DEVELOPMENT OF MIXED TIME PARTITION PROCEDURES
FOR THERMAL ANALYSIS OF STRUCTURES*

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ABSTRACT

The computational methods used to predict and optimize the thermal-structural behavior of aerospace vehicle structures are reviewed. In general, two classes of algorithms, implicit and explicit, are used in transient thermal analysis of structures. Each of these two methods has its own merits. Due to the different time scales of the mechanical and thermal responses, the selection of a time integration method can be a difficult yet critical factor in the efficient solution of such problems.

Therefore mixed time integration methods for transient thermal analysis of structures are being developed. This proposed methodology would be readily adaptable to existing computer programs for structural thermal analysis.

1. INTRODUCTION

Over the last two decades, significant attention has been devoted to the development of lightweight, durable thermal protection systems (TPS) for future space transportation systems. Research programs are currently under way at the Langley Research Center to investigate various metallic TPS concepts [1]. One of the proposed candidates is the titanium multiwall tile (see [2] and references therein for a discussion). Early design procedures of the TPS concept involved both analytical and experimental studies. In particular, a degree of confidence has been established in the TPS concept due to the design studies by Jackson and Dixon [3] and Blair et. al. [4].

A titanium multiwall tile consists of alternating layers of superplastically formed dimpled sheets and flat septum sheets of titanium foil. As described in reference [3], this multiwall concept impedes all three modes of

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heat transfer----conduction, radiation and convection. The superplastically
formed dimpled sheets and the long thin conduction path tend to minimize heat
conduction. The flat septum sheets of titanium foil impede radiation. The
small individual volumes created by the dimpled layers virtually eliminate air
convection. The optimal design of such thermal protection systems requires
effective techiques in coupled thermal and stress analyses. Finite element
methods offer the greatest potential in modeling such complicated problems.
However, the resulting semi-discrete equations may involve many thousand
degrees of freedom. Since the problem to be solved is transient and non-
linear, the selection of an appropriate time integration method is an essen-
tial step in the solution of such a complicated problem. Adelman and Haftka
[5] recently conducted a survey study on the performance of explicit and
implicit algorithms for transient thermal analysis of structures. Calcula-
tions were carried out using the SPAR finite element computer program [6] and
a special purpose finite element program incorporating the GEARB and GEARIB
algorithms. Based upon their studies, they concluded that, generally, implic-
it algorithms are preferable to explicit algorithms for "stiff" problems,
though non-convergence and/or wide-banding of the resulting matrix equations
may decrease the advantage of the implicit methods.

These difficulties are similar to those found in fluid-structure prob-
lems. Over the past few years, several remedies have been proposed for these
difficulties. Belytschko and Mullen [7] have proposed an explicit-implicit
method where the mesh is partitioned into domains by nodes and the partitions
are simultaneously integrated by explicit and implicit methods. Hughes and
Liu [8] have proposed an alternate implicit-explicit finite element method
where the mesh is partitioned into domains by elements and this element parti-
tion concept simplifies the computer-implementation and enhances its compati-
bility with the general purpose finite element software.

Although the implicit-explicit method has been proven to be very success-
ful in some fluid-structure interaction problems (see e.g., [8-10]), the size
and complexity of the program are increased because of the addition of the
implicit method. To overcome these difficulties, Belytschko and Mullen [11]
have proposed an E^m-E partition, in which explicit time integration is used
throughout. However, different time steps within different parts of the mesh
can be employed simultaneously. Partitioned and adaptive algorithms for ex-
plicit time integration have also been proposed by Belytschko [12].

Recently, Liu and Belytschko [13] put forward a general mixed time
implicit-explicit partition procedure within a linear context. It incorpo-
rates the mentioned algorithms as special cases and is shown to have better
stability properties than those in E^m-E partition [11]. Similar concepts can
also be used in transient conduction forced-convection analysis (see Liu and
Lin [14]).

In the present paper, we extend these implicit-explicit concepts (nodes
and elements) to transient thermal analysis of structures where different time
integration methods with different time steps can be used in each element
group. The aim of this approach is to achieve the attributes of the various
time integration methods.

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For example, in transient structural analysis, explicit methods require the size of the time step to be proportional to the length of the shortest element, while in transient thermal analysis, explicit methods require the step size to be proportional to the the square of the length of the shortest element. So it is more advantageous to employ this mixed time implicit-explicit technique for transient thermal analysis of structures since the E^m-E partition proposed in [11,12] is often inefficient for this kind of problem though it is very efficient in structural analysis.

In section 2 we review the finite element formulation for transient heat conduction. In section 3 we describe the mixed time integration procedures viz two element groups "A" and "B" . A family of integration partitions can then be deduced by selecting the appropriate definitions for the quantities of "A" and "B". Five useful partitions which are of practical importance are presented. In section 4 the stability characteristic of the algorithm is discussed. In section 5 we generalize the mixed time methods described in section 3 to NUMEG element groups. A computational algorithm for this mixed time implicit-explicit integration is also presented. Numerical results are presented in section 6 and conclusions and suggestions for further research are presented in section 7.

2. FINITE ELEMENT FORMULATION FOR TRANSIENT HEAT CONDUCTION

We consider a body Ω enclosed by surface Γ which consists of two parts: Γ_g and Γ_q . The Cartesian coordinates of the body will be denoted by x_i .

The governing equations for transient heat conduction are:

$$\theta,_{ii} = \frac{1}{c^2} \dot{\theta} \qquad\qquad \text{in } \Omega \qquad\qquad (2.1)$$

$$\theta = g \qquad\qquad \text{for } x_i \text{ in } \Gamma_g \qquad\qquad (2.2)$$

$$\theta,_i n_i + h\theta = q \qquad\qquad \text{for } x_i \text{ in } \Gamma_q \qquad\qquad (2.3)$$

and

$$\theta = \theta_o \qquad\qquad \text{for } x_i \text{ in } \Omega \text{ and } t = 0 . \qquad\qquad (2.4)$$

Here a comma designates a partial derivative with respect to x_i ; a superscript dot designates time (t) derivative; n_i is the component of the outward unit normal vector; C^2 is the thermal diffusivity (the ratio of thermal conductivity to specific heat times density); θ is the temperature; h is the convective heat transfer coefficient; and g, q and θ_o are given functions. Repeated indices denote summations over the appropriate range.

The variational or weak form of equations (2.1)-(2.4) is:

$$(\dot{\theta},v) + a(\theta,v) = (q,v) \qquad\qquad \text{in } \Gamma_q \qquad\qquad (2.5)$$

where v is the test function; and

$$(\dot{\theta},v) = \int_\Omega \frac{1}{c^2} \dot{\theta} \, v d\Omega \qquad\qquad (2.6)$$

$$a(\theta,v) = \int_\Omega \theta,_i v,_i d\Omega + \int_{\Gamma_q} h\theta v \, d\Gamma \qquad (2.7)$$

and

$$(q,v)_{\Gamma_q} = \int_{\Gamma_q} qv \, d\Gamma \qquad (2.8)$$

The finite element equations are obtained by approximating the trial functions by shape functions (N_i) so that

$$v = \sum_{i=1}^{NEQ} N_i(x_j)d_i(t) \qquad (2.9)$$

$$g = \sum_{i=NEQ+1}^{NUMNP} N_i(x_j)g_i(x_j,t) \qquad (2.10)$$

and

$$\theta = v + g \qquad (2.11)$$

Here NUMNP is the total number of nodal points used in the finite element mesh and NEQ is the number of trial functions used (for this particular case it is equal to the number of equations to be solved).

The resulting semidiscrete equation for transient heat conduction is then:

$$\underset{\sim}{M}\dot{\underset{\sim}{\theta}} + \underset{\sim}{K}\underset{\sim}{\theta} = \underset{\sim}{F} \qquad (2.12)$$

with initial condition

$$\underset{\sim}{\theta}(0) = \underset{\sim}{\theta}_o \qquad (2.13)$$

where

$$\underset{\sim}{M} = [M_{ij}] = (N_i, N_j) = \int_\Omega \frac{1}{c^2} N_i N_j d\Omega \qquad (2.14)$$

$$\underset{\sim}{K} = [K_{ij}] = a(N_i,N_j) = \int_\Omega N_{i,k} N_{j,k} d\Omega + \int_{\Gamma_q} h N_i N_j \, d\Gamma \qquad (2.15)$$

and

$$\underset{\sim}{F} = [F_i] = (q,N_i)_{\Gamma_q} - (N_k,N_i)\dot{g}_k - a(N_k,N_i)g_k \qquad (2.16)$$

$\underset{\sim}{M}$ and $\underset{\sim}{K}$ are assumed to be symmetric and positive definite. The thermal parameters, c^2 and h, are in general temperature dependent. However for discussion purposes herein, they are assumed constant throughout.

3. MIXED TIME PARTITION PROCEDURES

In this section, mixed time integration methods are employed to solve equations (2.12) and (2.13). For the purpose of describing these mixed time integration techniques we subdivide the mesh into element groups A and B, each of which is to be integrated by a different method. Let n be the time step number, $\underset{\sim}{\theta}_n$, $\underset{\sim}{v}_n$ and $\underset{\sim}{F}_n$ be approximations to $\underset{\sim}{\theta}(t_n)$, $\dot{\underset{\sim}{\theta}}(t_n)$ and $\underset{\sim}{F}(t_n)$ respectively. Let $m\Delta t$ and Δt be the time steps used for element group A and

element group B respectively, where m is an integer and is greater or equal to 1. A time step cycle (mΔt) can then be defined by an increment of m substeps with a time step of Δt each, so that one time step cycle is defined by step n to step n+m. The portions of the matrices obtained by assembling element group A and element group B are denoted by superscripts "A" and "B", respectively. Hence it follows that any global matrix is the sum of the two matrices, cf. $\underset{\sim}{M} = \underset{\sim}{M}^A + \underset{\sim}{M}^B$ and $\underset{\sim}{K} = \underset{\sim}{K}^A + \underset{\sim}{K}^B$. Nodes associated with <u>only</u> element group B are denoted by superscript "B", whereas those which are in contact with at least one element of group A are denoted by superscript "A"; nodes which are connected to both group A and group B are designated by "C", so "C" is a subset of "A". To simplify the presentation, we further denote those element matrices associated with at least one node C by superscript "C", so $\underset{\sim}{M}^C$ and $\underset{\sim}{K}^C$ are subsets of $\underset{\sim}{M}^B$ and $\underset{\sim}{K}^B$ respectively. However, in actual computer implementation this element group is not necessary. We further denote $\underset{\sim}{M}^R = \underset{\sim}{M}^A + \underset{\sim}{M}^C$ and $\underset{\sim}{K}^R = \underset{\sim}{K}^A + \underset{\sim}{K}^C$.

Similarly, all vectors are then partitioned accordingly into "A" and "B" parts, cf. $\underset{\sim}{\theta} = (\theta^A \theta^B)^T$, $\underset{\sim}{V} = (\underset{\sim}{V}^A \underset{\sim}{V}^B)^T$ and $\underset{\sim}{F} = (\underset{\sim}{F}^A \underset{\sim}{F}^B)^T$. The superscript "T" denotes the transpose. The vector $\underset{\sim}{\theta}$ is sometimes redefined by augmented matrices, $\underset{\sim}{\theta} = \theta^{*A} + \theta^{*B}$ where $\theta^{*A} = (\theta^A 0)^T$ and $\theta^{*B} = (0 \; \theta^B)^T$. Similar definitions are used for $\underset{\sim}{V}$ and $\underset{\sim}{F}$. Any nonzero terms in $\underset{\sim}{F}^A$ obtained in a computation of $\underset{\sim}{F}^{*B}$ are neglected; they are assumed to be zero.

As an example, consider a one dimensional mesh depicted in figure 1, it consists of 8 nodes and 7 elements. Then the set of nodes "A" will be 1,2,3, 4,5; the set of nodes "B" will be 6,7,8; and the set of nodes "C" will be 5.
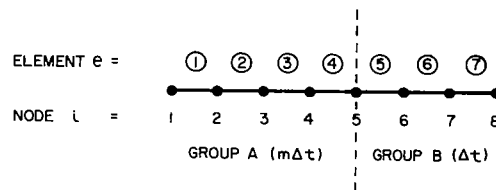
ELEMENT e = ① ② ③ ④ ⑤ ⑥ ⑦

NODE ι = 1 2 3 4 5 6 7 8

GROUP A (mΔt)　　GROUP B (Δt)

Figure I.

Let $\underset{\sim}{M}^e$, $\underset{\sim}{K}^e$ and $\underset{\sim}{F}^e$ be the eth element mass, stiffness and force contributions to the global arrays respectively; then

$$\underset{\sim}{M}^A = \sum_{e=1}^{4} \underset{\sim}{M}^e \qquad \underset{\sim}{K}^A = \sum_{e=1}^{4} \underset{\sim}{K}^e \qquad \underset{\sim}{F}^A = \sum_{e=1}^{4} \underset{\sim}{F}^e$$

$$\underset{\sim}{M}^B = \sum_{e=5}^{7} \underset{\sim}{M}^e \qquad \underset{\sim}{K}^B = \sum_{e=5}^{7} \underset{\sim}{K}^e \qquad \underset{\sim}{F}^B = \sum_{e=5}^{7} \underset{\sim}{F}^e$$

$$\underset{\sim}{M}^C = \underset{\sim}{M}^5 \qquad \underset{\sim}{K}^C = \underset{\sim}{K}^5 \qquad \underset{\sim}{F}^C = \underset{\sim}{F}^5$$

and

$$\underset{\sim}{M}^R = \sum_{e=1}^{5} \underset{\sim}{M}^e \qquad \underset{\sim}{K}^R = \sum_{e=1}^{5} \underset{\sim}{K}^e \qquad \underset{\sim}{F}^R = \sum_{e=1}^{5} \underset{\sim}{F}^e$$

If we let P_i^{*B} be the ith component of the global assembled vector $\underset{\sim}{P}^{*B}$ then

$$\underset{\sim}{P}^{*B} = \left(0,0,0,0,P_5 \equiv 0,P_6,P_7,P_8\right)$$

With these definitions, the mixed time partition is given as follows.

● Governing equation
   for j=0,m;

$$\underset{\sim\sim}{M}\underset{\sim n+j}{V} + \underset{\sim}{K}^R \hat{\underset{\sim}{\theta}}_{n+j}^{*A} + \underset{\sim}{K}^B \hat{\underset{\sim}{\theta}}_{n+j}^{*B} = \underset{\sim n+j}{F} \tag{3.1}$$

and

   for j=1,...,m-1;

$$\underset{\sim}{M}^B \underset{\sim n+j}{V}^{*B} + \underset{\sim}{K}^B \hat{\underset{\sim}{\theta}}_{n+j}^{*B} + \underset{\sim}{K}^C \hat{\underset{\sim}{\theta}}_{n+j}^{*A} = \underset{\sim n+j}{F}^{*B} \tag{3.2}$$

where $\hat{\underset{\sim n+j}{\theta}}^{*x}$ is a suitable extrapolator (and/or interpolator) of $\underset{\sim}{\theta}^{*x}$ (and/or $\underset{\sim}{\theta}_{n+m}^{*x}$) for x=A and B. In actual computation, equation (3.2) is implicitly included in equation (3.1), and for j=1,...,m-1 no quantities of A are being solved. A family of integration partitions can then be deduced from equations (3.1) and (3.2) if $\underset{\sim}{M}$ is assumed to be lumped. Some members which are of practical importances are shown in table 1.

Table 1

Designation	Time Integration In Element Group A	Time Integration In Element Group B	Extrapolator/Interpolator	
			Node A	Node B
E-E	explicit with Δt	explicit with Δt	$\underset{\sim n}{\theta}^{*A}$	$\underset{\sim n}{\theta}^{*B}$
mE-E	explicit with $m\Delta t$	explicit with Δt	$\underset{\sim n}{\theta}^{*A}$	$\underset{\sim n}{\theta}^{*B}$
mE-I	implicit with $m\Delta t$	explicit with Δt	$\underset{\sim n+m}{\theta}^{*A}$	$\underset{\sim n}{\theta}^{*B}$
E-I	implicit with Δt	explicit with Δt	$\underset{\sim n+1}{\theta}^{*A}$	$\underset{\sim n}{\theta}^{*B}$
I-I	implicit with Δt	implicit with Δt	$\underset{\sim n+1}{\theta}^{*A}$	$\underset{\sim n+1}{\theta}^{*B}$

For purposes of describing the computer implementation and stability analysis, the modified generalized trapezoidal rule will be used to carry out the time temporary discretization of equations (3.1) and (3.2) though other implicit integration methods can also be used; they are:

● Modified generalized trapezoidal rule
   for j=1,...,m;

$$\tilde{\underset{\sim n+j}{\theta}}^A = \underset{\sim n}{\theta}^A + (1-\alpha)j\Delta t \underset{\sim n}{V}^A \tag{3.3}$$

for $1 < j < m$ define the set "C" only,

$$\tilde{\underset{\sim}{\theta}}{}^{B}_{n+j} = \underset{\sim}{\theta}{}^{B}_{n+j-1} + (1-\alpha)\Delta t \; \underset{\sim}{V}{}^{B}_{n+j-1} \tag{3.4}$$

$$\underset{\sim}{\theta}{}^{A}_{n+m} = \tilde{\underset{\sim}{\theta}}{}^{A}_{n+m} + \alpha m \, \Delta t \; \underset{\sim}{V}{}^{A}_{n+m} \tag{3.5}$$

and

$$\underset{\sim}{\theta}{}^{B}_{n+j} = \tilde{\underset{\sim}{\theta}}{}^{B}_{n+j} + \alpha \Delta t \; \underset{\sim}{V}{}^{B}_{n+j} \tag{3.6}$$

In the above equations, α is a free parameter which governs the stability and accuracy of the method. We now illustrate some useful partitions which have been depicted in table 1.

Example 1: E-E partition

In this case, $m=1$, $\hat{\underset{\sim}{\theta}}{}^{x}_{n+1} \equiv \tilde{\underset{\sim}{\theta}}{}^{x}_{n+1}$ for $x=A$ and B. Equations (3.1) and (3.2) reduce to:

$$\underset{\sim}{M}\underset{\sim}{V}_{n+1} + \underset{\sim}{K}\tilde{\underset{\sim}{\theta}}_{n+1} = \underset{\sim}{F}_{n+1} \tag{3.7}$$

$$\tilde{\underset{\sim}{\theta}}_{n+1} = \underset{\sim}{\theta}_{n} + (1-\alpha)\Delta t \; \underset{\sim}{V}_{n} \tag{3.8}$$

and

$$\underset{\sim}{\theta}_{n+1} = \tilde{\underset{\sim}{\theta}}_{n+1} + \alpha\Delta t \; \underset{\sim}{V}_{n+1} \tag{3.9}$$

Equations (3.7) to (3.9) represent the predictor-corrector explicit algorithms with equation (3.8) as the predictor and equation (3.9) as the corrector.

Example 2: mE-E partition
In this case, $m > 1$, $\hat{\underset{\sim}{\theta}}{}^{A}_{n+j} \equiv \tilde{\underset{\sim}{\theta}}{}^{A}_{n+j}$ and $\hat{\underset{\sim}{\theta}}{}^{B}_{n+j} \equiv \tilde{\underset{\sim}{\theta}}{}^{B}_{n+j}$. Equations (3.1) to (3.6) reduce to:

PREDICTOR PHASE:

equation (3.3) $\hspace{6cm}$ (3.10)

and

equation (3.4) $\hspace{6cm}$ (3.11)

GOVERNING EQUATIONS:

equation (3.1) $\hspace{6cm}$ (3.12)

and

equation (3.2) $\hspace{6cm}$ (3.13)

CORRECTOR PHASE:

equation (3.5) $\hspace{8cm}$ (3.14)

and

equation (3.6) $\hspace{8cm}$ (3.15)

Example 3 mE-I partition

In this case, $m > 1$; in equation (3.1), $\hat{\theta}^A_{\sim n+j} \equiv \theta^A_{\sim n+m}$ for element group A only, $\hat{\theta}^A_{\sim n+j} \equiv \tilde{\theta}^A_{\sim n+m}$ for the portion which is related to K^C for $j=0$ and m. This is done automatically if element group A is defined to be the implicit element group and element group B is defined to be the explicit element group. $\hat{\theta}^A_{\sim n+j} \equiv \tilde{\theta}^A_{\sim n+j}$ for $1 < j < m$, and $\hat{\theta}^B_{\sim n+j} \equiv \tilde{\theta}^B_{\sim n+j}$ for $1 < j < m$. Equations (3.1) to (3.6) reduce to:

PREDICTOR PHASE:

equation (3.3) $\hspace{8cm}$ (3.16)

and

equation (3.4) $\hspace{8cm}$ (3.17)

GOVERNING EQUATIONS:

for $j=0,m$;

$$M V_{\sim n+j} + K^A \theta^{*A}_{\sim n+j} + K^B \tilde{\theta}_{\sim n+j} = F_{\sim n+j} \tag{3.18}$$

and

for $j=1,\ldots,m-1$;

$$M^B V^{*B}_{\sim n+j} + K^B \tilde{\theta}^{*B}_{\sim n+j} + K^C \tilde{\theta}^{*A}_{\sim n+j} = F^{*B}_{\sim n+j} \tag{3.19}$$

CORRECTOR PHASE:

equation (3.5) $\hspace{8cm}$ (3.20)

equation (3.6) $\hspace{8cm}$ (3.21)

Example 4: E-I partition

This is a special case of example 3. Equations (3.1) to (3.6) reduce to:

$$M V_{\sim n+1} + K^A \theta_{\sim n+1} + K^B \tilde{\theta}_{\sim n+1} = F_{\sim n+1} \tag{3.22}$$

$$\tilde{\theta}_{\sim n+1} = \theta_{\sim n} + (1-\alpha)\Delta t\, V_{\sim n} \tag{3.23}$$

$$\theta_{\sim n+1} = \tilde{\theta}_{\sim n+1} + \alpha \Delta t\, V_{\sim n+1} \tag{3.24}$$

Equations (3.22) to (3.24) represent the implicit-explicit algorithms developed by Hughes and Liu (see e.g., [8,10]) in which equation (3.23) is the predictor and equation (3.24) is the corrector.

Example 5: I-I partition

In this case, $m=1$, $\theta_{n+1}^A \equiv \hat{\theta}_{n+1}^A$ and $\hat{\theta}_{n+1}^B \equiv \theta_{n+1}^B$. Equations (3.1) to (3.6) reduce to the usual implicit formulation and it is:

$$\underset{\sim}{M} V_{n+1} + \underset{\sim}{K} \theta_{n+1} = F_{n+1} \tag{3.25}$$

$$\tilde{\theta}_{n+1} = \theta_n + (1-\alpha)\Delta t\, V_n \tag{3.26}$$

$$\theta_{n+1} = \tilde{\theta}_{n+1} + \alpha \Delta t\, V_{n+1} \tag{3.27}$$

4. STABILITY CRITERION

Our aim in this section is to deduce the stability characteristic of these mixed time partition algorithms. It suffices to restrict ourselves to the case in which $F=0$ and all mass matrices are lumped for purposes of stability analysis. An energy balance technique (see [8] for a discussion) is employed to carry out the stability analysis. To simplify the subsequent writing, the following notations will be used.

$$[x_{n+m}] = x_{n+m} - x_n \tag{4.1}$$

$$\langle x_{n+m} \rangle = (x_{n+m} + x_n)/2 \tag{4.2}$$

$$[x_{n+j}] = x_{n+j+1} - x_{n+j} \tag{4.3}$$

and   for $j=0,1,\ldots,m-1$

$$\langle x_{n+j} \rangle = (x_{n+j+1} + x_{n+j})/2 \tag{4.4}$$

We have not made a complete satisfactory stability analysis of these mixed time partition procedures. However, if we assume:

1. $\langle V_{n+m}^{*A} \rangle^T \underset{\sim}{K}^C \langle V_{n+m}^{*A} \rangle / z^{*B^T} \underset{\sim}{K}^C z^{*B} \geqslant 1/m^2$ where

$$z^{*B} = V_n^{*B} + \sum_{j=1}^{m-2} \langle V_{n+j}^{*B} \rangle \tag{4.5}$$

2. $\langle V_n^{*B} \rangle^T \underset{\sim}{K}^C \langle V_n^{*B} \rangle / V_n^{*A^T} \underset{\sim}{K}^C V_n^{*A} \geqslant (1+(m-1)\alpha)^2 \tag{4.6}$

3. $\langle V_{n+j}^{*B} \rangle^T \underset{\sim}{K}^C \langle V_{n+j}^{*B} \rangle / V_n^{*A^T} \underset{\sim}{K}^C V_n^{*A} \geqslant (1-\alpha)^2$ for

    $j=1,\ldots,m-1 \tag{4.7}$

and let

4. $S_n = V_n^{*A^T} \underset{\sim}{M}^{*R} V_n^{*A} + V_n^{*B^T} \underset{\sim}{M}^{*B} V_n^{*B} \geqslant 0 \tag{4.8}$

5. $P_{n+m}^A = \langle V_{n+m}^{*A} \rangle^T \underset{\sim}{K}^A \langle V_{n+m}^{*A} \rangle \geqslant 0 \tag{4.9}$

6.
$$P_{n+j}^B = \langle V_{n+j}^{*B} \rangle^T \, K^{\bar{B}} \, \langle V_{n+j}^{*B} \rangle \geqslant 0 \qquad (4.10)$$

the energy expression of these mixed time partition procedures can be shown to be:

$$S_{n+m} \leqslant S_n - 2m\Delta t \, P_{n+m}^A - 2\Delta t \, \sum_{j=0}^{m-1} P_{n+j}^B \qquad (4.11)$$

Here, $K^{\bar{B}} = K^B - K^C$ and the stability is governed by M^{*R} and M^{*B} provided $\alpha \geqslant 1/2$. Let

$$Q_j^X = M^X - 1/2 j \Delta t \, K^X \qquad (4.12)$$

and

$$W_j^X = M^X + (\alpha - 1/2) j \Delta t \, K^X \qquad (4.13)$$

the definitions of M^{*R} and M^{*B} for the five cases discussed in section 3 are:

Example 1: E-E partition

$$M^{*R} = Q^R \quad \text{and} \quad M^{*B} = Q_1^B \qquad (4.14)$$

Example 2: mE-E partition

$$M^{*R} = Q_m^R \quad \text{and} \quad M^{*B} = Q_1^B \qquad (4.15)$$

Example 3: mE-I partition

$$M^{*R} = W_m^A + Q_m^C \quad \text{and} \quad M^{*B} = Q_1^B \qquad (4.16)$$

Example 4: E-I partition

$$M^{*R} = W_1^A + Q_1^C \quad \text{and} \quad M^{*B} = Q_1^B \qquad (4.17)$$

Example 5: I-I partition

$$M^{*R} = W_1^R \quad \text{and} \quad M^{*R} = W_1^B \qquad (4.18)$$

These mixed time partition procedures are stable if $\alpha \geqslant 1/2$ and M^{*R} and M^{*B} are both positive definite. A summary of the results is as follows:

Example 1: E-E partition

$$\Omega_{crit}^A = \Omega_{crit}^B \leqslant 2 \qquad (4.19)$$

Example 2: mE-E partition

$$\Omega_{crit}^{mA} = \Omega_{crit}^{mC} \leqslant 2 \quad \text{and} \quad \Omega_{crit}^B \leqslant 2 \qquad (4.20)$$

Example 3: mE-I partition

$$\Omega^{mC}_{crit} < 2 \quad \text{and} \quad \Omega^{B}_{crit} < 2 \qquad (4.21)$$

Example 4: E-I partition

$$\Omega^{C}_{crit} < 2 \quad \text{and} \quad \Omega^{B}_{crit} < 2 \qquad (4.22)$$

Example 5: I-I partition

$$\text{unconditionally stable} \qquad (4.23)$$

In equations (4.19) to (4.22), Ω^{jx} is defined to be $j\Delta t\ \lambda^{x}_{crit}$ where λ^{x}_{crit} denotes a typical eigenvalue of the eigenproblem

$$\underset{\sim}{M}^{x}\dot{\underset{\sim}{\theta}} + \underset{\sim}{K}^{x}\underset{\sim}{\theta} = \underset{\sim}{0} \qquad (4.24)$$

5. IMPLEMENTATION ASPECTS

In this section, we generalize the mixed time integration methods described in section 3 to NUMEG element groups. Different time integration methods (implicit/explicit) with different time steps can be used in each element group. Let Δt_{NEG} and T_{NEG} be the element group time step and element group time respectively for NEG = 1,...,NUMEG. There are NUMEL elements in each element group. We denote Δt as the minimum time step amount for all these element groups. In this formulation all element group time steps are required to be integer multiples of Δt and the time steps for adjacent groups are integer multiples of each other. Furthermore, no two implicit groups with different time steps can be adjacent to each other. In addition, for each implicit group that element group time step must be greater than those of the adjacent explicit groups. The main advantage of this m_1 implicit – m_2 explicit – m_3 implicit –... etc. technique is to minimize the semi-bandwidth of complicated problems especially in the three-dimensional case. To illustrate the idea, consider the one dimensional mesh shown in figure 2. It consists of NUMEG element groups and NUMNP nodes. In this case NUMEG is equal to 4 and NUMNP is equal to 12. We assumed that node 1 is an essential boundary condition node and hence the number of equations, NEQ, is equal to 11. The essence of the present development can be deduced graphically by considering
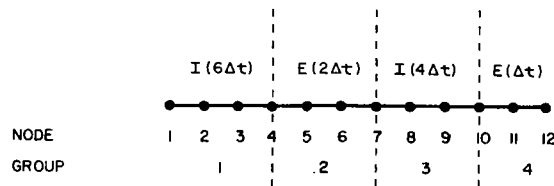
Figure 2.

the solution procedures of the matrix equations. The "active column equation solver" is the key to the success of this technique (see[8,13] for a description of this equation solver). The profile of the effective stiffness matrix $\overset{*}{\underset{\sim}{K}}$ of this one dimensional mesh is shown in figure 3. We can observe from figure 3 the following:

Group 1: implicit with $\Delta t_1 = 6\Delta t$, five words of storage (1-5), 3 elements and 3 equations.

Group 2: explicit with $\Delta t_2 = 2\Delta t$, two words of storage (6-7), 3 elements and 2 equations.

Group 3: implicit with $\Delta t_3 = 4\Delta t$, seven words of storage (8-14), 3 elements and 4 equations.

Group 4: explicit with $\Delta t_4 = \Delta t$, two words of storage (15-16), 2 elements and 2 equations.
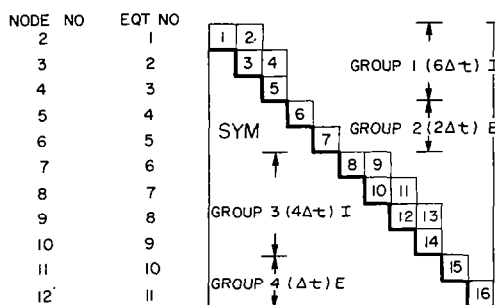

Figure 3.

The equation systems of each element group are uncoupled and hence each group can be integrated at its own group time step. For example, we assume the effective stiffness matrix $\overset{*}{\underset{\sim}{K}}$ is formed and factorized once. In a time interval of $6\Delta t$, group 1 will be integrated implicitly once, group 2 will be integrated explicitly three times, group 3 will be integrated implicitly once and group 4 will be integrated explicitly six times. In order to handle the forward reduction and backsubstitution and update procedures automatically, we required two arrays Δt_{NODE} and T_{NODE}; each has a dimension of NUMNP. Δt_{NODE} array contains the nodal time steps of each node. Nodes associated with only one element group NEG are assigned a time step of Δt_{NEG}, whereas those which are in common to other element groups are assigned to have the maximum time step from the adjacent groups. T_{NODE} array contains the nodal time of each node. From these two arrays (Δt_{NODE} and T_{NODE}) and the boundary condition

codes, another time step array Δt_{NEQ} and an equation time array T_{NEQ} of the equation systems can then be generated. Both Δt_{NEQ} and T_{NEQ} have dimensions of NEQ. We require further a master time T_M which is incremented by the smallest time step Δt . For this particular example the Δt_{NODE} and Δt_{NEQ} arrays are:

$$\Delta t_{NODE}=\left(6\Delta t,6\Delta t,6\Delta t,6\Delta t,2\Delta t,2\Delta t,4\Delta t,4\Delta t,4\Delta t,4\Delta t,\Delta t,\Delta t\right)$$

and

$$\Delta t_{NEQ}=\left(6\Delta t,6\Delta t,6\Delta t,2\Delta t,2\Delta t,4\Delta t,4\Delta t,4\Delta t,4\Delta t,\Delta t,\Delta t\right)$$

The T_{NODE} and T_{NEQ} arrays are incremented by time steps of Δt_{NODE} and Δt_{NEQ} respectively. With these definitions, the generalized mixed time integration is to proceed over the time interval $[0,T_{max}]$. The procedures are as follows:

1.   Initialization

    Set T_M, T_{NEG}, T_{NODE} and $T_{NEQ} = 0$

2.   Determine $\underset{\sim}{V}_0$

$$\underset{\sim}{V}_0 = \underset{\sim}{M}^{-1}\left(\underset{\sim}{F}_0 - \underset{\sim}{K}\underset{\sim}{\theta}_0\right) \tag{5.1}$$

3.   Form and factorize $\underset{\sim}{K}^*$

$$\underset{\sim}{K}^* = \sum_{NEG=1}^{NUMEG} \underset{\sim}{K}^{*NEG} \tag{5.2}$$

where

$$\underset{\sim}{K}^{*NEG} = \underset{\sim}{M}^{NEG} + \alpha\Delta t_{NEG}\underset{\sim}{K}^{NEG} \quad \text{if implicit} \tag{5.3}$$

and

$$\underset{\sim}{K}^{*NEG} = \underset{\sim}{M}^{NEG} \quad \text{which is a diagonal matrix if explicit} \tag{5.4}$$

4. $T_M \leftarrow T_M + \Delta t$; set effective force $\underset{\sim}{F}^*$ equal to zero

5.   Loop on element groups $NEG=1,\ldots,NUMEG$
    If $T_{NEG} + \Delta t_{NEG} > T_M$ go to 5a

6.   Loop on elements $e=1,\ldots,NUMEL$

6a.   Define predictor values $\underset{\sim}{\tilde{\theta}}^e$

    If $T^e_{NODE} + \Delta t^e_{NODE} \leqslant T_M$

    then $\underset{\sim}{\tilde{\theta}}^e_{NODE} = \underset{\sim}{\theta}^e_{NODE} + (1-\alpha)\Delta t_{NODE} \underset{\sim}{V}^e_{NODE}$ \tag{5.5}

    If $T^e_{NODE} + \Delta t^e_{NODE} > T_M$

$$\text{then } \tilde{\underset{\sim}{\theta}}^{e}_{\text{NODE}} = \underset{\sim}{\theta}^{e}_{\text{NODE}} + (1-\alpha)\left(T_M - T^{e}_{\text{NODE}}\right)\underset{\sim}{v}^{e}_{\text{NODE}} \tag{5.6}$$

6b.   Form element effective force $\underset{\sim}{f}^{*e}$

$$\underset{\sim}{f}^{*e} = \underset{\sim}{M}^{e}\tilde{\underset{\sim}{\theta}}^{e} + \alpha\Delta t_{\text{NEG}}\underset{\sim}{F}^{e} \quad \text{if implicit} \tag{5.7}$$

and

$$\underset{\sim}{f}^{*e} = \underset{\sim}{M}^{e}\tilde{\underset{\sim}{\theta}}^{e} + \alpha\underset{\sim}{W}\left(\underset{\sim}{F}^{e} - \underset{\sim}{K}^{e}\tilde{\underset{\sim}{\theta}}^{e}\right) \quad \text{if explicit} \tag{5.8}$$

where

$$\underset{\sim}{W} = \text{diagonal matrix with } \Delta t^{e}_{\text{NODE}} \text{ along the diagonals} \tag{5.9}$$

6c.   Sum up effective force from element contributions

$$\underset{\sim}{F}^{*} \leftarrow \underset{\sim}{F}^{*} + \underset{\sim}{f}^{e} \tag{5.10}$$

6d.   End of element loop

5a.   End of element group loop

7.   Solve for $\underset{\sim}{\theta}$, i.e., forward reduction and backsubstitution

$$\underset{\sim}{\theta} = \underset{\sim}{K}^{*-1}\underset{\sim}{F}^{*} \tag{5.11}$$

7a.   Loop on equation number N=1,...,NEQ

   If $T^{N}_{\text{NEQ}} + \Delta t^{N}_{\text{NEQ}} > T_M$ go to 7b

   Forward reduction and backsubstitution for equation N

7b.   End of equation number loop

8.   Update $\underset{\sim}{V}$ and $\underset{\sim}{\theta}$

8a.   Loop on N=1,...,NUMNP

   If $T^{N}_{\text{NODE}} + \Delta t^{N}_{\text{NODE}} > T_M$ go to 8b

   $\underset{\sim}{\theta}^{N} \leftarrow$ solution from step 7
   $\tilde{\underset{\sim}{V}}^{N} \leftarrow \left(\underset{\sim}{\theta}^{N} - \tilde{\underset{\sim}{\theta}}^{N}\right)/\alpha\Delta t^{N}_{\text{NODE}}$

8b.   End of nodal number loop

9.   If $T_M > T_{\text{MAX}}$ stop, otherwise go to 4

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‡ "←" means "is replaced by"

6. NUMERICAL EXAMPLES

The stability and accuracy of these mixed time partition procedures are confirmed by the following one dimensional heat conduction problem. The finite element mesh consists of a rod kept at a temperature of 0.0 for all time at the left end and insulated at the right end, and subjected to a constant initial temperature of 0.1. The finite element model consists of (from left to right) 10 elements, each with a length $\ell_1 = 10.0$, 10 elements, each with a length of $\ell_2 = 100.0$, and 10 elements, each with a length of $\ell_1 = 10.0$. The thermal diffusivity, C^2, is set to 200.0. A total of five computer runs are being made. They are (1) explicit, (2) explicit-implicit, (3) 10 explicit-explicit, (4) 10 explicit-implicit, and (5) implicit. All analyses are run with $\Delta t = 0.25$ which is the critical time step based on $\ell_1 = 10.0$. For the m explicit-implicit cases, the implicit elements are the middle 10 elements ($\ell_2 = 100.0$). The results obtained from these five analyses are compared to the analytical solution. They are depicted in figure 4. The temperature reported is at $x = 100.0$. The solution time ratios (normalized by the implicit time) for the above five cases are 0.892, 0.971, 0.578, 0.638 and 1.000 respectively.
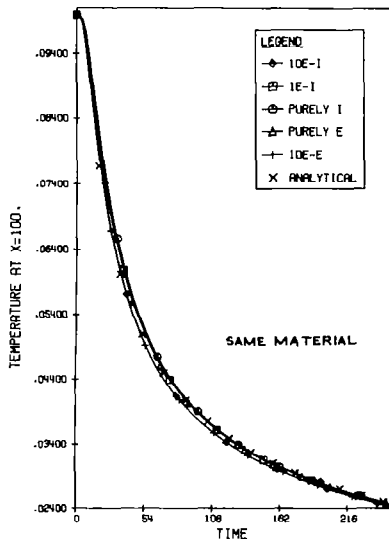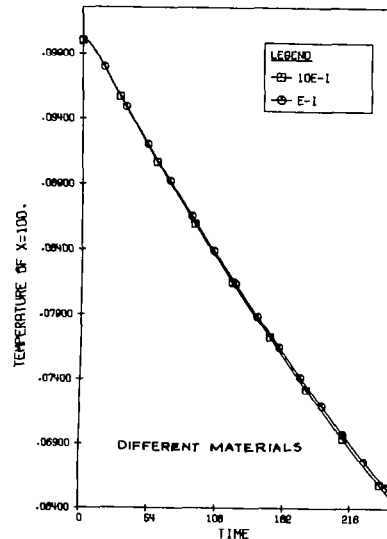

Figure 4.

Figure 5.

In order to demonstrate the advantages of this mixed time technique C^2 is raised to 2.0×10^5 for the implicit elements. A time step of $\Delta t = 0.25$ is used for the explicit-implicit and 10 explicit-implicit runs. This problem would therefore not be stable with a 10E-E partition. The results are presented in figure 5.

All the above calculations performed with $\alpha = 0.5$ and lumped mass matrices are assumed throughout.

7. CONCLUSION

In this paper, we have developed a family of mixed time partition procedures for transient thermal analysis of structures. Both the stability criterion and the implementaton aspects of these methods are described. Numerical corroboration of the stability and accuracy of these techniques is also presented. The implementation procedures of these new algorithms are straightforward and are recommended for inclusion in current thermal analysis computer programs.

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