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THE TRANSMISSION OR SCATTERING OF ELASTIC WAVES BY AN INHOMOGENEITY OF SIMPLE GEOMETRY - A COMPARISON OF THEORIES

Y. C. Sheu and L. S. Fu
Department of Engineering Mechanics

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
Lewis Research Center
Cleveland, Ohio 44135

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The Transmission or Scattering of Elastic Waves by an Inhomogeneity of Simple Geometry - A Comparison of Theories

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ABSTRACT

The extended method of equivalent inclusion developed earlier by Fu (August, 1981), (February, 1982) is applied to study the specific wave problems (i) the transmission of elastic waves in an infinite medium containing a layer of inhomogeneity, and (ii) the scattering of elastic waves in an infinite medium containing a perfect spherical inhomogeneity. The eigenstrains are expanded as a geometric series and method of integration for the inhomogeneous Helmholtz operator given by Fu and Mura (1982) is adopted.

The purpose of the study is to compare results, obtained by using limited number of terms in the eigenstrain expansion, with exact solutions for the layer problem and that for a perfect sphere. Two parameters are singled out for this comparison: the ratio of elastic moduli, $f=(\lambda'+2\mu')/(\lambda+2\mu)$, and the ratio of the mass densities, $h=\rho'/\rho$. General trend for three different situations are shown: (i) $\Delta\rho=0$, $(\Delta\lambda+2\Delta\mu)\neq 0$, (ii) $\Delta\rho\neq 0$, $(\Delta\lambda+2\Delta\mu)=0$ and (iii) $\Delta\rho\neq 0$, $(\Delta\lambda+2\Delta\mu)\neq 0$. In the last case, material systems given in Truell, Elbaum and Chick (1969) are studied.

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CHAPTER 1

INTRODUCTION

The scattering of elastic waves by an inhomogeneity embedded in an infinite homogeneous isotropic elastic medium has been studied by numerous investigators. An "inhomogeneity" is a region in which different material properties from its surrounding medium exist. Eshelby [1,2] developed the method of equivalent inclusion to determine the elastic field of an ellipsoidal inclusion. An "inclusion" is considered to be a region which has the same geometric shape and dimension as the inhomogeneity but the same material properties as its surrounding medium after an eigenstrain is imposed within that region. Mal and Knopoff [3] appeared to be the first in applying Eshelby's result to form the scattering theory of a single sphere. Subsequently, Gubernatis [4] also used Eshelby's result to study the long-wave scattering of elastic waves for an ellipsoidal inhomogeneity. In the above studies Eshelby's solution for the static displacements was used as the first approximation in the iteration.

Wheeler and Mura [5] first applied the concept of eigenstrain to study composite materials, in which they considered the difference in elastic moduli between the inhomogeneity and matrix. Subsequently, Fu [6] presented a formulation for the elastodynamics field of two ellipsoidal inhomogeneities embedded in an infinite elastic medium subjected to plane time-harmonic waves. He [7] later gave a complete formulation in extending the method of equivalent inclusion to dynamic elasticity and gave some results for three-layered and five-layered media subjected to plane time-harmonic longitudinal waves. The scattering of plane waves by an ellipsoidal inhomogeneity is presented in [8,9].

The scattering of a plane compressional wave by a spherical inhomogeneity in an infinite elastic medium has been studied by Ying and Truell [10] and by Pao and Mow [11]. They used the method of separation of variables to solve the wave equation which describes the incident, reflected (scattered) and refracted waves inside and outside the spherical inhomogeneity. Because the solutions are expressed by a spherical coordinate and the inside and outside scattering field can match exactly along the boundary of the spherical inhomogeneity, the results by this method are considered as exact solutions. Some numerical results of the elastic scattering cross section were shown by Johnson and Truell [12], and some dynamical stress concentration factors around a spherical cavity were found by Pao and Mow [14].

This research is concerned with numerical calculations according to the extended method of equivalent inclusion by Fu [6-9], and a comparison with the exact solutions. Two cases are studied here. The first case is concerned with the three-layered problem and the other case is concerned with the spherical inhomogeneity problem. Results by the direct integration method for the three-layered problem and the method of separation of variables for the spherical inhomogeneity problem are compared with those obtained by the extended method of equivalent inclusion applied on the two cases.

CHAPTER 2

WAVE MOTION IN AN ELASTIC MEDIUM

2.1. Governing Equation

Consider the infinitesimal element of an elastic body with mass density ρ in the absence of body forces, the equations of motion are

$$\sigma_{jk,k} = \rho \ddot{U}_j \tag{1}$$

where a dot indicates a differentiation with respect to time while a subscript comma indicates spatial differentiation.

If the elastic material is linear and homogeneous, the stress-strain relation is

$$\sigma_{jk} = C_{jkrs} \varepsilon_{rs} \tag{2}$$

where C_{jkrs} are the elastic constant.

If the elastic material is isotropic, then the independent elastic constants are reduced to two and

$$C_{jkrs} = \lambda \, \delta_{jk} \, \delta_{rs} + \mu(\delta_{jr} \, \delta_{ks} + \delta_{js} \, \delta_{kr}) \tag{3}$$

where λ, μ are Lame's constants and δ_{jk} are kronecker's delta.

Substituting eqs. (2.1.2)(2.1.3) into eqs. (2.1.1) the equations of motion in terms of the displacements are obtained as follows:

$$(\lambda+\mu) \, U_{j,ji} + \mu \, U_{i,jj} = \rho \, \ddot{U}_i \tag{4}$$

or, in vector notation as:

$$(\lambda+\mu) \, \nabla\nabla \cdot \vec{U} + \mu \nabla^2 \vec{U} = \rho \, \ddot{\vec{U}} \tag{5}$$

where ∇ is the vector differential operator.

The displacement vector $\vec{U}$ can be decomposed as

$$\vec{U} = \nabla \psi + \nabla \times \vec{\phi} , \quad \nabla \cdot \vec{\phi} = 0. \tag{6}$$

where the ψ and $\vec{\phi}$ are the scalar and vector displacement potentials, respectively.

A substitution of eqs. (2.1.6) into eqs. (2.1.5) leads to

$$\nabla[(\lambda+2\mu)\nabla^2\psi - \rho \ddot{\psi}] + \nabla \times[\mu \nabla^2\vec{\phi} - \rho \ddot{\vec{\phi}}] = 0 \tag{7}$$

and the ψ and $\vec{\phi}$ satisfy the wave equations

$$\nabla^2\psi = \frac{1}{v_L^2} \ddot{\psi} \tag{8}$$

$$\nabla^2\vec{\phi} = \frac{1}{v_T^2} \ddot{\vec{\phi}} \tag{9}$$

in which $v_L^2 = (\lambda+2\mu)/\rho$ and $v_T^2 = \mu/\rho$ are the velocities of longitudinal waves and shear waves, respectively.

2.2. Initial and Boundary Conditions

In the above section the general wave equations in the absence of body forces have been mentioned without considering external forces. Usually the external force can be treated as an internal or surface source (incident wave), or in the form of initial conditions (impulses) and boundary conditions (displacement or force or mixed boundary conditions). Here the external force from sources, i.e. incident waves, will be discussed.

If the incident waves are assumed to be plane waves, in general they are composed of compressional and shear waves. When the incident waves move in an elastic medium, the compressional and shear waves will propagate independently. But they can not travel independently if there is an inhomogeneity in the elastic medium. When the incident waves impinge on the inhomogeneity, compressional and shear waves will be reflected back into the matrix while the same types of waves will be refracted into the inhomogeneity. Both the reflected and refracted waves must satisfy the general wave equations. If the elastic inhomogeneity is bounded to the matrix at all times, then the tractions and displacements must be continuous at the interface between the inhomogeneity and matrix.

2.3. Time-harmonic Wave Problem

If the incident waves are time-harmonic, then the reflected and refracted waves will also be time-harmonic waves of the same angular frequency. Therefore, the displacement of all the waves can be represented by

$$\vec{U}(\vec{r},t) = U(\vec{r})\ \exp(-i\omega t)\vec{e}_u \tag{1}$$

where ω is the angular frequency and $\vec{e}_u$ is the unit vector in the direction of wave propagation.

The scalar and vector displacement potentials are

$$\psi(\vec{r},t) = \psi(\vec{r})\ \exp(-i\omega t) \tag{2}$$

$$\vec{\phi}(\vec{r},t) = \phi(\vec{r})\ \exp(-i\omega t)\vec{e}_\phi \tag{3}$$

where $\vec{e}_\phi$ is the unit vector in the direction of shear wave.

Substituting eqs. (2.3.2)(2.3.3) into eqs. (2.1.8)(2.1.9), the wave equations are reduced to

$$(\nabla^2 + \alpha^2)\ \psi(\vec{r}) = 0 \tag{4}$$

$$(\nabla^2 + \beta^2)\ \phi(\vec{r}) = 0 \tag{5}$$

where $\alpha = \omega/v_L$, $\beta = \omega/v_T$ are the wavenumber of longitudinal and shear waves, respectively.

CHAPTER 3

INHOMOGENEITY PROBLEMS

3.1. Transmission and reflection of waves in an infinite three-layered medium

A plane compressional incident wave which is simple harmonic is assumed to propagate in a three-layered medium. The geometry and material properties of the medium are shown in Fig. 1. The displacement of the incident wave is

$$\vec{U}_z^{(i)} = U_o \exp(i\underline{k}\cdot\underline{r} - i\omega t)\vec{e}_z \tag{1}$$

where U_o is the amplitude and k, ω are wavenumber and angular frequency, respectively. $\vec{e}_z$ is the unit vector in the positive direction of z-axis, i.e. plane wave propagation.

3.1.1. Integration of the governing differential equation

When the plane compressional incident wave impinges on the interface between the elastic inhomogeneity (medium II) and its surrounding medium (medium I), a compressional wave is reflected back into the medium I, while a compressional wave is transmitted into region III[13]. For convenience of the ensuing discussion, the displacements and stresses associated with the incident, reflected, refracted and transmitted waves will be designated by the superscript (i), (r), (f) and (t). Details for incident compressional plane waves along the +z axis are given below. The displacement of these waves are

$$U_z^{(i)} = U_o \exp(i\alpha_1 z - i\omega t) \tag{1}$$

$$U_z^{(r)} = -A\exp(-i\alpha_1 z - i\omega t) \tag{2}$$

$$U_z^{(t)} = B \exp(i\alpha_1 z - i\omega t) \tag{3}$$

$$U_z^{(f)} = U^{(f)}(z) \exp(-i\omega t) \tag{4}$$

where U_o, A, B are the amplitude of the incident, reflected and transmitted waves, respectively. $\alpha_1{}^2 = \omega^2/v_L^2 = \rho_1\omega^2/(\lambda_1+2\mu_1)$, where the subscript 1 denotes the matrix.

The displacement function $U^{(f)}(z)$ should satisfy the wave equation

$$(\nabla^2 + \alpha_2{}^2)U(z) = 0 \tag{5}$$

where $\alpha_2{}^2 = \omega^2/v_L^2 = \dfrac{\rho_2\omega^2}{(\lambda_2+2\mu_2)}$ and the subscript 2 refers to the inhomogeneity.

For the simple case chosen, $U^{(f)}(z)$ are obtained as

$$U^{(f)}(z) = C\cos\alpha_2 z + D\sin\alpha_2 z \tag{6}$$

where C, D are constants.

The stresses associated with the above displacements are

$$\sigma_{zz}^{(i)} = i\alpha_1(\lambda_1+2\mu_1)\ U_o \exp(i\alpha_1 z - i\omega t) \tag{7}$$

$$\sigma_{zz}^{(r)} = -i\alpha_1(\lambda_1+2\mu_1)\ A \exp(-i\alpha_1 z - i\omega t) \tag{8}$$

$$\sigma_{zz}^{(t)} = i\alpha_1(\lambda_1+2\mu_1)\ B \exp(i\alpha_1 z - i\omega t) \tag{9}$$

$$\sigma_{zz}^{(f)} = \alpha_2(\lambda_2+2\mu_2)(-C\sin\alpha_2 z + D\cos\alpha_2 z) \tag{10}$$

The stresses and displacements must be continuous at the interface. Thus at $z = -\delta/2$ the continuity conditions are

$$U_z^{(i)} + U_z^{(r)} = U_z^{(f)} \tag{11}$$

$$\sigma_{zz}^{(i)} + \sigma_{zz}^{(r)} = \sigma_{zz}^{(f)} \tag{12}$$

and at $z = \delta/2$ the continuity conditions are

$$U_z^{(t)} = U_z^{(f)} \tag{13}$$

$$\sigma_{zz}^{(t)} = \sigma_{zz}^{(f)} \tag{14}$$

These continuity conditions give four simultaneous equations in terms of four unknown coefficients A, B, C and D. After solving the simultaneous equations, the unknowns A, B, C, D are found to be:

$$A=U_o\exp(-i\alpha_1\delta)\left[1 - \frac{\cos\frac{\alpha_2\delta}{2}}{\cos\frac{\alpha_2\delta}{2} - i\sqrt{fh}\,\sin\frac{\alpha_2\delta}{2}} - \frac{\sin\frac{\alpha_2\delta}{2}}{\sin\frac{\alpha_2\delta}{2} + i\sqrt{fh}\,\cos\frac{\alpha_2\delta}{2}}\right] \tag{15}$$

$$B=U_o\exp(-i\alpha_1\delta)\left[\frac{\cos\frac{\alpha_2\delta}{2}}{\cos\frac{\alpha_2\delta}{2} - i\sqrt{fh}\,\sin\frac{\alpha_2\delta}{2}} - \frac{\sin\frac{\alpha_2\delta}{2}}{\sin\frac{\alpha_2\delta}{2} + i\sqrt{fh}\,\cos\frac{\alpha_2\delta}{2}}\right] \tag{16}$$

$$C=U_o\exp(-i\alpha_1\delta/2)\ /\ \left(\cos\frac{\alpha_2\delta}{2} - i\sqrt{fh}\,\sin\frac{\alpha_2\delta}{2}\right) \tag{17}$$

$$D=-U_o\exp(-i\alpha_1\delta/2)\ /\ \left(\sin\frac{\alpha_2\delta}{2} + i\sqrt{fh}\,\cos\frac{\alpha_2\delta}{2}\right) \tag{18}$$

where $f = (\lambda_2+2\mu_2)/(\lambda_1+2\mu_1)$ and $h = \rho_2/\rho_1$.

The displacements and stresses in the whole medium can be obtained by using equations (1-4) and equations (7-10), respectively.

3.1.2. Extended Method of Equivalent Inclusion

The equations of motion for the inhomogeneity and the matrix are:

$$\sigma_{jk,k} = \rho_1\,\ddot{U}_j \qquad \text{matrix} \tag{1}$$

$$\sigma_{jk,k} = \rho_2\,\ddot{U}_j \qquad \text{inhomogeneity} \tag{2}$$

The stress-strain relations are:

$$\sigma_{jk} = C^{(1)}_{jkrs}\ \varepsilon_{rs} \qquad \text{matrix} \tag{3}$$

$$\sigma_{jk} = C^{(2)}_{jkrs}\ \varepsilon_{rs} \qquad \text{inhomogeneity} \tag{4}$$

where ρ_1, $C^{(1)}_{jkrs}$ and ρ_2, $C^{(2)}_{jkrs}$ are mass density and the elastic constants for the matrix and the inhomogeneity, respectively.

Replacing the inhomogeneity with an equivalent inclusion which has the same material properties as the surrounding matrix after an eigen-strain is imposed in the inclusion, the governing equations are equations (1,3) and

$$\sigma_{jk,k} = \rho_1\ \ddot{U}_j \qquad \text{inclusion} \tag{5}$$

$$\sigma_{jk} = C^{(1)}_{jkrs}\ \varepsilon^{e}_{rs} \qquad \text{inclusion} \tag{6}$$

$$\varepsilon^{e}_{rs} = \varepsilon_{rs} - \varepsilon^{*}_{rs} \tag{7}$$

where ε_{rs}, ε^{e}_{rs}, ε^{*}_{rs} are the total strain, elastic strain and eigen-strain, respectively.

The equivalence conditions derived and given in [7,9] are:

$$\Delta C_{jkrs}\ \varepsilon^{(m)}_{rs} + C^{(1)}_{jkrs}\ \varepsilon^{*(I)}_{rs} = -\ \Delta C_{jkrs}\ \varepsilon^{(a)}_{rs} \qquad \text{inside the inclusion} \tag{8}$$

$$\Delta\rho\ \ddot{U}^{(m)}_{j} + C^{(1)}_{jkrs}\ \varepsilon^{*(II)}_{rs,k} = -\ \Delta\rho\ \ddot{U}^{(a)}_{j} \qquad \text{inside the inclusion} \tag{9}$$

where $\Delta C_{jkrs} = C^{(2)}_{jkrs} - C^{(1)}_{jkrs}$, $\Delta\rho = \rho_2 - \rho_1$ and

$$C^{(1)}_{jkrs}\ \varepsilon^{*}_{rs,k} = C^{(1)}_{jkrs}\ \varepsilon^{*(I)}_{rs,k} + C^{(1)}_{jkrs}\ \varepsilon^{*(II)}_{rs,k} \tag{10}$$

$$U_j = U_j^{(a)} + U_j^{(m)} \tag{11}$$

where the superscripts "m" and "a" denote field that is induced by the presence of mis-match and that is applied.

Equations (3.1.2.5-3.1.2.11) are the general equations for the equivalent inclusion with no restrictions on the geometric shape of the inclusion, i.e. these equations can be applied to an inclusion of arbitrary geometry.

For the three-layered problem, the equivalence conditions are reduced to

$$\frac{1}{0!}\Delta\rho\omega^2 \sum_{n=o}^{m} [\phi_n[0]A_n + \phi_n^{(1)}[0]B_n] + (\lambda_1 + 2\mu_1)A_o = -\Delta\rho\omega^2 H_o$$

$$\frac{1}{1!}\Delta\rho\omega^2 \sum_{n=o}^{m} [\phi_n^{(1)}[0]A_n + \phi_n^{(2)}[0]B_n] + (\lambda_1 + 2\mu_1)A_1 = -\Delta\rho\omega^2 H_1 \qquad (12)$$

....

$$\frac{1}{m!}\Delta\rho\omega^2 \sum_{n=o}^{m} [\phi_n^{(m)}[0]A_n + \phi_n^{(m+1)}[0]B_n] + (\lambda_1 + 2\mu_1)A_m = -\Delta\rho\omega^2 H_m$$

and

$$\frac{1}{0!}(\Delta\lambda+2\Delta\mu) \sum_{n=o}^{m} [\phi_n^{(1)}[0]A_n+\phi_n^{(2)}[0]B_n] + (\lambda_1+2\mu_1)B_o = -(\Delta\lambda+2\Delta\mu)E_o$$

$$\frac{1}{1!}(\Delta\lambda+2\Delta\mu) \sum_{n=o}^{m} [\phi_n^{(2)}[0]A_n+\phi_n^{(3)}[0]B_n] + (\lambda_1+2\mu_1)B_1 = -(\Delta\lambda+2\Delta\mu)E_1 \qquad (13)$$

....

$$\frac{1}{m!}(\Delta\lambda+2\Delta\mu) \sum_{n=o}^{m} [\phi_n^{(m+1)}[0]A_n+\phi_n^{(m+2)}[0]B_n] + (\lambda_1+2\mu_1)B_m = -(\Delta\lambda+2\Delta\mu)E_m$$

where:
$$\phi_n(z) = \frac{1}{2i\alpha_1}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} (z')^n e^{i\alpha_1|z-z'|} dz'$$

$$\phi_n^{(k)}(z) = \frac{d^k}{dz^k}\phi_n(z)$$

$$\Delta\lambda = \lambda_2 - \lambda_1$$

$$\Delta\mu = \mu_2 - \mu_1$$

$$H_m = \frac{(i\alpha_1)^m}{m!} U_o$$

$$E_m = \frac{(i\alpha_1)^{m+1}}{m!} U_o$$

The eqs. (3.1.2.12) and (3.1.2.13) give 2(m+1) simultaneous equations, having matrix dimensions of 2(m+1) x 2(m+1) if matrix representation is used.

The displacement and strain fields inside and outside the equivalent inclusion are:

$$U_z(z,t) = U_z(z) \exp(-i\omega t) \tag{14}$$

$$U_z(z) = \sum_{n=o}^{m} [A_n\phi_n(z) + B_n\phi_n'(z)] + U_o e^{i\alpha_1 z} \tag{15}$$

$$\varepsilon_{zz}(z,t) = \varepsilon_{zz}(z) \exp(-i\omega t) \tag{16}$$

$$\varepsilon_{zz}(z) = \sum_{n=o}^{m} [A_n\phi_n'(z) + B_n\phi_n''(z)] + i\alpha_1 U_o e^{i\alpha_1 z} \tag{17}$$

and the stress fields are:

$$\sigma_{zz}(z,t) = \sigma_{zz}(z) \exp(-i\omega t) \tag{18}$$

$$\sigma_{zz}(z) = (\lambda_1 + 2\mu_1)\ \varepsilon_{zz}(z) \qquad \text{matrix} \tag{19}$$

$$\sigma_{zz}(z) = (\lambda_2 + 2\mu_2)\ \varepsilon_{zz}(z) \qquad \text{inclusion} \tag{20}$$

Evaluation and integration of the ϕ_n-integrals are shown in Appendix I.

3.2. Scattering of waves by a spherical inhomogeneity

A plane compressional incident wave which is simple harmonic is assumed to move in the positive z-axis in an infinite elastic medium where an elastic spherical inhomogeneity is embedded. The configuration of the whole domain is shown in Fig. 2. The displacement of the incident wave is

$$\vec{U}_z^{(i)} = U_o \exp(i\alpha z - i\omega t)\vec{e}_z \tag{1}$$

where U_o is the amplitude and α, ω are the compressional wavenumber and angular frequency, respectively, and $\vec{e}_z$ is the unit vector in the positive direction of the z-axis.

In this section two methods, the separation of variables approaches and the extended equivalent inclusion method, are used to determine the displacements and stresses of the reflected (scattered) waves. Then, two measurable physical quantities, differential scattering cross section and the total scattering cross section far from the inhomogeneity, will be expressed in terms of the scattered asymptotic values. The differential scattering cross section $dP(\omega)/d\Omega$ [4] is a measure of the fraction of incident power scattered into a particular direction, where $d\Omega$ is the differential element of solid angle. The total scattering cross section $P(\omega)$ is the ratio of the average power flux scattered into all directions to the average intensity of the incident fields.

3.2.1. Method of separation of variables

In this section, the work by Ying and ruell [10] and that by Pao and Mow [11] will be briefly introduced.

When a plane compressional incident wave impinges on the surface of the elastic inhomogeneity, scattering occurs. Both compressional and shear waves are reflected back into the matrix (Medium I) while the same types of waves are refracted into the inhomogeneity (Medium II). The potentials, displacements and stresses associated with the incident, reflected and refracted waves are denoted by the superscript (i), (r) and (f). The wave equations in terms of potentials are

$$(\nabla^2 + \alpha_1{}^2)\ \psi^{(i)} = 0 \qquad (1)$$

$$(\nabla^2 + \beta_1{}^2)\ \phi^{(i)} = 0$$

$$(\nabla^2 + \alpha_1{}^2)\ \psi^{(r)} = 0 \qquad (2)$$

$$(\nabla^2 + \beta_1{}^2)\ \phi^{(r)} = 0$$

$$(\nabla^2 + \alpha_2{}^2)\ \psi^{(f)} = 0 \qquad (3)$$

$$(\nabla^2 + \beta_2{}^2)\ \phi^{(f)} = 0$$

where $\alpha_1{}^2 = \rho_1\omega^2/(\lambda_1+2\mu_1)$, $\beta_1{}^2 = \rho_1\omega^2/\mu_1$, $\alpha_2{}^2 = \rho_2\omega^2/(\lambda_2+2\mu_2)$,

$\beta_2{}^2 = \rho_2\omega^2/\mu_2$.

The displacement of the incident wave is

$$U_z^{(i)} = U_o \exp(i\alpha_1 z - i\omega t) \qquad (4)$$

and the associated potential functions are

$$\psi^{(i)} = \frac{U_o}{\alpha_1} \exp(i\alpha_1 z) \tag{5}$$

$$\phi^{(i)} = 0 \tag{6}$$

$\psi^{(i)}$ can further be represented in terms of a spherical coordinate function

$$\psi^{(i)} = \frac{U_o}{\alpha_1} \sum_{n=o}^{\infty} (2n+1)i^n \, j_n(\alpha_1 r) \, P_n(\cos\theta) \tag{7}$$

where $j_n(x)$ are the spherical Bessel functions of the first kind, and $P_n(x)$ are Legendre polynomials.

Because of axisymmetry of this problem, displacements, stresses and potential functions are independent of the spherical coordinate Φ. Therefore, after solving the wave equations, the potentials of the reflected wave are [12]:

$$\psi^{(r)} = \sum_{n=o}^{\infty} (-i)^{n+1} \, a(2n+1)A_n \, h_n(\alpha_1 r)P_n(\cos\theta) \tag{8}$$

$$\phi^{(r)} = \sum_{n=o}^{\infty} (-i)^{n+1} \, a(2n+1)B_n \, h_n(\beta_1 r)P_n(\cos\theta) \tag{9}$$

where $h_n(x)$ are the spherical Hankel functions of the first kind.

For the refracted wave inside the inhomogeneity the potentials are [12]:

$$\psi^{(f)} = \sum_{n=o}^{\infty} (-i)^{n+1} \, a(2n+1)C_n j_n(\alpha_2 r)P_n(\cos\theta) \tag{10}$$

$$\phi^{(f)} = \sum_{n=o}^{\infty} (-i)^{n+1} \, a(2n+1)D_n j_n(\beta_2 r)P_n(\cos\theta) \tag{11}$$

The resultant waves in the medium I and medium II are

$$\psi_1 = \psi^{(i)} + \psi^{(r)} \tag{12}$$

$$\phi_1 = \phi^{(i)} + \phi^{(r)} \tag{13}$$

$$\psi_2 = \psi^{(f)} \tag{14}$$

$$\phi_2 = \phi^{(f)} \tag{15}$$

The general representation of displacements and stresses in terms of the potential functions ψ, ϕ can be found in [15]:

$$U_r = \frac{\partial\psi}{\partial r} + \frac{1}{r}(D_\theta \phi) \tag{16}$$

$$U_\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta} - \frac{\partial}{\partial\theta}(D_r\phi) \tag{17}$$

$$\sigma_{rr} = 2\mu[-\frac{\beta^2}{2}\psi - \frac{2}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}D_\theta\psi + \frac{1}{r}D_\theta(\frac{\partial\phi}{\partial r} - \frac{\phi}{r}) \tag{18}$$

$$\sigma_{\theta\theta} = 2\mu[-\frac{\beta^2-2\alpha^2}{2}\psi + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}D_\theta\psi - \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} \tag{19}$$

$$+ \frac{\cot\theta}{r}\frac{\partial}{\partial\theta}(D_r\phi) - \frac{1}{r}D_\theta\frac{\partial\phi}{\partial r}]$$

$$\sigma_{r\theta} = \mu\frac{\partial}{\partial\theta}[\frac{2}{r}\frac{\partial\psi}{\partial r} - \frac{2}{r^2}\psi + \beta^2\phi + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{2}{r^2}\phi + \frac{2}{r^2}D_\theta\psi] \tag{20}$$

$$\sigma_{\phi\phi} = 2\mu[-\frac{\beta^2-2\alpha^2}{2}\psi + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{D_\theta\phi}{r^2} - \frac{\cot\theta}{r}\frac{\partial}{\partial\theta}D_r\phi] \tag{21}$$

where $D_r = \frac{1}{r}\frac{\partial}{\partial r}$ and $D_\theta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})$. For the incident field, $\psi = \psi^{(i)}$, $\phi=0$, $\alpha=\alpha_1$, $\beta=\beta_1$. For the reflected field, $\psi = \psi^{(r)}$, $\phi=\phi^{(r)}$, $\alpha=\alpha_1$, $\beta=\beta_1$. For the refracted field, $\psi=\psi^{(f)}$, $\phi=\phi^{(f)}$, $\alpha=\alpha_2$, $\beta=\beta_2$.

Consider the boundary conditions at r=a

$$U_r^{(i)} + U_r^{(r)} = U_r^{(f)} \tag{22}$$

$$U_\theta^{(i)} + U_\theta^{(r)} = U_\theta^{(f)} \tag{23}$$

$$\sigma_{rr}^{(i)} + \sigma_{rr}^{(r)} = \sigma_{rr}^{(f)} \tag{24}$$

$$\sigma_{r\theta}^{(i)} + \sigma_{r\theta}^{(r)} = \sigma_{r\theta}^{(f)} \tag{25}$$

Substituting eqs. (7-11) and eqs. (16-21) into the above continuity equations will form four simultaneous equations. After solving these equations, the unknowns A_n, B_n, C_n, D_n will be obtained. Thus, the elastic field inside and outside the inhomogeneity can be determined.

The total scattering cross section far from the inhomogeneity is [12]:

$$P(\omega) = 4\pi a^2 \sum_{n=o}^{\infty} (2n+1)[|A_n|^2 + n(n+1) \frac{\alpha_1}{\beta_1} |B_n|^2] \tag{26}$$

and $P(\omega)$ can be normalized as

$$P(\omega) = 4 \sum_{n=o}^{\infty} (2n+1)[|A_n|^2 + n(n+1) \frac{\alpha_1}{\beta_1} |B_n|^2] \tag{27}$$

3.2.2. Extended Method of Equivalent Inclusion

The eqs.(3.1.2.1 - 3.1.2.11) are general equations by this method and can be applied to the inhomogeneity problem of arbitrary geometry [7-9]. The equivalence conditions are recorded here as follows:

$$\frac{1}{0!} \Delta\rho\omega^2 [f_{sj}[0]A_j + f_{sjk}[0]A_{jk} + \ldots + F_{skj}[0]B_{kj} + F_{skj\ell}[0]B_{kj\ell} + \ldots]$$

$$+A_s = - \Delta\rho\omega^2 H_s \tag{1}$$

$$\frac{1}{1!}\,\Delta\rho\omega^2[f_{sj,p}[0]A_j + f_{sjk,p}[0]A_{jk}+\ldots+F_{skj,p}[0]B_{kj} + F_{skj\ell,p}[0]B_{kj\ell}+\ldots]$$

$$+A_{sp} = -\Delta\rho\omega^2 H_{sp} \tag{2}$$

....

$$\frac{1}{0!}\,\Delta\lambda\delta_{st}[d_{mmj}[0]A_j + d_{mmjk}[0]A_{jk}+\ldots+D_{mmjk}[0]B_{jk}+D_{mmijk}[0]B_{ijk}+\ldots]$$

$$+\frac{1}{0!}\,2\Delta\mu[d_{stj}[0]A_j + d_{stjk}[0]A_{jk}+\ldots+D_{stjk}[0]B_{jk}+D_{stjki}[0]B_{jki}+\ldots]$$

$$+(\lambda_1\delta_{st}B_{mm}+2\mu_1 B_{st}) = -(\Delta\lambda\delta_{st}E_{mm} + 2\Delta\mu E_{st}) \tag{3}$$

$$\frac{1}{1!}\,\Delta\lambda\delta_{st}[d_{mmj,p}[0]A_j+d_{mmjk,p}[0]A_{jk}+\ldots+D_{mmjk,p}[0]B_{jk}+D_{mmijk,p}[0]B_{ijk}+\ldots]$$

$$+\frac{1}{1!}2\Delta\mu[d_{stj,p}[0]A_j+d_{stjk,p}[0]A_{jk}+\ldots+D_{stjk,p}[0]B_{jk}+D_{stjki,p}[0]B_{jki}+\ldots]$$

$$+(\lambda_1\delta_{st}B_{mmp}+2\mu_1 B_{stp}) = -(\Delta\lambda\delta_{st}E_{mmp} + 2\Delta\mu E_{stp}) \tag{4}$$

....

where: $\Delta\rho = \rho_2 - \rho_1$, $\Delta\lambda = \lambda_2 - \lambda$, $\Delta\mu = \mu_2 - \mu_1$

$H_s = U_o$ $\quad s=3$

$= 0$ $\quad$ otherwise

$H_{sp} = i\alpha U_o$ $\quad s=3 \quad p=3$

$= 0$ $\quad$ otherwise

$E_{mn} = i\alpha U_o$ $\quad m=3 \quad n=3$

$= 0$ $\quad$ otherwise

$E_{mnp} = -U_o\alpha^2$ $\quad m=3 \quad n=3 \quad p=3$

$= 0$ $\quad$ otherwise

$$4\pi\rho_1\omega^2 f_{js}[\vec{r}] = -[\beta_1^{\,2}\phi\delta_{js} + \psi_{,js} - \phi_{,js}]$$

$$4\pi\rho_1\omega^2 f_{jsk}[\vec{r}] = -[\beta_1^{\,2}\phi_k\delta_{js} + \psi_{k,js} - \phi_{k,js}]$$

$$4\pi\rho_1\omega^2 F_{skj}[\vec{r}] = -[\lambda_1\alpha_1^{\,2}\psi_{,s}\delta_{kj} + 2\mu_1\beta_1^{\,2}\phi_{,k}\delta_{sj} - 2\mu_1\psi_{,skj} + 2\mu_1\phi_{,kjs}]$$

$$4\pi\rho_1\omega^2 F_{skj\ell}[\vec{r}] = -[\lambda_1\alpha_1^{\,2}\psi_{\ell,s}\delta_{kj} + 2\mu_1\beta_1^{\,2}\phi_{\ell,k}\delta_{sj} - 2\mu_1\psi_{\ell,skj} + 2\mu_1\phi_{\ell,kjs}]$$

$$4\pi\rho_1\omega^2 d_{mnj}[\vec{r}] = -\beta_1^{\,2}[\phi_{,n}\delta_{jm} + \phi_{,m}\delta_{jn} + \psi_{,jmn} - \phi_{,jmn}]$$

$$4\pi\rho_1\omega^2 d_{mnjk}[\vec{r}] = -\beta_1^{\,2}[\phi_{k,n}\delta_{jm} + \phi_{k,m}\delta_{jn} + \psi_{k,jmn} - \phi_{k,jmn}]$$

$$4\pi\rho_1\omega^2 D_{mnjk}[\vec{r}] = 2\mu_1[\psi_{,kjmn} - \phi_{,kjmn}] - \mu_1\beta_1^{\,2}[\phi_{,jn}\delta_{km} + \phi_{,mj}\delta_{kn}] - \lambda_1\alpha_1^{\,2}\psi_{,mn}\delta_{jk}$$

$$4\pi\rho_1\omega^2 D_{mnijk}[\vec{r}] = 2\mu_1[\psi_{k,ijmn} - \phi_{k,ijmn}] - \mu_1\beta_1^{\,2}[\phi_{k,in}\delta_{jm} + \phi_{k,mi}\delta_{jn}] - \lambda_1\alpha_1^{\,2}\psi_{k,mn}\delta_{ij}$$

where the ϕ- and ψ-integrals and their derivatives are evaluated for a sphere by using the method given in [8] and are listed in Appendix II.

From Appendix II, $f_{sjk}[0]$, $F_{sjk}[0]$, $f_{sj,p}[0]$, $F_{skj\ell,p}[0]$, $d_{stj}[0]$, $D_{stjki}[0]$, $d_{stjk,p}[0]$ and $D_{stjk,p}[0]$ are equal to zero. Then eqs. (3.2.2.1 - 3.2.2.4) can therefore be much simplified.

If one assumes a two term expansion in the eigenstrains, then eqs. (3.2.2.1) and eqs. (3.2.2.4) will form 21 simultaneous equations while eqs. (3.2.2.2) and eqs. (3.2.2.3) will form another 15 simultaneous equations. The two sets of simultaneous equations are uncoupled.

Solving the two set simultaneous equations for A_j, A_{jk}, B_{jk} and $B_{jk\ell}$, the displacements of the scattered wave can be written as [8,9]

$$U_m(\vec{r},t) = [\vec{f}_{mj}A_j + \vec{f}_{mjk}A_{jk} + \vec{F}_{mkj\ell}B_{kj\ell} + \vec{F}_{mkj}B_{kj}]\exp(-i\omega t) \tag{5}$$

where: $$4\pi\rho_1\omega^2\vec{f}_{mj} = -\beta_1^2\vec{\phi}\,\delta_{mj} + \vec{\psi}_{,mj} - \vec{\phi}_{,mj}$$

$$4\pi\rho_1\omega^2\vec{f}_{mj\ell} = -\beta_1^2\vec{\phi}_\ell\delta_{mj} + \vec{\psi}_{\ell,mj} - \vec{\phi}_{\ell,mj}$$

$$4\pi\rho_1\omega^2\vec{F}_{mkj} = -[\lambda_1\alpha_1^2\vec{\psi}_{,m}\delta_{kj} + 2\mu_1\beta_1^2\vec{\phi}_{,k}\delta_{mj} - 2\mu_1\vec{\psi}_{,mkj} + 2\mu_1\vec{\phi}_{,mkj}]$$

$$4\pi\rho_1\omega^2\vec{F}_{mkj\ell} = -[\lambda_1\alpha_1^2\vec{\psi}_{\ell,m}\delta_{kj} + 2\mu_1\beta_1^2\vec{\phi}_{\ell,k}\delta_{mj} - 2\mu_1\vec{\psi}_{\ell,mkj} + 2\mu_1\vec{\phi}_{\ell,mkj}]$$

where: $$\vec{\psi}[\vec{r}] = \iiint_\Omega \frac{\exp(i\alpha_1 R)}{R}\,dV' \qquad \vec{r} \text{ outside } \Omega$$

$$\vec{\psi}_k[\vec{r}] = \iiint_\Omega x'_k \frac{\exp(i\alpha_1 R)}{R}\,dV' \qquad \vec{r} \text{ outside } \Omega$$

$$\vec{\psi}_{k\ell\ldots s}[\vec{r}] = \iiint_\Omega x'_k x'_\ell \ldots x'_s \frac{\exp(i\alpha_1 R)}{R}\,dV' \qquad \vec{r} \text{ outside } \Omega$$

$$\vec{\psi}_{,k}[\vec{r}] = \frac{\partial}{\partial x_k}\psi[\vec{r}]$$

$$\vec{\psi}_{,k\ell\ldots s}[\vec{r}] = \frac{\partial}{\partial x_k \partial x_\ell \ldots \partial x_s}\psi[\vec{r}]$$

$$\vec{\phi}[\vec{r}] = \iiint_\Omega \frac{\exp(i\beta_1 R)}{R}\,dV' \qquad \vec{r} \text{ outside } \Omega$$

$$\vec{\phi}_k[\vec{r}] = \iiint_\Omega x'_k \frac{\exp(i\beta_1 R)}{R}\,dV' \qquad \vec{r} \text{ outside } \Omega$$

$$\vec{\phi}_{k\ell\ldots s}[\vec{r}] = \iiint_{\Omega} x'_k x'_\ell \ldots x'_s \frac{\exp(i\beta_1 R)}{R}\, dV' \qquad \vec{r} \text{ outside } \Omega$$

$$\vec{\phi}_{,k}[\vec{r}] = \frac{\partial}{\partial x_k}\, \phi[\vec{r}]$$

$$\vec{\phi}_{,k\ell\ldots s}[\vec{r}] = \frac{\partial}{\partial x_k \partial x_\ell \ldots x_s}\, \vec{\phi}[\vec{r}]$$

As $|\vec{r}| \to \infty$ the far field value of U_m can be expressed as [9]:

$$U_m = C_m \frac{\exp i\alpha_1 r}{r} + D_m \frac{\exp i\beta_1 r}{r} \tag{6}$$

and the associated far field stress are

$$\begin{aligned} \sigma_{mn} = {} & i\lambda_1 \alpha_1 \frac{\exp i\alpha_1 r}{r} C_k \ell_k \delta_{mn} \\ & + i\mu_1 [\alpha_1 \frac{\exp i\alpha_1 r}{r} (C_m \ell_n + C_n \ell_m) \\ & + \beta_1 \frac{\exp i\beta_1 r}{r} (D_m \ell_n + D_n \ell_m)] \end{aligned} \tag{7}$$

where ℓ_k are direction cosines.

The differential scattering cross section is therefore [9]

$$\frac{dP(\omega)}{d\Omega} = \left|\frac{C_m}{U_o}\right|^2 + \frac{\alpha_1}{\beta_1}\left|\frac{D_m}{U_o}\right|^2 \tag{8}$$

and the total scattering cross section is

$$P(\omega) = \int \frac{dP(\omega)}{d\Omega}\, d\Omega \tag{9}$$

and $P(\omega)$ can be normalized as

$$P(\omega) = \frac{1}{\pi a^2} \int \frac{dP(\omega)}{d\Omega}\, d\Omega \tag{10}$$

Some important volume integral calculations are shown in Appendix II.

CHAPTER 4

COMPARISON OF COMPUTATIONAL RESULTS

In this chapter numerical results are presented and compared for the three-layered and spherical inhomogeneity problems. Numerical results were made known by Truell and his co-workers for a perfect spherical inhomogeneity. In order to compare with these results, the same material properties are used and listed in Table 1.

Table 1. Material Properties

Material	Compressional wave velocity (m/s)	shear wave velocity (m/s)	Mass density (g/cm^3)
Stainless Steel	5790	3100	7.90
Mg	5770	3050	1.74
Al	6568	3149	2.70
Ge	5285	3376	5.36
Polyethylene	1950	540	0.90
Be	12890	8880	1.87

When the extended equivalent inclusion method is used, there are complex matrices, with dimensions depending upon the dimensionless wavenumber and the differences in elastic constant and mass density between the inhomogeneity and the matrix, need to be solved. In some cases, the dimension of the matrix are so large that the numerical error can be relatively high. In order to get the best results, the IMSL(International Mathematical and Statistical Libraries) subroutine LEQ2C is used to solve matrices with double precision. The routine applies iterative improvement until the solution is accurate to machine precision. If the matrix is too ill-conditioned to get effective iterative improvement, a terminal error is produced.

4.1. Three-Layered Medium Problem

The input parameters for both methods are the dimensionless wavenumber $\alpha_1\delta$, the relative ratio of elastic constants f and mass density h, where f is $(\lambda_2+2\mu_2)/(\lambda_1+2\mu_1)$ and h is ρ_2/ρ_1. The displacement amplitude U_o and the stress amplitude $(\lambda_1+2\mu_1)\alpha_1 U_o$ given in the preceding figures are nondimensionalized.

The calculation of the exact solutions is simple and the dimension of the matrix is just 4x4. The calculation of the equivalent inclusion method is relatively complicated. The solution is an infinite series summation, and N_f(the accepted number of terms to get convergent results) depends on $\alpha_1\delta$, f and h. In the calculation of the displacements and stresses from Eqs. (3.1.2.14 - 3.1.2.20), the summation is considered to be acceptable until the ratio of the current term to the current partial term is less than 0.5%.

In order to make detailed comparison, three cases are studied here. The first case considers the difference in elastic constants, i.e. f≠1 and h=1. Fig. 3-7 and Fig. 8-12 display the displacement amplitude and stress amplitude vs $\alpha_1\delta$, respectively. The second case considers the difference in mass density, i.e. f=1 and h≠1. Fig. 13-17 and Fig. 18-22 display the displacement amplitude and stress amplitude vs $\alpha_1\delta$. The last case considers the difference both in elastic constants and mass density. Fig. 23-27 and Fig. 28-32 display the displacement amplitude and stress amplitude vs $\alpha_1\delta$. From the results, it is found that the extended method of equivalent inclusion gives excellent results which can be treated as the exact solutions. The values of N_f in getting the convergent displacement and stress amplitude in the third case are listed in Table 2.

Table 2. The Value of N_f for the Three-Layered Problem

$\alpha_1\delta$	Ge in Al f=1.285 h=1.985	Al in Ge f=0.778 h=0.504	Polyethylene in Be f=90.790 h=2.078	Mg in St f=0.219 h=0.220	St in Mg f=4.572 h=4.540
0.5	6	4	6	6	6
1.0	6	6	6	6	8
2.0	8	6	6	8	8
4.0	12	8	8	10	12
6.0	16	10	8	14	14
8.0	18	14	8	16	18
10.0	22	16	10	20	20

4.2. Spherical Inhomogeneity Problem

The input parameters are the dimensionless wave number $\alpha_1 a$, the compressional and shear wave velocity and the mass density of the inhomogeneity and the matrix.

For the method of separation of variables, the results of the scattering cross section were plotted in Truell's paper but the specific values for different $\alpha_1 a$ are not listed.†

For the equivalent inclusion method, there are two independent series summation. The first one is to get the inside function value in the eqs. (3.2.2.1 - 3.2.2.4), which are related to $\alpha_1 a$ and β_1/α_1 only. The second one is to get the outside function value in the eqs. (3.2.2.5 - 3.2.2.10), which also depends on $\alpha_1 a$ and β_1/α_1.

From the three-layered problem, it is known that the value of N_f

†The data shown in this section are calculated from the computer program kindly supplied by Dr. J. Gubernatis.

depends on $\alpha_1\delta$, f and h. This is also true for the current problem. Besides, it is found that the value of N_f also depends on β_1/α_1. It should be noted that the dimension of the matrix is proportional to the value of N_f. Therefore, for large N_f, the dimension of the matrix will be very large and the derivation and numerical computation to get the scattering cross section is very lengthy and time-consuming. Instead of finding the scattering cross section for large $\alpha_1 a$, it is hoped to find the accepted value in what range of $\alpha_1 a$ when $N_f=1$. Later, the result is called one-term solution when N_f is equal to 1. The eqs. (3.2.2.1 - 3.2.2.4) are reduced to:

$$\Delta\rho\omega^2[f_{sj}[0]A_j + F_{skj}[0]B_{kj}] + A_s = -\Delta\rho\omega^2 H_s \tag{1}$$

$$\Delta\lambda\delta_{st}[d_{mmj}[0]A_j + D_{mmjk}[0]B_{jk}] + 2\Delta\mu[d_{stj}[0]A_j + D_{stjk}[0]B_{jk}]$$
$$+ (\lambda_1\delta_{st}B_{mm} + 2\mu_1 B_{st}) = -(\Delta\lambda\delta_{st}E_{mm} + 2\Delta\mu E_{st}) \tag{2}$$

From the formulas in Appendix II, it is found that $F_{skj}[0]$ and $d_{stj}[0]$ are equal to zero, which make the eqs. (4.2.2.1) and (4.2.2.2) uncoupled, i.e. the difference in mass density and the difference in elastic constants will have no coupled effects. After some manipulation, it is found that the nonzero variables are A_3, B_{11}, B_{22} and B_{33} with B_{11} equal to B_{22} by symmetry. For low $\alpha_1 a$, the closed form solutions of these variables can be obtained and the closed form of the scattering cross section can also be obtained.

Fig. 33-36 display the scattering cross section vs $\alpha_1 a$ from two method. It is found if $\alpha_1 a$ is less than 1, the tendency of the one term solution is good compared to the exact solution. For certain material system the one-term solution represents a good approximation up to medium frequency range. The accuracy, however, is not good enough to produce relative errors less than 1% though $\alpha_1 a$ goes down to 0.01. The Fig. 37-40 display the comparison for the $\alpha_1 a$ between 0.01 and 0.1. It is therefore necessary to take more terms in the series for higher $\alpha_1 a$ if this approach is to be used.

CHAPTER 5

DISCUSSION AND CONCLUSION

It is worth studying the numerical characteristics of the extended method of equivalent inclusion. By doing this, we will know what kind of inclusion this method can be applied to. Because of the simplicity and low computer time, the three-layered problem is chosen first for studying. Fig. 41-50 display the displacement and stress amplitude vs $\alpha_1\delta$, f and h, respectively. In these figures, the curves both for N(number of terms) = 6,12 and for the exact solution are shown. From these figures, it is found N_f will increase while $\alpha_1\delta$ or h increases or f decreases. Besides, the value of N_f for convergent stress amplitude is larger than the value for convergent displacement amplitude. It is also found that the displacement amplitude is less than the exact solution and the stress amplitude is larger than the exact solution at the larger $\alpha_1\delta$ if N is less than N_f.

Though the acceptable numerical results of the equivalent inclusion method in spherical inhomogeneity are not obtained in this study, the one-term solution supplies a very good tendency at low wavenumber. It is expected when the two-term solution (N=2) are applied, the accuracy at low dimensionless wavenumber will be very good and it will also supply good tendency in higher dimensionless wavenumber. And if N_f is large enough for different inclusion, the value from this method should be very close to the exact solution as in the three-layered problem.

From the above study, it is found N_f will be large as the $\alpha_1\delta$ (or $\alpha_1 a$) or h or β_1/α_1 is very large or f is very small. Therefore, the dimension of the complex matrices will also be very large, which will probably cause the numerical singularity (algorthmic singularity) in high $\alpha_1 a$ range. The numerical singularity may be overcome after some numerical improvement skills are applied or other numerical methods are used. It is suggested that numerical scheme be carefully studied for large h, β_1/α_1 or small f at high wavenumber range.

APPENDIX I

EVALUATION OF SOME INTEGRALS FOR THE THREE-LAYERED PROBLEM

I.1. $\int_{-h}^{h} (z')^n e^{i\alpha|z-z'|} dz'$

$$= (-1)^n e^{i\alpha z} R_n(\alpha h) \qquad \text{for } z \geq h$$

$$= [(-1)^n e^{i\alpha z} + e^{-i\alpha z}] e^{i\alpha h} R_n(\alpha h) - \sum_{r=o}^{n} [1 + (-1)^r] \frac{n! z^{n-r}}{(n-r)!(i\alpha)^{r+1}}$$

$$\text{for } -h \leq z \leq h$$

$$= e^{-i\alpha z} R_n(\alpha h) \qquad \text{for } z \leq -h$$

where: $R_n(\alpha h) = e^{i\alpha h} \sum_{r=o}^{n} \frac{(-1)^r n! h^{n-r}}{(n-r)!(i\alpha)^{r+1}}$

$$- (-1)^n e^{-i\alpha h} \sum_{r=o}^{n} \frac{n! \, h^{n-r}}{(n-r)! \, (i\alpha)^{r+1}}$$

I.2. $\alpha^{n+2}\phi_n(z) = \frac{\alpha^{n+2}}{2i\alpha} \int_{-\delta/2}^{\delta/2} (z')^n e^{i\alpha|z-z'|} dz' \qquad -\frac{\delta}{2} \leq z \leq \frac{\delta}{2}$

$$= i^{-2} \left[\frac{(-1)^n e^{i\alpha z} + e^{-i\alpha z}}{2} e^{\frac{i\alpha\delta}{2}} L_n(\alpha\delta) - \sum_{m=o}^{n} \frac{1+(-1)^m}{2} \frac{n! \, (\alpha z)^n}{(n-m)! \, (i\alpha z)^m} \right]$$

where: $L_n(\alpha\delta) = \sum_{m=o}^{n} \frac{n! (\frac{\alpha\delta}{2})^n}{(n-m)!(-\frac{i\alpha\delta}{2})^m}$

I.3. $\alpha^{n-k+2}\phi_n^{(k)}(z) = \alpha^{n-k+2} \frac{d^k}{dz^k} \phi_n(z) \qquad -\frac{\delta}{2} \leq z \leq \frac{\delta}{2}$

$$= i^{k-2} \left[\frac{(-1)^n e^{i\alpha z} + (-1)^k e^{-i\alpha z}}{2} e^{\frac{i\alpha\delta}{2}} L_n(\alpha\delta) - \sum_{m=o}^{n-k} \frac{1+(-1)^m}{2} \frac{n! \, (\alpha z)^n}{(n-m-k)!(i\alpha z)^{n+k}} \right]$$

I.4. $\alpha^{n-k+2}\phi_n^{(k)}[0]$ $\qquad z=0$

(1) both n and k are not odd integer

$$\alpha^{n-k+2}\phi_n^{(k)}[0] = i^{k-2}\, e^{\frac{i\alpha\delta}{2}} L_n(\alpha\delta) \qquad \text{for } n < k$$

$$= i^{k-2}\left[e^{\frac{i\alpha\delta}{2}} L_n(\alpha\delta) - \frac{n!}{i^n}\right] \qquad \text{for } n \geq k$$

(2) both n and k are odd integer

$$\alpha^{n-k+2}\phi_n^{(k)}[0] = -\, i^{k-2} e^{\frac{i\alpha\delta}{2}} L_n(\alpha\delta) \qquad \text{for } n < k$$

$$= -i^{k-2}\left[e^{\frac{i\alpha\delta}{2}} L_n(\alpha\delta) + \frac{n!}{i^n}\right] \qquad \text{for } n \geq k$$

(3) only one of n and k is odd integer

$$\alpha^{n-k+2}\phi_n^{(k)}[0] = 0$$

I.5. $$\alpha^{n+2}\phi_n(z) = \frac{(-1)^n e^{i\alpha z}}{2i^2} R_n(\alpha\delta) \qquad \text{for } z \geq \frac{\delta}{2}$$

$$= \frac{e^{-i\alpha z}}{2i^2} R_n(\alpha\delta) \qquad \text{for } z \leq -\frac{\delta}{2}$$

I.6. $$\alpha^{n-k+2}\phi_n(z) = \frac{(-1)^n (i)^k e^{i\alpha z}}{2i^2} R_n(\alpha\delta) \qquad \text{for } z \geq \frac{\delta}{2}$$

$$= \frac{(-i)^k e^{-i\alpha z}}{2i^2} R_n(\alpha\delta) \qquad \text{for } z \leq -\frac{\delta}{2}$$

where $$R_n(\alpha\delta) = e^{\frac{i\alpha\delta}{2}} \sum_{m=o}^{n} \frac{n!\,(\frac{\alpha\delta}{2})^n}{(n-m)!\,(-\frac{i\alpha\delta}{2})^m}$$

$$-(-1)^n e^{-\frac{i\alpha\delta}{2}} \sum_{m=o}^{n} \frac{n!\,(\frac{\alpha\delta}{2})^n}{(n-m)!\,(\frac{i\alpha\delta}{2})^m}$$

APPENDIX II

EVALUATION OF SOME VOLUME INTEGRALS FOR THE SPHERICAL INHOMOGENEITY PROBLEM

From Ref. [6,8], we obtain:

II.1. $\Psi(\vec{r}) = \iiint_{\Omega} \frac{\exp(i\alpha R)}{R}\, dV'$ $\qquad \vec{r}$ inside Ω

$$= \sum_{n=o}^{\infty} \sum_{\ell=o}^{n} \sum_{k=o}^{n-\ell} \frac{(-1)^n}{\ell!\,k!\,(n-\ell-k)!}\, x^{\ell}\, y^{k}\, z^{n-\ell-k}$$

$$\iiint_{\Omega} \frac{\partial^n}{\partial x'^{\ell}\, \partial y'^{k}\, \partial z'^{(n-\ell-k)}} \frac{\exp\, i\alpha r'}{r'}\, dx'\, dy'\, dz'$$

$$= \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \underbrace{x_p x_q \; \ldots \; x_v}_{\text{nth power of } \overline{x}}$$

$$\left[\sum_{m=o}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} C^{(n),m}_{pq\ldots v} + i \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \alpha^{2m-1}}{(2m-1)!} S^{(n),m}_{pq\ldots v} \right]$$

where: $R = |\vec{r} - \vec{r}\,'|$

$$C^{(n),m}_{pq\ldots v} = \iiint_{\Omega} \rho(\vec{r}\,') \frac{\partial^n (r')^{2m-1}}{\partial x'_p \partial x'_q \ldots \partial x'_v}\, dV'$$

$$S^{(n),m}_{pq\ldots v} = \iiint_{\Omega} \rho(\vec{r}\,') \frac{\partial^n (r')^{2m-2}}{\partial x'_p \partial x'_q \ldots \partial x'_v}\, dV'$$

$$\rho(\vec{r}\,') = 1$$

II.2. $\Psi_{k\ell\ldots s}(\vec{r}) = \iiint_{\Omega} x'_k x'_{\ell} \ldots x'_s \frac{\exp(i\alpha R)}{R}\, dV'$ $\qquad \vec{r}$ inside Ω

$$= \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \underbrace{x_p x_q \ldots x_v}_{\text{nth power of } \overline{x}}$$

$$\left[\sum_{m=o}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} C^{(n),m}_{pq\ldots v} + i \sum_{m=1}^{\infty} \frac{(-1)^{m-1} \alpha^{2m-1}}{(2m-1)!} S^{(n),m}_{pq\ldots v} \right]$$

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where: $$C^{(n),m}_{pq\ldots v} = \iiint_\Omega \rho(\vec{r}\,') \frac{\partial^n (r')^{2m-1}}{\partial x'_p \partial x'_q \ldots \partial x'_v} dV'$$

$$S^{(n),m}_{pq\ldots v} = \iiint_\Omega \rho(\vec{r}\,') \frac{\partial^n (r')^{2m-2}}{\partial x'_p \partial x'_q \ldots \partial x'_v} dV'$$

$$\rho(\vec{r}\,') = x'_k x'_\ell \ldots x'_s$$

In what follows we develop these formula specially for a sphere of radius a:

II.3. $C^{(n),m}_{pq\ldots v}$ and $S^{(n),m}_{pq\ldots v}$ for $\rho(\vec{r}) = 1$ and Ω is sphere

(1) $$C^{(o),m} = 4\Pi \frac{a^{2m+2}}{2m+2}$$

$$S^{(o),m} = 4\Pi \frac{a^{2m+1}}{2m+1}$$

(2) $$C^{(2),m}_{pq} = 0 \qquad \text{for } m=0$$

$$= \frac{4\Pi}{3}(2m-1)a^{2m}\delta_{pq}$$

$$S^{(2),m}_{pq} = \frac{4\Pi}{3}(2m-2)a^{2m-1}\delta_{pq}$$

(3) $$C^{(4),m}_{pquv} = 0 \qquad \text{for } m=0,1$$

$$= (2m-1)(2m-3)2m\ a^{2m-2}\ \frac{4\Pi}{5} \qquad p=q=u=v$$

$$= (2m-1)(2m-3)2m\ a^{2m-2}\ \frac{4\Pi}{15} \qquad \text{for any two equal pairs of indices}$$

$$= 0 \qquad \text{otherwise}$$

$$S^{(4),m}_{pquv} = (2m-2)(2m-4)(2m-1)a^{2m-3}\ \frac{4\Pi}{5} \qquad p=q=u=v$$

$$= (2m-2)(2m-4)(2m-1)\ a^{2m-3}\ \frac{4\Pi}{15} \qquad \text{for any two equal pairs of indices}$$

$$= 0 \qquad \text{otherwise}$$

(4) $C_p^{(1),m} = C_{pqu}^{(3),m} = S_p^{(1),m} = S_{pqu}^{(3),m} = 0$

II.4. $C_{pqu\ldots v}^{(n),m}$ and $S_{pq\ldots v}^{(n),m}$ for $\rho(\vec{r}\,') = x_i'$ and Ω is sphere

(1) $C_p^{(1),m} = \frac{(2m-1)}{2m+2}\, a^{2m+2} \cdot \frac{4\Pi}{3}\, \delta_{pi}$

$S_p^{(1),m} = \frac{2m-2}{2m+1}\, a^{2m+1}\, \frac{4\Pi}{3}\, \delta_{pi}$

(2) $C_{pqu}^{(3),m} = 0$ for $m=0$

$= (2m-1)(2m-3)a^{2m}\ \frac{4\Pi}{5}$ $p=q=u=i$

$= (2m-1)(2m-3)a^{2m}\ \frac{4\Pi}{15}$ for any two equal pairs of indices

$= 0$ otherwise

$S_{pqu}^{(3),m} = (2m-2)(2m-4)a^{2m-1}\ \frac{4\Pi}{5}$ $P=q=u=i$

$= (2m-2)(2m-4)a^{2m-1}\ \frac{4\Pi}{15}$ for any two equal pairs of indices

$= 0$ otherwise

(3) $C_{pquvw}^{(5),m} = 0$ for $m=0,1$

$= (2m-1)(2m-3)(2m-5)2m\ a^{2m-2} \cdot D$

$D = \frac{4\Pi}{7}$ $i=p=q=u=v=w$

$D = \frac{4\Pi}{35}$ $\begin{cases} i=p & q=u=v=w \\ p=q & i=u=v=w \end{cases}$ p,q,u,v,w can change the order arbitrarily

$D = \frac{4\Pi}{105}$ $i=p$ $q=u$ $v=w$

$D = 0$ otherwise

$$S^{(5),m}_{pquvw} = (2m-2)(2m-4)(2m-6)(2m-1)a^{2m-3} \cdot D$$

$$D = \frac{4\Pi}{7} \qquad i=p=q=u=v=w$$

$$D = \frac{4\Pi}{35} \qquad \begin{matrix} i=p \;\; q=u=v=w \\ p=q \;\; i=u=v=w \end{matrix}$$

$$D = \frac{4\Pi}{105} \qquad i=p \;\; q=u \;\; v=w$$

$$D = 0 \qquad \text{otherwise}$$

(4) $$C^{(o),m} = S^{(o),m} = C^{(2),m}_{pq} = S^{(2),m}_{pq} = C^{(4),m}_{pquv} = S^{(4),m}_{pquv} = 0$$

II.5.

(1) $$\psi[0] = \sum_{m=o}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} C^{(o),m} + i \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\alpha^{2m-1}}{(2m-1)!} S^{(o),m}$$

(2) $$\psi,_{pq}[0] = (-1)^2 \left[\sum_{m=o}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} C^{(2),m}_{pq} + i \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\alpha^{2m-1}}{(2m-1)!} S^{(2),m}_{pq}\right]$$

(3) $$\psi,_{pquv}[0] = (-1)^4 \left[\sum_{m=o}^{\infty} \frac{(-1)^m \alpha^{2m}}{(2m)!} C^{(4),m}_{pquv} + i \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\alpha^{2m-1}}{(2m-1)!} S^{(4),m}_{pquv}\right]$$

(4) $$\psi,_{p}[0] = \psi,_{pqu}[0] = \psi,_{pquvw}[0] = 0$$

II.6. $$\tilde{\psi}(\vec{r}) = \iiint_\Omega \frac{\exp(i\alpha R)}{R}\, dV' \qquad \vec{r} \text{ outside } \Omega$$

$$= \sum_{n=o}^{\infty} \sum_{\ell=o}^{n} \sum_{k=o}^{n-\ell} \frac{(-1)^n}{\ell!\; k!\,(n-\ell-k)!} \cdot \frac{\partial^n \left(\frac{\exp\, i\alpha r}{r}\right)}{\partial x^\ell \partial y^k \partial z^{n-\ell-k}} \cdot$$

$$\iiint (x')^\ell (y')^k (z')^{n-\ell-k}\, dx'dy'dz'$$

for $|\vec{r}| \to \infty$ and Ω is sphere

$$\tilde{\psi}(\vec{r}) = \sum_{n=o}^{\infty} \sum_{\ell=o,2}^{n} \sum_{k=o,2}^{n-\ell} F(n,\ell,k,a,\alpha,r)$$

where: $F(n,\ell,k,a,\alpha,r) = (-i\alpha)^n \frac{\exp(i\alpha r)}{r} (e_x)^\ell (e_y)^k (e_z)^{n-\ell-k}$

$$\cdot \frac{4\Pi a^{n+3}}{(n+3)(n+1)} \quad \frac{\frac{n}{2}!}{(n!)(\frac{\ell}{2}!)(\frac{k}{2}!)(\frac{n-\ell-k}{2}!)}$$

e_x, e_y, e_z are direction cosines

II.7. $\overset{>}{\psi}_{,m}(\vec{r}) = \frac{\partial}{\partial x_m} \overset{>}{\psi}(\vec{r})$

$$\overset{>}{\psi}_{,mj}(\vec{r}) = \frac{\partial^2}{\partial x_m \partial x_j} \overset{>}{\psi}(\vec{r})$$

for $|\vec{r}| \to \infty$ and Ω is sphere

$$\overset{>}{\psi}_{,m}(\vec{r}) = \sum_{n=o,2}^{\infty} \sum_{\ell=o,2}^{n} \sum_{k=o,2}^{n-\ell} (i\alpha) e_m \cdot F(n,\ell,k,a,\alpha,r)$$

$$\overset{>}{\psi}_{,mj}(\vec{r}) = \sum_{n=o,2}^{\infty} \sum_{\ell=o,2}^{n} \sum_{k=o,2}^{n-\ell} (i\alpha)^2 e_m e_j \cdot F(n,\ell,k,a,\alpha,r)$$

REFERENCES

1. Eshelby, J.D., "The Determination of the Elastic Field of an Ellipsoidal Inclusion and Related Problems," Proc. Roy. Soc., A241, 376-396 (1957).

2. Eshelby, J.D., in Progress in Solid Mechanics, Vol. 2, ed. by Sneedon and Hill, North-Holland, Amsterdam, 89-140 (1961).

3. Mal, A.K. and Knopoff, L., "Elastic Waves in Two-Component Systems," J. of Inst. Maths. Applics., 3, 376-387 (1967).

4. Gubernatis, J., "Long-Wave Approximation for the Scattering of Elastic Waves from Flaws with Application to Ellipsoidal Voids and Inclusions," J. of Appl. Phys., Vol. 50, 4046-4058 (1979).

5. Wheeler, P. and T. Mura, "Dynamical Equivalence of Composite Material and Eigenstrain Problems," J. of Appl. Mech., Vol. 40, 498-502 (1973).

6. Fu, L.S., "Fundamental Aspects in the Quantitative Ultrasonic Determination of Fracture Toughness: General Equations," NASA Contractor Report #3445 (July 1981).

7. Fu, L.S., "Method of Equivalent Inclusion in Dynamical Elasticity," Proceedings AF/DARPA Progress Review on Quantitative NDE, Plenum Press, (1982). Presented at Boulder, Colorado (Aug., 1981).

8. Fu, L.S. and Mura, T., "Volume Integrals of Ellipsoids Associated with the Inhomogeneous Helmholtz Equation," Wave Motion, 4(2), 141-149 (1982).

9. Fu, L.S. and Mura, T., "An Eigenstrain Approach to the Scattering of a Single Ellipsoidal Inhomogeneity," J. Appl. Mech., Trans. ASME, to appear.

10. Ying, C. and Truell, R., "Scattering of a Plane Longitudinal Wave by a Spherical Obstacle in an Isotropically Elastic Solid," J. of Appl. Phys., Vol. 27, 1086-1097 (1956).

11. Pao, Y. and Mow, C., "Scattering of Plane Compressional Waves by a Spherical Obstacle," J. of Appl. Mech., Vol. 34, 493-499 (1963).

12. Johnson, G. and Truell, R., "Numerical Computations of Elastic Scattering Cross Sections," J. of Appl. Phys., Vol. 36, 3466-3475 (1965).

13. Achenbach, J.D., Wave Propagation in Elastic Solids, North-Holland Publishing Co. (1975).

14. Pao, Yih-Hsing and Chao-Chow Mow, Diffraction of Elastic Waves and Dynamical Stress Concentrations, Crane, Russak & Company, Inc. (1973).

15. Graff, K.F., Wave Motion in Elastic Solids, Ohio State University Press (1975).

16. Truell, R., Elbaum, C. and Chick, B., Ultrasonic Methods in Solid State Physics, Academic Press (1969).

17. Fu, L.S., "Fundamental Aspects in the Quantitative Ultrasonic Determination of Fracture Toughness: The Scattering of a Single Ellipsoidal Inhomogeneity," NASA Contractor Report, NASA-CR-xxx; OSURF Technical Report No. 102, (February, 1982).

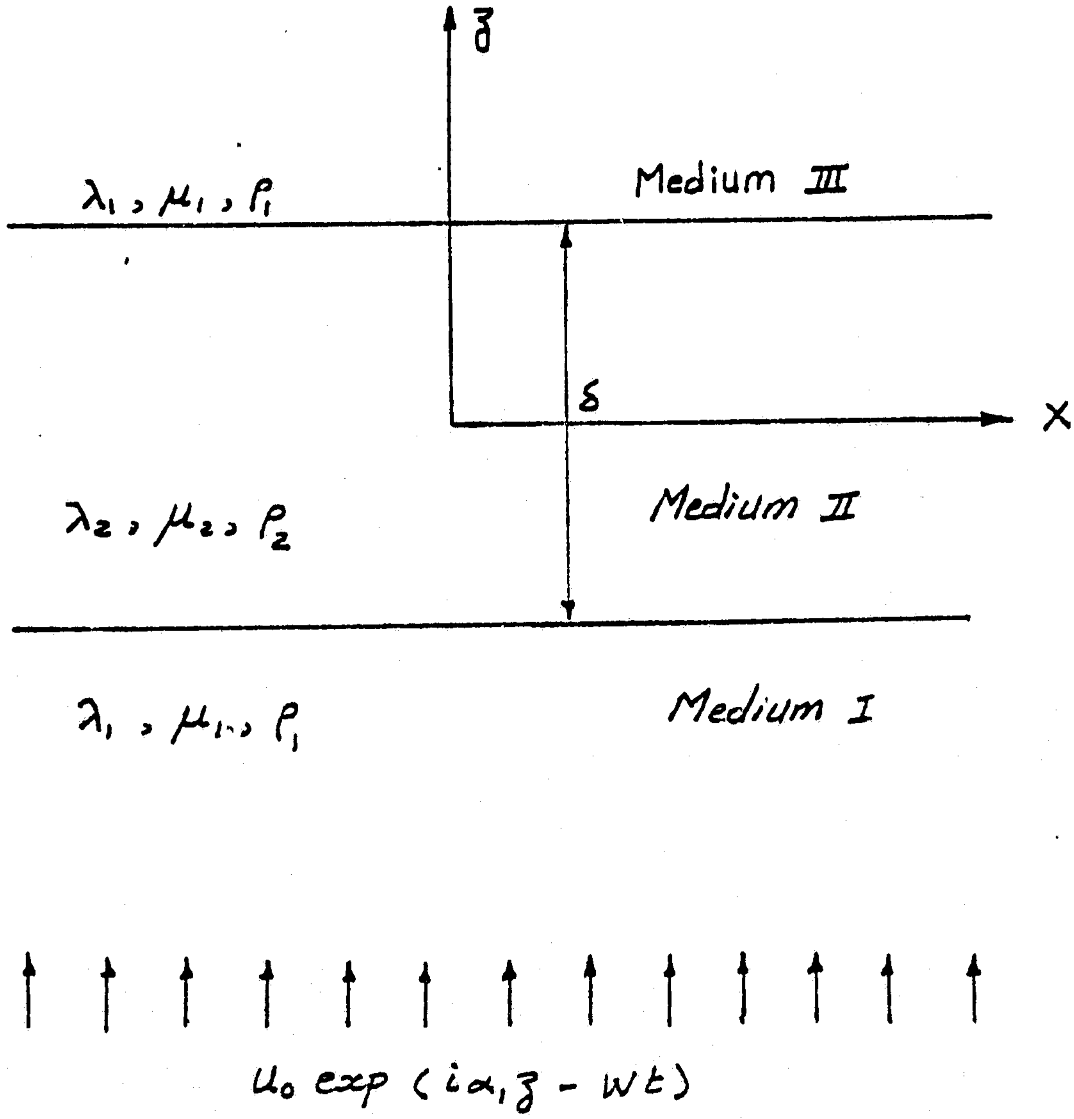

Fig. 1 Geometry and material properties of the three-layered medium

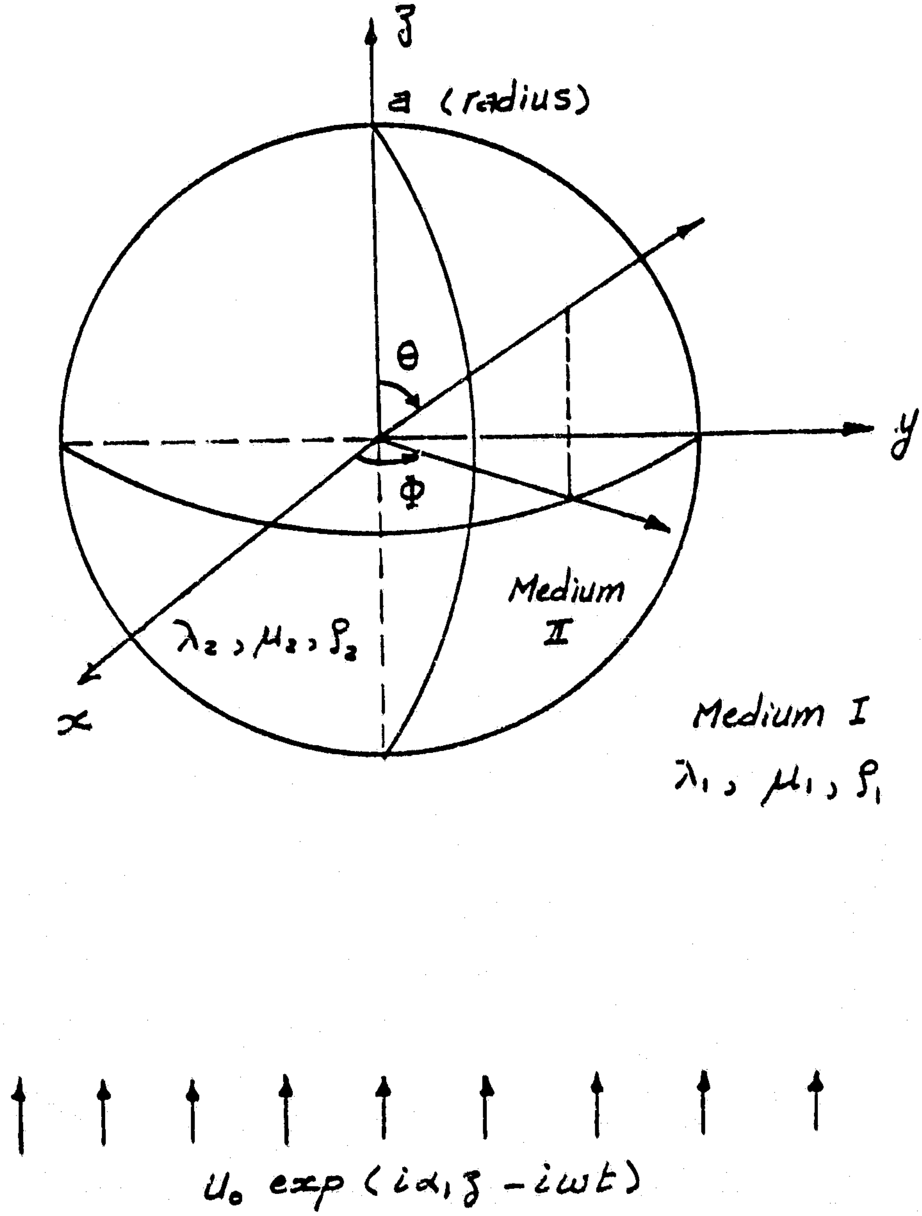

Fig. 2 Geometry and material properties of an elastic spherical inhomogeneity in an infinite elastic medium

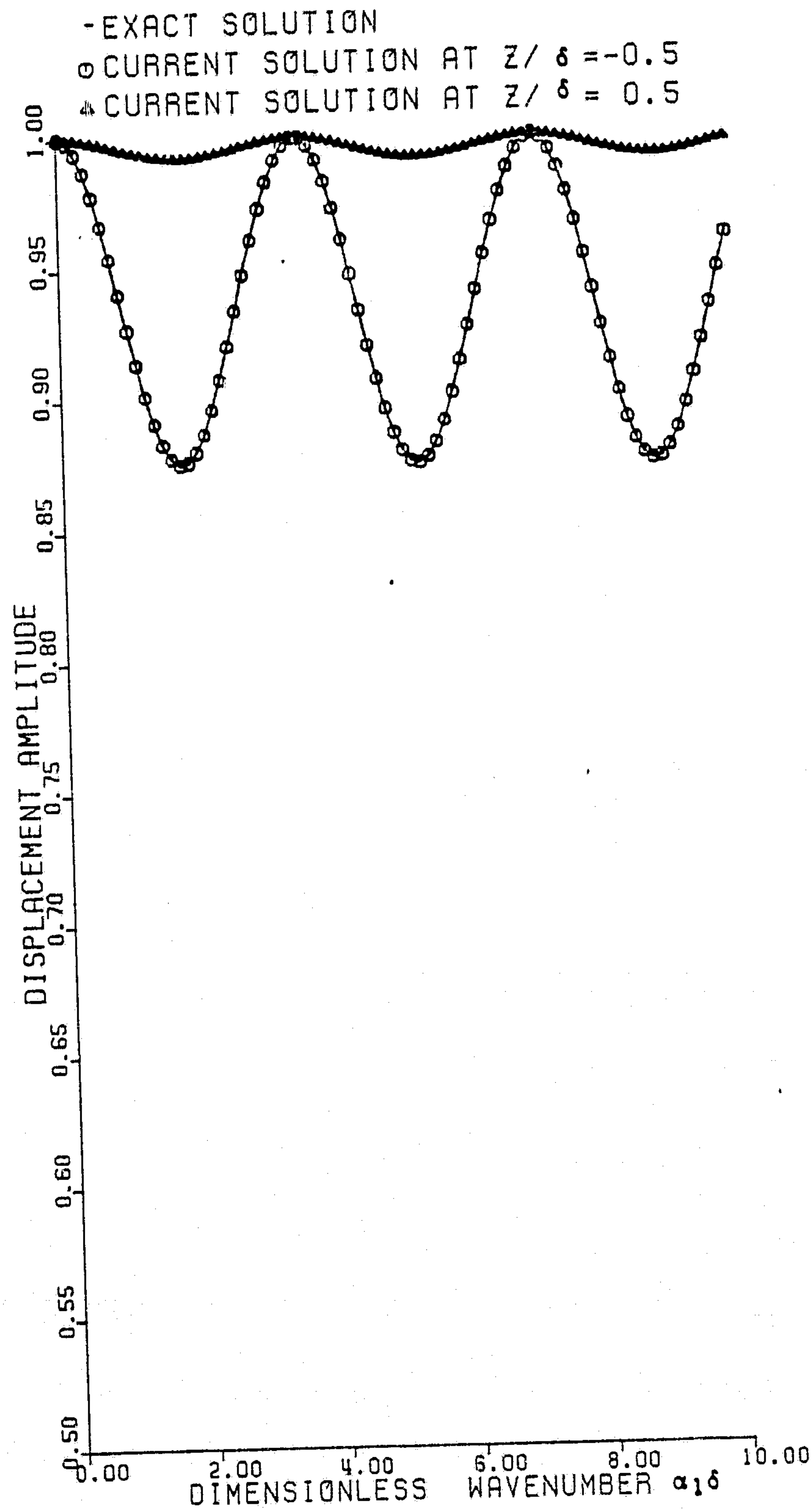

Fig. 3 Displacement amplitude as a function of $\alpha_1\delta$ for the three-layered problem, Ge in Al, with h=1.0

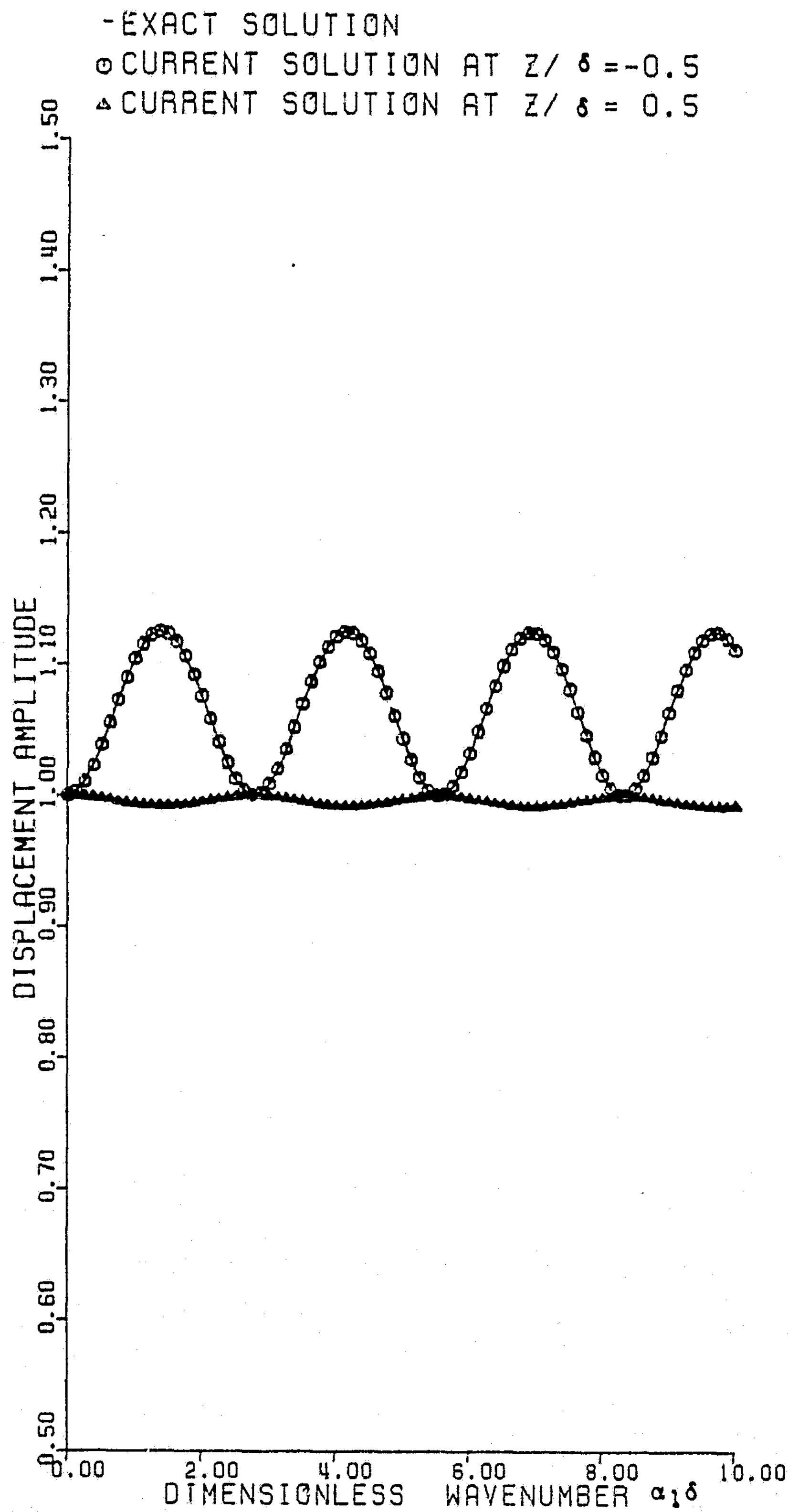

Fig. 4 Displacement amplitude as a function of $\alpha_1\delta$ for the three-layered problem, Al in Ge, with h=1.0

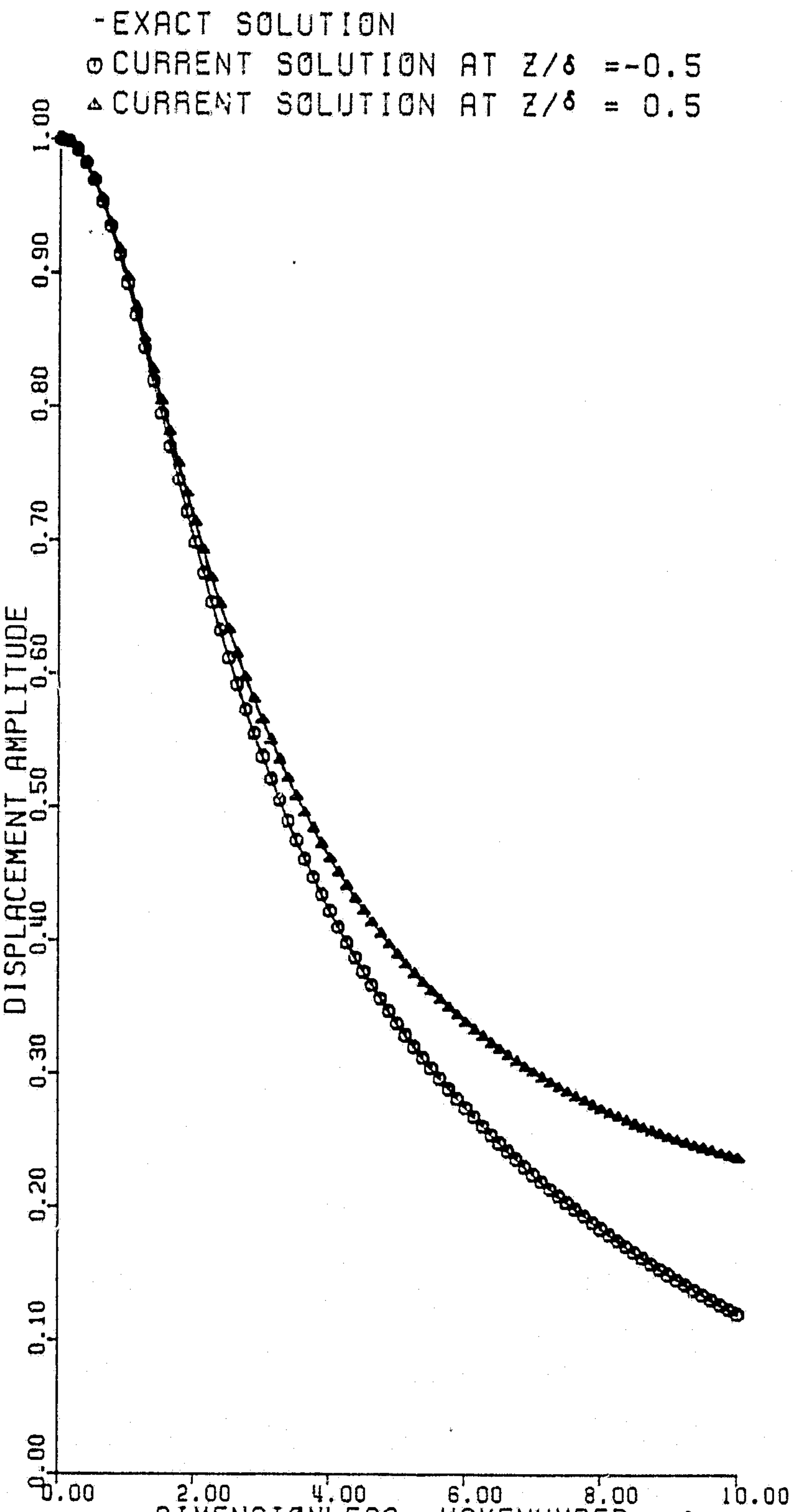

Fig. 5 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem .Be in Polythylene, with h=1.0

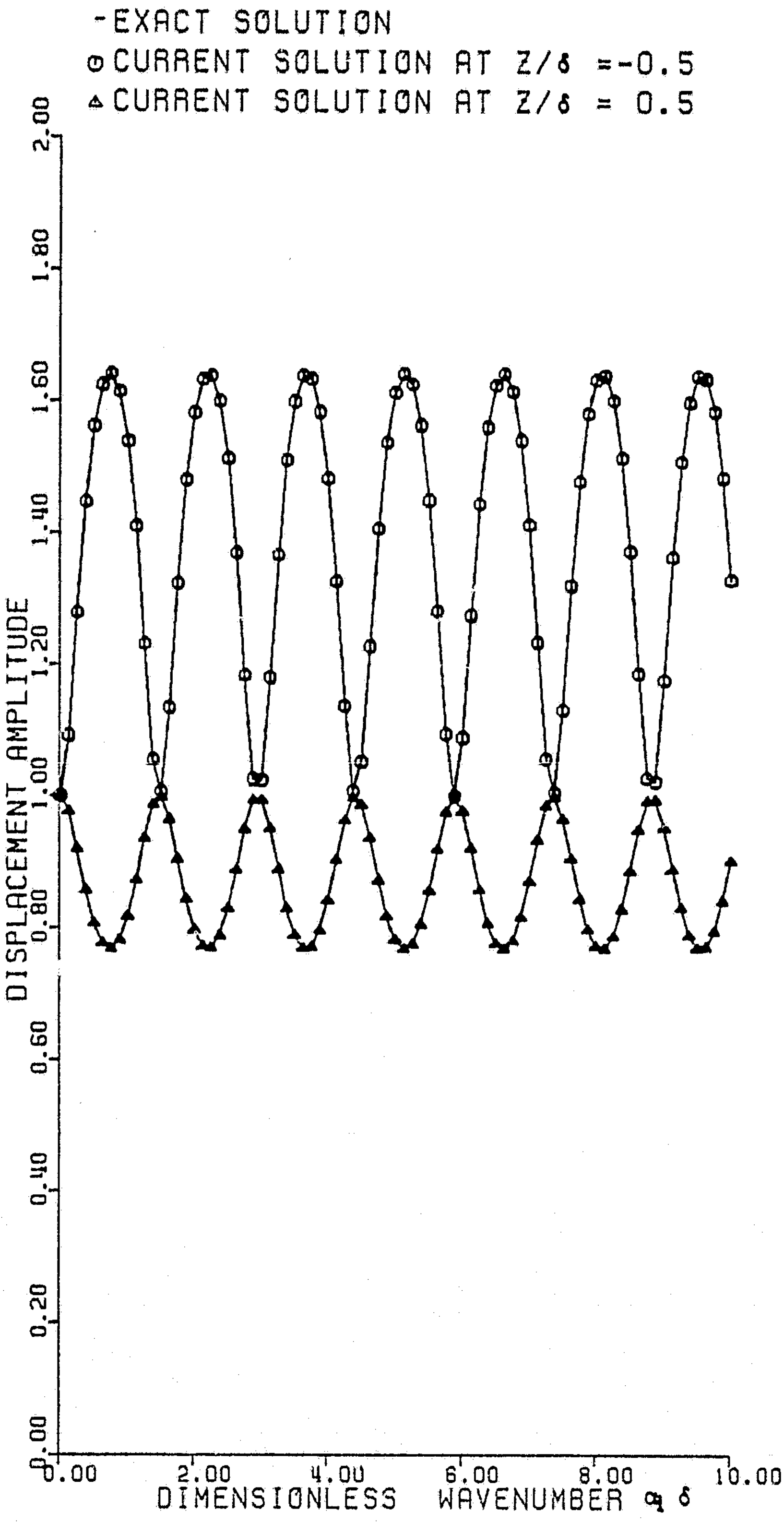

Fig. 6 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Mg in Stainless Steel, with h=1.0

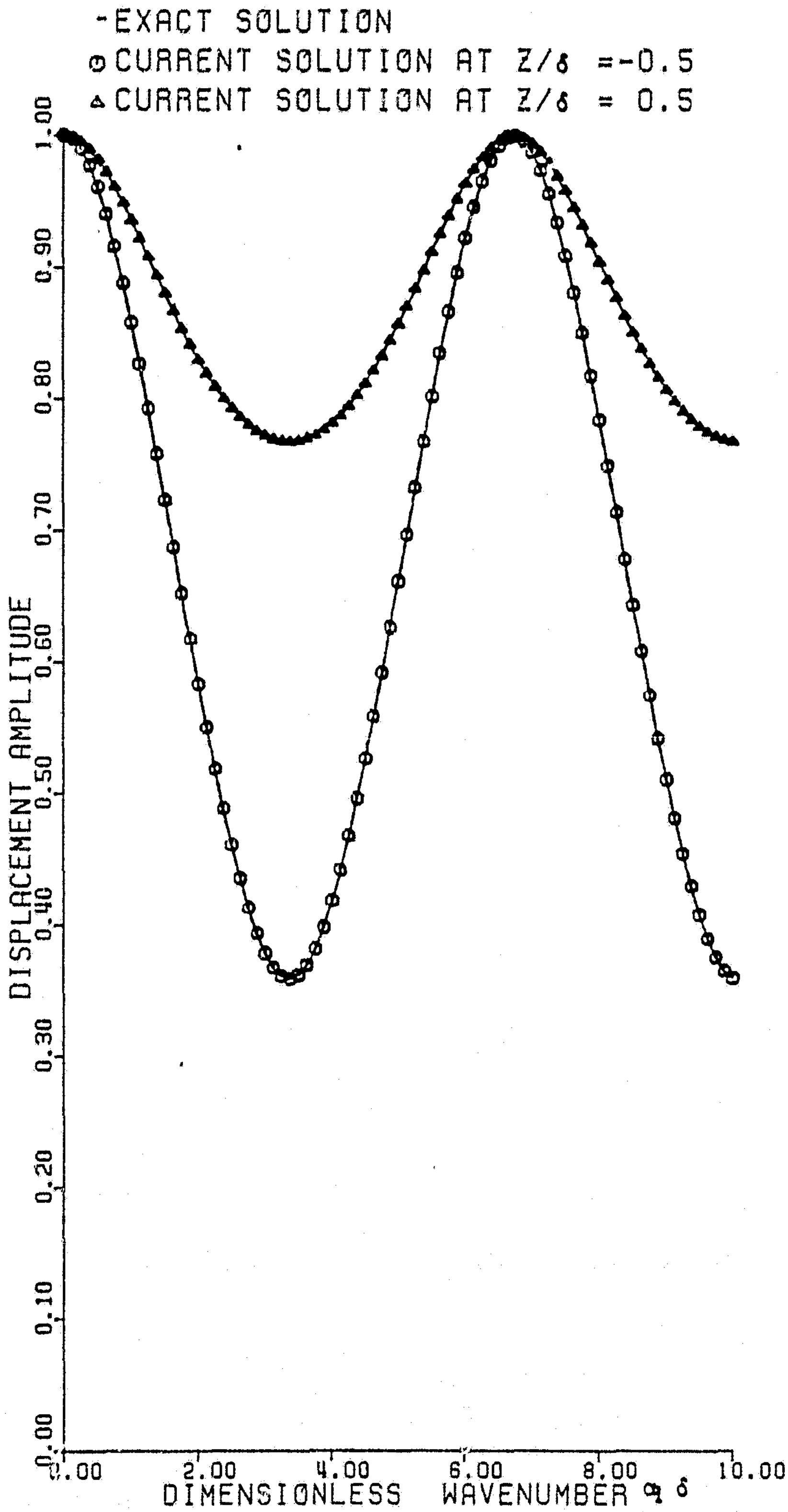

Fig. 7 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Stainless Steel in Mg, with h=1.0

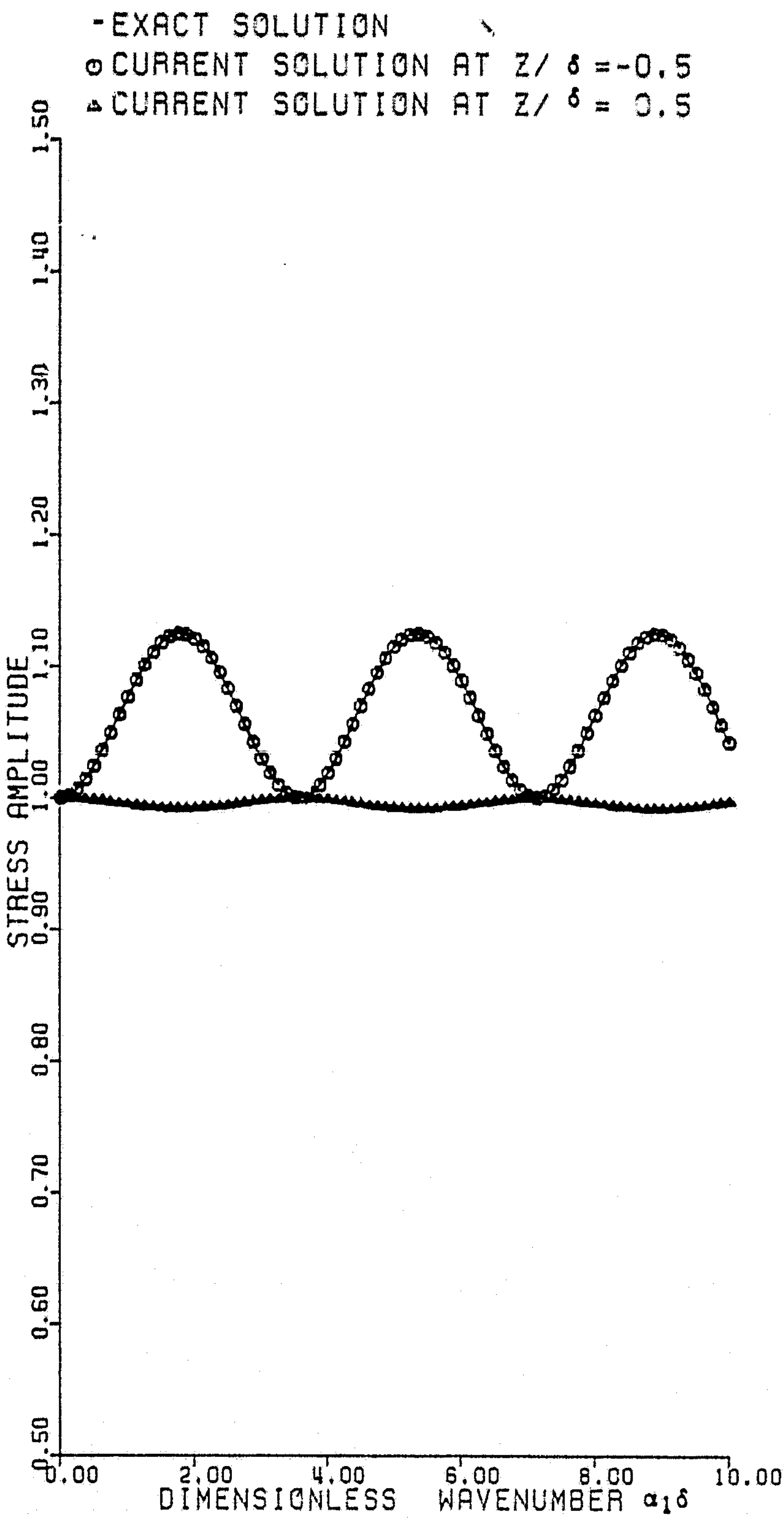

Fig. 8 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Ge in Al, with h=1.0

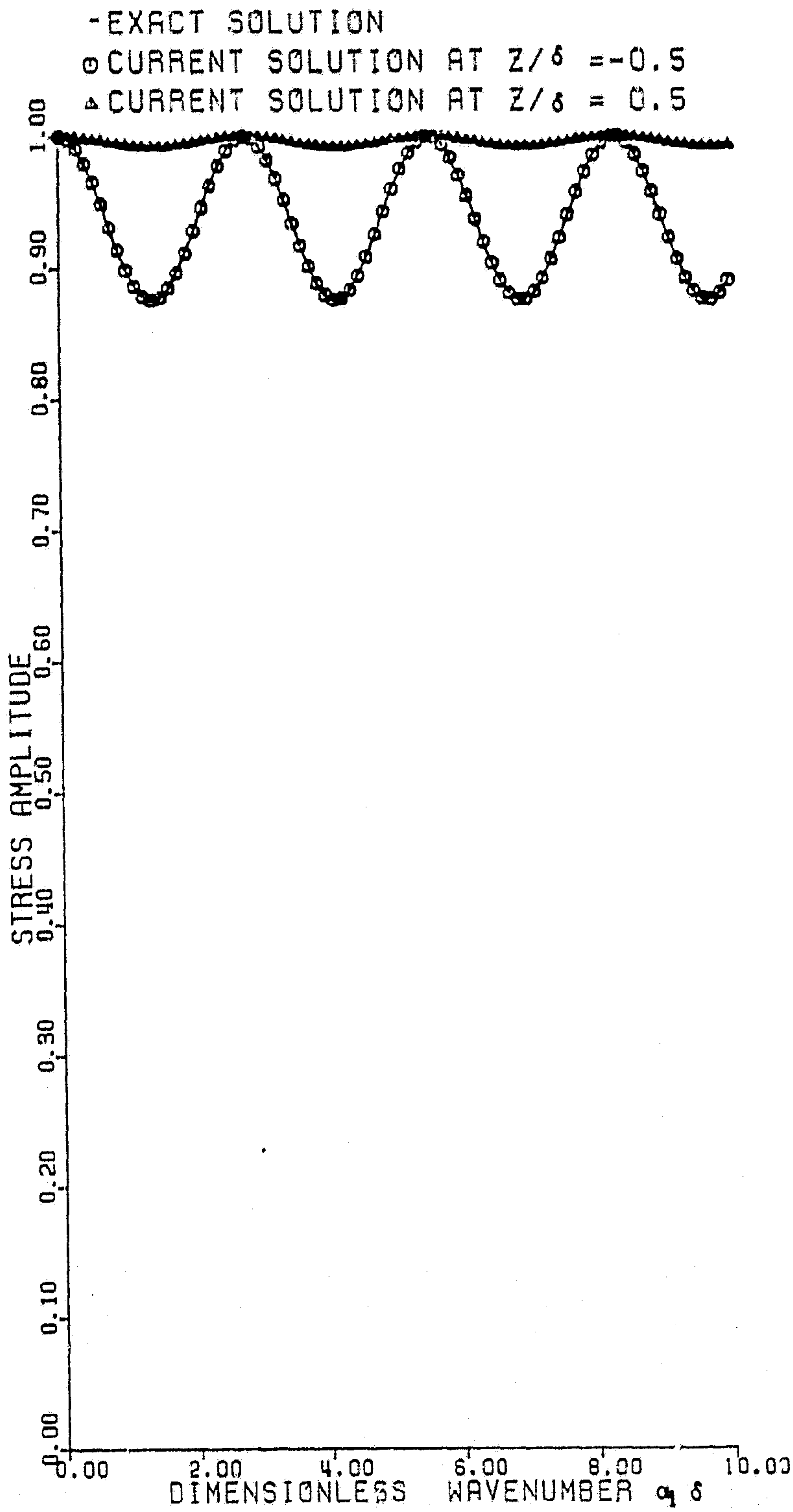

Fig. 9 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Al in Ge, with h=1.0

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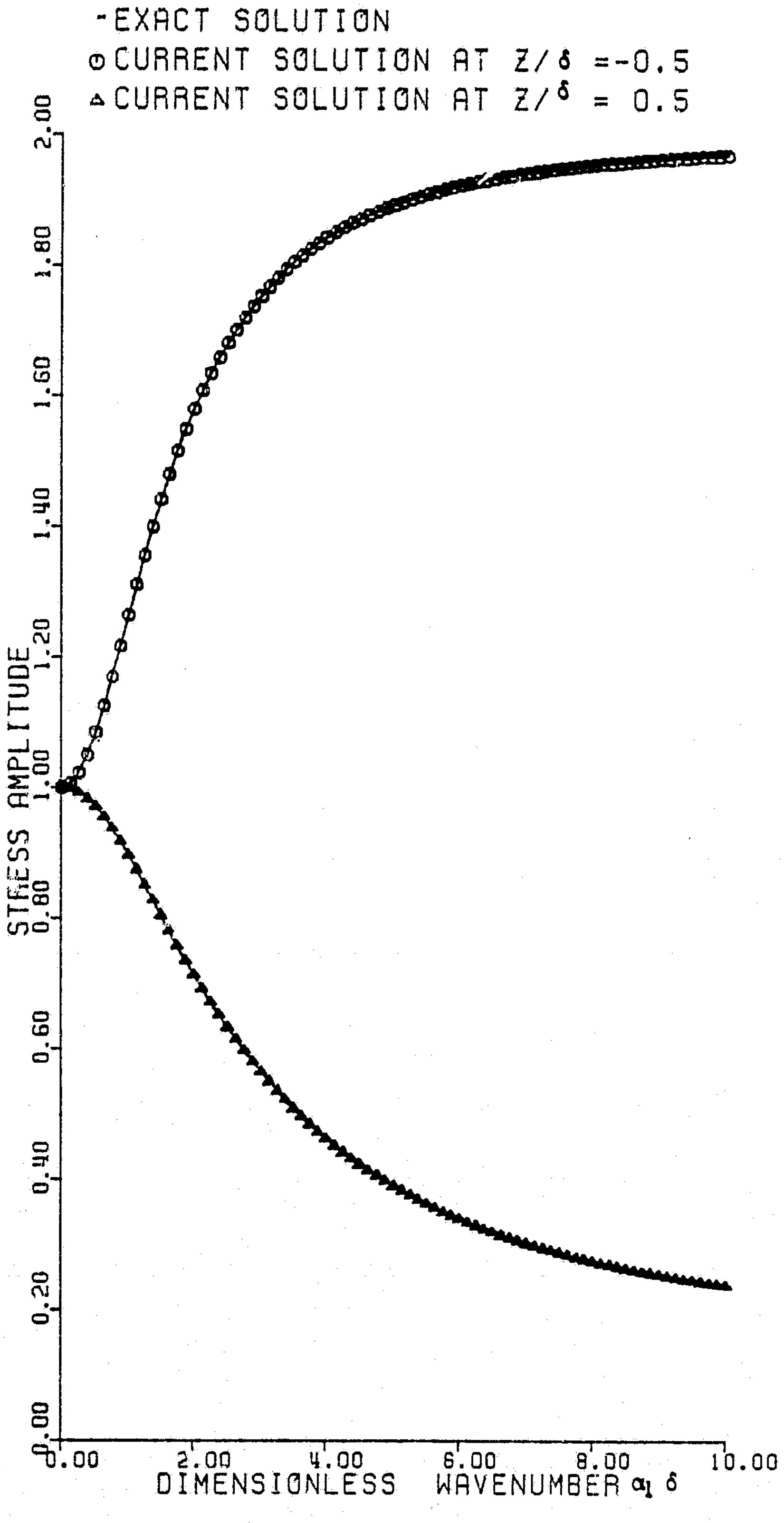

Fig. 10 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Be in Polythylene, with h=1.0

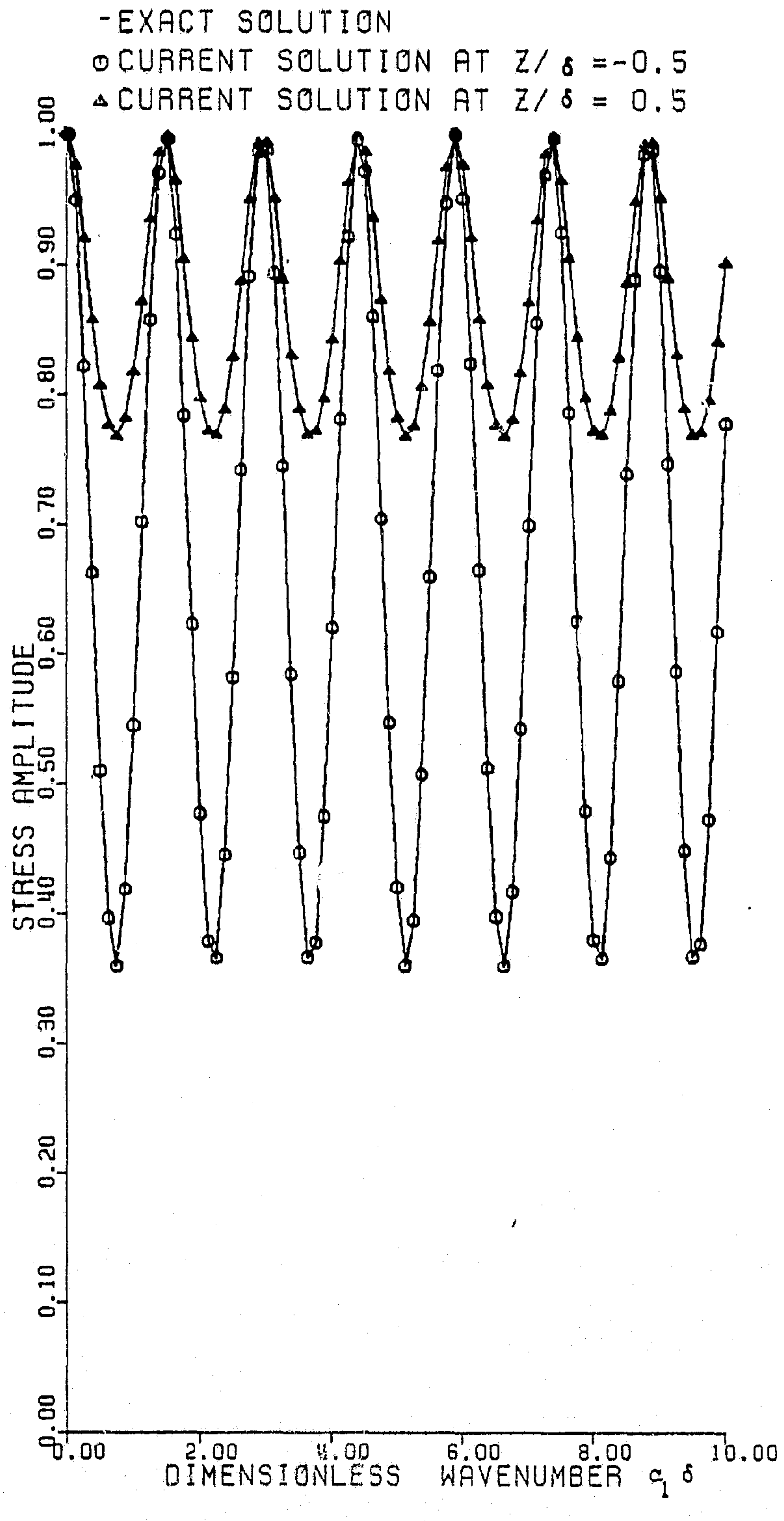

Fig. 11 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Mg in Stainless Steel, with h=1.0

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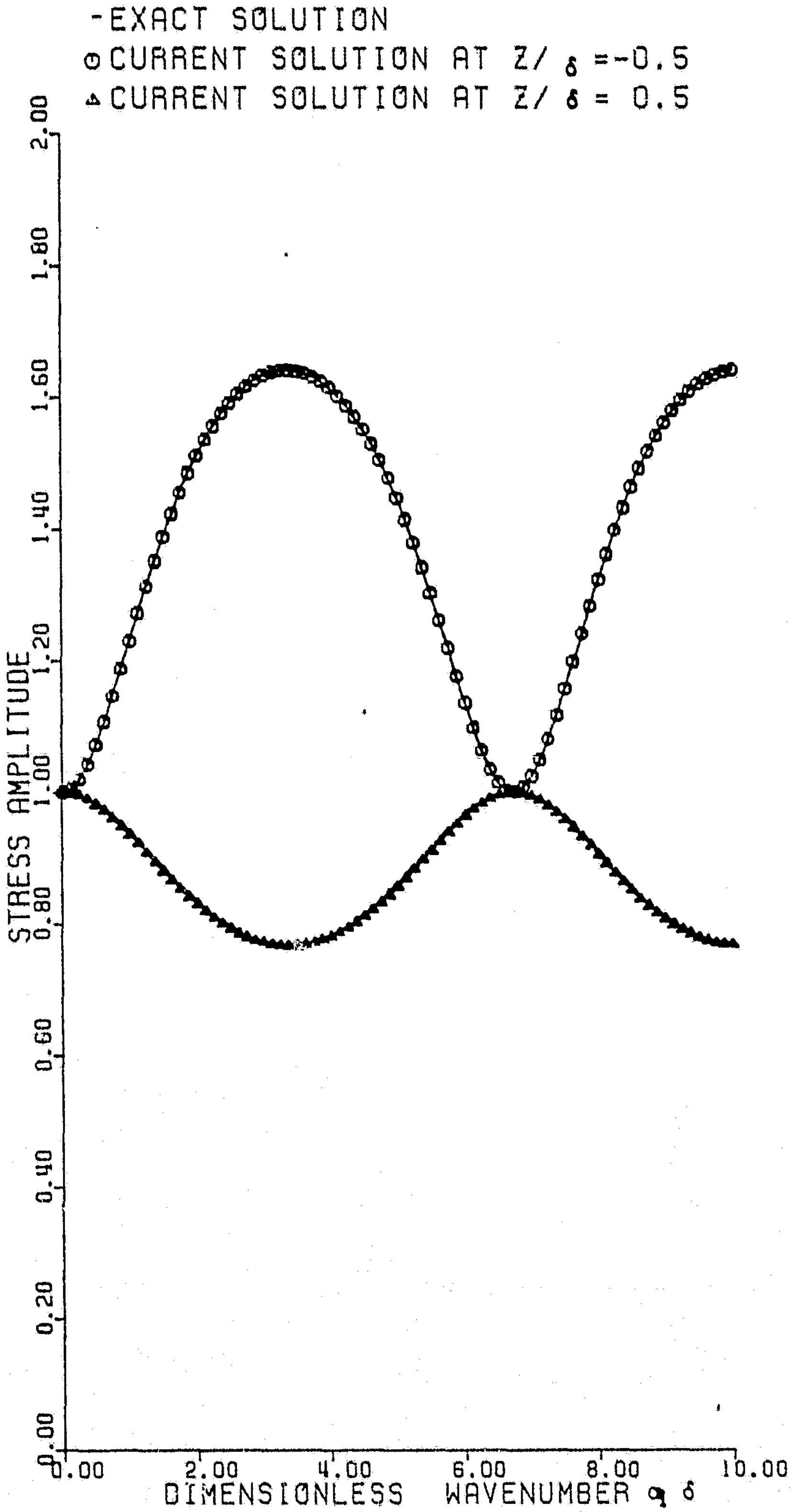

Fig. 12 Stress amplitude as a function of $q_1\delta$ for the three-layered problem, Stainless Steel in Mg, with h=1.0

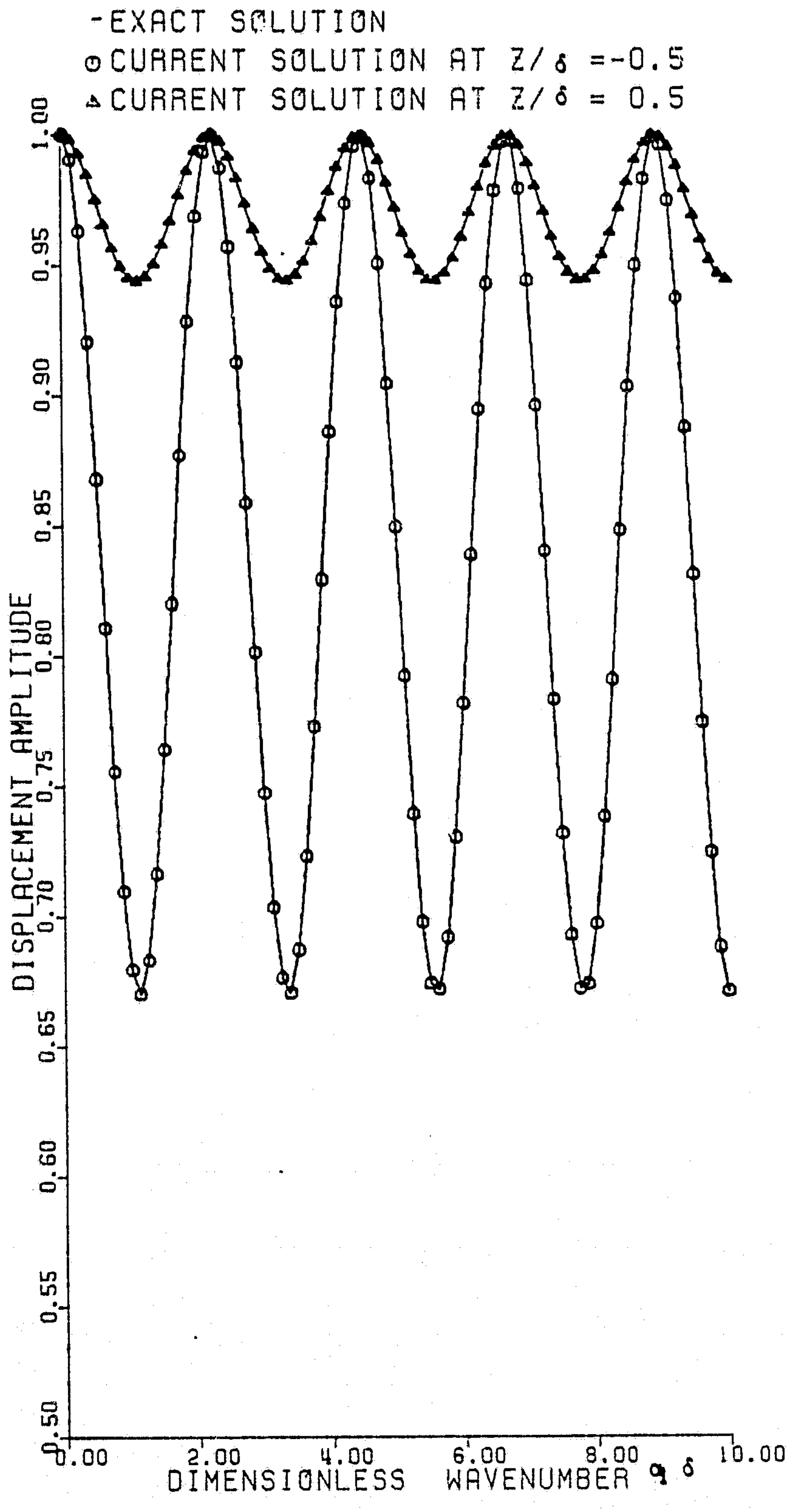

Fig. 13 Displacement amplitude as a function of $q\,\delta$ for the three-layered problem, Ge in Al, with f=1.0

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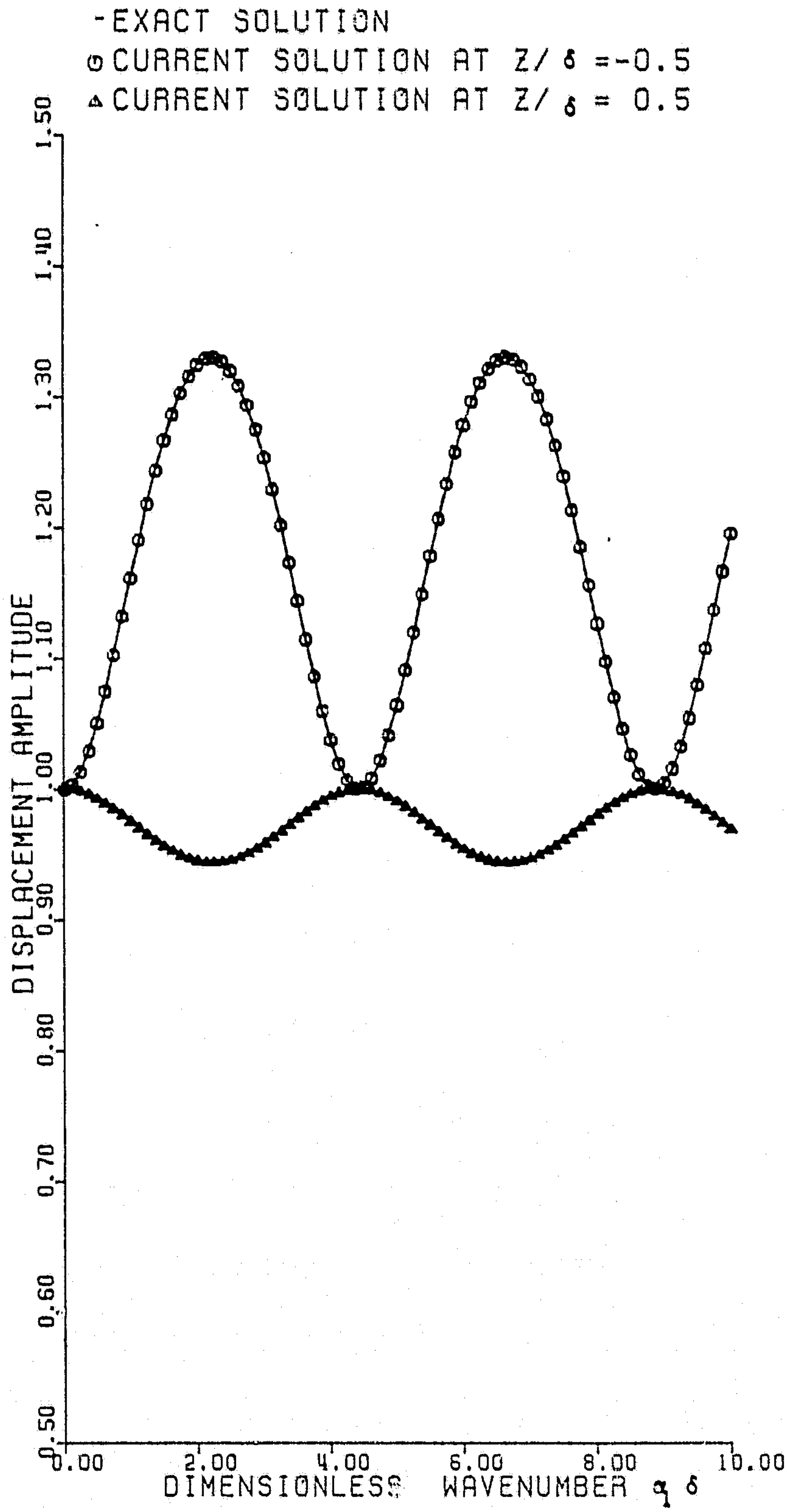

Fig. 14 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Al in Ge, with f=1.0

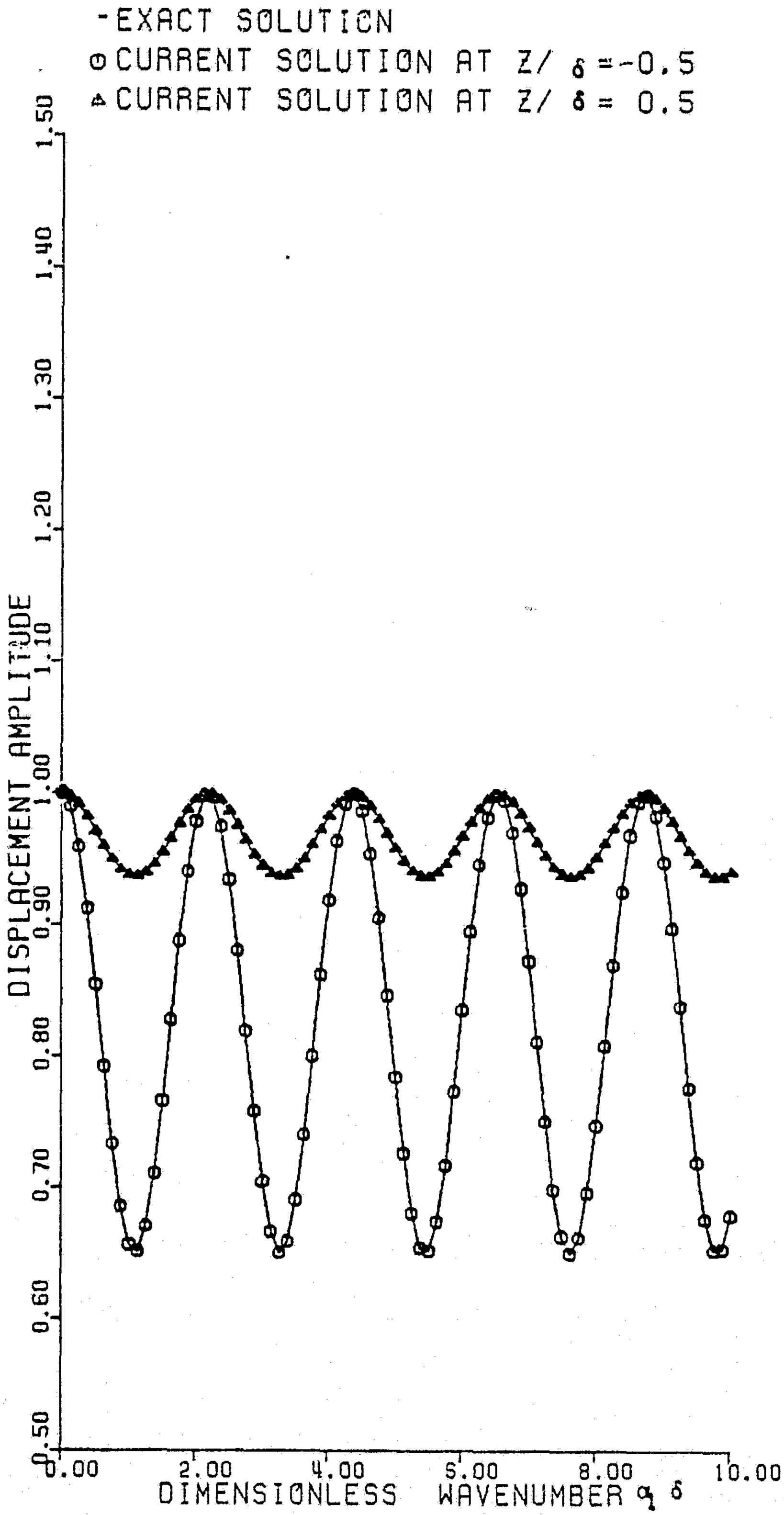

Fig. 15 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Be in Polythylene, with f=1.0

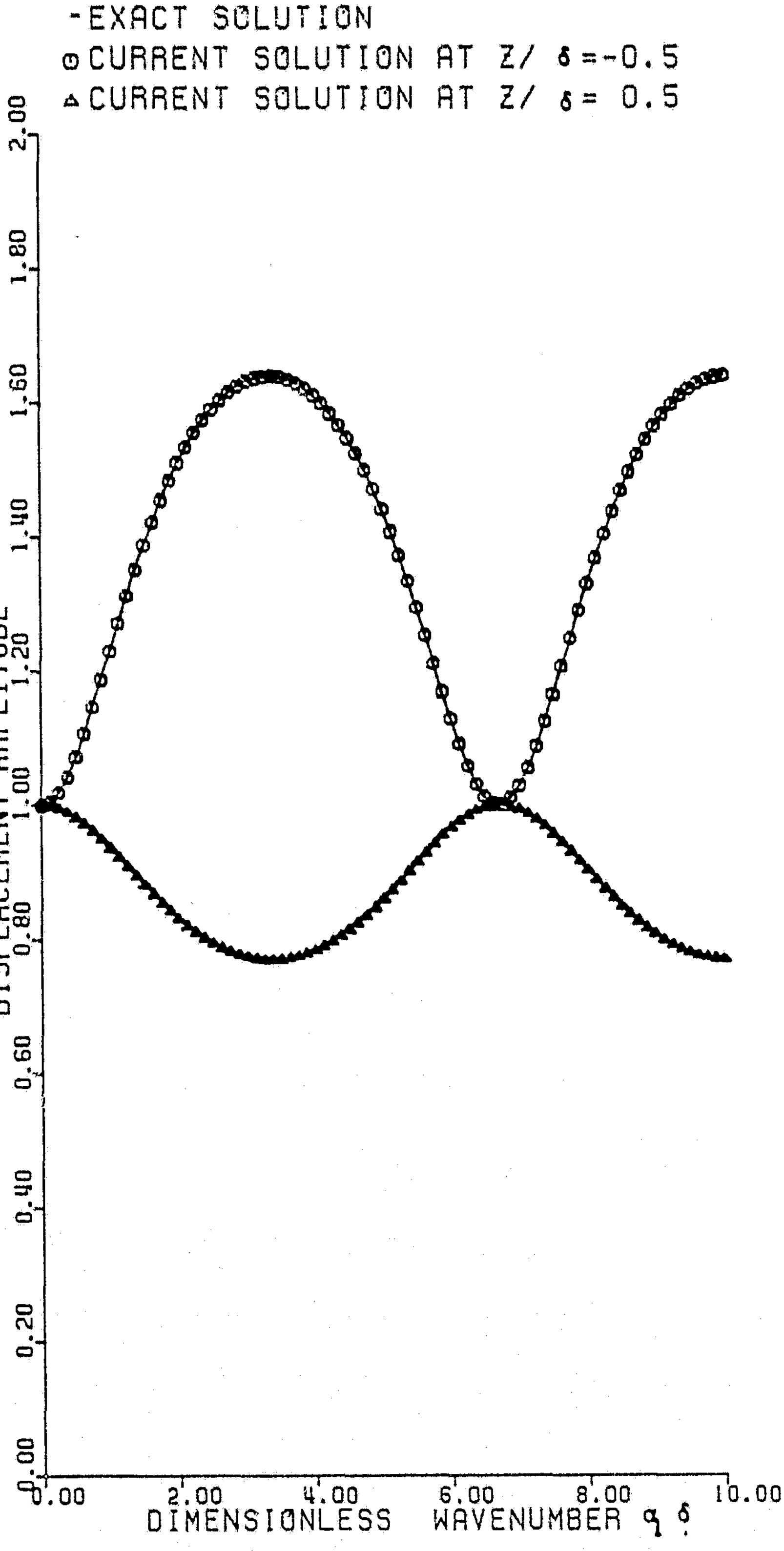

Fig. 16 Displacement amplitude as a function of $q\delta$ for the three-layered problem, Mg in Stainless Steel, with f=1.0

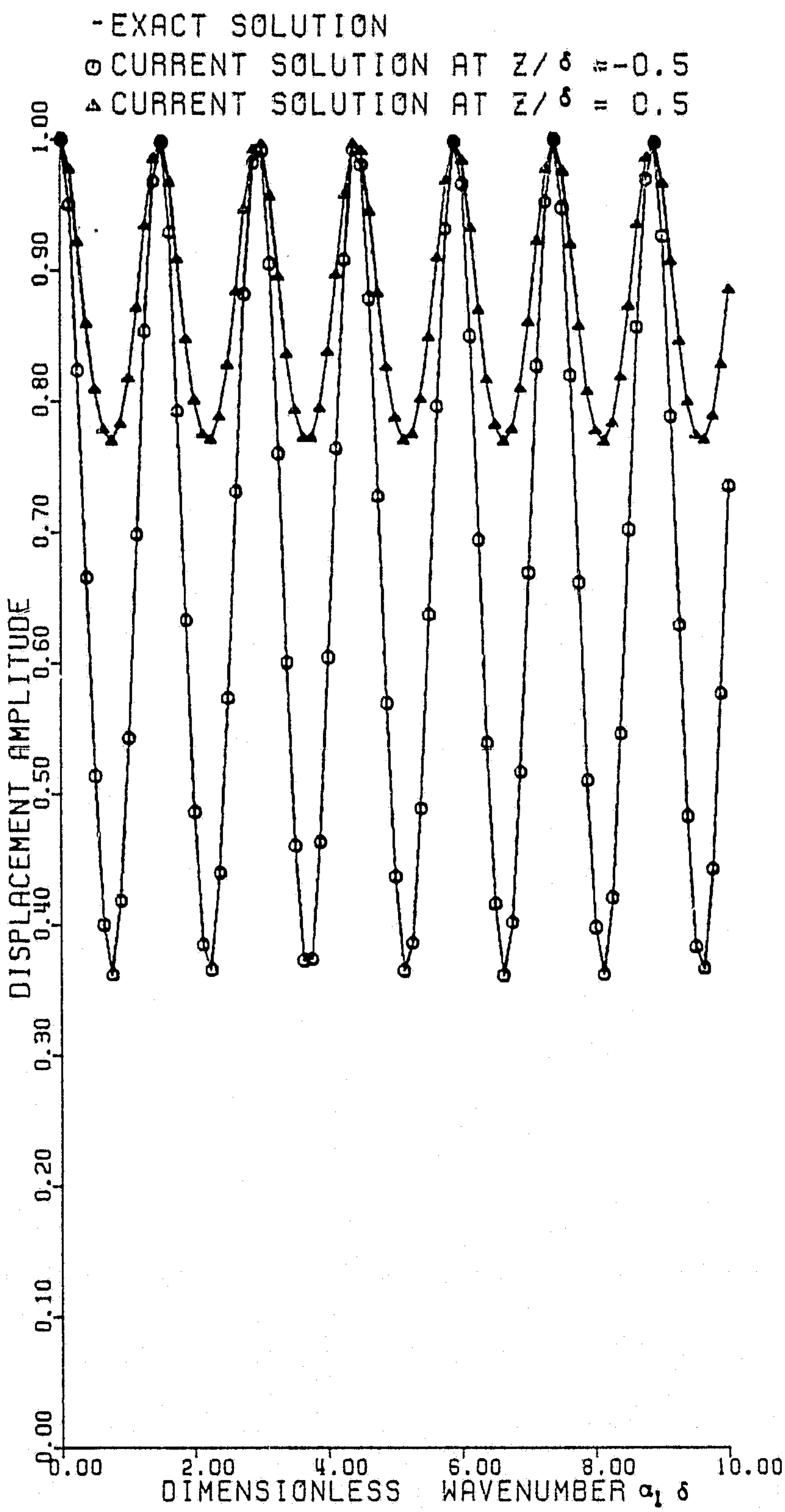

Fig. 17 Displacement amplitude as a function of $\alpha_1\delta$ for the three-layered problem, Stainless Steel in Mg, with f=1.0

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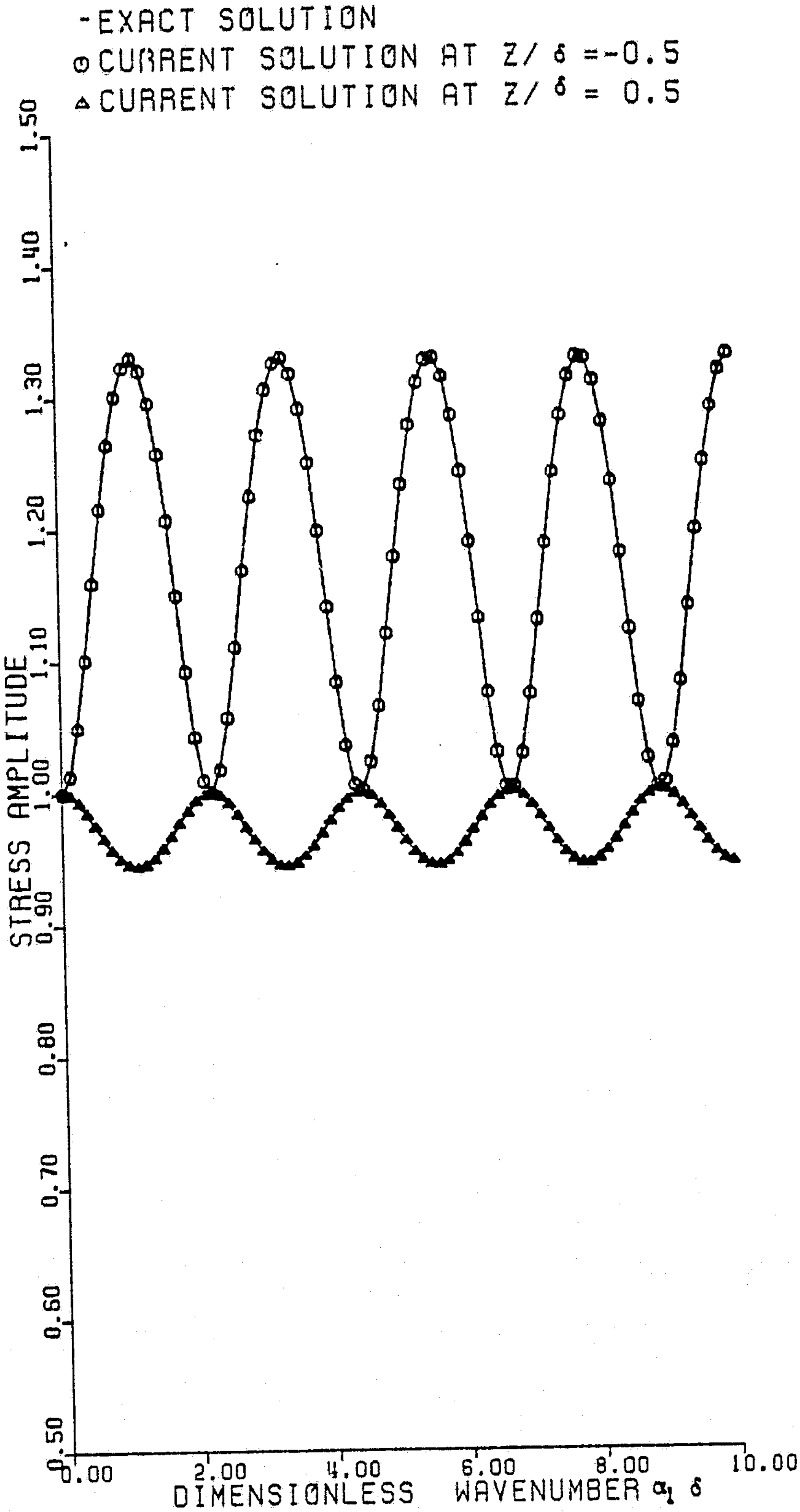

Fig. 18 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Ge in Al, with f=1.0

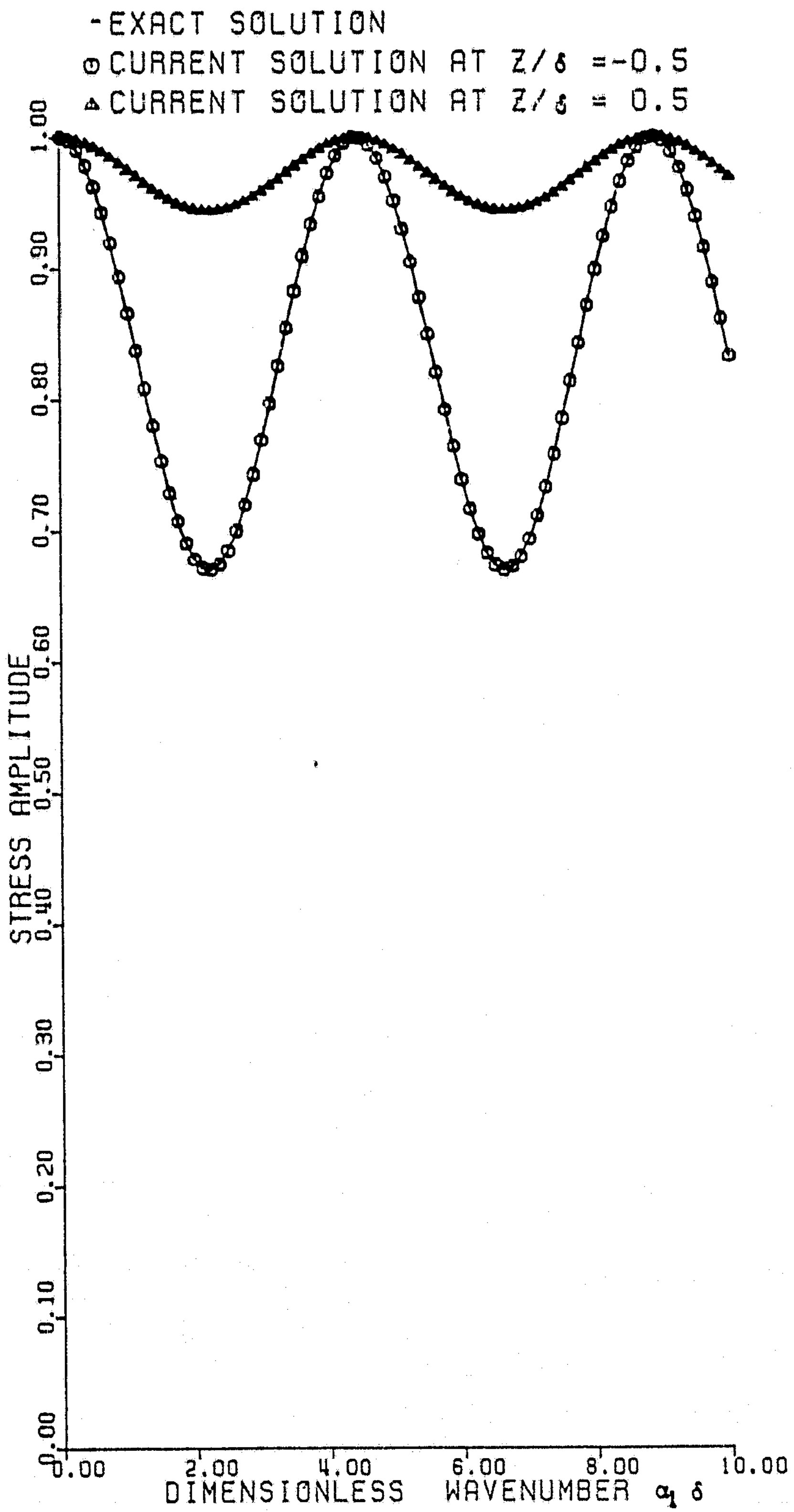

Fig. 19 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Al in Ge, with f=1.0

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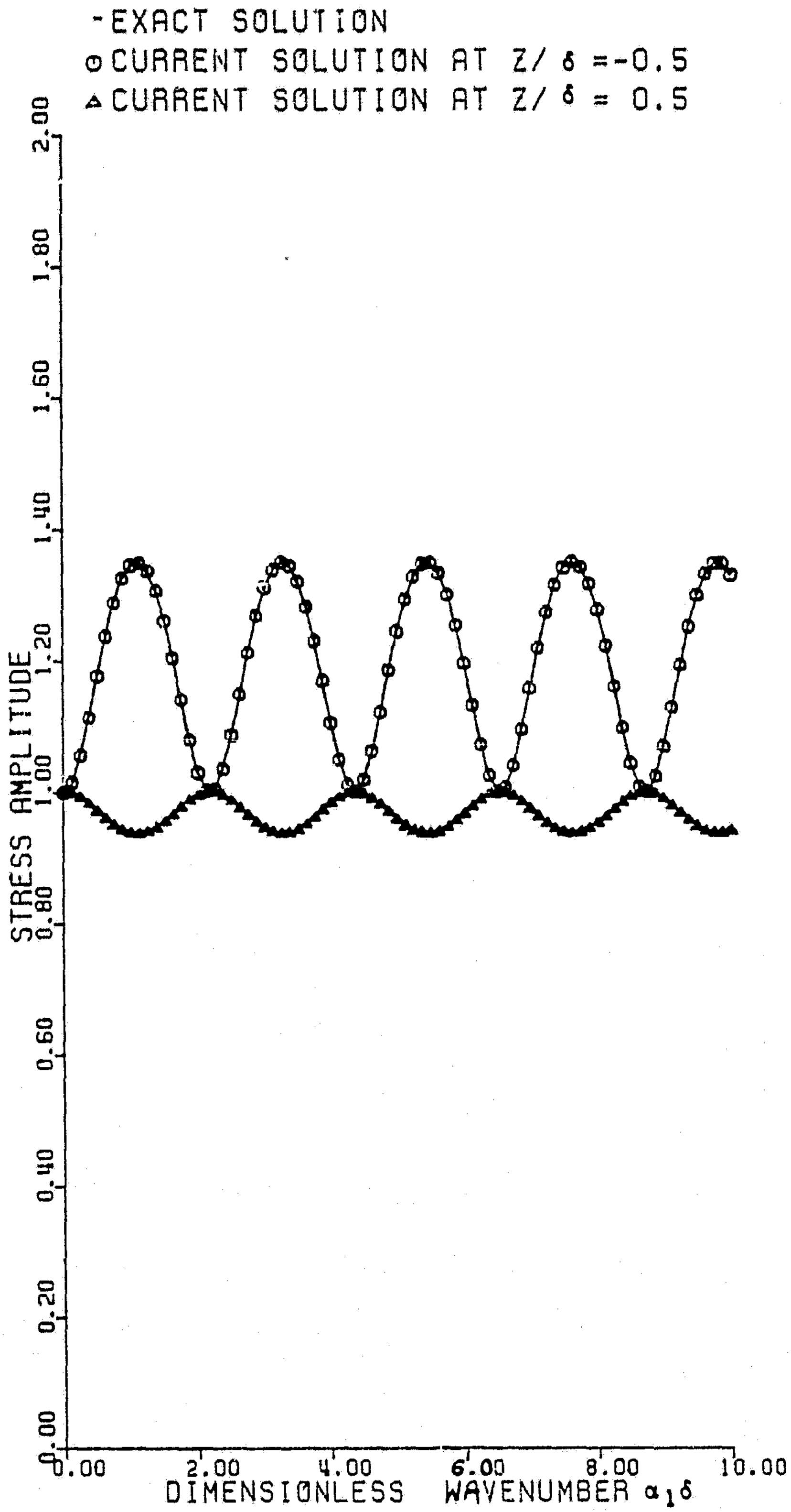

Fig. 20 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Be in Polythylene, with f=1.0

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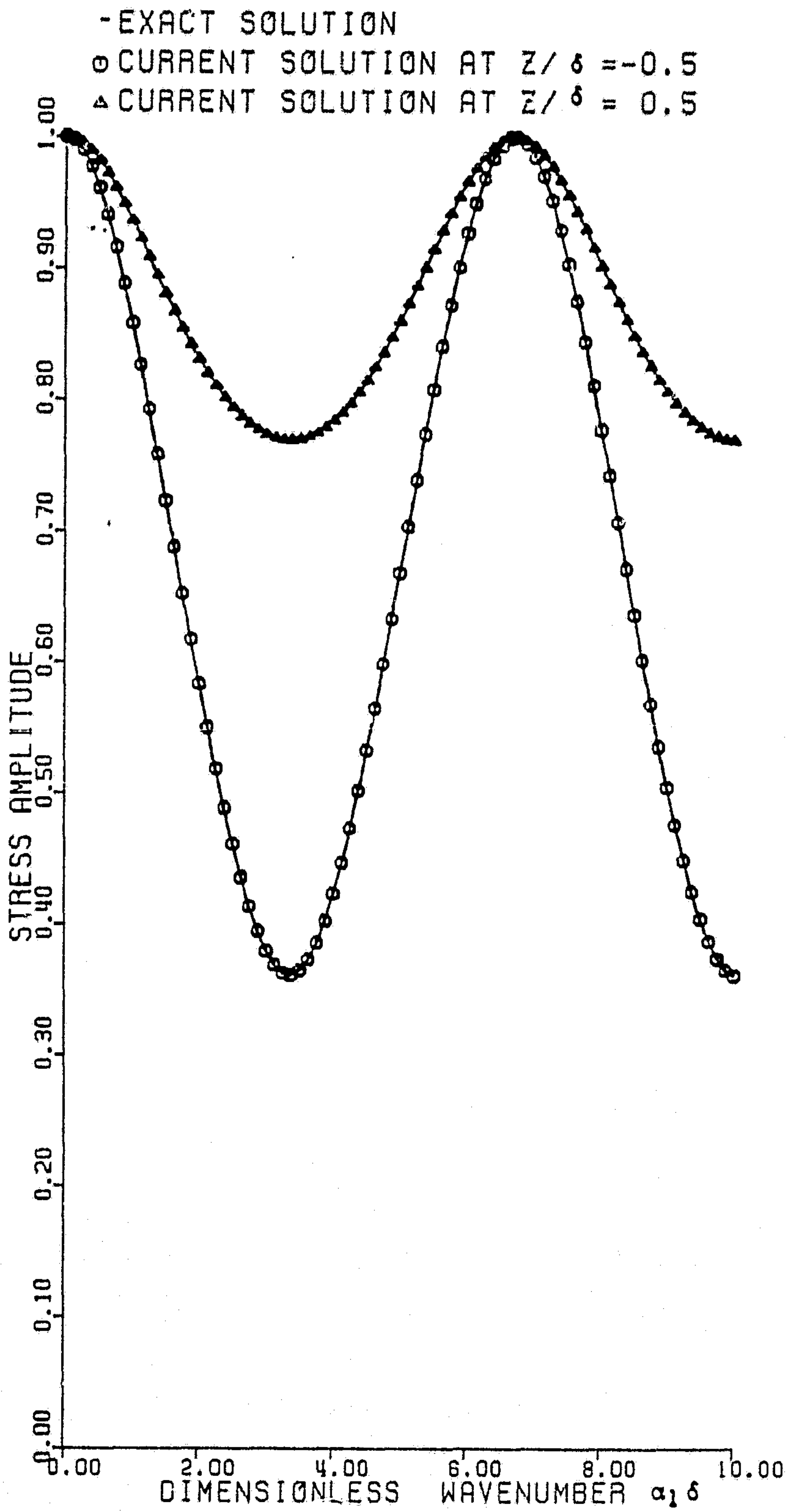

Fig. 21 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Mg in Stainless Steel, with f=1.0

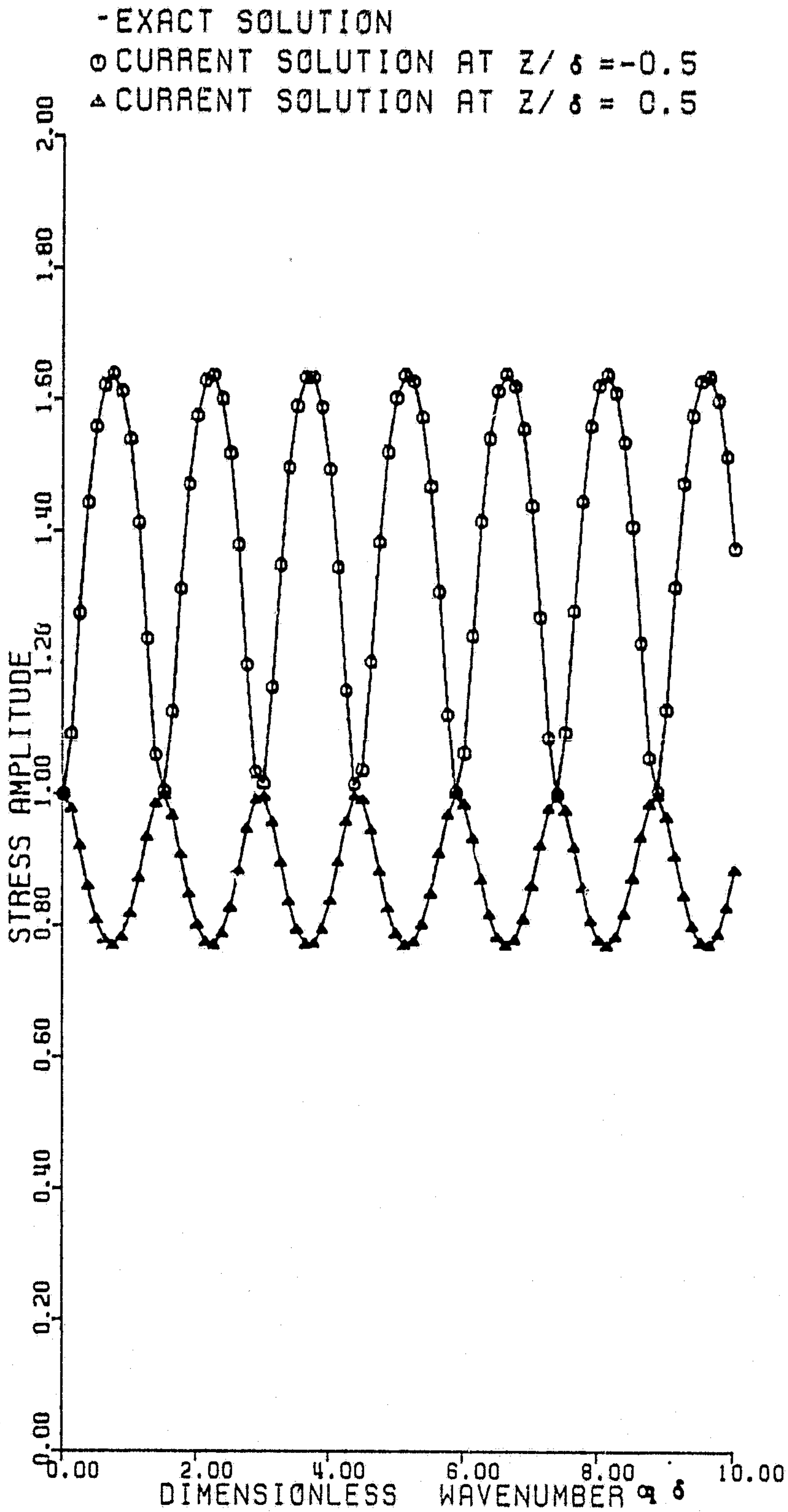

Fig. 22 Stress amplitude as a function of $\alpha\delta$ for the three-layered problem, Stainless Steel in Mg, with f=1.0

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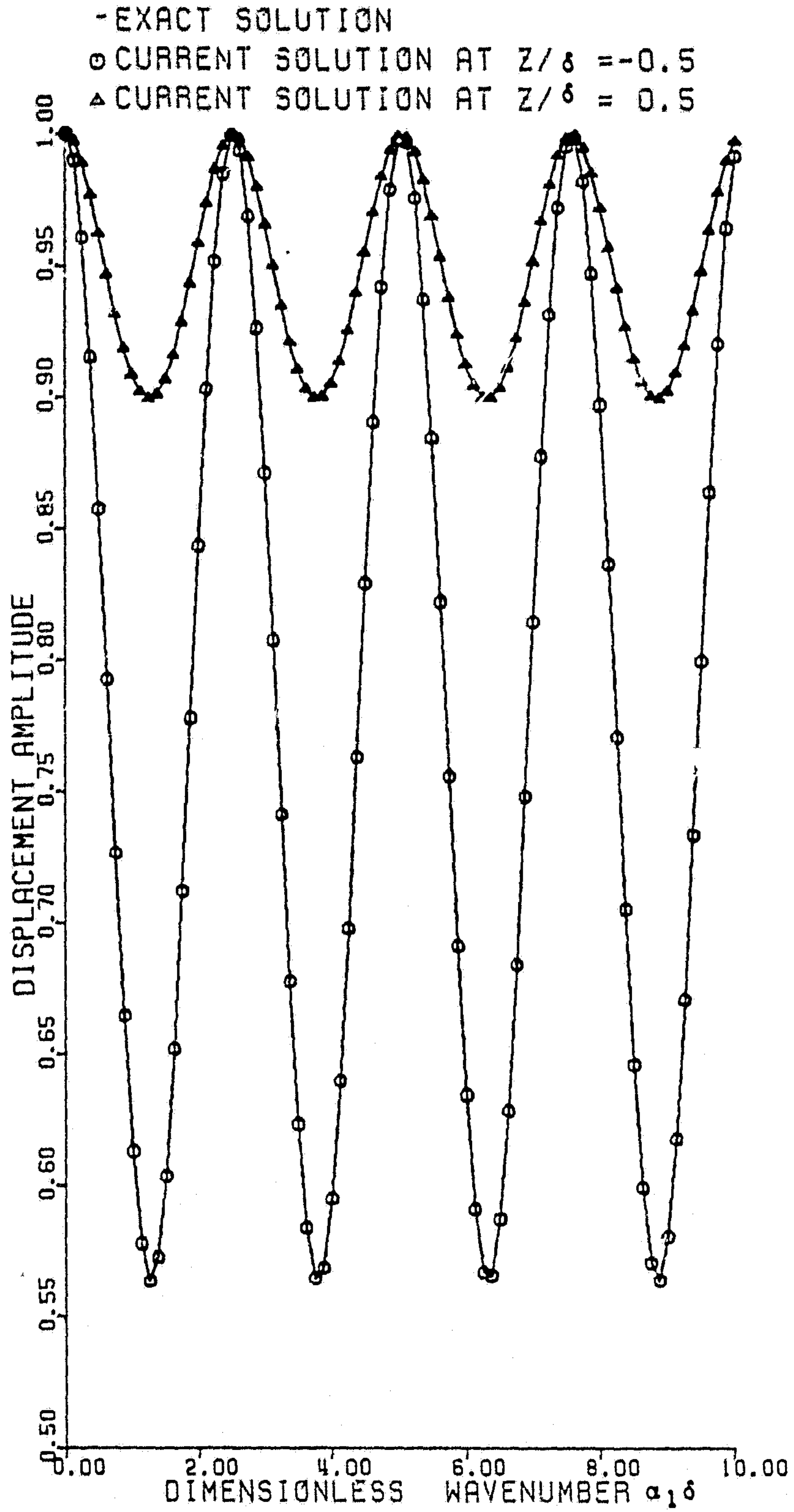

Fig. 23 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Ge in Al

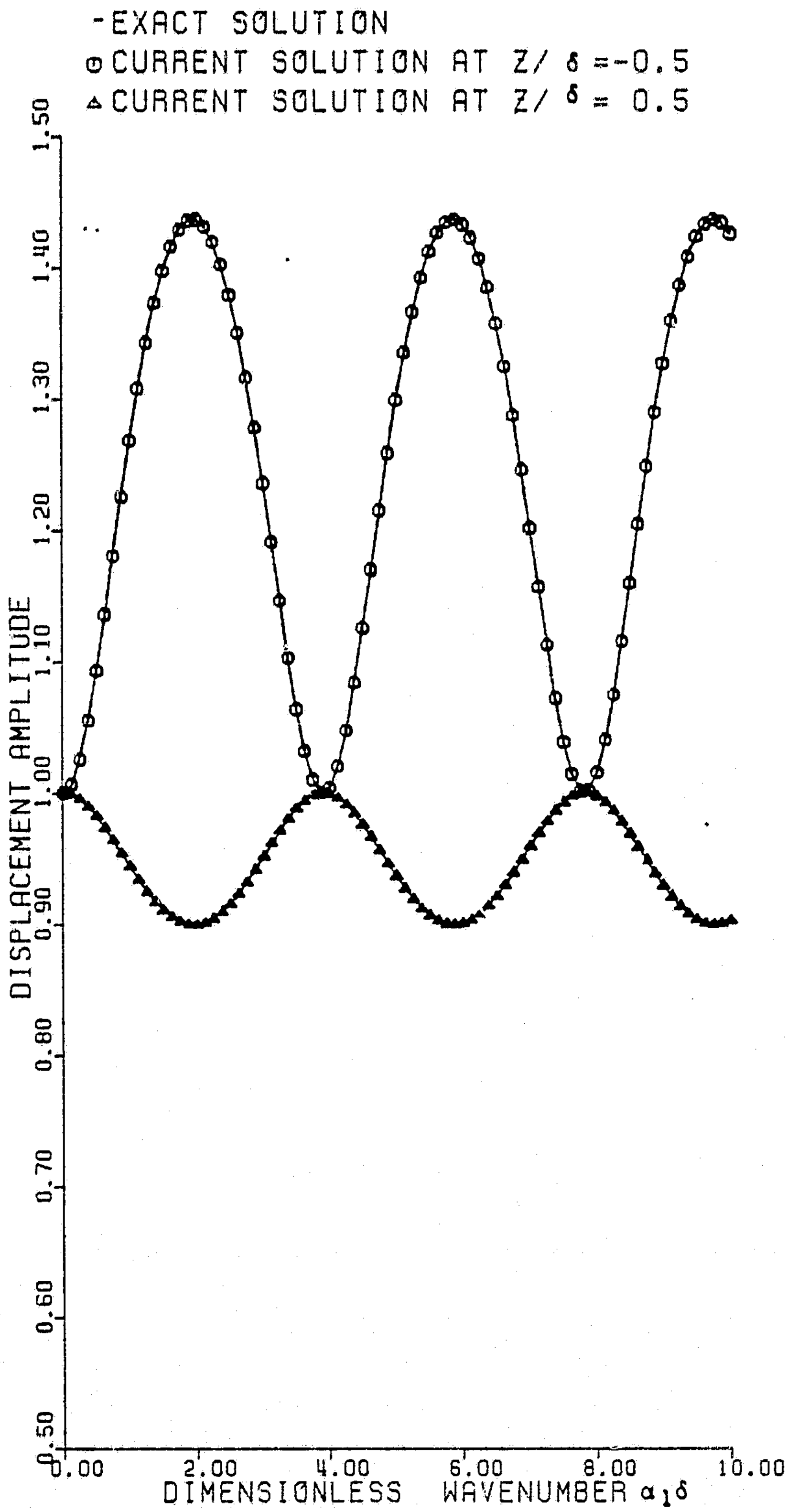

Fig. 24 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Al in Ge

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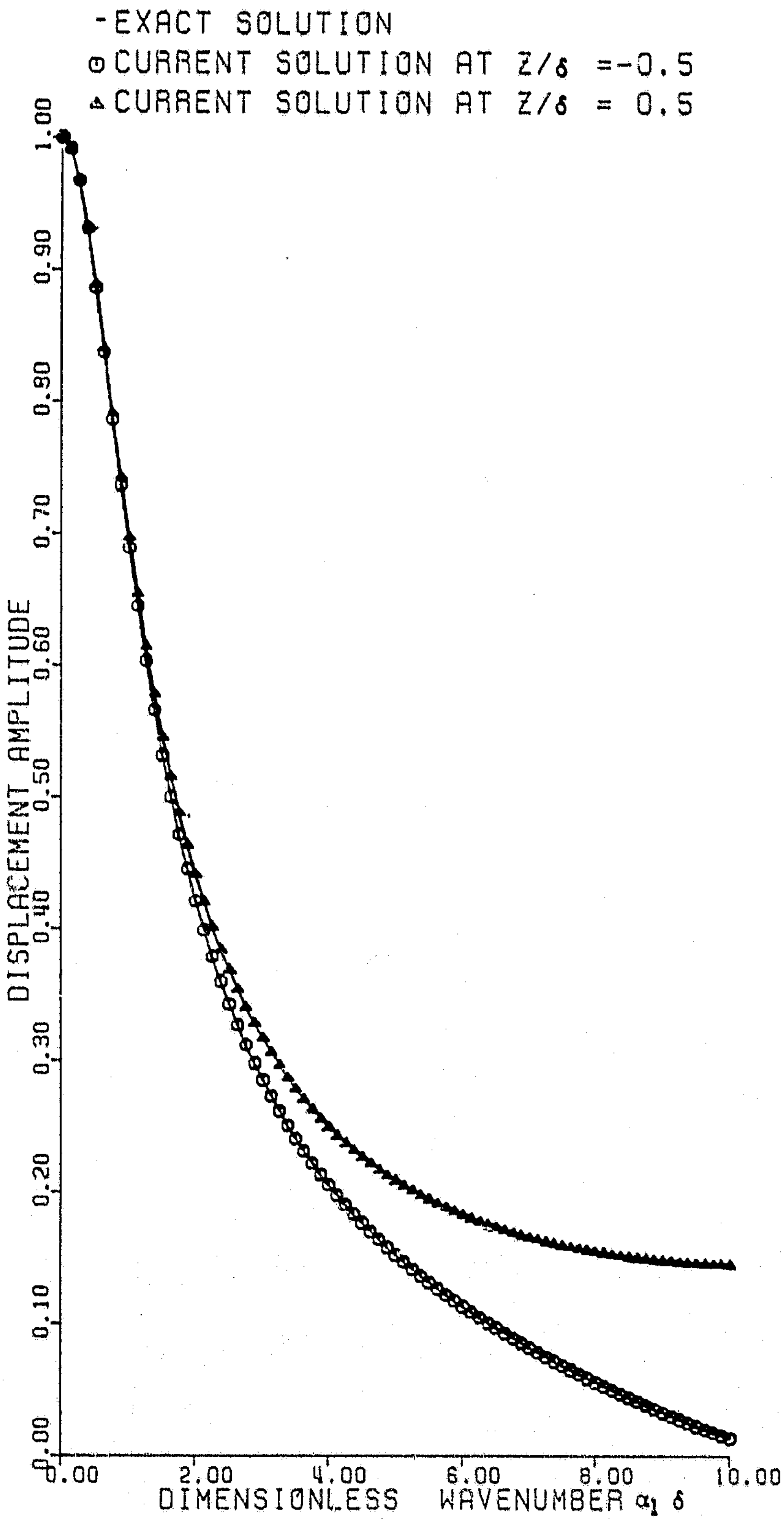

Fig. 25 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Be in Polythylene

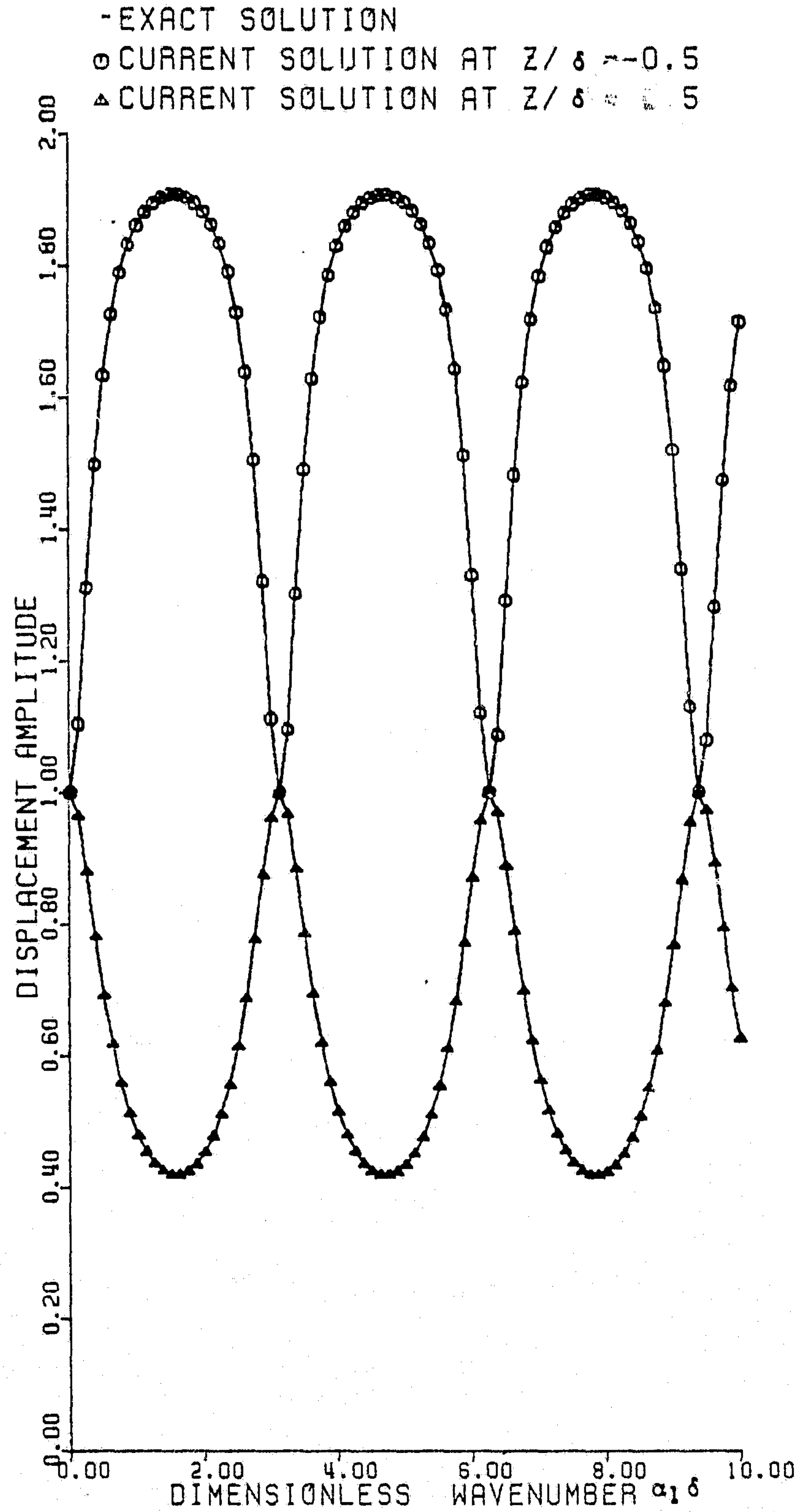

Fig. 26 Displacement amplitude as a function of $\alpha_1\delta$ for the three-layered problem, Mg in Stainless Steel

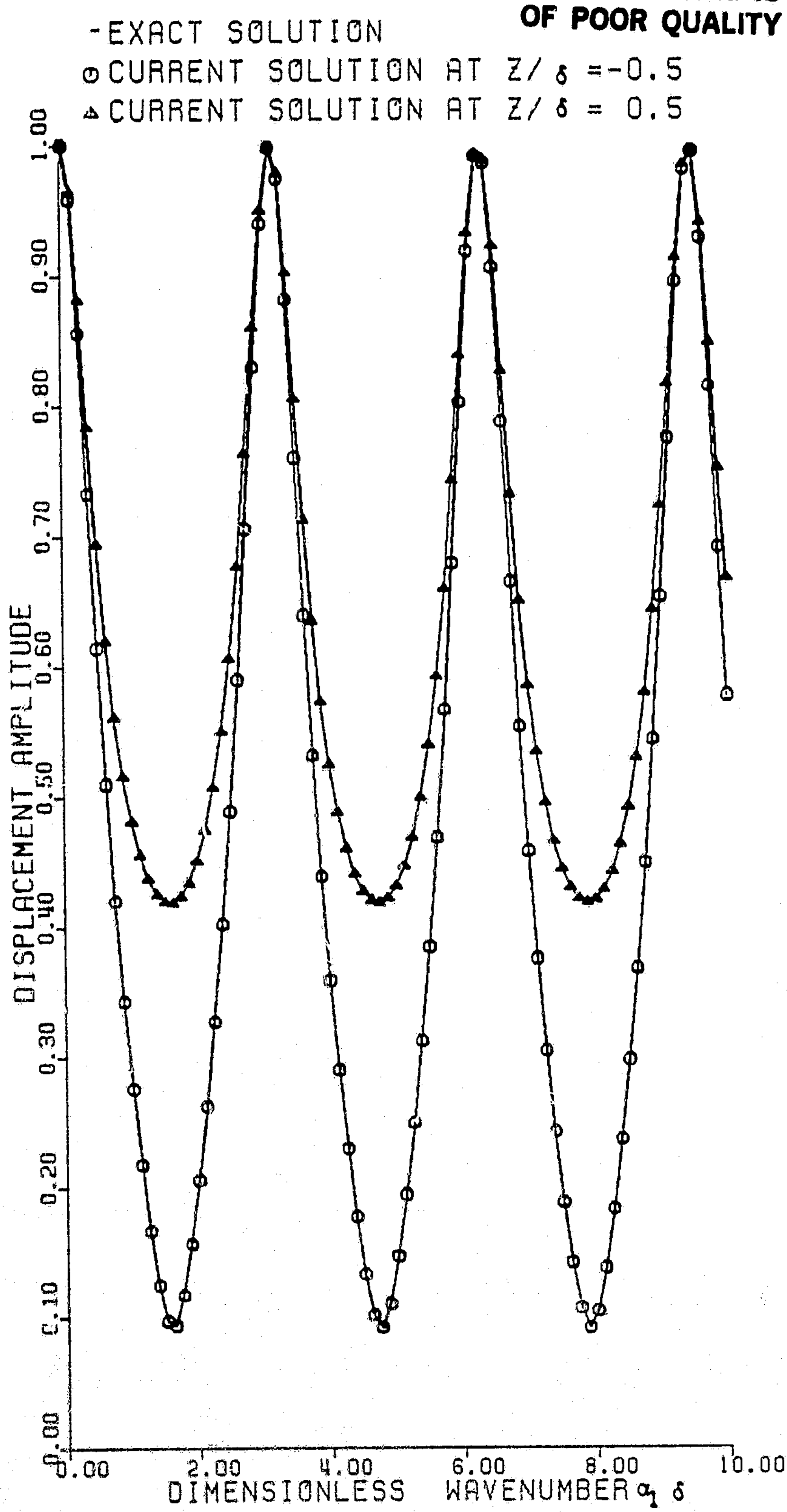

Fig. 27 Displacement amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Stainless Steel in Mg

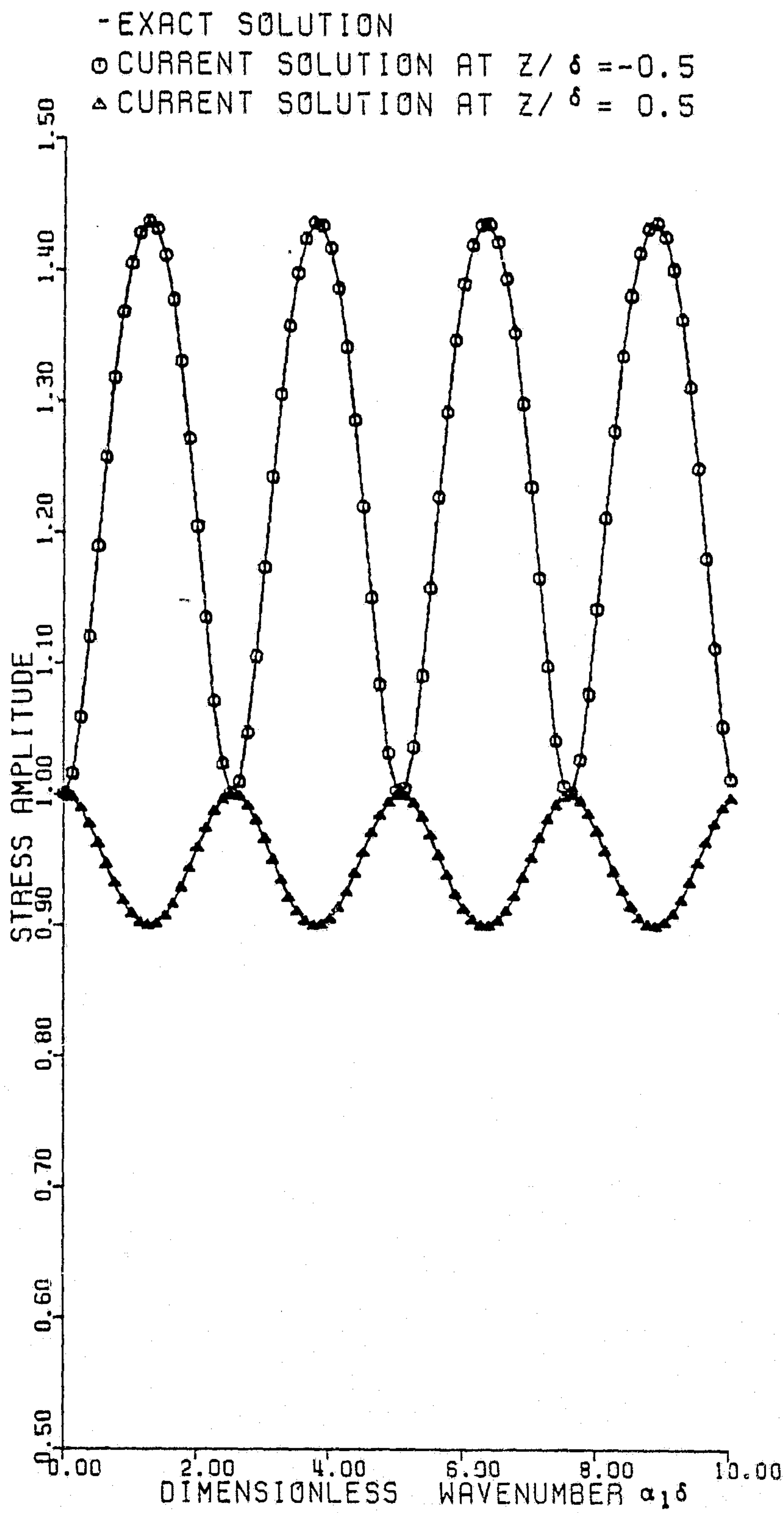

Fig. 28 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Ge in Al

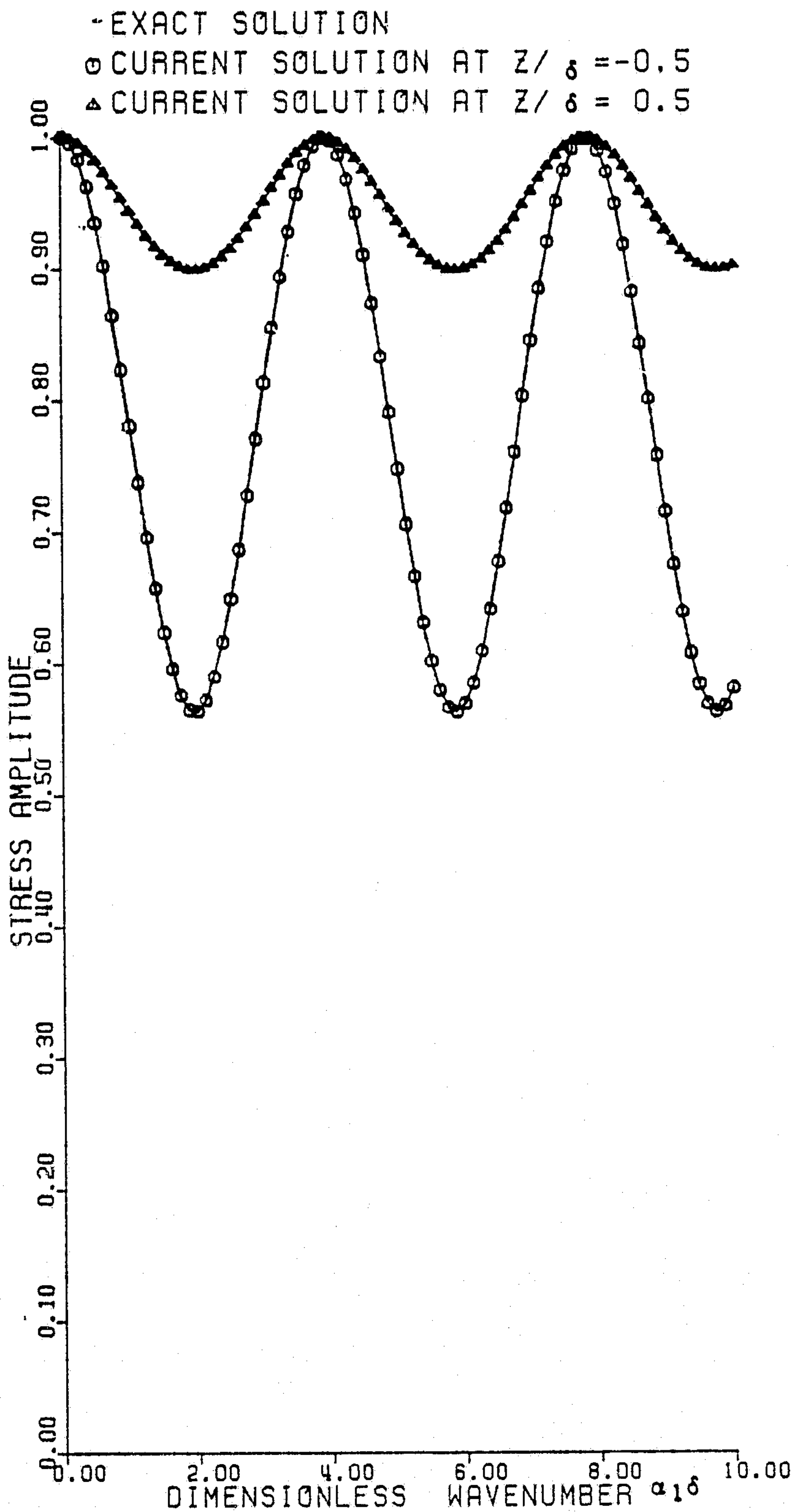

Fig. 29 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Al in Ge

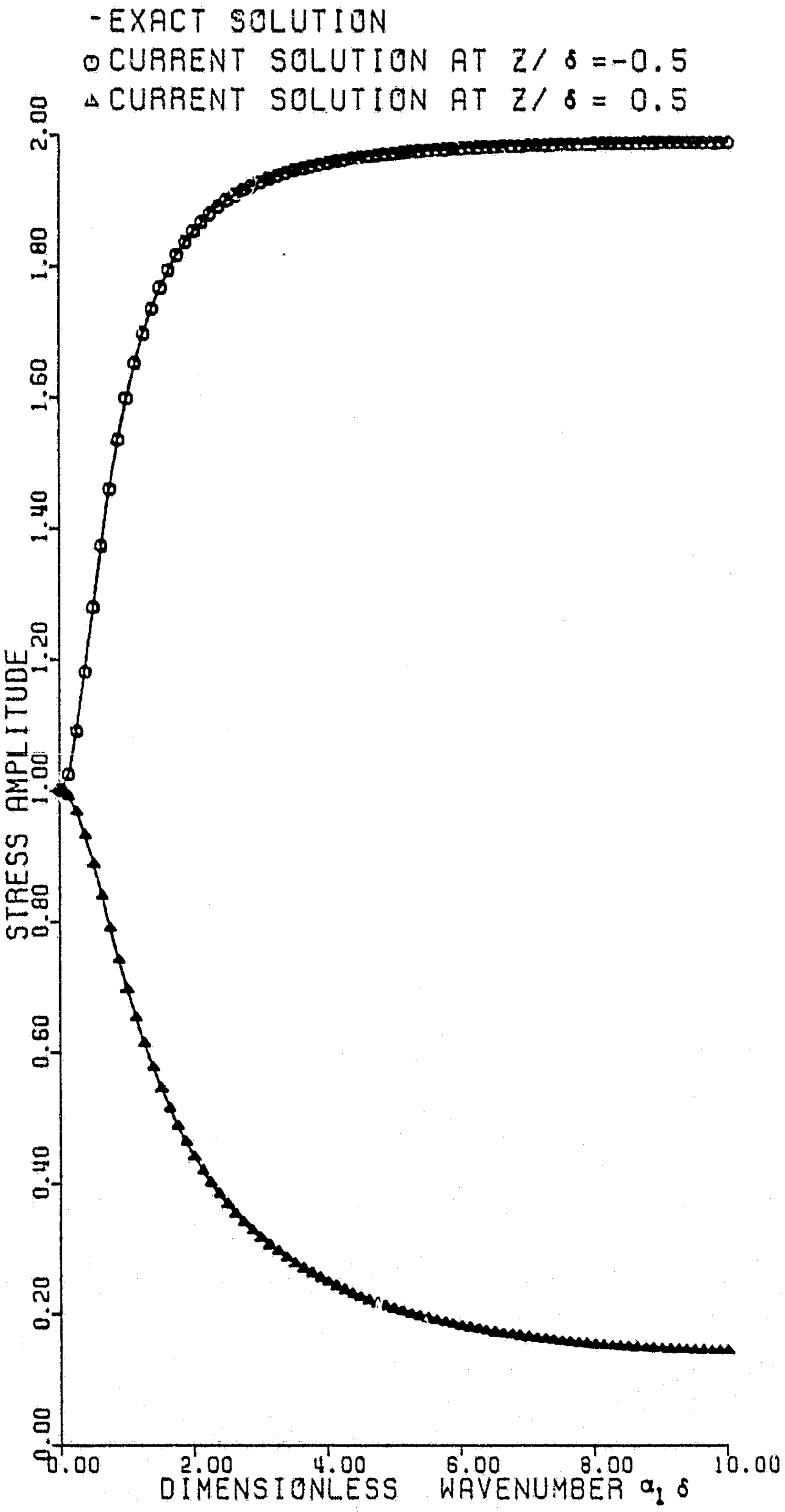

Fig. 30 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Be in Polythylene

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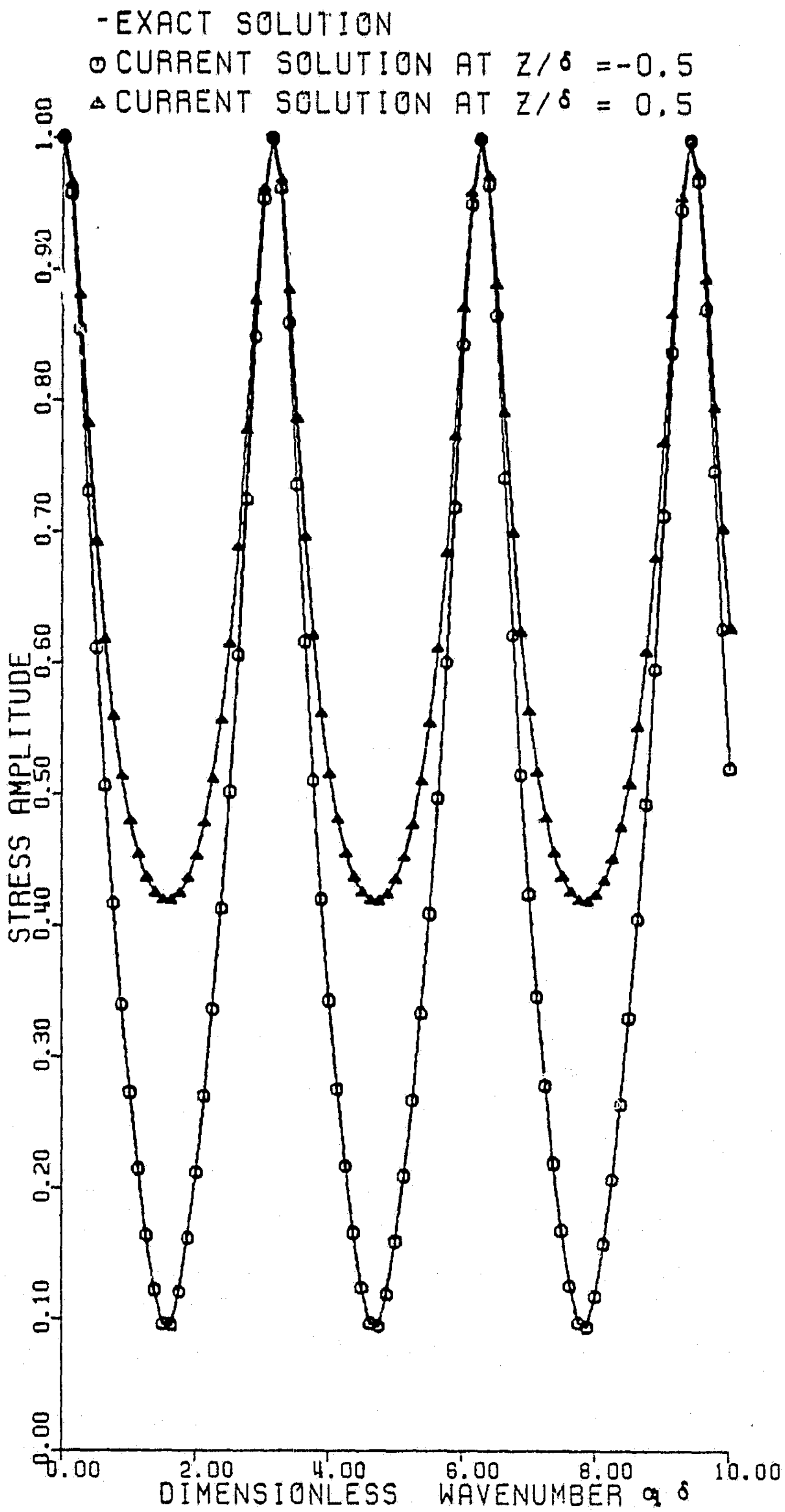

Fig. 31 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Mg in Stainless Steel

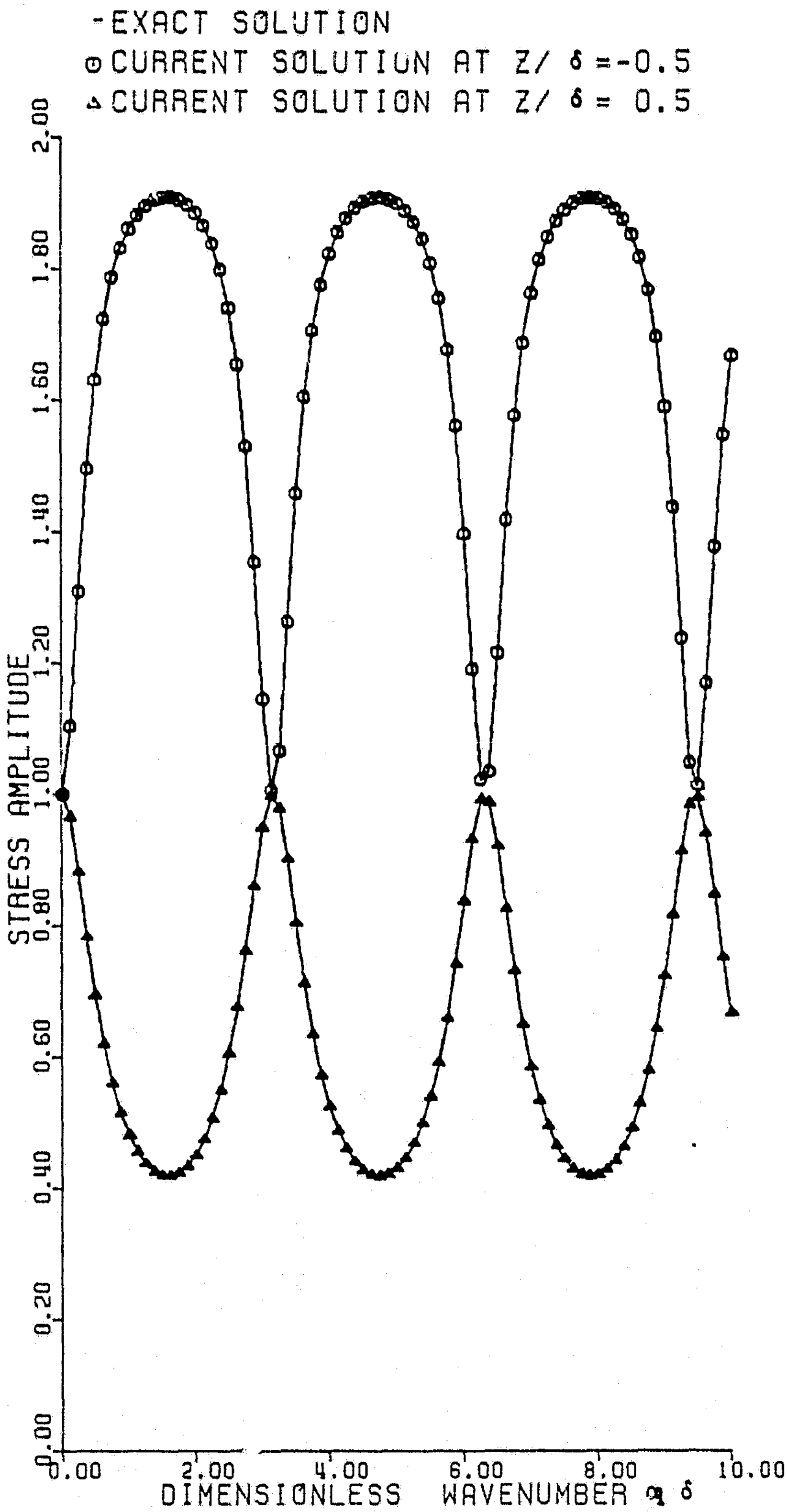

Fig. 32 Stress amplitude as a function of $\alpha_1 \delta$ for the three-layered problem, Stainless Steel in Mg

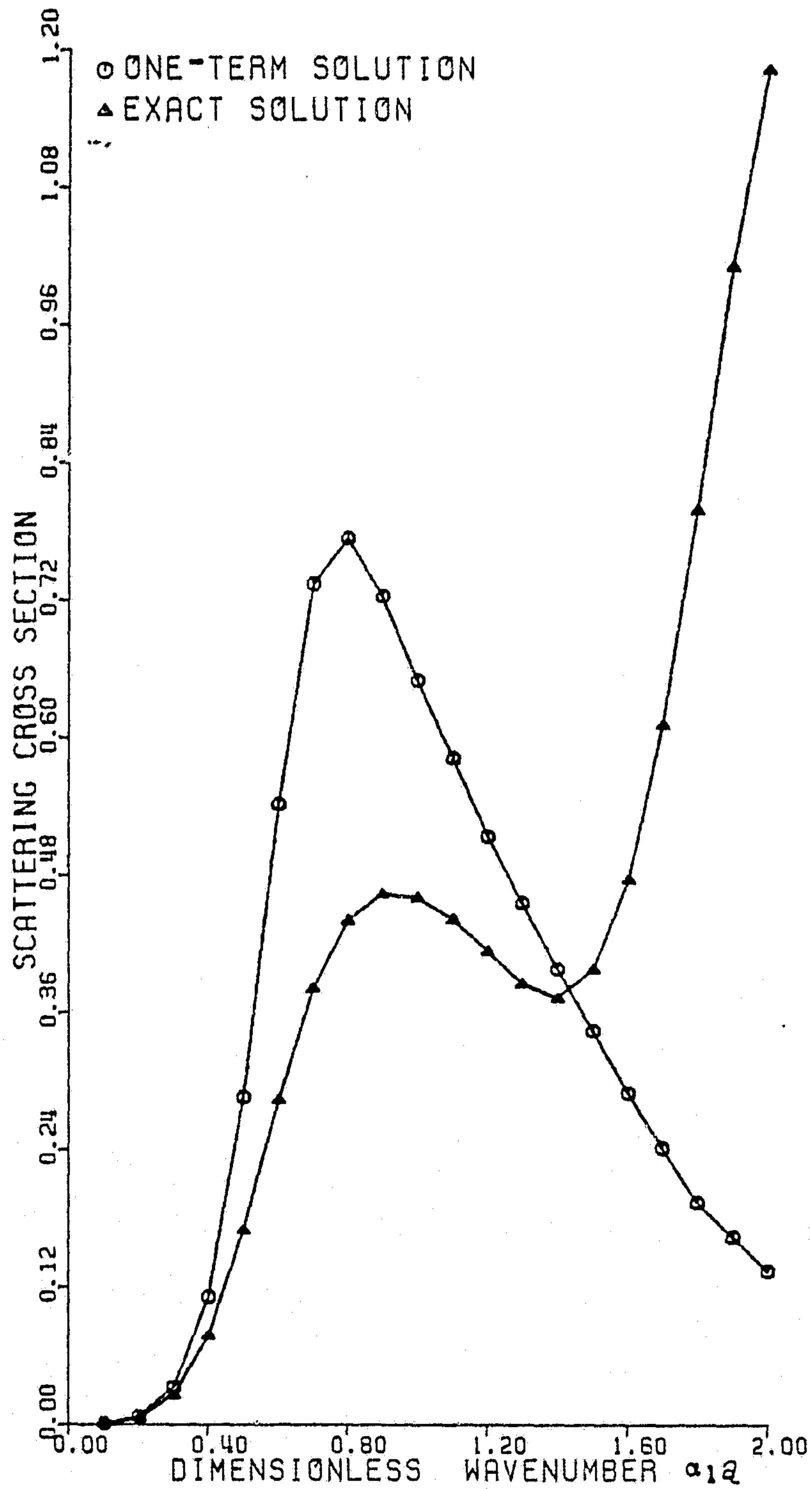

Fig. 33 Scattering cross section as a function of $\alpha_1 a$ for the spherical inhomogeneity problem , Ge in Al

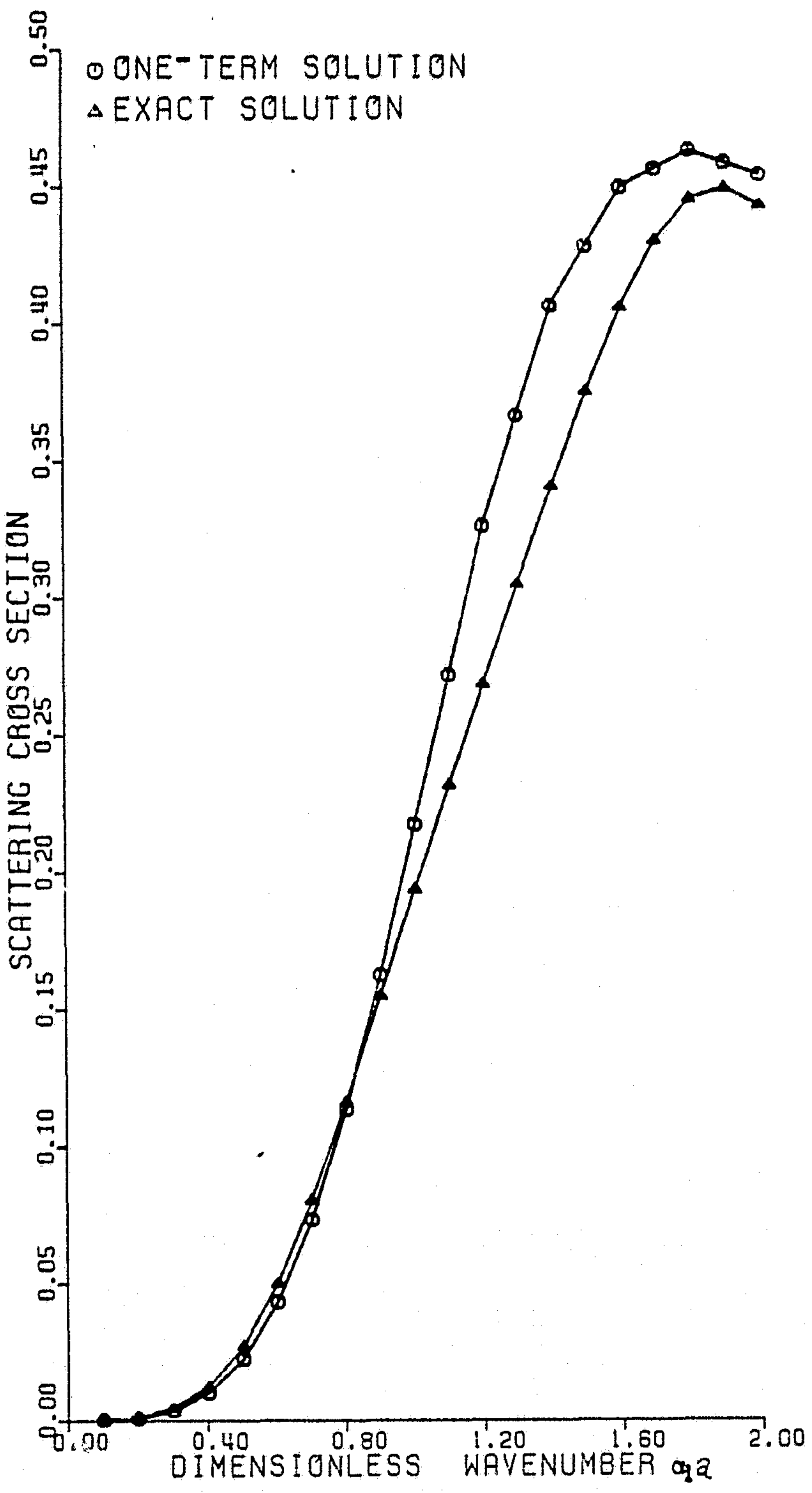

Fig. 34 Scattering cross section as a function of $\alpha_1 a$ for the spherical inhomogeneity problem, Al in Ge

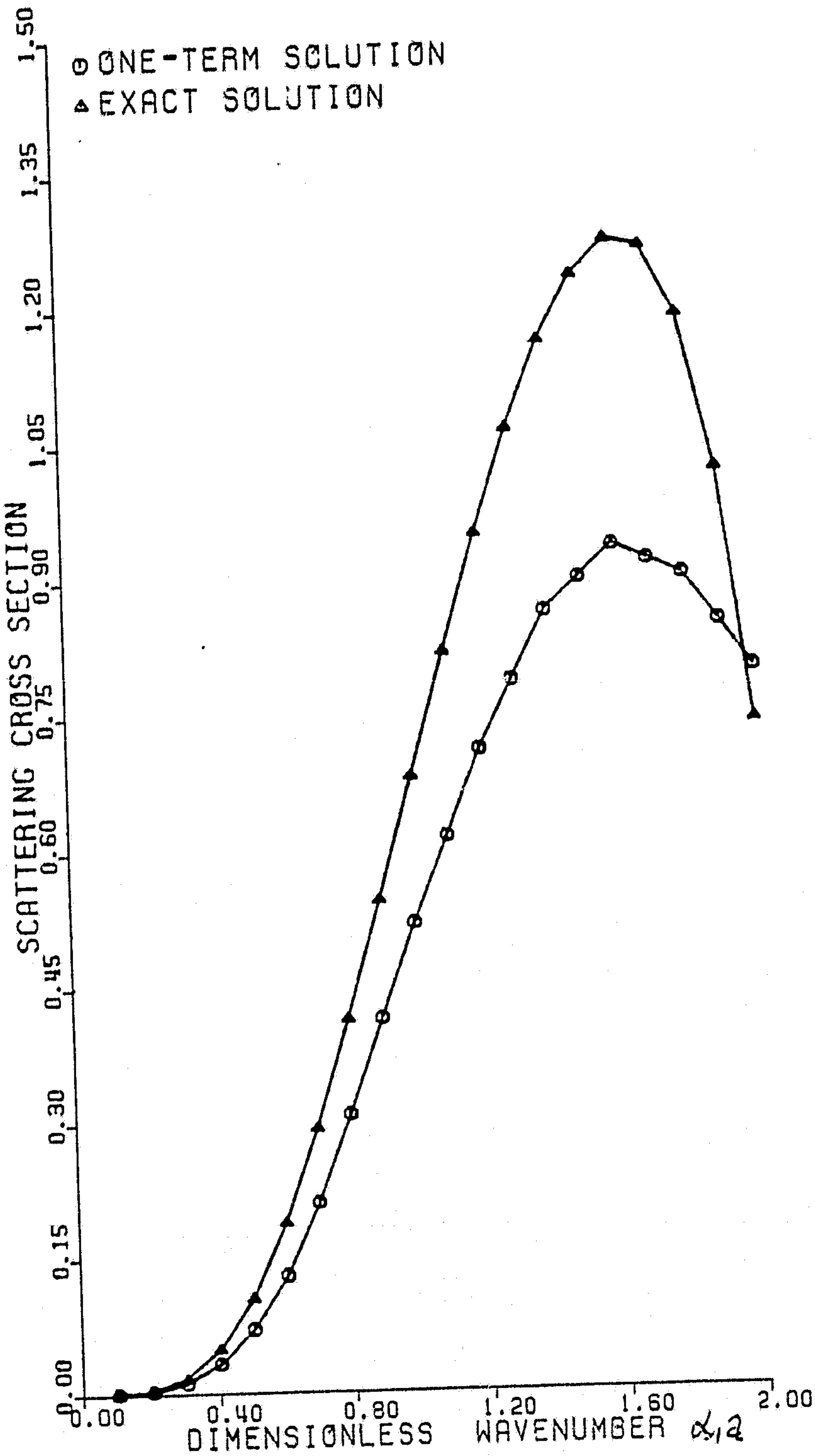

Fig. 35 Scattering cross section as a function of α_1 a for the spherical inhomogeneity problem, Mg in Stainless Steel

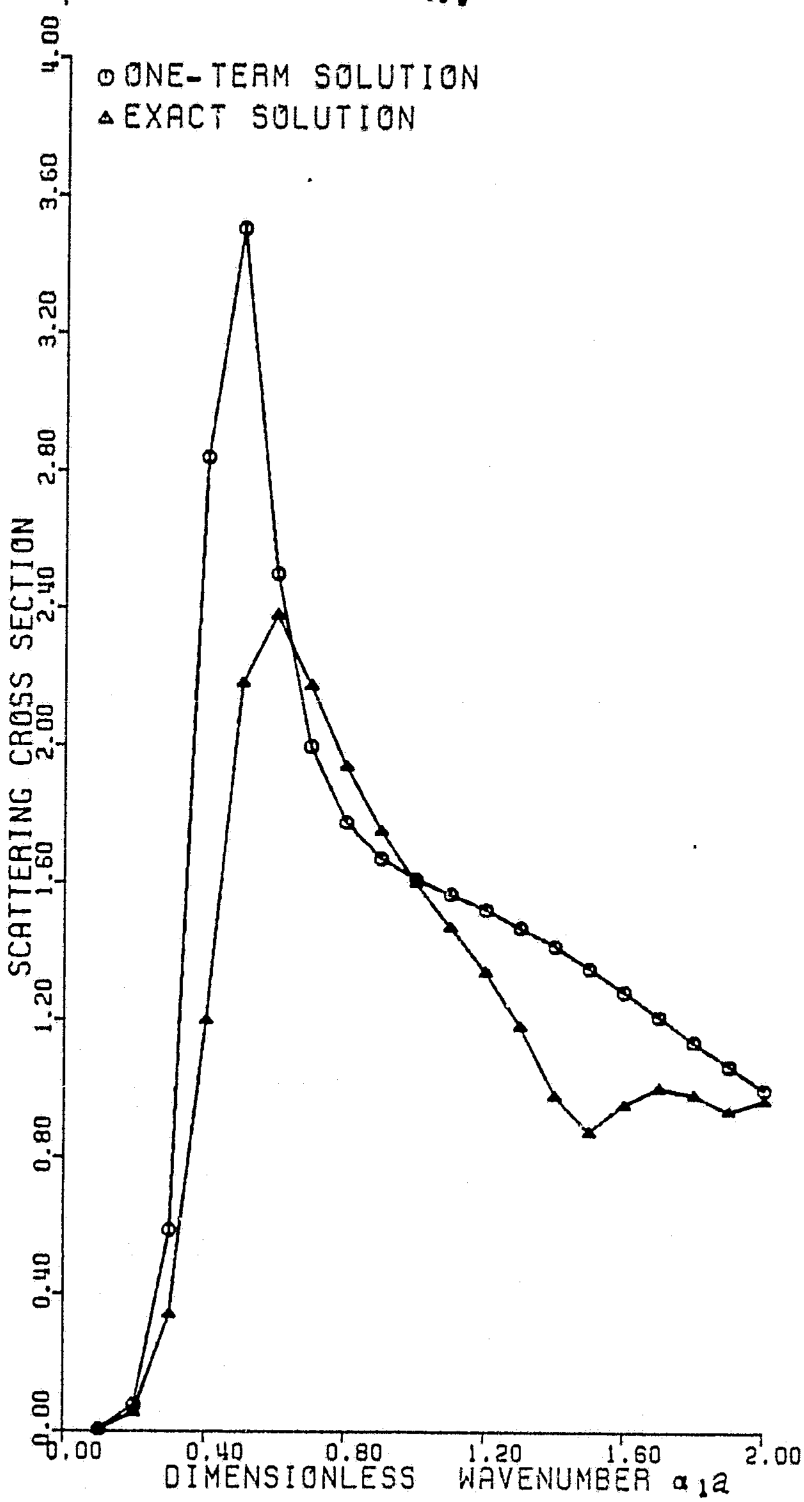

Fig. 36 Scattering cross section as a function of $\alpha_1 a$ for the spherical inhomogeneity problem, Stainless Steel in Mg

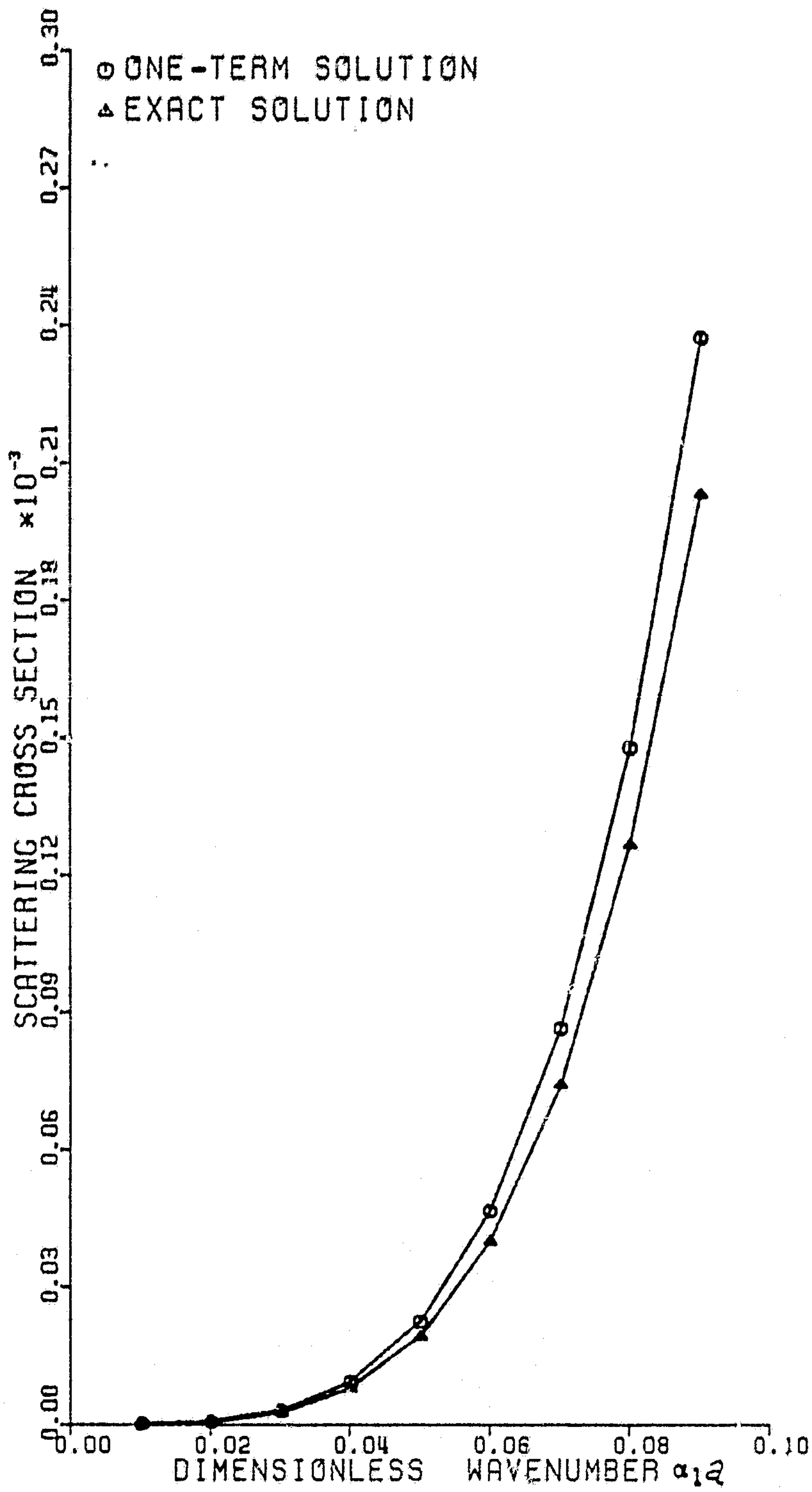

Fig. 37 Scattering cross section as a function of low $\alpha_1 a$ for the spherical inhomogeneity problem, Ge in Al

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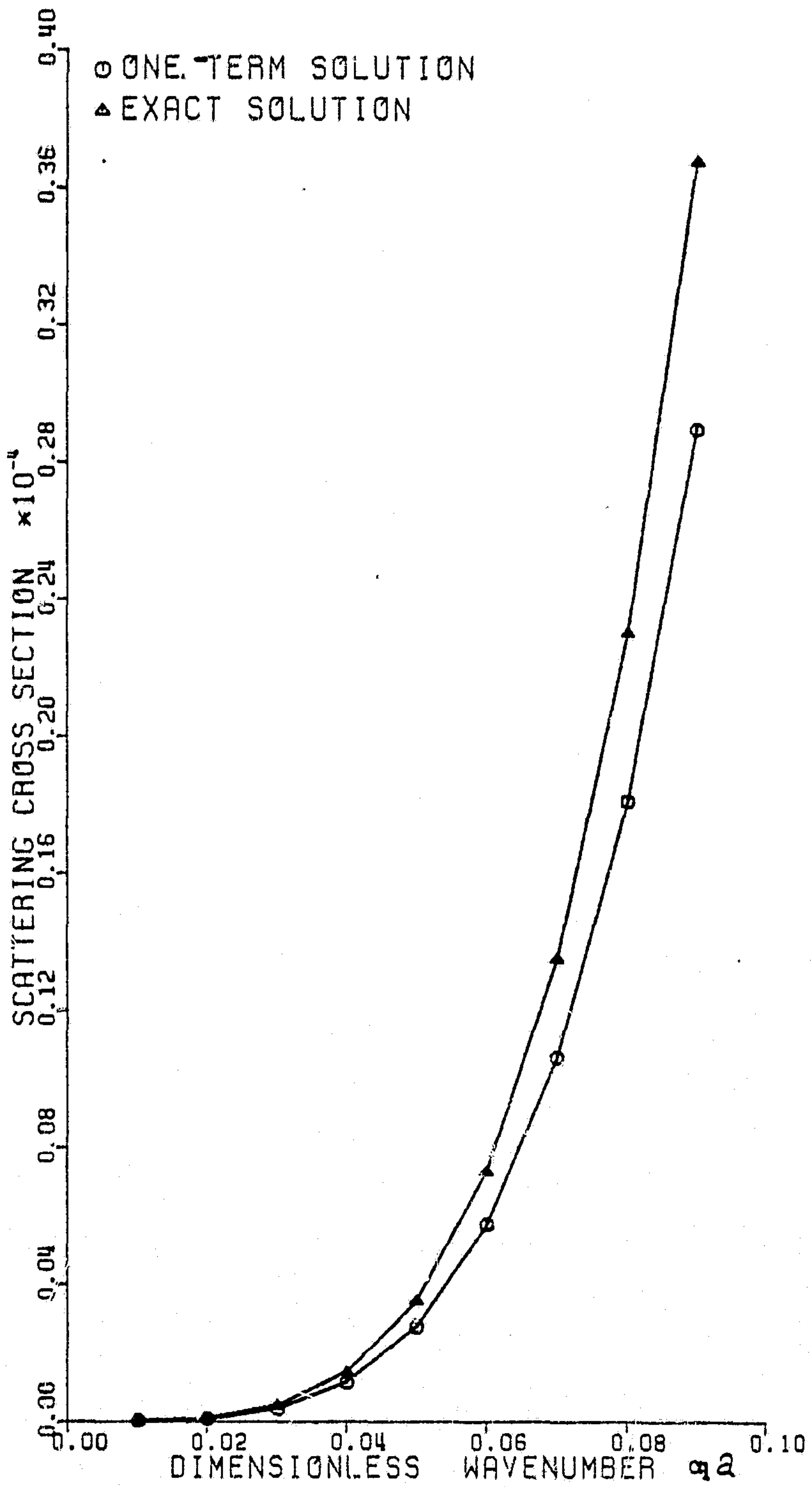

Fig. 38 Scattering cross section as a function of low $\alpha_1 a$ for the spherical inhomogeneity problem ,Al in Ge

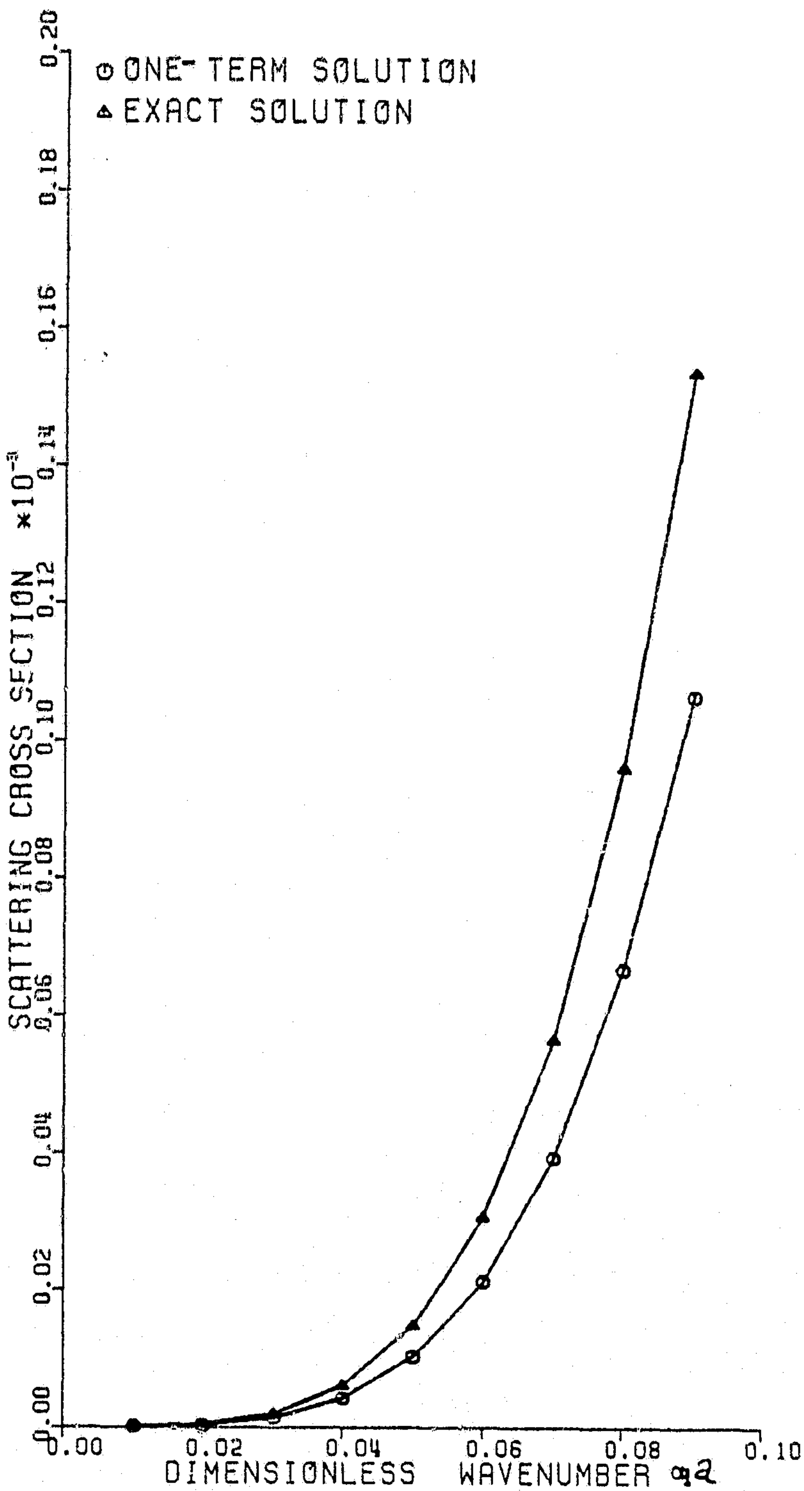

Fig. 39 Scattering cross section as a function of low $\alpha_1 a$ for the spherical inhomogeneity problem, Mg in Stainless Steel

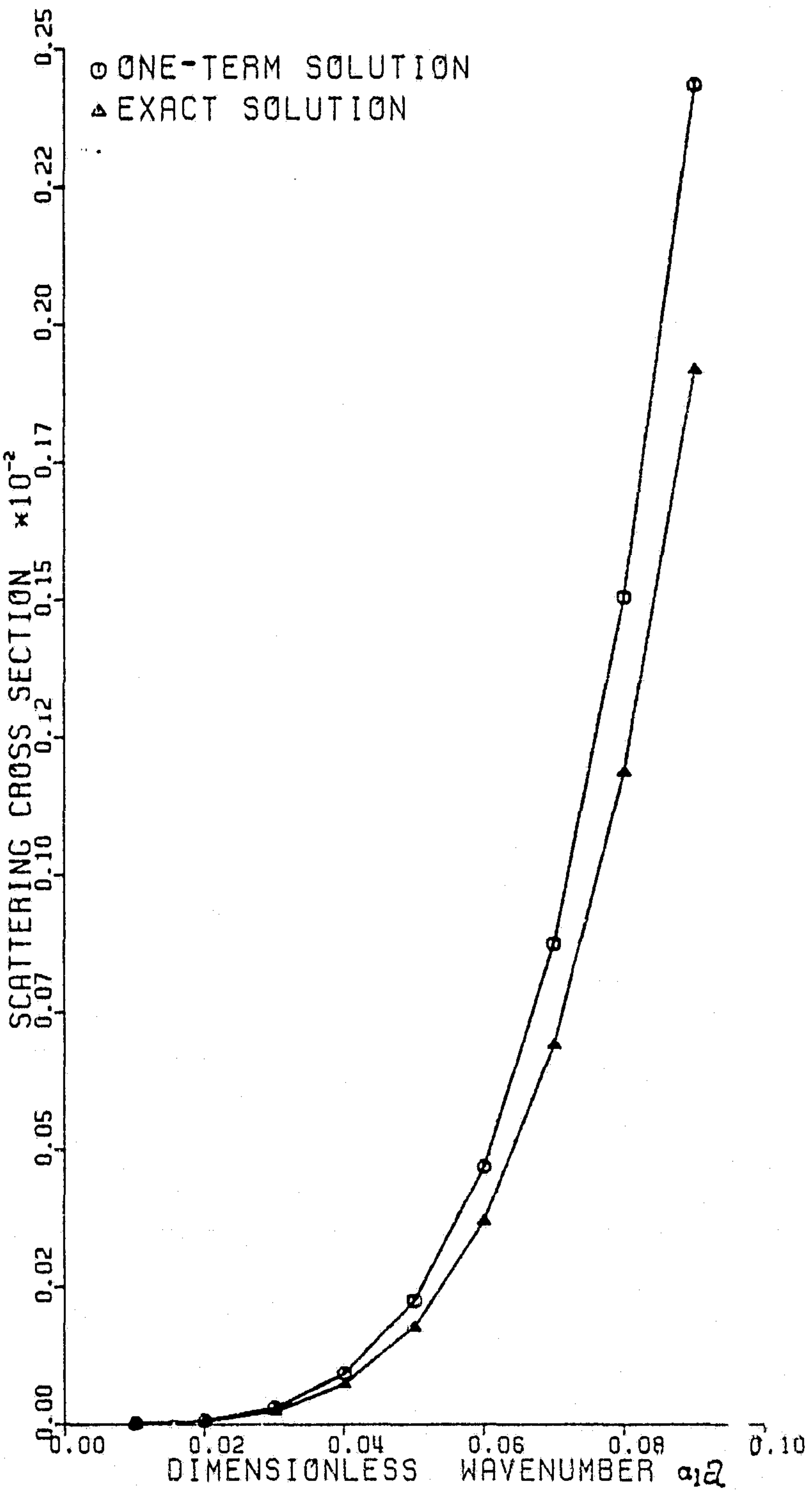

Fig. 40 Scattering cross section as a function of low $\alpha_1 a$ for the spherical inhomogeneity problem, Stainless Steel in Mg

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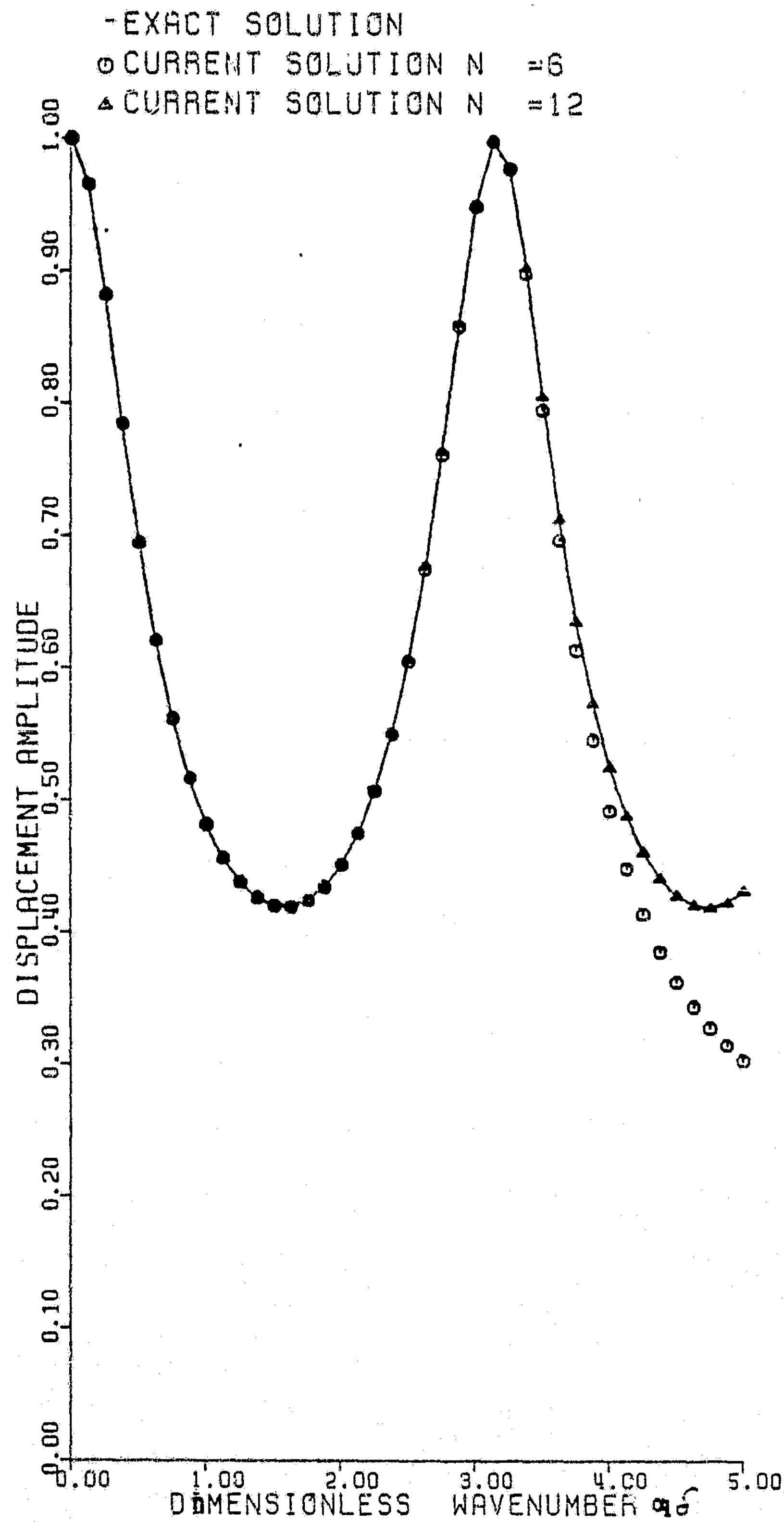

Fig. 41 Displacement amplitude at the point $z/\delta = 0.5$ as a function of $\alpha_1\delta$ for the three-layered problem, Stainless Steel in Mg

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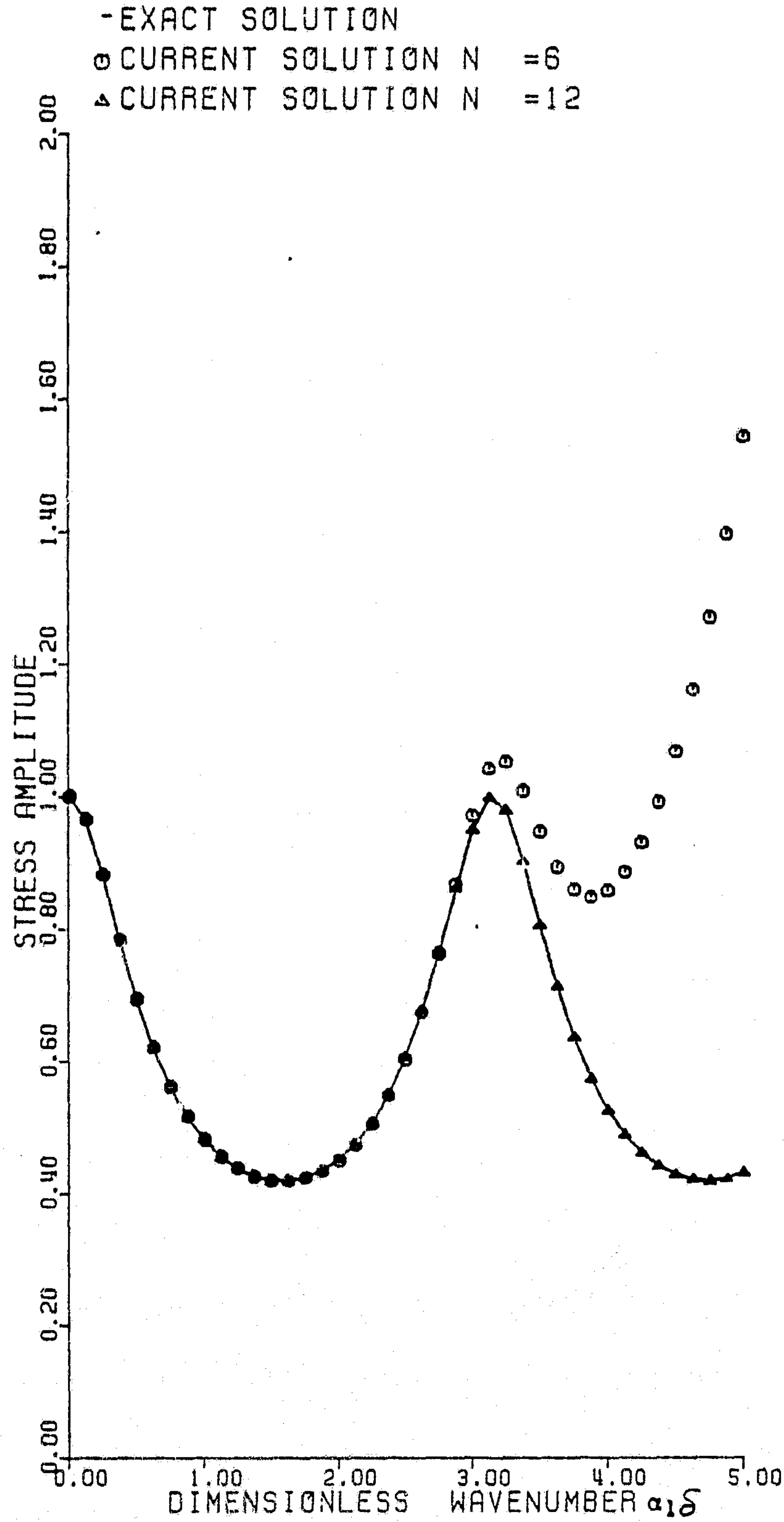

Fig. 42 Stress amplitude at the point $z/\delta=0.5$ as a function of $\alpha_1\delta$ for the three-layered problem, Stainless Steel in Mg

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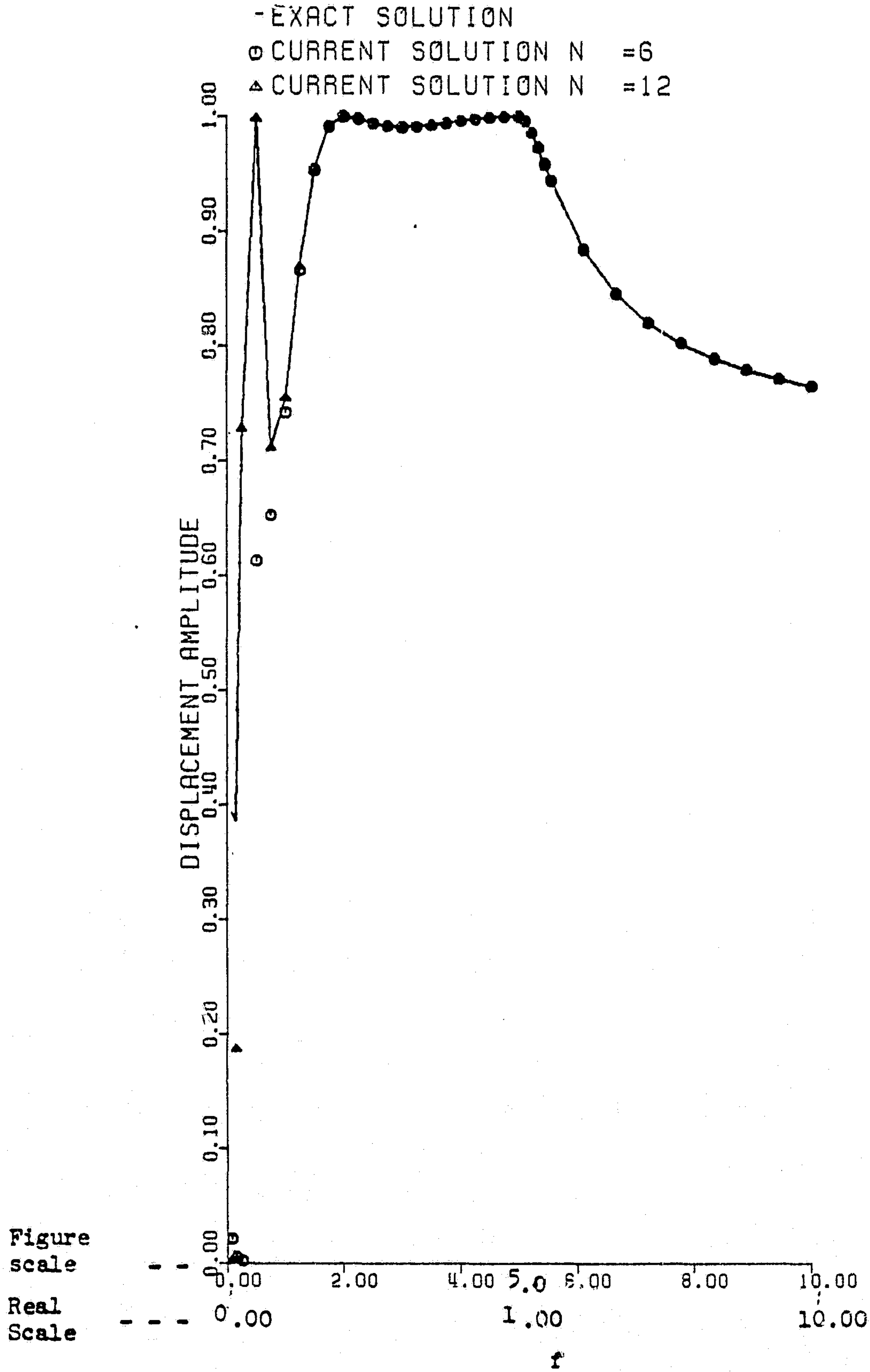

Fig. 43 Displacement amplitude at the point $z/\delta=0.5$ as a function of f for the three-layered problem at the wavenumber $\alpha_1\delta=2.0$ and h=1.

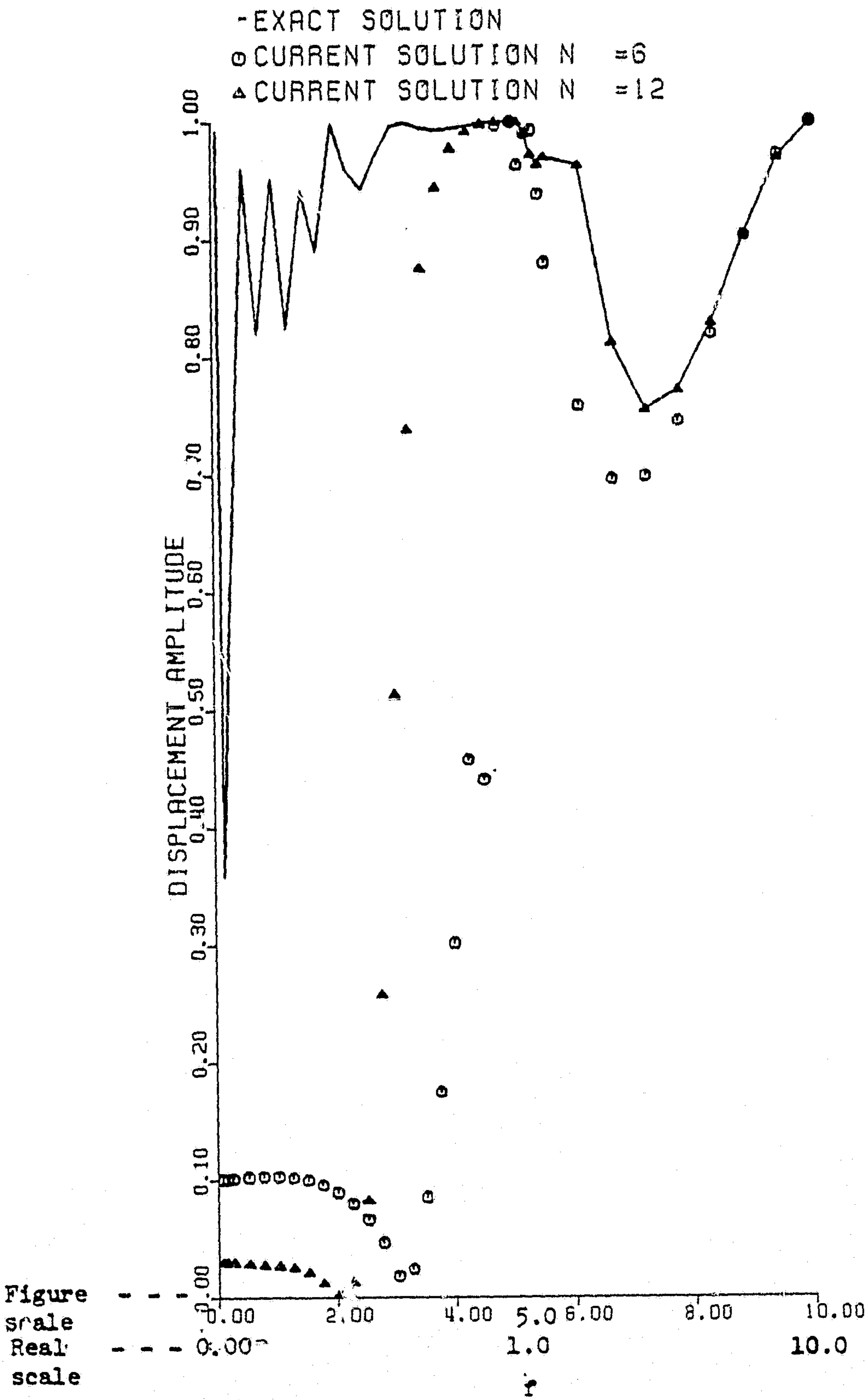

Fig. 44 Displacement amplitude at the point $z/\delta=0.5$ as a function of f for the three-layered problem at the wavenumber $\alpha_1\delta=10.0$ and h=1.0

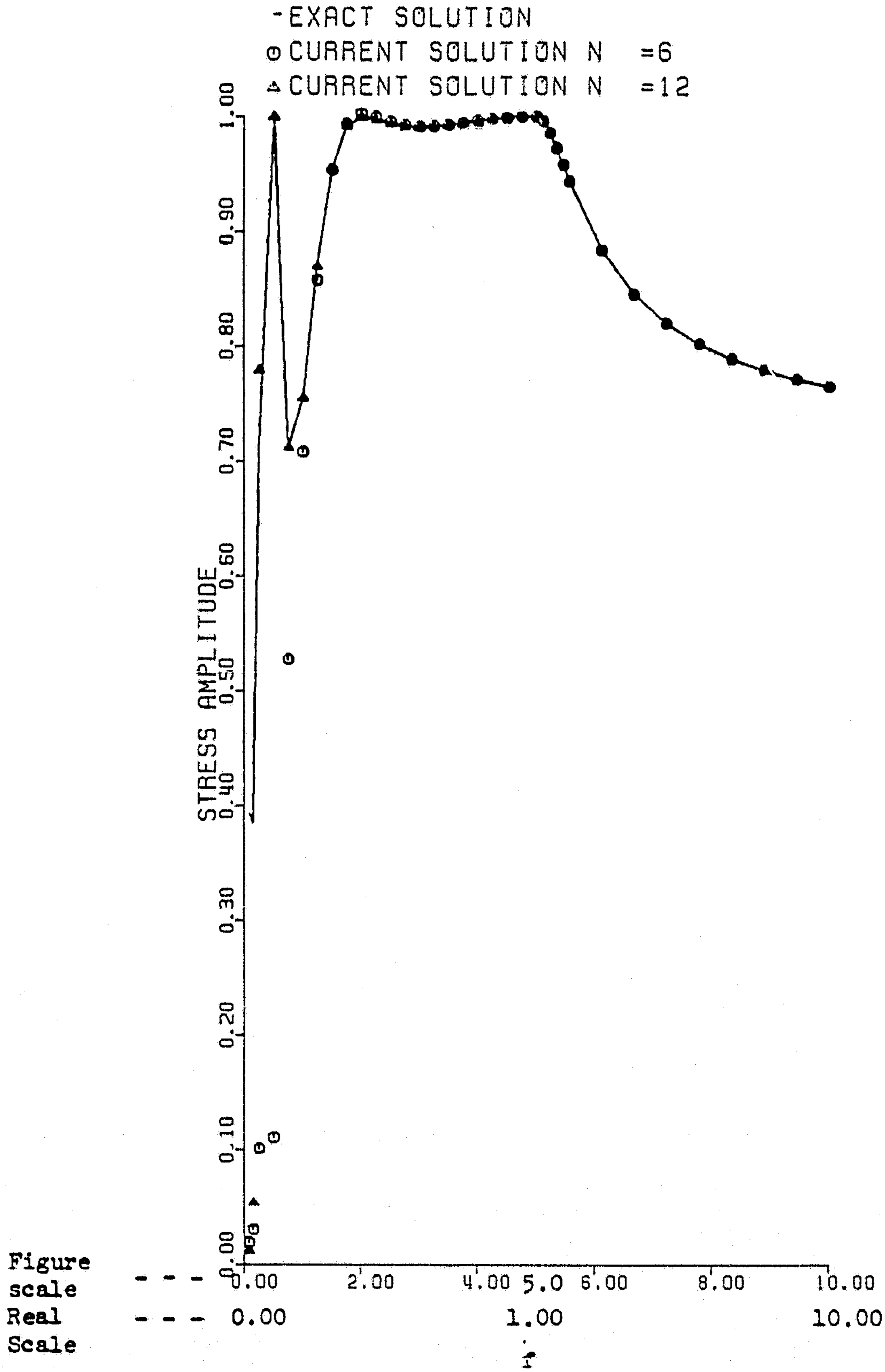

Fig. 45 Stress amplitude at the point $z/\delta=0.5$ as a function of f for the three-layered problem at the wavenumber $\alpha/\delta=2.0$ and h=1.0

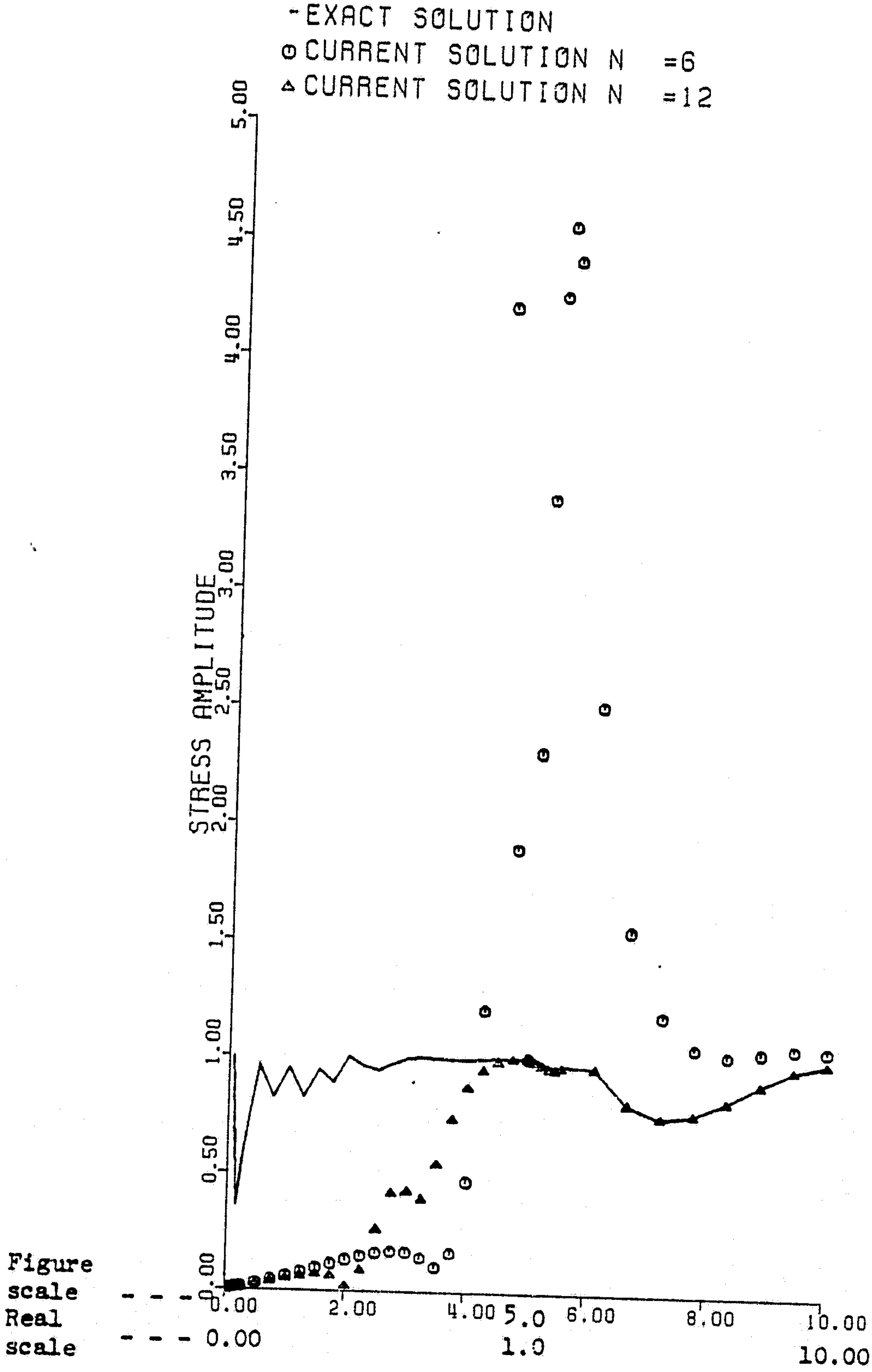

Fig. 46 Stress amplitude at the point $z/\delta = 0.5$ as a function of f for the three-layered problem at the wavenumber $\alpha/\delta = 10.0$ and h=1.0

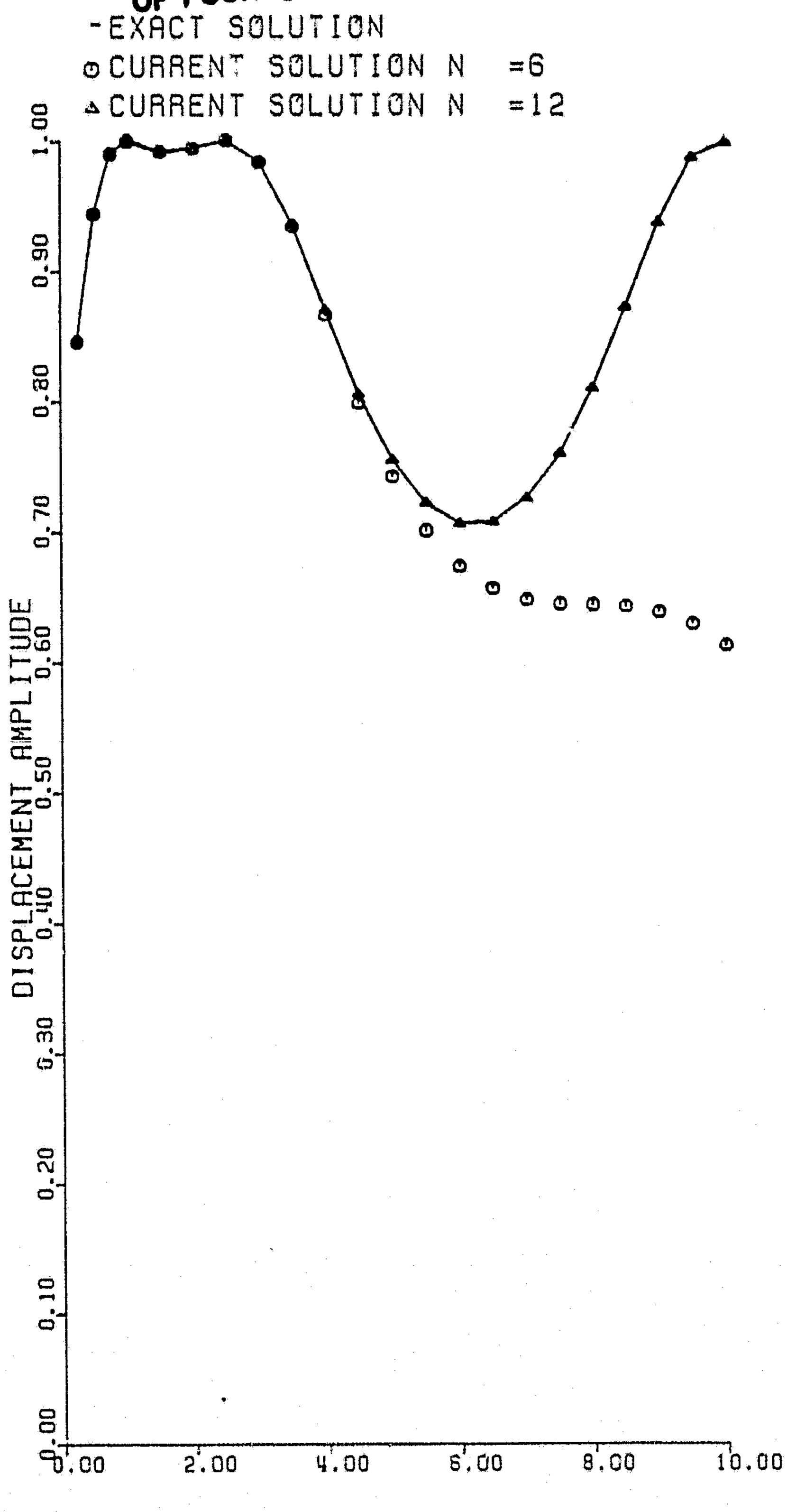

Fig. 47 Displacement amplitude at the point $z/\delta=0.5$ as a function of h for the three-layered problem at the wavenumber $\alpha_1\delta=2.$ and f=1.0

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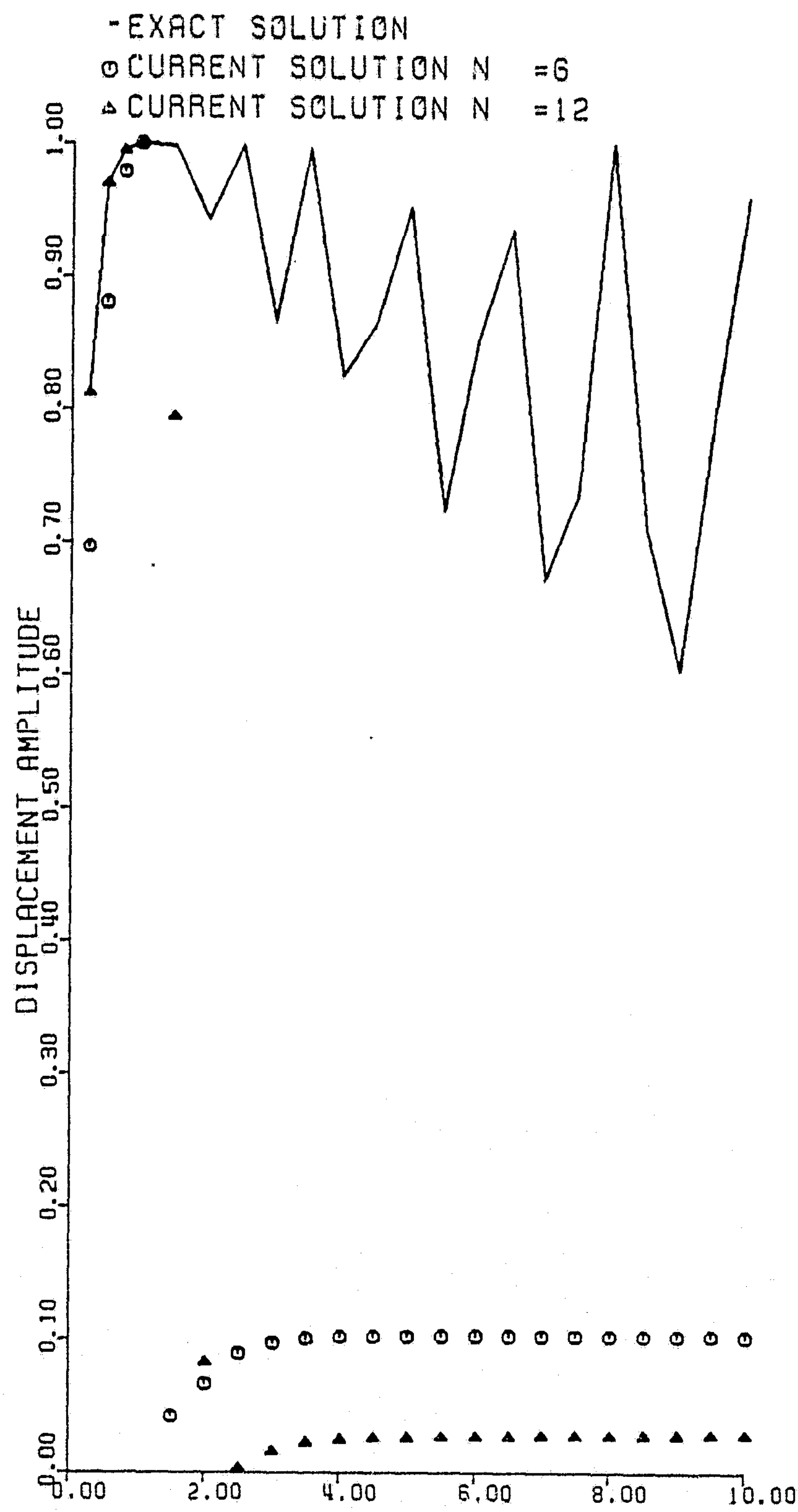

Fig. 48 Displacement amplitude at the point $z/\delta = 0.5$ as a function of h for the three-layered problem at the wavenumber $\alpha_1 \delta = 10.0$ and f=1.0

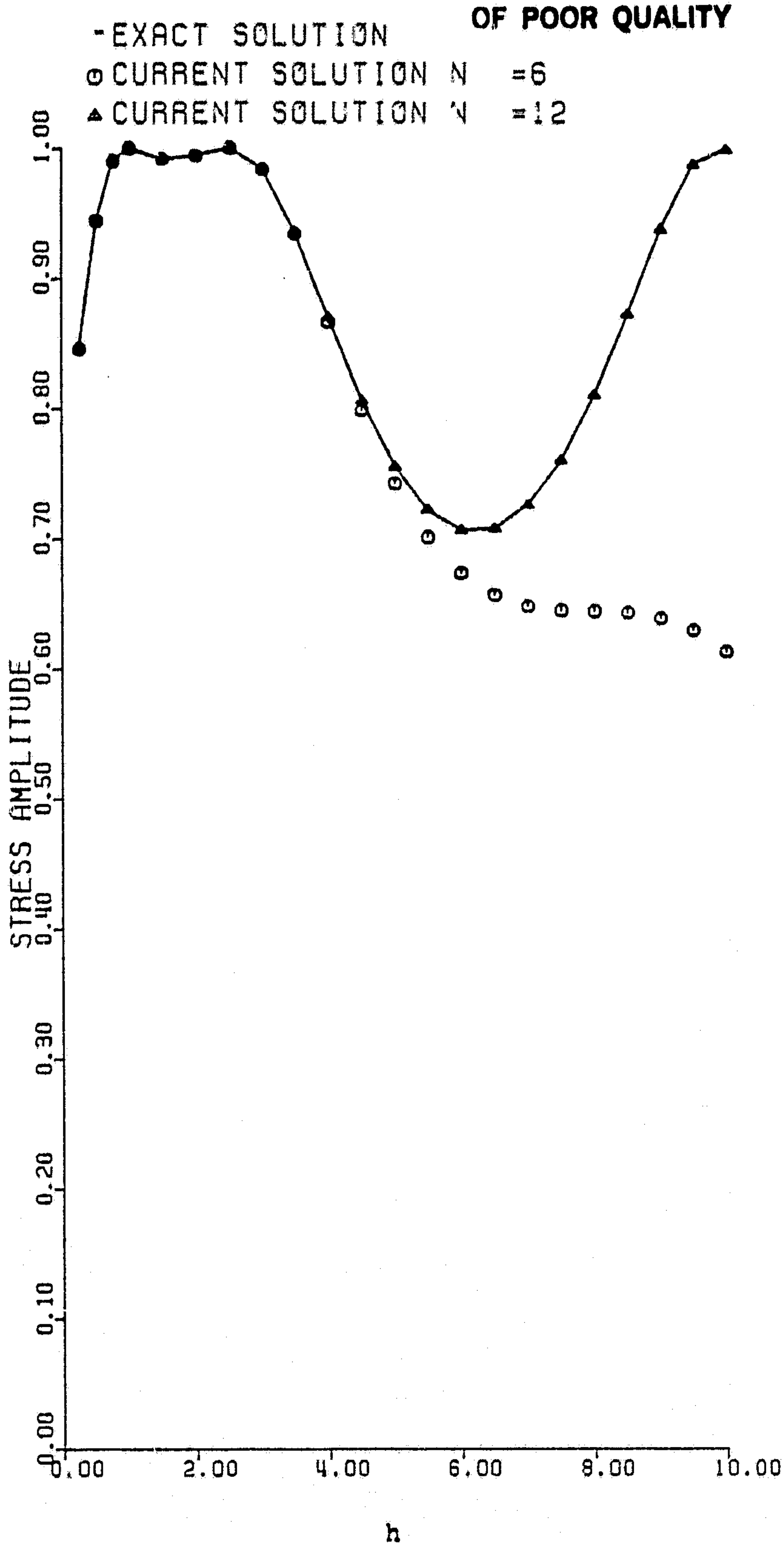

Fig. 49 Stress amplitude at the point $z/\delta = 0.5$ as a function of h for the three-layered problem at the wavenumber $\alpha/\delta = 2.0$ and f=1.0

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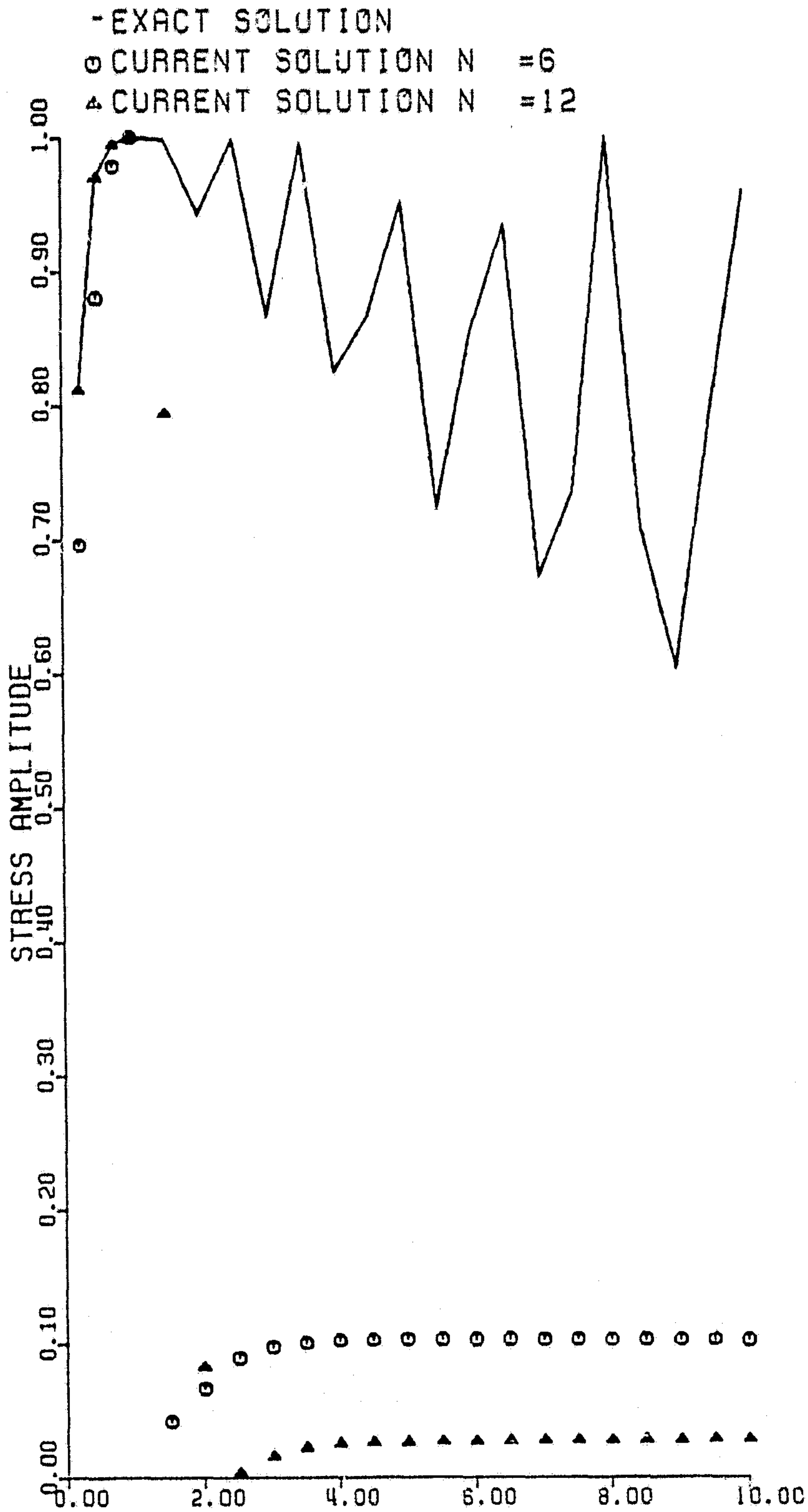

Fig. 50 Stress amplitude at the point $z/\delta = 0.5$ as a function of h for the three-layered problem at the wavenumber $\alpha_1\delta = 10.0$ and f=1.0