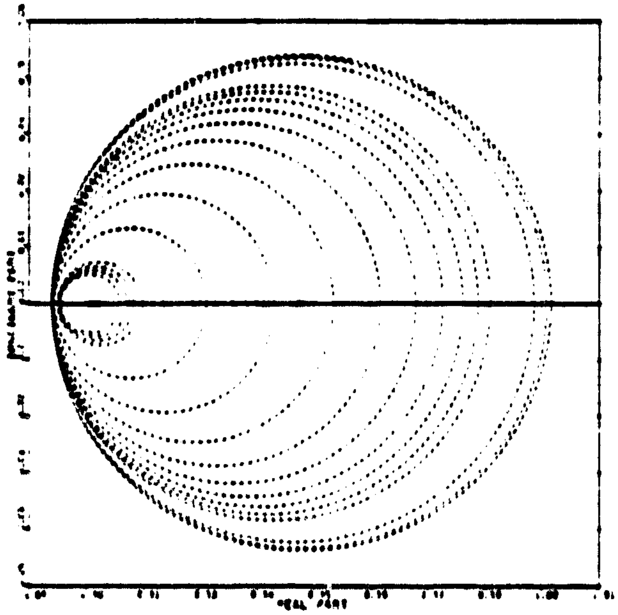

Figure 6.

CARDIAD PLOT: 2,1 ENTRY: IDENTITY COMPENSATOR 3/07/90

INPUTS = 2, SYSTEM ORDER = 2, ANALYSIS TYPE = 1, FORM = 1

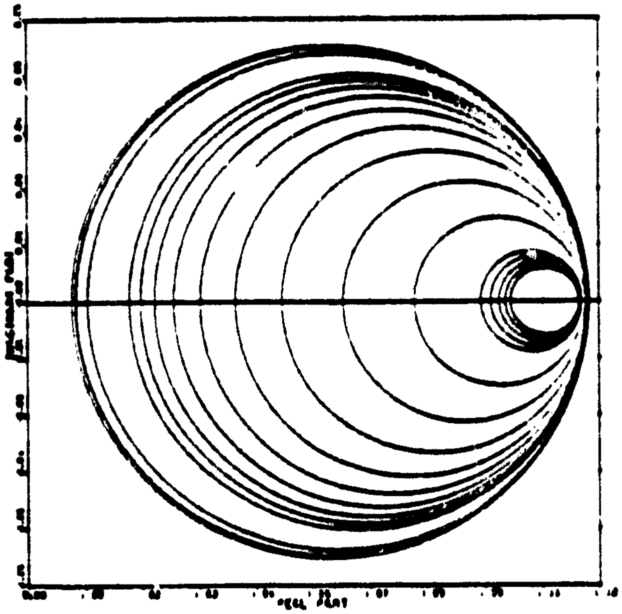

Figure 7.

CARDIAD PLOT: 1,2 ENTRY: IDENTITY COMPENSATOR 3/07/90

INPUTS = 2, SYSTEM ORDER = 2, ANALYSIS TYPE = 1, FORM = 1

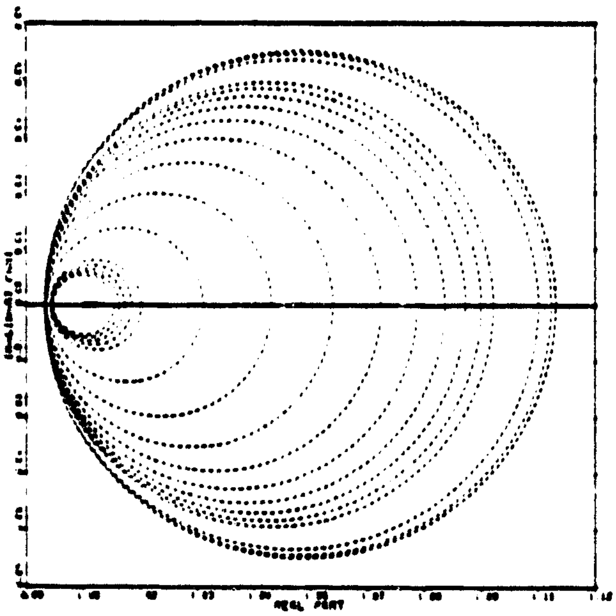

Figure 8.

CARDIAD PLOT 2,1 ENTRY: LCL 2 OF COMP ALTS .9 1/08/90

INPUTS = 2, SYSTEM ORDER = 2, ANALYSIS TYPE = 1, FORM = 1

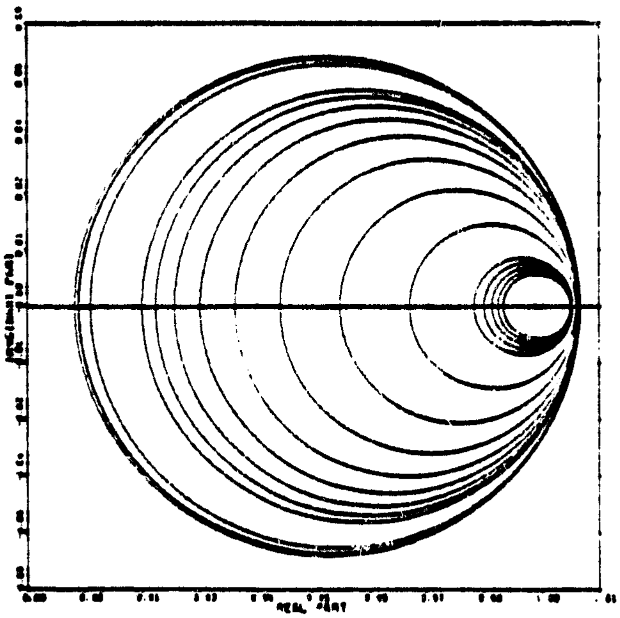

Figure 9.

CARDIAD PLOT: 1,2 ENTRY: LCL 2 OF COMP ALTS .8 1/08/90

INPUTS = 2, SYSTEM ORDER = 2, ANALYSIS TYPE = 1, FORM = 1

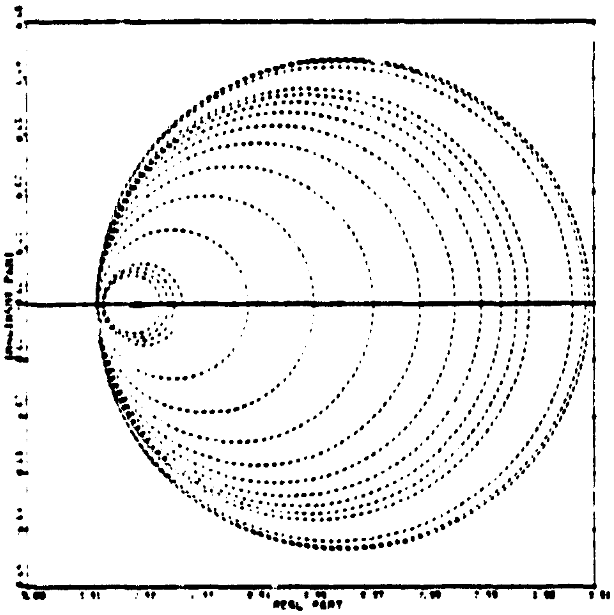

Figure 10.

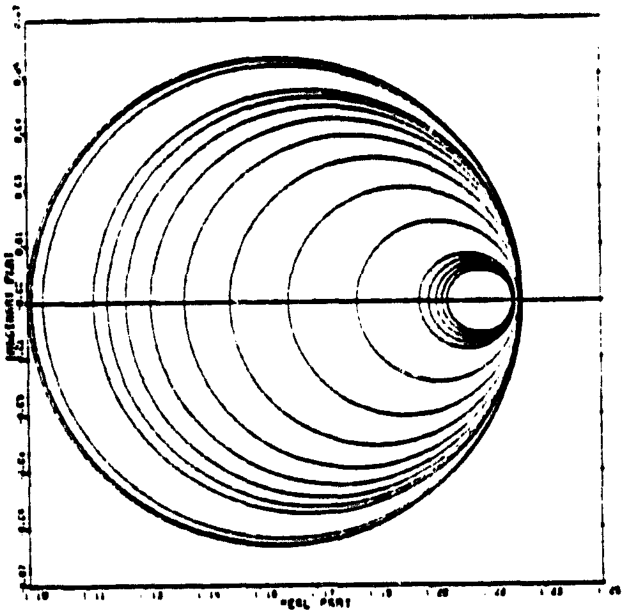

Figure 11.

Figure 3. Figure 9 shows that column two fails to be dominant at all s; and Figures 10 and 11 show both columns failing dominance—relative to (33)-(34). When the dominance condition fails, the individual loop stability arguments based upon Rosenbrock's theorem [7] fail; and this is an indication of robustness difficulty to be expected in decoupling the plant.

Conclusions

This paper has discussed the use of procedures developed by Schafer and Sain to forecast stability robustness problems in decoupling control systems and has illustrated the use of the Cruz-Perkins comparison sensitivity idea to carry out design.

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APPENDIX C

Reprint

"Quotient Signal Flowgraphs: New Insights"

M.K. Sain

Proceedings IEEE International Conference on Circuits and Computers

Page 417

October 1980

QUOTIENT SIGNAL FLOWGRAPHS: NEW INSIGHTS

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SUMMARY

When large scale, interconnected systems can be described in terms of signal flow graphs, there is available a natural algebraic way in which to regard generalized model "order reduction". The basic idea is to regard the node variables as abelian group valued and to consider the mappings from node to node as homomorphisms of groups. Then variable simplification on node variables can be established by projections onto quotient groups. If the node-to-node homomorphisms are correctly related to the kernels of these projections, then such a construction induces a new set of unique node-to-node homomorphisms on the cosets of original node variables. One feature of the resulting quotient signal flow graph is that it preserves the connection structure of the original system. Another feature is that the projections induced on node-to-node homomorphisms are interchangeable with basic flow graph operations. [1, 2]. This presentation reviews the notions above, extends them to the feedback case, and discusses the possibility for generalizations.

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APPENDIX D

Reprint

"Exterior Algebra and Simultaneous Pole-Zero Placement"

V. Seshadri

Proceedings Conference on Information Sciences and Systems

1981

EXTERIOR ALGEBRA AND SIMULTANEOUS POLE-ZERO PLACEMENT

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ABSTRACT

The ubiquity of the determinant of return difference in time-invariant multivariable linear system studies can be attributed to the multivariable feedback loop control prevalent in such systems. As a link between open-loop and closed-loop characteristic polynomials, as the fundamental entity leading to most generalized Nyquist studies, and indeed as the key quantity in solving the generalized loop, this determinant is of essential algebraic importance. Consequently, exterior algebras designed explicitly for the analysis of such calculations as determinants can be of assistance in discussions of multivariable systems with feedback control. In this paper the basic usage of an exterior algebra, in determinantal constructions relating to the poles and zeros of individual transfer functions in the closed loop transfer matrix, is illustrated. When such exterior algebras are formed over the input and output vector spaces, the map from the input space to the output space induces a morphism over the algebras; this induced exterior morphism plays a significant role in the simultaneous placement of poles and individual zeros of the transfer matrix at desired locations. It is also expected that the compact expressions for closed-loop individual zeros, rendered transparent by the exterior algebraic structure, would be of general interest inasmuch as they enhance the designer's ability to shape the transient responses of individual system outputs.

INTRODUCTION

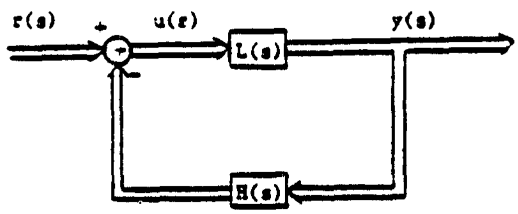

Figure 1. Output Feedback Structure

Consider the system of Figure 1, which consists of a stabilized $m \times m$ plant transfer matrix $L(s)$ and an $m \times m$ output feedback matrix $H(s)$ over the field $F(s)$ of rational functions in s with coefficients from the field F. Note that $r(s)$, $u(s)$ and $y(s)$ are the reference, input and output m-vectors, belonging to $F(s)$-vector spaces R, U and Y respectively. The design of a feedback compensator $H(s)$ as in the above often leads to the following equation (the dependence on s is dropped henceforth for notational simplicity):

$$(I + LH)y = Lr. \tag{1}$$

Figure 1 may be viewed as follows. Once the loop has been closed through H, there results a closed-loop transfer matrix, having in its various rows and columns the individual closed-loop transfer functions. In numerous practical applications, as for example [1], specifications are given in terms of the response of individual outputs to individual references. This means that the zeros of individual closed-loop transfer functions are of considerable importance in design.

Relatively little seems to have been written on this subject. The reasons for the paucity of literature in this area may become clearer if one were to look, again, at Equation (1). Whereas it is generally acknowledged that the return-difference matrix $(I + LH)$ plays a key role in determining the effect of feedback connections on the output responses [2], the explicit nature of the relationship and the precise way in which the return-difference matrix enters the dynamics of the feedback control problem have been difficult to study. This is because the return-difference matrix is generally expressed in the feedback problem as the inverse of a matrix sum.

Building on the work of Sain [3] one can construct an exterior algebraic structure [4,5,6] suited to represent classical adjoints and determinants and hence, by definition, the inverse of the return difference matrix. It has already been demonstrated that this exterior algebraic structure is a useful and easily applied construction for pole assignment in an important class of minimal design problems [7,8,9].

The present paper may roughly be divided into three parts. The first part introduces the exterior algebra. The presentation is extremely brief due to limitations of space; for more details the reader is referred to Greub [4]. The second part of the paper considers the multivariable control problem with full output feedback. Based on the exterior morphisms induced over the algebras, expressions are obtained for the pole polynomial and individual zero polynomials of the closed-loop transfer matrix relating the output vector to the reference vector. In the last part of this paper the above expressions are used, in concert with unity rank feedback, to place poles and individual zeros simultaneously. Depending on the number of specifications given, one may be able to either place the poles and zeros precisely or make a least squares approximate placement. An example of this is included, involving solution using a computer program.

THE EXTERIOR ALGEBRA

Consider the $F(s)$-vector space U of inputs, where $F(s)$ is the field of rational functions in s with coefficients from the field F. An element of the vector space U would be an input vector $u(s)$; in numerical calculations, $u(s)$ would be represented by a column vector whose elements are rational functions in s. An exterior algebra ΛU can be constructed over the vector space U [4]. The bilinear operator introduced by this construction is commonly called the exterior product or the "wedge" product $\wedge$, and operates as

$$(\alpha_1 a_1 + \alpha_2 a_2) \wedge a_3 = \alpha_1 a_1 \wedge a_3 + \alpha_2 a_2 \wedge a_3 \quad (2)$$

$$a_1 \wedge (\alpha_3 a_3 + \alpha_4 a_4) = \alpha_3 a_1 \wedge a_3 + \alpha_4 a_1 \wedge a_4, \quad (3)$$

where a_1, a_2, a_3, a_4 belong to the algebra ΛU, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are field elements from $F(s)$. Furthermore, the operator $\wedge$ is skew-symmetric.

Now consider the map L from the input vector space U to the output vector space Y. If we construct the exterior algebras ΛU and ΛY over the vector spaces U and Y respectively, the map L induces a unique morphism $L^\wedge$ over the algebras [5], which is just a sequence of maps $L_k^\wedge$ over the k^{th} exterior spaces, as shown in Figure 2.

$$\begin{array}{ccccccccccc}
\Lambda U & & F(s) & U & \Lambda_2 U & & \Lambda_k U & & \Lambda_m U \\
\downarrow L^\wedge & : & \downarrow L_0^\wedge = 1 & \downarrow L_1^\wedge = L & \downarrow L_2^\wedge & \cdot\cdot & \downarrow L_k^\wedge & \cdot\cdot & \downarrow L_m^\wedge \\
\Lambda Y & & F(s) & Y & \Lambda_2 Y & \cdot\cdot & \Lambda_k Y & \cdot\cdot & \Lambda_m Y
\end{array}$$

Figure 2. Induced Exterior Morphism $L^\wedge$.

In terms of numerical calculations, the map L is represented by the feedforward transfer matrix, each of whose elements is a rational function in s; the k^{th} exterior map $L_k^\wedge$ is obtained by forming minors of order k from the matrix L according to a predetermined sequence dependent on the choice of basis in exterior spaces [10]. Also, it has been assumed here that the number of inputs is the same as the number of outputs and equals m; thus $L_m^\wedge$ would be represented in numerical calculations by the determinant of the matrix L.

MULTIVARIABLE CONTROL WITH FULL OUTPUT FEEDBACK

Let us now consider expressions for individual closed-loop transfer functions in a general case such as shown in Figure 1. Here L is the feedforward matrix and H is the output feedback matrix. The feedback matrix H is full, in general, and is represented as

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdot\cdot & h_{1m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ h_{m1} & h_{m2} & \cdot\cdot & h_{mm} \end{bmatrix} . \quad (4)$$

Starting with the equation relating the output vector y to the reference vector r, namely,

$$(I + LH)y = Lr \quad (1)$$

one can use the "wedge" operator $\wedge$ associated with the exterior algebras [10] in order to isolate the closed loop transfer function relating the arbitrary a^{th} output y_a to the arbitrary b^{th} reference r_b, namely y_a/r_b, as

$$\frac{y_a}{r_b} = \frac{\det(I+L\ddot{H}) - \det(I+L\bar{H})}{\det(I+LH)} \quad (5)$$

where $\ddot{H}$ and $\bar{H}$ are identical to the feedback matrix H except that the a^{th} columns, corresponding to feedback of the a^{th} output y_a, are sparse as shown below.

$$\ddot{H} = \begin{bmatrix} h_{11} & \cdot\cdot & h_{1a-1} & 0 & h_{1a+1} & \cdot\cdot & h_{1m} \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ h_{b-11} & \cdot\cdot & h_{b-1a-1} & 0 & h_{b-1a+1} & \cdot\cdot & h_{b-1m} \\ h_{b1} & \cdot\cdot & h_{ba-1} & 1 & h_{ba+1} & \cdot\cdot & h_{bm} \\ h_{b+11} & \cdot\cdot & h_{b+1a-1} & 0 & h_{b+1a+1} & \cdot\cdot & h_{b+1m} \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ h_{m1} & \cdot\cdot & h_{ma-1} & 0 & h_{ma+1} & \cdot\cdot & h_{mm} \end{bmatrix} \quad (6)$$

$$\bar{H} = \begin{bmatrix} h_{11} & \cdot\cdot & h_{1a-1} & 0 & h_{1a+1} & \cdot\cdot & h_{1m} \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ h_{m1} & \cdot\cdot & h_{ma-1} & 0 & h_{ma+1} & \cdot\cdot & h_{mm} \end{bmatrix} \quad (7)$$

Thus the a^{th} column of $\bar{H}$ is all zero while the a^{th} column of $\ddot{H}$ has a single 1 in the b^{th} row.

We can now expand the numerator and denominator of Equation (5) in terms of sums of traces [4] as

$$\frac{y_a}{r_b} = \frac{\{[1+tr(L\ddot{H})+tr(L_2^\wedge \ddot{H}_2^\wedge)+..+tr(L_m^\wedge \ddot{H}_m^\wedge) -1-tr(L\bar{H})-tr(L_2^\wedge \bar{H}_2^\wedge)-..-tr(L_m^\wedge \bar{H}_m^\wedge)]\}}{1+tr(LH)+tr(L_2^\wedge H_2^\wedge)+..+tr(L_m^\wedge H_m^\wedge)} \quad (8)$$

The numerator of Equation (8) above can be rearranged [10], based on the linearity of the trace operator, so that the closed loop transfer function relating the a^{th} output to the b^{th} reference is expressed as

$$\frac{y_a}{r_b} = \frac{\{tr[L(\ddot{H}-\bar{H})]+tr[L_2^\wedge(\ddot{H}_2^\wedge-\bar{H}_2^\wedge)] +..+tr[L_m^\wedge(\ddot{H}_m^\wedge-\bar{H}_m^\wedge)]\}}{1+tr[LH]+tr[L_2^\wedge H_2^\wedge]+..+tr[L_m^\wedge H_m^\wedge]} . \quad (9)$$

The above closed-loop expression for the transfer function y_a/r_b contains m terms in the numerator and $m+1$ terms in the denominator; this could mean a lot of terms if m is large. If, however, m is moderate, say $m = 5$ corresponding to five references and five outputs, the number of terms in Equation (9) would actually be quite small. In either case the number of terms is likely to be reduced because it depends on the number of individual feedback loops that are closed and, ultimately, on the rank of the feedback matrix H. This fact is used in later sections, where the extreme case of unity rank feedback is considered in order to obtain simple expressions for the closed loop pole polynomial and individual zero polynomials.

Let us now consider, in more detail, the matrices in the numerator of (9). Because $\hat{H}$ and $\bar{H}$ are almost identical, the matrix

$$\hat{H}_{\hat{k}} - \bar{H}_{\hat{k}} \tag{10}$$

will be sparse. Specifically, the matrix (10) may be calculated via first obtaining a new matrix $H_{(b)(a)}$ of order $m-1$ from H by striking out the b^{th} row and a^{th} column of H, and then calculating

$$(H_{(b)(a)})_{\widehat{k-1}}. \tag{11}$$

This is consistent with the result obtained in an earlier paper [11] that the zeros of the transfer function relating the a^{th} output to the b^{th} reference <u>cannot</u> be moved by means of feedback from the a^{th} output or to the b^{th} reference.

Corresponding to this reduction of (10) to form the reduced matrix $(H_{(b)(a)})_{\widehat{k-1}}$, the matrix $L_{\hat{k}}$ to which this matrix is to be multiplied may also be reduced in size to include only those elements which are involved in the matrix product and in the trace calculations of Equation (9). Recall that the elements of $L_{\hat{k}}$ are all possible minors of order k formed from the feedforward matrix L. The part of $L_{\hat{k}}$ relevant here consists of those elements which result from minors of L of order k that include l_{ab}, l_{ab} being the feedforward transfer function relating the a^{th} output to the b^{th} reference [10]. Thus the product involves the reduced feedforward matrix $(L_{\hat{k}})_{ab}$ and the reduced feedback matrix $(H_{(b)(a)})_{\widehat{k-1}}$ so that the closed loop transfer function relating the a^{th} output to the b^{th} reference may be expressed as

$$\frac{y_a}{r_b} = \frac{\{l_{ab}+tr[(L_{\hat{2}})_{ab}H_{(b)(a)}]+tr[(L_{\hat{3}})_{ab}(H_{(b)(a)})_{\hat{2}}]+..+tr[(L_{\hat{k}})_{ab}(H_{(b)(a)})_{\widehat{k-1}}]\}}{1+tr[LH]+tr[L_{\hat{2}}H_{\hat{2}}]+..+tr[L_{\hat{m}}H_{\hat{m}}]} . \tag{12}$$

<u>ROOT PLACEMENT UNDER RANK-ONE FEEDBACK</u>

In the previous section, expressions were obtained for the arbitrary individual closed loop transfer function, that is, for elements of the closed loop transfer matrix, with feedback from the m outputs to the m comparison points. In this section, we use these expressions in order to design a constant output feedback matrix H so as to place the poles and certain zeros of interest. In addition, we treat the special case where the output feedback matrix is restricted to unity rank by predefining its structure in dyadic form. Whereas this restriction reduces the number of roots that can be placed arbitrarily, it has the strong advantage of resulting in a bilinear relationship between the feedback matrix and the closed loop characteristic polynomial, thus simplifying the calculation of the feedback matrix; hence it has attracted considerable attention in recent years [12,13].

Our approach to using rank-one feedback involves the simultaneous placement of poles and certain individual zeros of interest. The general expressions for the closed loop poles and individual closed loop zeros, in the case of output feedback by means of a matrix H, have already been derived in the previous section; Figure 1 is relevant here. Let the feedback matrix H be expressed in dyadic form as

$$H = fg^T \tag{14}$$

where f and g belong to R^m, that is,

$$f = (f_1..f_m)^T,\ g = (g_1..g_m)^T. \tag{15}$$

The closed loop characteristic polynomial of the system may be expressed in terms of the open loop characteristic polynomial as

$$CLCP = det[I + LH]OLCP. \tag{16}$$

In the previous section, we saw that $det[I+LH]$ may be written as a sum of traces as

$$det[I+LH]=1+tr(LH)+tr(L_{\hat{2}}H_{\hat{2}})+..+tr(L_{\hat{m}}H_{\hat{m}}) \tag{17}$$

where $L_{\hat{k}}$ is the k^{th} exterior map induced by the map L. Based on the above equation and on the assumption that the feedforward matrix L is adjusted so that its common denominator, den L, is the open loop characteristic polynomial, we can rewrite Equation (16) for the closed loop characteristic polynomial (CLCP) as

$$CLCP = [1+tr(LH)+..+tr(L_{\hat{m}}H_{\hat{m}})]denL. \tag{18}$$

Because the feedback matrix H has rank one, the induced exterior maps $H_{\hat{k}}$, $k = 2, \ldots, m$, are all zero, so that the above expression simplifies as shown below, using the dyadic description of H in (14).

$$CLCP = [1+tr(LH)]den\ L = den\ L + g^T(num\ L)f, \tag{19}$$

where num L is the numerator of the fe. orward transf.r matrix L. Rewriting the expression (19) in order to state explicitly the dependency on s, and calling the CLCP equivalently as the pole polynomial p(s), we thus have

$$p(s) = den\ L(s) + g^T[num\ L(s)]f. \tag{20}$$

Hence in order to place the closed loop poles of the system we would need to find suitable values

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for the components g and f of the feedback matrix H so that the right-hand side of Equation (20) is identical to $p(s)$. Assuming that we are interested in placing n poles, we would have n bilinear equations in $2m$ unknowns $g_1,\ldots,g_m$ and $f_1,\ldots,f_m$. The approach to solving this problem, if we were to place only poles and ignore the individual zeros, would depend on the value of n in comparison to m. However we are interested in placing some individual zeros also in addition to the poles, and hence we will defer solution of the problem until we have rewritten the expression for closed loop zeros in a convenient form so that the problem may be approached in a comprehensive manner.

Just as the expression for the closed loop pole polynomial was simplified considerably because the feedback is of unity rank, so also is the expression for the individual zero polynomial $z_{ab}(s)$ which is the numerator of the transfer function relating the closed loop output y_a to the reference r_b. Specifically, if the plant has m references and m outputs with feedback from all outputs to all comparison points, the expression for the numerator of the closed loop transfer function y_a/r_b with rank-one feedback becomes

$$\left(\frac{y_a}{r_b}\right)_{num} = L_{ab}+tr[(\hat{L_2})_{ab}H_{(b)(a)}]. \qquad (21)$$

where we recall that $(\hat{L_2})_{ab}$ is a sub-matrix of $\hat{L_2}$ obtained by including only those minors of order two which involve L_{ab}; also, $H_{(b)(a)}$ is a sub-matrix of the feedback matrix H, obtained by deleting from H the b^{th} row (corresponding to the reference r_b) and the a^{th} column (corresponding to the output y_a), as discussed in the previous section. Because we multiplied the denominator of the closed loop y_a/r_b expression by den L to get the pole polynomial $p(s)$, so also do we multiply the numerator, as described in Equation (21), by den L and obtain the individual closed loop numerator polynomial or zero polynomial. Thus

$$z_{ab}(s) = n_{ab}(s)+tr[(\hat{L_2})_{ab}H_{(b)(a)}]\text{den } L(s) \qquad (22)$$

where $z_{ab}(s)$ and $n_{ab}(s)$ are the zero polynomials of the closed loop and open loop transfer functions, respectively, relating the a^{th} output to the b^{th} reference. Just as the feedback matrix H has been expressed as the outer product of m-length vectors f and g, we can similarly express the sub-matrix $H_{(b)(a)}$ as the outer product of $(m-1)$-length vectors, as

$$H_{(b)(a)} = f_{(b)}g_{(a)}^T,$$

where $\quad f_{(b)} = (f_1\cdots f_{b-1}f_{b+1}f_m)^T$

and $\quad g_{(a)} = (g_1\cdots g_{a-1}g_{a+1}\cdots g_m)^T.$

Then

$$z_{ab}(s) = n_{ab}(s)+g_{(a)}^T(\hat{L_2})_{ab}(s)f_{(b)}\text{den}L(s). \qquad (23)$$

Equation (23) above is the desired expression for the closed loop zero polynomial of y_a/r_b, bilinear in the arguments $g_{(a)}$ and $f_{(b)}$. Depending on the number of zeros that are of particular interest and need to be placed, we will have corresponding equations for the zero polynomial, similar to (23).

RECURSIVE POLE AND ZERO PLACEMENT

Equations (20) and (23) for the pole polynomial and the individual zero polynomial, respectively, may now be used to place poles and zeros of closed loop transfer functions in specified locations. Each placement can be done exactly if the number of desired poles plus zeros, $n+z$, does not exceed $2m-1$ [14]; on the other hand, if this sum does exceed $2m-1$, a least squares approximate placement may be made. Weighting is also possible to reflect the relative importance of some roots over others.

Assume that we wish to place n poles, at $\lambda_1,\ldots,\lambda_n$. Also assume that we wish to place a total of z zeros, at $\mu_1,\ldots,\mu_z$, this total being the sum of different numbers of zeros associated with the various transfer function numerators of interest. The poles and the zeros must then satisfy the pole polynomial (20) and an appropriate zero polynomial of the form (23) respectively. Thus we have n equations

$$\begin{aligned} p(\lambda_1) &= \text{den}L(\lambda_1)+g^T[\text{num}L(\lambda_1)]f = 0 \\ &\vdots \\ p(\lambda_n) &= \text{den}L(\lambda_n)+g^T[\text{num}L(\lambda_n)]f = 0 \end{aligned} \qquad (24)$$

for the n poles, and z equations

$$\begin{aligned} z_{a_1b_1}(\mu_1) &= n_{a_1b_1}(\mu_1) \\ &+g_{(a_1)}^T(\hat{L_2})_{a_1b_1}(\mu_1)f_{(b_1)}\text{den}L(\mu_1) = 0 \\ z_{a_zb_z}(\mu_z) &= n_{a_zb_z}(\mu_z) \\ &+g_{(a_z)}^T(\hat{L_2})_{a_zb_z}(\mu_z)f_{(b_z)}\text{den}L(\mu_z) = 0 \end{aligned} \qquad (25)$$

for the z zeros. The subscripts $a_1b_1,\ldots,a_zb_z$ are meant to emphasize that the individual zeros to be placed are associated with different transfer function numerators in general, though more than one may be associated with the same numerator.

Equations (24) and (25) are $n+z$ bilinear equations in the $2m$ unknowns $g_1,\ldots,g_m,f_1,\ldots,f_m$. The equations are bilinear in that for a given g the equations are linear in f, and for a given f the equations are linear in g. Treating g as a constant, the equations can be formulated as a set of linear equations in f as

$$Pf = c \qquad (26)$$

where P is a constant $(n+z) \times m$ matrix, and c is the $(n+z)$-vector

$$c = [-\mathrm{den}L(\lambda_1),\ldots,-\mathrm{den}L(\lambda_n),$$

$$-n_{a_1 b_1}(\mu_1),\ldots,-n_{a_z b_z}(\mu_z)]^T.$$

Alternatively, treating f as a constant, the equations can be formulated as a set of linear equations in g as

$$Qg = c \tag{27}$$

where Q is a constant $(n+z) \times m$ matrix, and c is the $(n+z)$-length vector defined above. Notice, in formulating Equations (26) and (27), that each placement equation contributes one row to P and to Q. The pole placement equations such as (24) contribute rows of length m. The zero placement equations such as (25) contribute rows of length m-1 only; a zero must be placed in these rows at the k^{th} position if the reference of interest is b_k in (26), or the output of interest is a_k in (27), to bring these rows up to length m.

Equations (26) and (27) may now be solved recursively in the least-squares sense by the following algorithm to minimize the error function

$$\begin{aligned} E &= \sum_{i=1}^{n} p^2(\lambda_i) + \sum_{j=1}^{z} z^2_{a_j b_j}(\mu_j) \\ &= ||Pf-c||^2 = ||Qg-c||^2. \end{aligned} \tag{28}$$

In addition, if some root placements are more important than others, this may be reflected in a diagonal, invertible matrix of weights

$$W = \mathrm{diag}(w_1 \ldots w_{n+z}) \tag{29}$$

so that the error function would be modified to

$$\begin{aligned} E_W &= \sum_{i=1}^{n} w_i p^2(\lambda_i) + \sum_{j=1}^{z} w_{n+j} z^2_{a_j b_j}(\mu_j) \\ &= ||W(Pf-c)||^2 = ||W(Qg-c)||^2. \end{aligned} \tag{30}$$

<u>Step one</u>. Assume an initial value of $g = g^{(1)}$. Then the least squares solution of Equation (26) for f is known to be

$$f^{(1)} = [P(g^{(1)})]^+ c \tag{31}$$

where + denotes a suitable pseudo-inverse. The matrix P^+ is simply calculated when the $(n+z) \times m$ matrix $P(g^{(1)})$ is of full rank m. If the latter condition is satisfied by a suitable selection of $g^{(1)}$,

$$[P(g^{(1)})]^+ = (P^T P)^{-1} P^T.$$

Thus $f^{(1)}$ can be calculated, and the least-squares error of Equation (26) is then given by

$$E_1 = ||P(g^{(1)})f^{(1)} - c||^2. \tag{32}$$

<u>Step two</u>. Set $f = f^{(1)}$ and obtain the least-squares solution of Equation (27) for g as

$$g^{(2)} = [Q(f^{(1)})]^+ c. \tag{33}$$

The least squares error of Equation (27) is then given by

$$E_2 = ||Q(f^{(1)})g^{(2)} - c||^2. \tag{34}$$

This process is repeated in the following manner. The updated value of g, namely $g^{(2)}$, is used in step one, and a new value of f, namely $f^{(2)}$, is calculated using (31). Then the least squares error E_3 is computed using $g^{(2)}$ and $f^{(2)}$. Next the updated value of f, namely $f^{(2)}$, is used in step two and a new value of g, namely $g^{(3)}$, is calculated using (33). The least squares error E_4 is now calculated using $f^{(2)}$ and $g^{(3)}$. This procedure is continued until the least-squares error E is sufficiently small. It has been shown [10] that the least-squares error function E is guaranteed to converge.

<u>EXAMPLE</u>

The transfer matrix, L(s), of a plant is given as

$$L(s) = \frac{\begin{bmatrix} s+6 & 2 & 2s+6 \\ 6 & -s+1 & 3s+9 \\ 3s+6 & 2s+4 & 0 \end{bmatrix}}{(s+2)(s+3)}$$

Let us assume that we wish to keep one of the poles at -2 while moving the other from -3 to -10; thus

$$\lambda_1 = -2,\ \lambda_2 = -10.$$

Also assume that we want to move the (2,2) zero, z_{22}, from its present location at +1 to -5 in the left half-plane while ensuring that the (1,1) zero, z_{11}, remains unaffected by feedback. Therefore we have two more specifications

$$z_{11}(s) = s+6,\ z_{22}(s) = s+5.$$

Assuming that we are looking for a dyadic constant feedback matrix

$$H = fg^T,$$

this problem was run on the digital computer, using the iterative algorithm outlined above, starting from an initial value of g as

$$g = (1\ 1\ 1)^T.$$

The program converged to a solution in five iterations, with a corresponding least-square error of the order of 10**(-20); the corresponding final values of f and g were

$$f = (.15,-1.87,1.57)^T,\ g = (.06,1.37,.69)^T$$

where we have rounded to the second decimal place for convenience. The feedback matrix H may be calculated as

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$$H = fg^T = \begin{bmatrix} .009 & .204 & .102 \\ -.116 & -2.571 & -1.286 \\ .097 & 2.164 & 1.082 \end{bmatrix}$$

We can check the positions of the poles and the individual zeros to see whether our specifications have been met. The pole polynomial

$$p(s) = \text{den } L(s) + g^T \text{ num } L(s)f$$

may be obtained, using the computed values of f and g, as

$$p(s) = (s^2+5s+6)+(6.92s+12.65)$$

$$= (s+10.07)(s+1.85).$$

Thus one of the poles is at -10.07 which is close to the desired value of -10. We wanted the other pole to remain at -2; it has moved a little, to -1.85, in the process of accommodating a least-squares error solution. The two individual zeros, $z_{11}(s)$ and $z_{22}(s)$, may also be calculated; they are found to be at -5.93 and -5.1 respectively, which are close to the desired values of -6 and -5.

CONCLUSIONS

In this paper we have considered multivariable output feedback, and looked at it from the viewpoint of the exterior morphisms induced over the plant map and the feedback map. From this vantage point, we have derived expressions for the individual closed loop transfer functions. It is found that the individual closed loop zeros are affected by the corresponding individual feedforward zeros and by the zeros of minors formed from the feedforward matrix while including these individual zeros. As far as feedback is concerned, it is found that the individual closed loop zeros of the transfer function relating the a^{th} output to the b^{th} reference, say, are not affected at all by feedback from the a^{th} output to the b^{th} comparison point. Rather, these zeros depend on the sub-matrix of feedback elements corresponding to feedback from outputs <u>other than</u> the a^{th} to comparison points <u>other than</u> the b^{th}, and on the exterior maps induced by this sub-matrix.

The expressions that have been derived for the individual closed loop zero polynomials and for the pole polynomial should themselves be of interest because many closed loop system specifications are yet given in terms of single channel excitations. In addition, however, the above-mentioned expressions have been used in this paper, in conjunction with unity-rank feedback, to derive an iterative algorithm for simultaneous pole- and individual zero-placement. A computer program has been written to implement this algorithm, and an example of root placement, using this program, appears in the latter part of this paper.

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APPENDIX E

Reprint

"Tensor Ideas for Nonlinear Modeling of a Turbofan Jet Engine: Preliminary Studies"

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TENSOR IDEAS FOR NONLINEAR

MODELING OF A TURBOFAN JET ENGINE: PRELIMINARY STUDIES*

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ABSTRACT

The importance of nonlinear models for model following methods so prevalent in the modern turbofan engine industry dictates the need for reliable techniques for nonlinear model generation. This paper reports upon a continuing investigation aimed at developing nonlinear differential models utilizing the notions of power series and algebraic tensors. Emphasis of the study is on an application of these ideas in nonlinear model generation using a real-time digital turbofan engine simulation.

INTRODUCTION

Model following control systems constitute a solid backbone for much of the control work in modern aviation. Basically, one has the plant, which may be an airframe or an engine—or both the airframe and the engine regarded as an integrated whole. Under certain conditions, such as temperatures, pressures, and compressor speeds, the plant may be said to satisfy certain nonlinear differential equations. Moreover, many of the variables in these equations are related to one another through complicated nonlinear maps. It is within this range of acceptable conditions that any realistic control system must carry on its work. Accordingly, when a control system receives a request to change important physical quantities within the plant, it must arrange to do so in such a way that the plant moves to the new condition without violating its identity, that is, without leaving the acceptable region of conditions anywhere along the way. For example, if altitude is to be changed, then this must be accomplished without stall. Or, if thrust is to be changed, it must be changed without permitting excessive increases in turbine inlet temperature.

The reason that model following control thinking is so useful in such situations is due to the fact that the models may be used to prescribe behaviors which are in consonance with the region of acceptable plant conditions. Scheduled over an operating envelope, such models can absorb a large part of control stress, and can free the feedback loop for its primary task of achieving accuracy in the presence of parametric uncertainties and disturbances.

This paper deals with studies on the use of algebraic tensors [1] for generating a family of nonlinear models. The main feature of the algebraic tensor involves the way that it gives ground on dimensionality in order to gain advantages of linearity. This provides an organized way of looking at expansion formula and provides a direct line with parameter identification techniques for linear equations. Basically, one has to design initial conditions and control signals in order to assure that the nonlinear model will outperform the linear model locally, as well as arranging that it yield a larger region of acceptable modeling. Such modeling exercises have been exhibited in [2] and [3] for representative off-simulator examples. Our purpose here, then, is to apply the techniques to nonlinear model generation using a real-time digital turbofan engine simulation, and to present some preliminary representative results.

ALGEBRAIC TENSORS IN MODELING

We illustrate now the use of algebraic tensors in nonlinear model building, utilizing ideas of power series and truncation approximations. To this end, let $x \in X$ be the n-vector of states and $u \in U$ be the m-vector of inputs, where X and U are real vector spaces. Consider the nonlinear ordinary differential equation

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$$\dot{x} = f(x,u); \tag{1}$$

we will assume without loss that the origin is an equilibrium point of (1). The function $f : X \times U \rightarrow X$, under certain technical assumptions such as analyticity in a neighborhood of $(0,0)$ in $X \times U$, may be expressed in a power series expansion of two variables. Due to the notational complexity of higher order mixed partial derivatives, only the first few terms of such an expansion may be easily recorded. However, if we employ the universal bilinear tensor product function $\otimes : X \times U \rightarrow X \otimes U$ [4,5], where the nm-dimensional real vector space $X \otimes U$ is the tensor product of X with U, we can express the right side of (1) in the compact form

$$f(x,u) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{j!k!} M_{jk} (\underbrace{x \otimes x \otimes \ldots \otimes x}_{j \text{ times}} \otimes \underbrace{u \otimes u \otimes \ldots \otimes u}_{k \text{ times}}) . \tag{2}$$

The main feature of (2) lies in the fact that the M_{jk} are linear maps, facilitating an orderly treatment of higher order terms in the expansion. Such elements are built up by iteration, as for example $(X \otimes (X \otimes U)) \otimes U$.

A nonlinear model of (1) can be obtained by truncation of (2). To allow for this (2) may be simplified by introducing the new linear map L_{jk} whose action is that of M_{jk} followed by the scalar multiplication by $1/j!k!$. Thus, a linear identification problem may be formulated via the equation

$$\dot{x} = [L_{10}\ L_{01}\ L_{20}\ L_{11}\ L_{02}\ L_{30}\ \cdots]\ x_T \tag{3}$$

where x_T is a vector partitioned into tensor product terms in an appropriate order, given by $(x, u, x \otimes x, x \otimes u, u \otimes u, x \otimes x \otimes x, \ldots)$. Consider the tensor product associated with the L_{12} map which is generated by elements in $X \otimes U \otimes U$. For the case of $n = 3$ and $m = 2$, the 12 monomials of $x \otimes u \otimes u$ consist of terms such as $x_1 u_1^2$, $x_1 u_1 u_2$, $x_1 u_2 u_1$, and so on. But due to the commutativity of scalar multiplication in the field there exist three redundant terms; elimination of these terms results in an object of dimension 9. In general, the number of distinct elements from each product is given by the combinatorial expression

$$p = \binom{n+q-1}{q} \cdot \binom{m+r-1}{r} \tag{4}$$

for q copies of x and r copies of u in the product. Construction of the system consists of stacking these monomials in the vector x_T to give a reduced-size version of (3), which amounts to a use of the symmetric tensor algebra [5]. It is important to note that now the matrix L_{jk} is of reduced size, corresponding to the reduced order structure.

To complete the construction of our approximate system, sinusoidal inputs are applied to (1), and the state solutions are sampled at h selected time points. These sampled values are loaded into the pxh matrix X_T. The first n+m rows of X_T are determined from the sampled values of x and u; the remaining $p-(n+m)$ rows contain monomials which are multiplies of the entries of those first n+m rows. The nxh matrix $\dot{X}$ is formed by loading derivatives estimates for $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ at the h time points. As an illustration, for an approximation retaining up to third degree tensor product terms we have

$$\dot{X} = [L_{10}\ L_{01}\ L_{20}\ L_{11}\ L_{02}\ L_{30}\ L_{21}\ L_{12}\ L_{03}]\ X_T . \tag{5}$$

The method employed here in solving for the coefficient matrix uses a singular value decomposition of the transpose of X_T to solve the minimal least-squares problem, returning the nxp partitioned matrix of the L_{jk}.

APPLICATION: JET ENGINE SIMULATOR

In the modeling discussions to follow attention will center around NASA's QCSEE ("Quixie")—Quiet, Clean, Short-haul Experimental Engine [6]. Following in the evolution of turbojet to turbofan engines in aircraft propulsion, the QCSE engine is an advanced turbofan designed specifically for powered-lift, short-haul aircraft. The engine incorporates several new concepts not all currently used on turbofans to achieve optimal efficiency as well as quiet, clean operation. Primary uses of QCSEE-type engines will be on short take off and landing (STOL) aircraft, promising brighter prospects for compact metropolitan airports.

An ideal of any propulsion simulation is to achieve absolute realism for use in flight simulators. To approach this ideal requires very detailed digital simulations in the form of complex computer programs. The goal of the QCSEE simulator program employed in this study has been to achieve real time propulsion simulation to be used in aircraft simulators with under-the-wing engine application.

For the analytical model to be discussed, the states and controls employed are as follows. Engine states are the combustor exit temperature and rotor dynamics in the form of the fan speed and compressor speed. Control inputs are the fuel metering valve position (which determines main burner fuel flow), nozzle area setting, and a fan pitch angle parameter for control of the variable pitch fan. Thus a three-state, three-control model will be formulated. Model formulations using more than three states are currently under investigation.

Engine operation for the model identification can take two basic approaches. The engine simulator may be run with the loop closed, that is, with the digital controller segment fully operative, while simply varying the power demand (or, equivalently, the "throttle") about some equilibrium point. Figure 1 illustrates this scheme where we represent the engine dynamics in terms of the states and their derivatives. The reference input power demand (PWRX) is depicted as a sinusoidal perturbation which in turn, with plant measurements y_m, determines the controller dynamics. An alternative approach for the simulator operation involves opening the loop, effectively deactivating the controller and independently inserting the individual control inputs. This situation is portrayed in Figure 2 where we insert a constant power demand and "turn-off" the controller by equating the controller state derivatives with zero. In this way sinusoidal inputs, u, may be inserted and engine states observed.

In the second approach mentioned above, which we will adopt here, nonlinearities of the plant are excited which might otherwise have been less pronounced had the controller been in the loop. For both operations the engine simulator is run into the steady state prior to any perturbations in order to establish an operating point. The initial conditions thus generated form the point of expansion for the series truncation approximation in the model formulation.

SAMPLE RESULTS

In this final section we offer an overview of the procedure for an identification using the QCSEE simulator. As mentioned in the preceding section, the operation of QCSEE for purposes of model generation in this study is of the type depicted in Figure 2. The simulator is run with a 100% power demand for several seconds to settle all transients. This produces some equilibrium value (x_o, u_o) where x and u each consist of three elements. Within the digital simulation program the control variables are manipulated so that a sinusoidal input with some amplitude and frequency is inserted into each input channel. Likewise, the state variables are perturbed from their equilibrium values and then sampled over some interval at evenly spaced points in time. The difference between these sampled values and the corresponding equilibrium values form the block of observed data for the identification procedure. The derivative values are also extracted directly from the simulator at the given sample times so that a truncation approximation, such as that given in (5), may be formulated. Ordering of the elements in x_T is of critical import for identification as well as simulation of the model; a complete algorithm for such an ordering procedure may be found in [2].

Validation studies of a model consist of comparing model responses to true responses of the state variables to perturbations in the initial states and input signal parameters about the point at which the model is identified. Moreover, a standard linear approximation model is normally identified by another method and also used in the comparison studies. All simulations here are done in the open loop. For example, observe the response curves given in Figures 3-5. The first plot represents a sample response for perturbations in the initial state values; Figure 3 shows the behavior of the compressor speed for a decrease to 25% of the perturbation used in the identification. Figure 4 represents the response of the compressor speed for a downward perturbation in the control signal amplitudes. Finally, Figure 5 exhibits the fan speed behavior for a 20% increase in frequencies in each signal.

Preliminary studies have resulted in several nonlinear models for specific identification points. To illustrate the type of simulations which result from such models, representative response curves have been presented with various input parameter sets for one such model. A final identification, that is, one with full validation studies, is currently under investigation.

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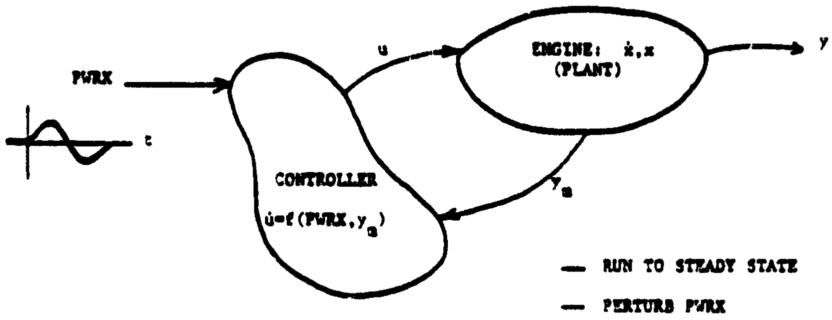

Figure 1 Closed Loop Identification

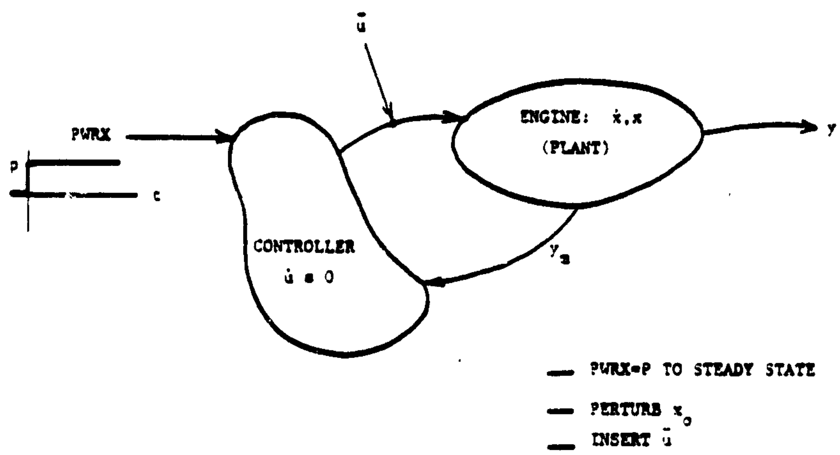

Figure 2 Open Loop Identification

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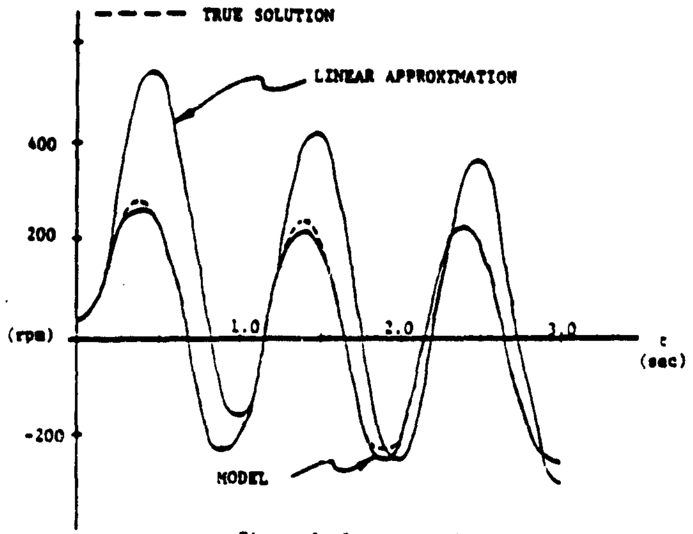

Figure 3 Compressor Speed

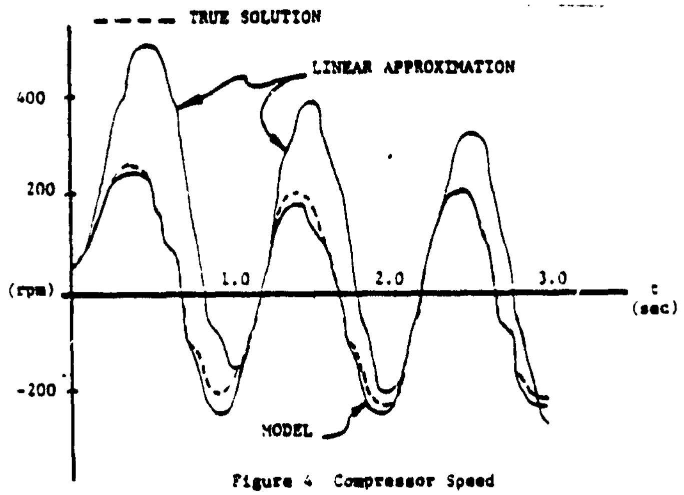

Figure 4 Compressor Speed

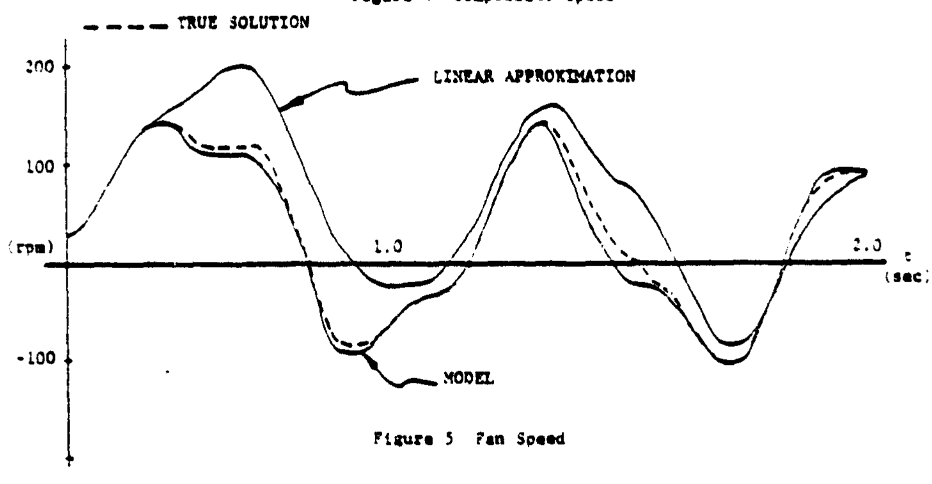

Figure 5 Fan Speed