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BOUNDARY LAYER EFFECTS IN COMPOSITE LAMINATES: FREE-EDGE STRESS SINGULARITIES

Final Report - Part VI

by

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16. Abstract

A rigorous mathematical model has been obtained for the boundary-layer free-edge stress singularity in angleplied and crossplied fiber composite laminates. The solution was obtained using the previously developed method (under the grant) consisting of complex-variable stress function potentials and unique eigenfunction expansions. The required order of the boundary-layer stress singularity is determined by solving the transcendental characteristic equation obtained from the homogeneous solution of the partial differential equations. Numerical results obtained show that the boundary-layer stress singularity depends only upon material elastic constants and fiber orientation of the adjacent plies. For angleplied and crossplied laminates the order of the singularity is weak in general.

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FOREWORD

This report describes a portion of the results obtained on NASA Grant NSG 3044. This work was done under subcontract to the University of Illinois, Urbana, with Prof. S.S. Wang as the Principal Investigator. The prime grantee was the Massachusetts Institute of Technology, with Prof. F.J. McGarry as the Principal Investigator and Dr. J.F. Mandell as a major participant. The NASA - LeRC Project Manager was Dr. C.C. Chamis.

Efforts in this project are primarily directed towards the development for finite element analyses for the study of flaw growth and fracture of fiber composites. This report presents study of boundary layer problems in composites based on an exact solution.

ABSTRACT

A study of boundary-layer stress singularities in multilayered fiber-reinforced composite laminates is presented. Based on Lekhnitskii's stress potentials and the theory of anisotropic elasticity, formulation of the problem leads to a pair of coupled governing partial differential equations. An eigen-function expansion method is developed to obtain the homogeneous solution for the governing P.D.E's. The order or strength of boundary-layer stress singularities is determined by solving the transcendental characteristic equation obtained from the homogeneous solution for the problem. Numerical examples of the singular strength (or singular eigenvalues) of boundary-layer stresses are given for angle-ply and cross-ply composites as well as the cases of more general composite lamination.

1. INTRODUCTION

The response of a multilayered fiber-reinforced composite laminate near its geometric boundaries has been a subject of intensive investigation during the last decade. Both experimental studies and approximate analytical solutions have indicated that complex stress states with rapid change of gradients occur along the edges of composite laminates, for example, Refs [1-15]. This phenomenon is considered to result from the presence and interactions of geometric discontinuities of the composite and materials discontinuities through the laminate thickness. The anomaly has been found to occur only within very local region near the geometric boundaries of a composite laminate, and is, therefore, frequently referred to as "boundary-layer effect" or "free-edge effect" — a problem unique to composite laminates and not observed in homogeneous solids in general. It has been shown further that the boundary-layer effect is three-dimensional in nature and not predictable by the classical laminate theory (CLT) [16,17]. The boundary-layer effect is apparently one of the most fundamental and important problems in the mechanics and mechanical behavior of composite laminates. The high stresses developed in the boundary-layer region coupled with the low interlaminar strength are certainly of critical significance in aggravating the failure of composite materials and structures. For example, boundary-layer stresses have been observed to be responsible for the initiation and growth of local heterogeneous damage in the forms of interlaminar (delamination) and intralaminar (transverse cracking) fracture in composite laminates under static loading [3,18]. They are considered to have even greater effects on the long term strength of composite laminates under cyclic fatigue loading [19,20].

While the significance of boundary-layer effects has long been recognized, research progress on this subject has been relatively slow. The situation is apparently caused by the inherent complexities involved in the problem: the strong anisotropy of mechanical properties of each individual ply, the abrupt change of materials properties through the laminate thickness, the geometric discontinuity along laminate boundaries, and the coupling between in-plane and transverse deformations and stresses near the edges of the composite laminate. According to Pagano [14], analytical studies to date may be roughly classified into two general categories: approximate theories and numerical solutions. The first approximate solution for finite-width composite laminates was proposed by Puppo, et al. [4] based on a laminate model containing anisotropic laminae and isotropic shear layers with the interlaminar normal stress being neglected throughout the laminate. Other approximate theories were also attempted to examine the problem such as the extension of the higher-order plate theory [21] by Pagano [10], the perturbation method by Hsu, et al. [12], and a boundary-layer theory by Tang, et al. [11]. Recently, Pagano [14,15] has developed an approximate theory based on assumed in-plane stresses and the use of Reissner's variational principle. Even though there is no singularity involved in the formulation, the approach has certain features significantly important in objectively determining detailed laminate stress fields. The study of edge stresses in composites by using a numerical (finite difference) method was apparently first made by Pipes, et al. [5]. Isakson and Levy [6] developed a finite-element scheme containing membrane elements, which closely resemble the laminate model of Puppo et al. [4]. Later finite element studies on this subject by Wang, et al. [13] and Herakovich, et al. [18] led to numerical solutions similar to that given by Pipes [5]. Due to the singular nature of the problem, a large number of elements, especially through the thickness direction, are required in conjunction with a lengthy extrapolation procedure in order to achieve satisfactory solutions even for a simple two or three layer laminate. Improved finite-element methods by using a more complex element stiffness formulation based upon Maxwell stress functions [7] and by hybrid-stress elements [22] have been able to achieve an expedient computation with significantly less elements. Unfortunately, the refinements do not guarantee [23] the convergence and accuracy of the numerical solutions because of the singular nature of the boundary-layer stress field. That is, with each more refined analysis, numerical values of the maximum interlaminar stresses are shown to rise with continuously decreasing element size. The quest, apparently, is to show that a stress singularity exists at the edge of a composite laminate.

From a linear elasticity point of view, it is well known that stress singularities are prevalent at the corners of geometric boundaries joining dissimilar materials (see, for example, [24-26]). Unfortunately, the search for the order of stress singularities for the boundary-layer region in a composite laminate containing anisotropic plies has not been successful to date, to the authors' knowledge. Since the singular boundary-layer stresses are extremely localized in nature, the precise nature of the boundary-layer effect will not be fully understood until the exact order of the stress singularities is defined. In this paper, the first in succession, a rigorous theoretical investigation of the free-edge stress singularity in composite laminates is presented.

In the next section, a mathematical model and basic equations for each lamina of the composite are presented. Based on the theory of anisotropic elasticity and Lekhnitskii's stress potentials [27], a pair of linear governing partial differential equations is derived. Appropriate near field boundary conditions, end loading conditions and interface continuity conditions are given also. The homogeneous solution for the problem is obtained in Section 3 by an eigenfunction expansion method. The solution procedure used to evaluate the exact order of the boundary-layer stress singularity is presented. Degenerated cases of commonly used cross-ply composite laminates are examined also. Numerical examples of determining the edge stress singularities for graphite-epoxy composite laminates with various fiber orientations are given in Section 4. As will be shown later, the existence of the free-edge stress singularity in composite laminates is proven mathematically in this paper. It settles, once and for all, the previous conjecture of boundary-layer stress singularities in composite materials, and provides a rigorous mathematical method for determining the exact value of the edge stress singularity, which is the fundamental basis for the boundary layer theory in composite materials and structures.

2. FORMULATION

2.1 Basic Equations

Consider a composite laminate composed of fiber-reinforced plies with constitutive equations described by generalized Hooke's law in the contracted notation as

$$\varepsilon_i = S_{ij}\sigma_j \qquad (i,j = 1,2,3,4,5,6), \tag{1}$$

where the repeated subscript indicates summation and S_{ij} is the compliance tensor. The engineering strains, ε_i, in Eq 1 are defined in a Cartesian coordinate system by

$$\varepsilon_1 = \varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_2 = \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_3 = \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\varepsilon_4 = \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \varepsilon_5 = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \varepsilon_6 = \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \tag{2}$$

where u, v and w are the components of displacements. The stresses, σ_i, are defined in an analogous manner in the Cartesian coordinate system.

The composite laminate considered here has a finite width and is subjected to surface tractions acting in planes normal to the generator of the lateral surface and not varying along the generator, i.e., along the z-axis (Fig. 1). The composite is assumed to be sufficiently long that, in the region far from the ends, the end effect is neglected by virtue of the Saint Venant principle. Consequently, the stresses in the laminate are independent of the z-coordinate. The case of a finite-width composite laminate subjected to a uniform axial strain, ε_z=e, along the z-axis has been intensively studied by many researchers [5-13]. The special case in which stresses and displacements are independent of z corresponds to the well known

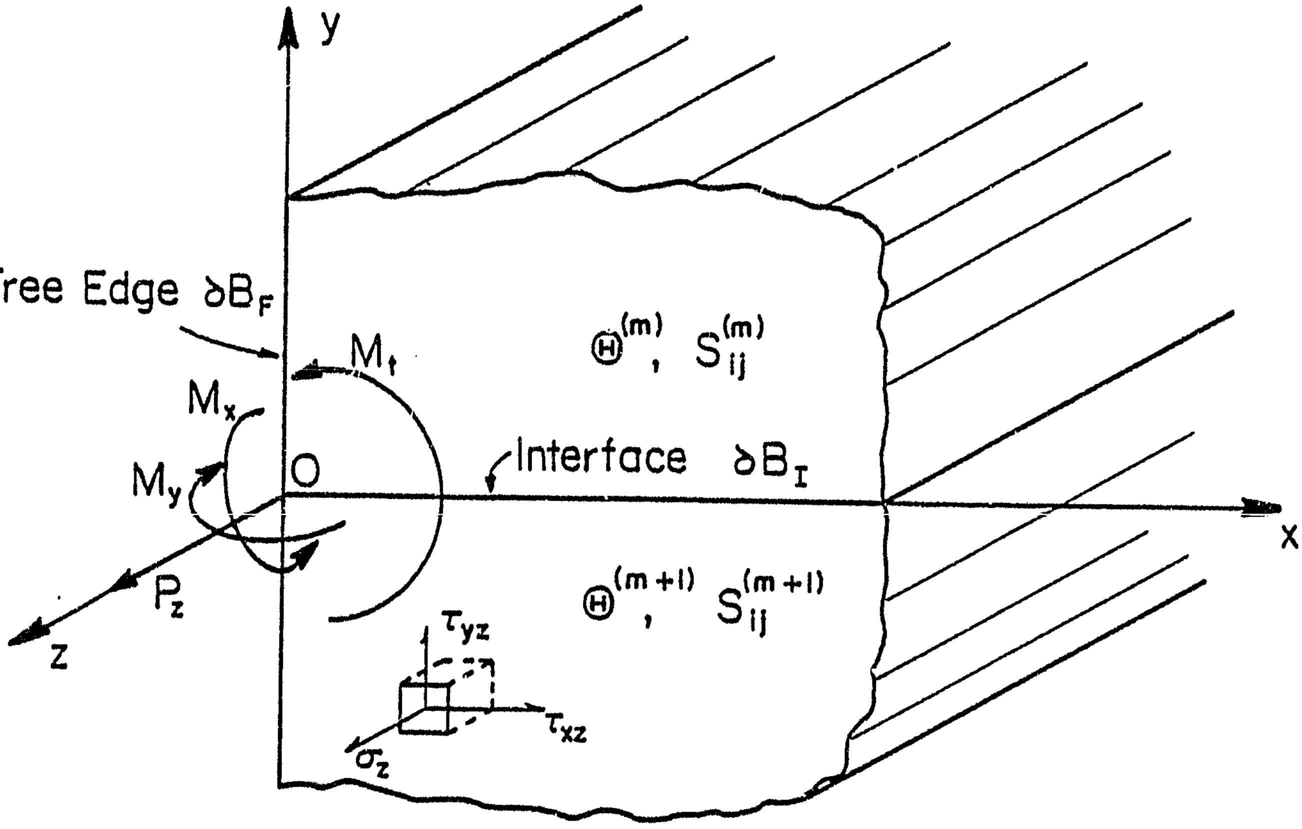

Fig. 1. Geometry and Coordinates of a Free Edge in Composite Laminates

generalized plane deformation [27]. Under these assumptions, the equations of equilibrium without body force read

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \qquad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0. \tag{3}$$

Following the procedure in [27], it can be shown after some mathematical manipulation that the general expressions for displacements and the stress component σ_z have the following form:

$$u = -\frac{1}{2} A_1 S_{33} z^2 - A_4 yz + U(x,y) + \omega_2 z - \omega_3 y + u_o, \tag{4a}$$

$$v = -\frac{1}{2} A_2 S_{33} z^2 + A_4 xz + V(x,y) + \omega_3 x - \omega_1 z + v_o, \tag{4b}$$

$$w = (A_1 x + A_2 y + A_3) S_{33} z + W(x,y) + \omega_1 y - \omega_2 x + w_o, \tag{4c}$$

$$\sigma_z = A_1 x + A_2 y + A_3 - S_{3j}\sigma_j / S_{33}, \qquad (j = 1,2,4,5,6). \tag{4d}$$

The unknown functions, U, V and W, depend on x and y only, and can be shown easily to obey the following relations:

$$\frac{\partial U}{\partial x} = \tilde{S}_{1j}\sigma_j + S_{13}(A_1 x + A_2 y + A_3), \tag{5a}$$

$$\frac{\partial V}{\partial y} = \tilde{S}_{2j}\sigma_j + S_{23}(A_1 x + A_2 y + A_3), \tag{5b}$$

$$\frac{\partial W}{\partial x} = \tilde{S}_{5j}\sigma_j + S_{53}(A_1 x + A_2 y + A_3) + A_4 y, \tag{5c}$$

$$\frac{\partial W}{\partial y} = \tilde{S}_{4j}\sigma_j + S_{43}(A_1 x + A_2 y + A_3) - A_4 x, \tag{5d}$$

$$\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \tilde{S}_{6j}\sigma_j + S_{63}(A_1 x + A_2 y + A_3), \tag{5e}$$

where

$$\tilde{S}_{ij} = S_{ij} - S_{i3}S_{j3}/S_{33}, \qquad (i,j = 1,2,4,5,6). \tag{5f}$$

It is obvious that the constants, u_o, v_o, w_o and ω_i $(i = 1,2,3)$ in Eqs 4a-4d characterize the rigid-body translations and rotations of the solid. A_1 and

A_2 represent the bending of the laminate in the x-z and y-z planes. A_3 characterizes the uniform axial extension of the composite laminate, and A_4, the relative angle of rotation about the z-axis.

2.2 Governing Partial Differential Equations

Introducing Lekhnitskii's stress potentials, $F(x,y)$ and $\Psi(x,y)$ [27], such that

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y},$$
$$\tau_{xz} = \frac{\partial \Psi}{\partial y}, \qquad \tau_{yz} = -\frac{\partial \Psi}{\partial x}, \tag{6}$$

one can show that the equations of equilibrium are satisfied identically. Eliminating U and V from Eqs 5a, 5b and 5e and W from Eqs 5c and 5d by differentiation, we obtain the following system of governing partial differential equations for the problem:

$$\begin{cases} L_3F + L_2\Psi = -2A_4 + A_1S_{34} - A_2S_{35}, & \text{(7a)} \\ L_4F + L_3\Psi = 0, & \text{(7b)} \end{cases}$$

where L_2, L_3 and L_4 are linear differential operators defined as

$$L_2 = \tilde{S}_{44}\frac{\partial^2}{\partial x^2} - 2\tilde{S}_{45}\frac{\partial^2}{\partial x \partial y} + \tilde{S}_{55}\frac{\partial^2}{\partial y^2}, \tag{7c}$$

$$L_3 = -\tilde{S}_{24}\frac{\partial^3}{\partial x^3} + (\tilde{S}_{25} + \tilde{S}_{46})\frac{\partial^3}{\partial x^2 \partial y} - (\tilde{S}_{14} + \tilde{S}_{56})\frac{\partial^3}{\partial x \partial y^2} + \tilde{S}_{15}\frac{\partial^3}{\partial y^3}, \tag{7d}$$

$$L_4 = \tilde{S}_{22}\frac{\partial^4}{\partial x^4} - 2\tilde{S}_{26}\frac{\partial^4}{\partial x^3 \partial y} + (2\tilde{S}_{12} + \tilde{S}_{66})\frac{\partial^4}{\partial x^2 \partial y^2} - 2\tilde{S}_{16}\frac{\partial^4}{\partial x \partial y^3} + \tilde{S}_{11}\frac{\partial^4}{\partial y^4}. \tag{7e}$$

2.3 Boundary and End Conditions

Assuming that the edges of a composite laminate, ∂B_F, are traction free and the interface of the mth and $(m+1)$th plies is a straight line meeting the traction-free edge at a right angle (Fig. 1), one can obtain the following boundary conditions along ∂B_F:

$$\sigma_x = \tau_{xy} = \tau_{xz} = 0. \tag{8}$$

The conditions at the ends of the composite laminate may have the form from the statically equivalent loads as

$$\iint_B \tau_{xz}\, dx\, dy = 0, \quad \iint_B \tau_{yz}\, dx\, dy = 0, \quad \iint_B \sigma_z\, dx\, dy = P_z,$$

$$\iint_B \sigma_z y\, dx\, dy = M_x, \quad \iint_B \sigma_z x\, dx\, dy = M_y, \quad \iint_B (\tau_{yz} x - \tau_{xz} y) dx\, dy = M_t, \tag{9}$$

where the integrals are taken over the entire area B of the cross section, and P_z, M_x, M_y and M_t are the applied force, bending moments and twisting moment acting on the ends, respectively.

2.4 Interface Continuity Conditions

Consider a portion of the laminate cross section composed of the mth and $(m+1)$th fiber-reinforced laminae, as shown in Fig. 1. Assuming that the plies are perfectly bonded along the interface ∂B_I, one can immediately establish the continuity conditions of the stresses and displacements along the interface as the following:

$$\sigma_x^{(m)} n_x^{(m)} + \tau_{xy}^{(m)} n_y^{(m)} = -\sigma_x^{(m+1)} n_x^{(m+1)} - \tau_{xy}^{(m+1)} n_y^{(m+1)}, \tag{10a}$$

$$\tau_{xy}^{(m)} n_x^{(m)} + \sigma_y^{(m)} n_y^{(m)} = -\tau_{xy}^{(m+1)} n_x^{(m+1)} - \sigma_y^{(m+1)} n_y^{(m+1)}, \tag{10b}$$

$$\tau_{xz}^{(m)}\, n_x^{(m)} + \tau_{yz}^{(m)}\, n_y^{(m)} = -\tau_{xz}^{(m+1)}\, n_x^{(m+1)} - \tau_{yz}^{(m+1)}\, n_y^{(m+1)}, \tag{10c}$$

and

$$u^{(m)} = u^{(m+1)}, \qquad v^{(m)} = v^{(m+1)}, \qquad w^{(m)} = w^{(m+1)}, \tag{10d-f}$$

where the superscripts denote the mth and $(m+1)$th plies in a composite laminate, and n_x and n_y are components of unit outward normal to the interface.

3. HOMOGENEOUS SOLUTION AND FREE-EDGE STRESS SINGULARITY

The governing equations, 7a and 7b, are coupled, linear partial differential equations with constant coefficients related to the anisotropic elastic constants of each individual lamina. With the aid of aforementioned near-field boundary conditions and interface continuity conditions, the homogeneous solution for the governing P.D.E.'s can be determined easily. The homogeneous boundary conditions and interface continuity conditions also provide the information for determining the important strength or order of the free-edge stress singularity in a composite laminate, which is the major concern in this paper.

According to Lekhnitskii [27], the homogeneous solution for the governing partial differential equations has the general form as

$$F(x,y) = \sum_{k=1}^{6} F_k(x + \mu_k y), \qquad \Psi(x,y) = \sum_{k=1}^{6} \eta_k F_k'(x + \mu_k y), \qquad (11a\text{-}b)$$

where the prime (') in Eq 11b denotes differentiation of the function $F_k(x + \mu_k y)$ with respect to its argument, and the coefficients μ_k are the roots of the following algebraic characteristic equation

$$\ell_4(\mu)\ell_2(\mu) - \ell_3^2(\mu) = 0, \qquad (12a)$$

and

$$\eta_k = - \ell_3(\mu_k)/\ell_2(\mu_k) = - \ell_4(\mu_k)/\ell_3(\mu_k), \qquad (12b)$$

where

$$\ell_2(\mu) = \tilde{S}_{55}\mu^2 - 2\tilde{S}_{45}\mu + \tilde{S}_{44}, \qquad (12c)$$

$$\ell_3(\mu) = \tilde{S}_{15}\mu^3 - (\tilde{S}_{14} + \tilde{S}_{56})\mu^2 + (\tilde{S}_{25} + \tilde{S}_{46})\mu - \tilde{S}_{24}, \qquad (12d)$$

$$\ell_4(\mu) = \tilde{S}_{11}\mu^4 - 2\tilde{S}_{16}\mu^3 + (2\tilde{S}_{12} + \tilde{S}_{66})\mu^2 - 2\tilde{S}_{26}\mu + \tilde{S}_{22}. \qquad (12e)$$

It can be shown that Eq 12a cannot have a real root (thus, μ_k have to appear as complex conjugates), and F_k are analytic functions of the complex variables $Z_k = x + \mu_k y = r(e^{i\theta} + \lambda_k e^{-i\theta})/(1 + \lambda_k)$ with $\lambda_k = (1 + i\mu_k)/(1 - i\mu_k)$ and r and θ being components of polar coordinates. Substituting the expressions of $F(x,y)$ and $\Psi(x,y)$, Eqs 11a and 11b, into Eqs 6a-6e, the homogeneous solutions for stresses σ_i may be expressed in terms of $F_k(Z_k)$ as

$$\sigma_x^{(h)} = \sum_{k=1}^{6} \mu_k^2 F_k''(Z_k), \tag{13a}$$

$$\sigma_y^{(h)} = \sum_{k=1}^{6} F_k''(Z_k), \tag{13b}$$

$$\tau_{yz}^{(h)} = -\sum_{k=1}^{6} \eta_k F_k''(Z_k), \tag{13c}$$

$$\tau_{xz}^{(h)} = \sum_{k=1}^{6} \mu_k \eta_k F_k''(Z_k), \tag{13d}$$

$$\tau_{xy}^{(h)} = -\sum_{k=1}^{6} \mu_k F_k''(Z_k). \tag{13e}$$

The expressions for displacement components may be obtained directly from Eqs 5, 7 and 13 with omission of the terms which are to be included in the particular solution,

$$u^{(h)} = \sum_{k=1}^{6} p_k F_k'(Z_k), \tag{14a}$$

$$v^{(h)} = \sum_{k=1}^{6} q_k F_k'(Z_k), \tag{14b}$$

$$w^{(h)} = \sum_{k=1}^{6} t_k F_k'(Z_k), \tag{14c}$$

where

$$p_k = \tilde{S}_{11}\mu_k^2 + \tilde{S}_{12} - \tilde{S}_{14}\eta_k + \tilde{S}_{15}\eta_k\mu_k - \tilde{S}_{16}\mu_k, \tag{14d}$$

$$q_k = \tilde{S}_{12}\mu_k + \tilde{S}_{22}/\mu_k - \tilde{S}_{24}\eta_k/\mu_k + \tilde{S}_{25}\eta_k - \tilde{S}_{26}, \tag{14e}$$

$$t_k = \tilde{S}_{14}\mu_k + \tilde{S}_{24}/\mu_k - \tilde{S}_{44}\eta_k/\mu_k + \tilde{S}_{45}\eta_k - \tilde{S}_{46}. \tag{14f}$$

We now choose the form of $F_k(Z_k)$ as

$$F_k(Z_k) = C_k Z_k^{\delta+2}/[(\delta + 1)(\delta + 2)], \tag{15}$$

where C_k and δ are arbitrary complex constants to be determined later. Substituting Eq 15 into Eqs 13 and 14 gives

$$\sigma_x^{(h)} = \sum_{k=1}^{3} [C_k\mu_k^2 Z_k^\delta + C_{k+3}\,\bar{\mu}_k^2\,\bar{Z}_k^\delta], \tag{16a}$$

$$\sigma_y^{(h)} = \sum_{k=1}^{3} [C_k Z_k^\delta + C_{k+3}\,\bar{Z}_k^\delta], \tag{16b}$$

$$\tau_{yz}^{(h)} = -\sum_{k=1}^{3} [C_k\eta_k Z_k^\delta + C_{k+3}\,\bar{\eta}_k\,\bar{Z}_k^\delta], \tag{16c}$$

$$\tau_{xz}^{(h)} = \sum_{k=1}^{3} [C_k\eta_k\mu_k Z_k^\delta + C_{k+3}\,\bar{\eta}_k\,\bar{\mu}_k\,\bar{Z}_k^\delta], \tag{16d}$$

$$\tau_{xy}^{(h)} = -\sum_{k=1}^{3} [C_k\mu_k Z_k^\delta + C_{k+3}\,\bar{\mu}_k\,\bar{Z}_k^\delta], \tag{16e}$$

and

$$u^{(h)} = \sum_{k=1}^{3} [C_k p_k Z_k^{\delta+1} + C_{k+3}\,\bar{p}_k\,\bar{Z}_k^{\delta+1}]/(\delta + 1), \tag{17a}$$

$$v^{(h)} = \sum_{k=1}^{3} [C_k q_k Z_k^{\delta+1} + C_{k+3}\,\bar{q}_k\,\bar{Z}_k^{\delta+1}]/(\delta + 1), \tag{17b}$$

$$w^{(h)} = \sum_{k=1}^{3} [C_k t_k Z_k^{\delta+1} + C_{k+3}\,\bar{t}_k\,\bar{Z}_k^{\delta+1}]/(\delta + 1), \tag{17c}$$

where the overbar denotes the complex conjugate of the associate variable. For convenience, we drop the superscript h associated with the above homogeneous solutions for stresses and displacements in this paper.

The homogeneous solutions are required to satisfy the homogeneous boundary conditions and interface continuity conditions. This leads to a standard eigenvalue problem for determining the values of δ. It is noted that δ generally appears as a set of complex conjugates, which enable to make Eqs 16 and 17 real functions by superposition. Furthermore, the value of δ is required to satisfy the condition

$$\mathrm{Re}[\delta] > -1 \tag{18}$$

to ensure the finiteness of displacement components at the origin, where Re represents the real part of δ.

To expedite further developments, we transform the stress and displacement components from Cartesian coordinates to polar coordinates. Thus, we have

$$\sigma_{\theta\theta} = \sum_{k=1}^{3} (C_k H_{1k} Z_k^{\delta} + C_{k+3} \overline{H}_{1k} \overline{Z}_k^{\delta}), \tag{19a}$$

$$\tau_{\theta z} = \sum_{k=1}^{3} (C_k H_{2k} Z_k^{\delta} + C_{k+3} \overline{H}_{2k} \overline{Z}_k^{\delta}), \tag{19b}$$

$$\tau_{\theta r} = \sum_{k=1}^{3} (C_k H_{3k} Z_k^{\delta} + C_{k+3} \overline{H}_{3k} \overline{Z}_k^{\delta}), \tag{19c}$$

$$\sigma_{rr} = \sum_{k=1}^{3} (C_k H_{4k} Z_k^{\delta} + C_{k+3} \overline{H}_{4k} \overline{Z}_k^{\delta}), \tag{19d}$$

$$\tau_{rz} = \sum_{k=1}^{3} (C_k H_{5k} Z_k^{\delta} + C_{k+3} \overline{H}_{5k} \overline{Z}_k^{\delta}), \tag{19e}$$

and

$$u_r = \sum_{k=1}^{3} [C_k H_{6k} Z_k^{\delta+1}/(\delta + 1) + C_{k+3}\, \overline{H}_{6k}\, \overline{Z}_k^{\delta+1}/(\delta + 1)], \tag{20a}$$

$$u_\theta = \sum_{k=1}^{3} [C_k H_{7k} Z_k^{\delta+1}/(\delta + 1) + C_{k+3}\, \overline{H}_{7k}\, \overline{Z}_k^{\delta+1}/(\delta + 1)], \tag{20b}$$

$$u_z = \sum_{k=1}^{3} [C_k H_{8k} Z_k^{\delta+1}/(\delta + 1) + C_{k+3}\, \overline{H}_{8k}\, \overline{Z}_k^{\delta+1}/(\delta + 1)], \tag{20c}$$

where Z_k are defined in the polar coordinates and H_{jk} ($j = 1,2,\ldots 8$) are functions of η_k, μ_k, p_k, q_k, t_k and θ given in Appendix 1.

The traction-free boundary conditions, Eqs 8a-8c, along the free-edges of the mth and $(m+1)$th plies in polar coordinates read

$$\sigma_{\theta\theta}^{(m)} = \tau_{\theta z}^{(m)} = \tau_{r\theta}^{(m)} = 0 \qquad \text{on} \quad \theta = \frac{\pi}{2}, \tag{21a}$$

$$\sigma_{\theta\theta}^{(m+1)} = \tau_{\theta z}^{(m+1)} = \tau_{r\theta}^{(m+1)} = 0 \qquad \text{on} \quad \theta = -\frac{\pi}{2}. \tag{21b}$$

The continuity conditions, Eqs 10a-10f, along the ply interface give

$$\{\sigma_{\theta\theta}^{(m)}, \tau_{\theta z}^{(m)}, \tau_{r\theta}^{(m)}, u_r^{(m)}, u_\theta^{(m)}, u_z^{(m)}\}$$
$$= \{\sigma_{\theta\theta}^{(m+1)}, \tau_{\theta z}^{(m+1)}, \tau_{r\theta}^{(m+1)}, u_r^{(m+1)}, u_\theta^{(m+1)}, u_z^{(m+1)}\}$$
$$\text{on} \quad \theta = 0. \tag{21c}$$

More explicitly, the homogeneous boundary conditions, Eqs 21a and 21b and the continuity conditions provides

$$\sum_{k=1}^{3} \{C_k^{(m)}\, H_{jk}^{(m)}(\tfrac{\pi}{2})\, [\Omega_k^{(m)}(\tfrac{\pi}{2})]^{\delta} + C_{k+3}^{(m)}\, \overline{H_{jk}^{(m)}(\tfrac{\pi}{2})}\, \overline{[\Omega_k^{(m)}(\tfrac{\pi}{2})]}^{\delta}\} = 0, \tag{22a}$$

$$\sum_{k=1}^{3}\{C_k^{(m+1)}H_{jk}^{(m+1)}(\tfrac{-\pi}{2})[\Omega_k^{(m+1)}(\tfrac{-\pi}{2})]^{\delta}+C_{k+3}^{(m+1)}\overline{H_{jk}^{(m+1)}(\tfrac{-\pi}{2})}\ \overline{[\Omega_k^{(m+1)}(\tfrac{-\pi}{2})]^{\delta}}\}=0, \tag{22b}$$

$$\sum_{k=1}^{3}\{[C_k^{(m)}\Gamma_{rk}^{(m)}+C_{k+3}^{(m)}\overline{\Gamma}_{rk}^{(m)}]-[C_k^{(m+1)}\Gamma_{rk}^{(m+1)}+C_{k+3}^{(m+1)}\overline{\Gamma}_{rk}^{(m+1)}]\}=0, \tag{22c}$$

$$(j=1,2,3;\ r=1,2,3,4,5,6),$$

where $H_{jk}(\frac{\pi}{2})$ and $H_{jk}(\frac{-\pi}{2})$ are values of H_{ij} evaluated at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$, respectively; $\Omega_k(\theta)$ are defined as

$$\Omega_k(\theta) = (e^{i\theta} + \lambda_k e^{-i\theta})/(1+\lambda_k), \tag{23}$$

and

$$\Gamma_{1k}=1,\quad \Gamma_{2k}=\eta_k,\quad \Gamma_{3k}=\mu_k,\quad \Gamma_{4k}=p_k,\quad \Gamma_{5k}=q_k,\quad \Gamma_{6k}=t_k. \tag{24}$$

Solving for $C_k^{(m)}$ from Eqs 22c in terms of $C_k^{(m+1)}$, one finds

$$C_k^{(m)} = a_{ks}C_s^{(m+1)} \qquad (k,s=1,2,\ldots 6). \tag{25}$$

Substituting Eq 25 into Eq 22a gives

$$\sum_{s=1}^{6}\left(C_s^{(m+1)}\sum_{k=1}^{3}\{H_{jk}^{(m)}(\tfrac{\pi}{2})\,a_{ks}\,[\Omega_k^{(m)}(\tfrac{\pi}{2})]^{\delta}+\overline{H_{jk}^{(m)}(\tfrac{-\pi}{2})}\,a_{(k+3)s}\,\overline{[\Omega_k^{(m)}(\tfrac{-\pi}{2})]^{\delta}}\}\right)=0. \tag{26}$$

Equations 22b and 26 constitute a system of homogeneous linear algebraic equations in $C_k^{(m+1)}$. The existence of a nontrival solution for $C_k^{(m+1)}$ requires the vanishing of the coefficient determinant

$$|\Delta(\delta)| = 0, \tag{27}$$

where $\Delta(\delta)$ is a six by six matrix involving δ in a transcendental form. Thus, Eq 27 is a transcendental characteristic equation for the standard eigenvalue problem. It has a very complicated structure as can be seen from the coefficients of $C_k^{(m+1)}$ in Eqs 22 and 26, and the detailed expression for $\Delta(\delta)$ is not given here. The investigation of the characteristic equation requires the employment of standard numerical techniques such as the Muller's method [28] with the aid of a digital computer. The eigenvalues δ_n obtained from the numerical solution of Eq 27 give important information concerning the behavior of the edge stresses and displacements. Due to the positive definiteness of strain energy of the elastic body and the condition in Eq. 18, the eigenvalue of δ_n bounded by

$$-1 < \mathrm{Re}[\delta_n] < 0 \tag{28}$$

characterizes the order of the inherent singularity of the boundary-layer or free-edge stresses in a composite laminate. Thus, for small values of r, the asymptotic stresses are proportional to $r^{\mathrm{Re}[\delta_n]}$, provided that δ_n satisfies Eq 28.

4. DEGENERATED CASES — CROSS-PLY COMPOSITE LAMINATES

In the case of a cross-ply composite laminate, i.e., laminae with 0° and 90° fiber orientations only, the in-plane stresses and the out-of-the-plane shear stresses are uncoupled by virtue of the material symmetry in each lamina. We shall concentrate our study on the four stress components of interest here, σ_x, σ_y, τ_{xy}, σ_z. The general expressions for displacements and σ_z may be simplified as

$$u = -\frac{1}{2} A_1 S_{33} z^2 + U(x,y) + \omega_2 z - \omega_3 y + u_o, \tag{29a}$$

$$v = -\frac{1}{2} A_2 S_{33} z^2 + V(x,y) + \omega_3 x - \omega_1 z + v_o, \tag{29b}$$

$$w = (A_1 x + A_2 y + A_3) S_{33} z + \omega_1 y - \omega_2 x + w_o, \tag{29c}$$

$$\sigma_z = A_1 x + A_2 y + A_3 = (S_{31}\sigma_x + S_{32}\sigma_y)/S_{33}. \tag{29d}$$

Following the same procedure shown in the previous section, the governing partial differential equations are uncoupled and may be written as

$$L_4 F(x,y) = 0, \tag{30a}$$

where L_4 is defined as before

$$L_4 = \tilde{S}_{22} \frac{\partial^4}{\partial x^4} + (2\tilde{S}_{12} + \tilde{S}_{66}) \frac{\partial^4}{\partial x^2 \partial y^2} + \tilde{S}_{11} \frac{\partial^4}{\partial y^4}. \tag{30b}$$

The homogeneous solution for Eq 30a may be obtained in a simplier form,

$$F(x,y) = \sum_{k=1}^{4} F_k(x + \mu_k y), \tag{31}$$

where μ_k are the roots of the following algebraic equation:

$$\ell_4(\mu) = \tilde{S}_{11}\mu^4 + (2\tilde{S}_{12} + \tilde{S}_{66})\mu^2 + \tilde{S}_{22} = 0. \tag{32}$$

The homogeneous stress and displacement solutions are then given as

$$\sigma_x = \sum_{k=1}^{4} \mu_k^2 F_k''(x + \mu_k y), \qquad \sigma_y = \sum_{k=1}^{4} F_k''(x + \mu_k y),$$

$$\tau_{xy} = -\sum_{k=1}^{4} \mu_k F_k''(x + \mu_k y), \tag{33}$$

$$U(x,y) = \sum_{k=1}^{4} p_k F_k'(x + \mu_k y), \qquad V(x,y) = \sum_{k=1}^{4} q_k F_k'(x + \mu_k y), \tag{34}$$

where

$$p_k = \tilde{S}_{11}\mu_k^2 + \tilde{S}_{12}, \qquad q_k = \tilde{S}_{21}\mu_k + \tilde{S}_{22}/\mu_k. \tag{35}$$

We shall choose the form of $F_k(x,y)$ as

$$F_k(Z_k) = \sum_{k=1}^{4} C_k Z_k^{\delta+2}/[(\delta+1)(\delta+2)], \tag{36}$$

where C_k and δ are, as before, arbitrary complex constants to be determined later. Imposing the homogeneous traction-free boundary conditions along the free edges and the continuity condition along the ply interface, one can procede with the same procedure outlined in the previous section. Then, the eigenvalues and eigenfunctions can be determined in a manner similar to those in the previous cases.

5. NUMERICAL EXAMPLES

From the structure of the governing p...cial differential equations and the homogeneous solution for the problem, it is clearly seen that the asymptotic stress and strain fields in the vicinity of the edge are governed by the singular terms with the strength of stress singularity δ_n determined from the eigenvalue analysis. Exam ning the structure of Eqs 22b and 26, it is obvious that the eigenvalue solutions, and therefore, the edge stress singularities, are related to laminar constitutive properties and fiber orientations of adjacent plies.

Consider a composite laminate with ply properties typical of those used in earlier studies [5] (values of $G_{Tz}=1.5G_{LT}$ and $\nu_{Lz}=.85\,\nu_{LT}$ have been found to make only a few % difference in δ_n):

$$
\begin{aligned}
&E_L = 20 \times 10^6 \text{ psi}, \qquad E_T = E_z = 2.1 \times 10^6 \text{ psi},\\
&G_{LT} = G_{Lz} = G_{Tz} = 0.85 \times 10^6 \text{ psi},\\
&\nu_{LT} = \nu_{Tz} = \nu_{Lz} = 0.21,
\end{aligned} \tag{37}
$$

where the subscripts, L, T, and z refer to the fiber, transverse, and thickness directions of an individual ply, respectively. The influence of material properties of composite plies on the boundary layer stresses may be related to the roots μ_k of the characteristic equation, Eq 12a. With the lamina properties given above, the roots of the characteristic equation for the graphite-epoxy lamina of different fiber orientations Θ are shown in Table 1. It appears that all the six roots μ_k are purely imaginary by virtue of the material properties of Eqs 37. Furthermore, the μ_k for $+\Theta$ ply is the same as those for $-\Theta$ due to the in-plane rotation of fiber directions.

Based on the material constants, μ_k, p_k, q_k and t_k obtained for the graphite-epoxy, the transcendental characteristic equation, Eq 27, can

Table 1

Roots μ_k of Characteristic Equations for Graphite-Epoxy Composite System with Fiber Orientation Θ

±Θ	$\mu_{1,2}$	$\mu_{3,4}$	$\mu_{5,6}$
15°	±0.88782 i	±1.10301 i	±1.56776 i
30°	±0.86222 i	±1.06956 i	±2.53630 i
45°	±0.80902 i	±1.03503 i	±3.44438 i
60°	±0.73316 i	±1.01323 i	±4.15870 i
75°	±0.66324 i	±1.00294 i	±4.61201 i

*Θ is the angle measured counterclockwise from the positive z-axis to the fiber direction be solved numerically to provide the eigenvalues for the homogeneous solution. For illustrative purposes, the first twelve non-integer eigenvalues associated with the stress solution for the free edge of a 45°/-45° graphite-epoxy are shown in Table 2. Eigenvalues δ_n smaller than -1 are excluded for the reasons given in the previous section. It is seen that there exists one and only one eigenvalue which satisfies the required constraint condition, Eq 28, for this case, i.e., $\delta_1 = -0.02557$. In fact, the eigenvalue δ_1 is the strength or order of the free-edge or boundary-layer stress singularity, which is of major concern in this study. The fact that there is only one δ_n which meets Eq 28 is observed in all cases with various other fiber orientations studied; the only difference is that, for each case, δ_1 possesses a different value. Higher-order eigenvalues occurring as integers (including zero) and as complex conjugates always exist and should be included for determining the complete solution when remote boundary conditions are matched by a numerical method to be discussed in an associated report [29].

For the commonly used (±Θ) angle-ply graphite-epoxy composite, as anticipated, the order of the boundary-layer stress singularity is a function of the fiber orientation Θ. Numerical results of δ_1 for each of the ±Θ fiber orientations are calculated and shown in a graphic form in Fig. 2. It is clearly seen from the Figure that the composite free edge associated with the laminate of an approximately (±51°) ply orientation possesses the strongest boundary-layer stress singularity. As the Θ changes to either directions, the order of the stress singularity δ_1 decreases rapidly. Its value converges to zero for the cases of $\Theta = 0°$ or $90°$, since the two plies become identical with orthotropic elastic properties.

Table 2

First Twelve Non-Integer Eigenvalues* for Free-Edge Stress Solutions in (±45°) Graphite-Epoxy Composite

-2.5575658 E-2
8.8147184 E-1 ± i 2.3400497 E-1
1.5115263 E 0 ± i 7.9281732 E-1
2.3389433 E 0 ± i 1.1158402 E 0
3.0913532 E 0 ± i 1.7360464 E 0
3.9520023 E 0 ± i 2.0287146 E 0
4.7440929 E 0 ± i 2.5683871 E 0
5.6021457 E 0 ± i 2.8588510 E 0
6.3962635 E 0 ± i 3.3652707 E 0
7.2565174 E 0 ± i 3.6575937 E 0
8.0497237 E 0 ± i 4.1479983 E 0
8.9120567 E 0 ± i 4.4407609 E 0

*Integers, 0,1,2,...n, are always eigenvalues obtained from Eq 27

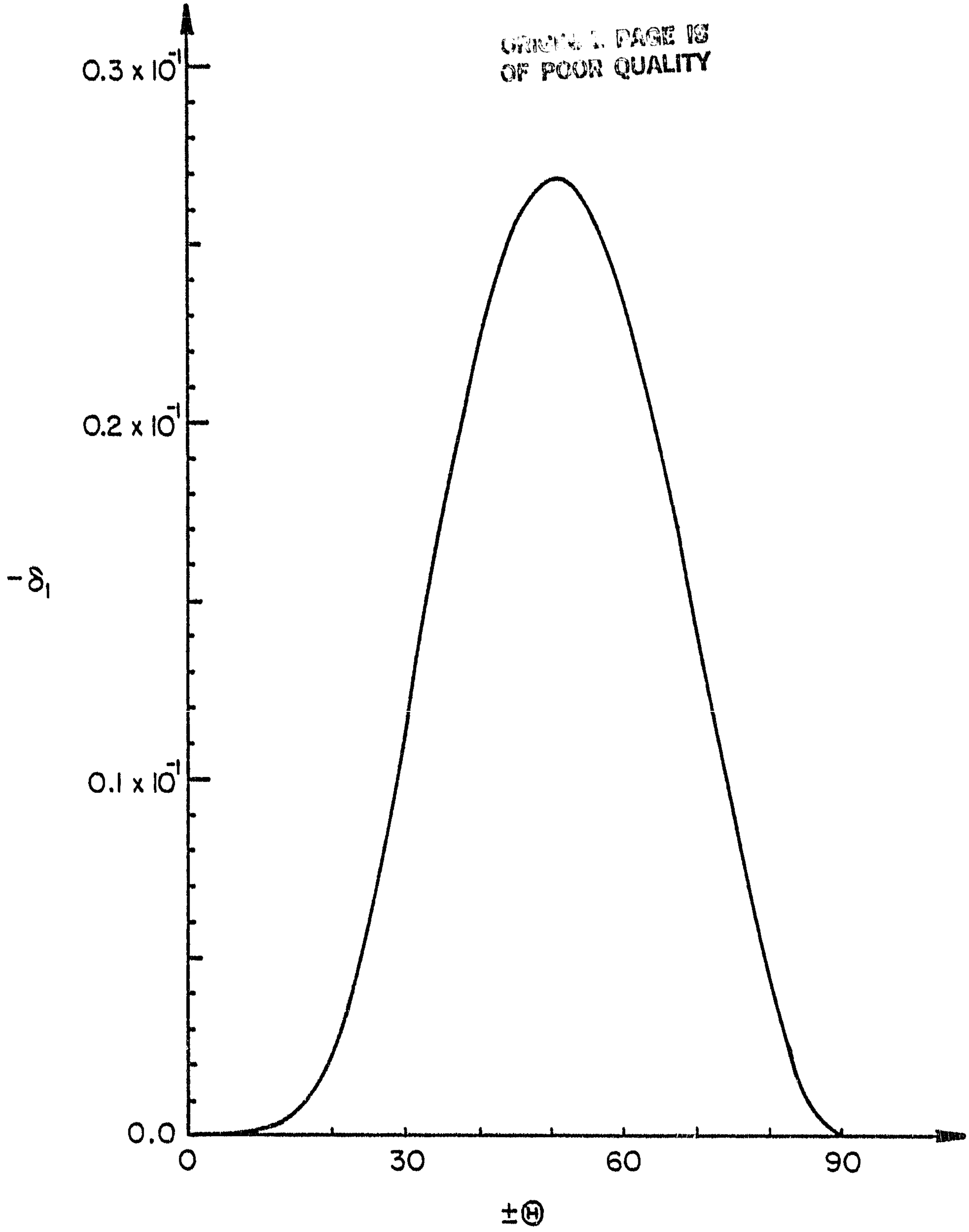

Fig. 2. Strength of Boundary-Layer Stress Singularity in (Θ/-Θ/-Θ/Θ) Graphite-Epoxy Composites

In the case of a free edge associated with two plies of more general fiber orientations, instead of the symmetric $+\Theta/-\Theta$ configuration, solutions for the eigenvalues δ_n are obtained also. To illustrate the nature of the eigenvalues for this case, we examine the free edge associated with $30°/\Theta$ fiber orientations in a graphite-epoxy composite, where Θ varies from $7.5°$ to $82.5°$. The first few non-integer eigenvalues for various Θ's in these cases are given in Table 3. The integers (including zero) are also eigenvalues, but not included in the Table. The case of $30°/30°$ graphite-epoxy is not included either since the two plies are identical. Again, it is observed from the Table that there exists only one δ_n which meets the requirement of Eq 28 and gives the dominant singular stress state at the edge of each of the $30°/\Theta$ composite laminate.

The degenerated case of cross-ply composite laminates which are discussed in the previous section is investigated also. The eigenvalues for the free-edge stresses in a graphite-epoxy composite with $0°/90°$ lamination are given in Table 4. The dominant stress singularity in the present case has an order of magnitude similar to those in $(\pm\Theta)$ angle-ply composites and in more general (Θ_1/Θ_2) laminates. It is noted that the orders of the boundary-layer stress singularity for both angle-ply and cross-ply composites are generally much weaker than those associated with other typical elastostatic singular stress problems such as elastic crack problems. The relatively weak singularity for the boundary-layer stresses introduces some unique features as well as difficulties for the evaluation of the boundary-layer effects in composite laminates, which will be discussed in an associated report [29].

Table 3

First Five Non-Integer Eigenvalues* for Free-Edge Stresses
Associated with (30°/Θ) Graphite-Epoxy Composite

Θ	δ_1	δ_2	δ_3	δ_4	δ_5
7.5°	-2.792991 E-3	9.887419 E-1 ±i 8.435637 E-2	1.930568 E 0 ±i 3.319540 E-1	3.095930 E 0 ±i 8.103959 E-2	3.862698 E 0 ±i 7.204410 E-1
15°	-1.986115 E-3	9.943591 E-1 ±i 6.113543 E-2	1.949288 E 0 ±i 2.882098 E-1	2.726071 E 0	3.441129 E 0 ±i 5.183480 E-1
22.5°	-6.725306 E-4	9.989196 E-1 ±i 2.920556 E-2	1.989893 E 0 ±i 1.661871 E-1	2.649382 E 0	3.342300 E 0 ±i 5.276117 E-1
30°	—	—	—	—	—
37.5°	-9.077774 E-4	1.001306 E 0 ±i 5.180632 E-3	1.984327 E 0	2.182540 E 0 ±i 1.910473 E-1	3.204760 ±i 8.009168 E-1
45°	-3.771122 E-3	9.624687 E-1	1.061916 E 0	1.821684 E 0	2.021985 ±i 4.926242 E-1
52.5°	-8.292252 E-3	9.291682 E-1	1.168571 E 0	1.621302 E 0	1.906136 E 0 ±i 6.212519 E-1
60°	-1.367358 E-2	9.010723 E-1	1.354059 E 0 ±i 1.864060 E-1	1.825695 E 0 ±i 6.942274 E-1	3.518516 E 0 ±i 1.413627 E 0
67.5°	-1.897304 E-2	8.810614 E-1	1.321793 E 0 ±i 3.224246 E-1	1.773577 E 0 ±i 7.358765 E-1	3.488780 E 0 ±i 1.428440 E 0
75°	-2.343868 E-2	8.719040 E-1	1.303673 E 0 ±i 3.820090 E-1	1.743059 E 0 ±i 7.550141 E-1	3.485411 E 0 ±i 1.425960 E 0
82.5°	-2.668523 E-2	8.778236 E-1	1.310388 E 0 ±i 3.866969 E-1	1.725286 E 0 ±i 7.501905 E-1	3.513604 E 0 ±i 1.387356 E 0

*Integers, 0,1,2,....n, are also eigenvalues.

Table 4

First Twelve Non-Integer Eigenvalues for Free-Edge Stresses in Cross-Ply Graphite-Epoxy Composite*

-3.33888 E-2
8.80268 E-1
1.41674 E 0 ± i 3.93303 E-1
1.65345 E 0 ± i 6.85523 E-1
2.83449 E 0 ± i 1.76219 E 0
3.75294 E 0 ± i 1.1853E E 0
4.29235 E 0 ± i 2.66884 E 0
5.70726 E 0 ± i 3.57190 E 0
5.79010 E 0 ± i 1.52461 E 0
7.12293 E 0 ± i 4.48145 E 0
7.81068 E 0 ± i 1.76401 E 0

*Integers, 0,1,2,3..., are also eigenvalues

6. SUMMARY AND CONCLUSIONS

A study of boundary-layer stress singularity in both angle-ply and cross-ply composite laminates has been presented. Formulation of the problem is based on Lekhnitskii's complex-variable stress functions and basic relationships in the anisotropic elasticity theory. An eigenfunction expansion method has been developed to obtain the homogeneous solution for the coupled governing partial differential equations for the problem. Angle-ply and cross-ply composites as well as more general laminates have been studied. The strength of boundary-layer stress singularity for each case has been determined to illustrate the fundamental nature of the edge effects in composite materials.

Based on the information obtained, the following conclusions may be drawn:

1. Boundary-layer or free-edge stress field in a composite laminate is inherently singular in nature due to the geometric and material discontinuities.
2. The order of boundary-layer stress singularity can be determined by solving for the transcendental characteristic equation obtained from the homogeneous solution of the governing partial differential equations.
3. The boundary-layer stress singularity depends only upon material's elastic constants and fiber orientations of adjacent plies in composite laminates.
4. For angle-ply and cross-ply composites as well as more general laminates the order of boundary-layer stress singularity is very weak in general. In a graphite-epoxy system, for example, δ_1 is much smaller than other kind of singular stress problems in elastostatics such as elastic crack problems.

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APPENDIX 1

Expressions for $H_{ij}(\Theta)$ in Equations 19 and 20

$$H_{1k} = (\mu_k \sin\theta + \cos\theta)^2$$

$$H_{2k} = -\eta_k(\mu_k \sin\theta + \cos\theta)$$

$$H_{3k} = -(\mu_k \sin\theta + \cos\theta)(\mu_k \cos\theta - \sin\theta)$$

$$H_{4k} = (\mu_k \cos\theta - \sin\theta)^2$$

$$H_{5k} = \eta_k(\mu_k \cos\theta - \sin\theta)$$

$$H_{6k} = p_k \cos\theta + q_k \sin\theta$$

$$H_{7k} = -p_k \sin\theta + q_k \cos\theta$$

$$H_{8k} = t_k$$