OLD DOMINION UNIVERSITY RESEARCH FOUNDATION

DEPARTMENT OF MATHEMATICAL SCIENCES
SCHOOL OF SCIENCES AND HEALTH PROFESSIONS
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA

(NASA-CR-169155) BURGERS APPROXIMATICN FOR TWO-DIMENSIONAL FLOW PAST AN ELLIPSE Final Report, 26 May 1981 - 25 May 1982 (Old Dominion Univ., Norfolk, Va.) 46 p HC A03/MF A01 CSCL 20D G3/34

N82-29559

Unclas 28498

BURGERS APPROXIMATION FOR TWO-DIMENSIONAL FLOW PAST AN ELLIPSE

By

J. Mark Dorrepaal, Principal Investigator

Final Report
For the period May 26, 1981 to May 25, 1982

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia

Under
Research Grant NAG1-197
Stephen F. Wornom, Technical Monitor
Transonic Aerodynamics Division

July 1982

DEPARTMENT OF MATHEMATICAL SCIENCES
SCHOOL OF SCIENCES AND HEALTH PROFESSIONS
OLD DOMINION UNIVERSITY
NORFOLK, VIRGINIA

BURGERS APPROXIMATION FOR TWO-DIMENSIONAL FLOW PAST AN ELLIPSE

By

J. Mark Dorrepaal, Principal Investigator

Final Report
For the period May 26, 1981 to May 25, 1982

Prepared for the
National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23665

Under
Research Grant NAG1-197
Stephen F. Wornom, Technical Monitor
Transonic Aerodynamics Division

Submitted by the
Old Dominion University Research Foundation
P.O. Box 6369
Norfolk, Virginia 23508-0369

July 1982

TABLE OF CONTENTS

BURGERS APPROXIMATION FOR TWO-DIMENSIONAL FLOW PAST AN ELLIPSE

By

J.M. Dorrepaal*

SUMMARY

This paper examines a linearization of the Navier-Stokes equation due to Burgers in which vorticity is transported by the velocity field corresponding to continuous potential flow. The governing equations are solved exactly for the two dimensional steady flow past an ellipse of arbitrary aspect ratio. The requirement of no slip along the surface of the ellipse results in an infinite algebraic system of linear equations for coefficients appearing in the solution. The system is truncated at a point which gives reliable results for Reynolds numbers R in the range $0 < R < 5$.

Predictions of the Burgers approximation regarding separation, drag and boundary layer behavior are investigated. In particular, Burgers linearization gives drag coefficients which are closer to observed experimental values than those obtained from Oseen's approximation. In the special case of flow past a circular cylinder, Burgers approximation predicts a boundary layer whose thickness is roughly proportional to $R^{-1/2}$. This is in agreement with the nonlinear theory despite the fact that the Burgers calculations are carried out using only moderate values of the Reynolds number. In the matter of separation, it is shown that standing eddies form on the downstream side of a circular cylinder at R = 1.12. Interestingly enough, this is the same value predicted by Skinner (1975) using singular perturbation techniques on the full nonlinear problem (see Van Dyke, 1975).

The linearizations due to Oseen and Burgers both give spatially uniform approximations to the flow past a finite obstacle. The main difference is that vorticity is transported around the obstacle in Burgers flow rather than through it. The results of this paper verify the superiority of Burgers approximation in modelling the flow near the obstacle at low to moderate Reynolds numbers.

*Associate Professor, Department of Mathematical Sciences, Old Dominion University, Norfolk, Virginia 23508.

LIST OF SYMBOLS

a	length scale
A	coefficient of e^{ζ} sin η in asymptotic expansion of $y(\zeta,\eta)$
A_n, B_n	real coefficients
A	the fluid domain in the x-y plane
$B_k^{(n)}(q)$	coefficient in the expansion of a normalized Mathieu function
B	boundary of an arbitrary finite obstacle
c_n	coefficient in asymptotic expansion of $Gek_n\ (z,q)$
C_{nk}	coefficient in infinite linear system of equations
C_D	drag coefficient
$D_k^{(n)}(q)$	power series in q
$D(\psi)$	differential operator dependent upon ψ
$E(\xi), E_n(\xi)$	function of ξ resulting from separation of variables
$f(z)$	an entire function of z
$f_o(x,y)$	function related to vorticity
$F(x,y)$	function related to vorticity
$g(\xi), h(\eta)$	arbitrary functions
$Gek_n\ (\xi,q)$	modified Mathieu function
$G(\zeta,\eta \mid \zeta',\eta')$	Green's function
$H(\eta)$	function of η resulting from separation of variables
$I_n(x)$	modified Bessel funcation of the first kind
k,m,n	positive integers
$\hat{k}$	unit vector perpendicular to the x-y plane
$K_n(x)$	modified Bessel function of the second kind
p	pressure

P	rear stagnation point on the circular cylinder
q	modulus of Mathieu function
$q(x,y), q(\xi,\eta)$	magnitude of potential flow velocity vector
(r,θ)	polar coordinates
R	Reynolds number
R_c	critical Reynolds number
$se_n(\eta, q)$	Mathieu function
$S_{nk}(\theta)$	coefficient in eigenfunction expansion
S	point of separation on circular boundary
$s_m^{(n)}$	coefficient in the expansion of $se_n(\eta, -\frac{1}{4}A^2R^2)$
T	point of minimum pressure on circular boundary
$\vec{v}$	fluid velocity
$\vec{v}_o$	convective velocity
$w(\theta), w(\eta)$	function related to the asymptotic behavior of the vorticity
W_n	vorticity coefficients
(x,y)	Cartesian coordinates
$y_n(\theta, q)$	Mathieu function before normalization
$Y_n(z)$	solution of the modified Mathieu equation
$\alpha(\xi,\eta), \alpha(\zeta,\eta)$ $\alpha(\mu,\eta)$	metric coefficients for respective coordinate systems
β_{nk}	coefficient in eigenvalue expansion
γ	Reynolds number exponent in least squares fit
Γ	a large positive number
δ_{kl}	Kroneckar delta
$\delta(x)$	delta function
ϵ_n	a function which is 1 if n is odd and 2 if n is even
(ζ,η)	modified elliptic coordinates

(λ, η)	elliptic coordinates
λ_o	particular value of λ
$\Lambda(\theta)$	coefficient in least squares fit
(μ, η)	modified elliptic coordinates
(ξ, η)	curvilinear coordinates
ξ_o	particular value of ξ
$\rho(\xi, \eta)$	the product $\alpha(\xi, \eta) \cdot q(\xi, \eta)$
σ, σ_n	separation eigenvalue
$\phi(\xi, \eta), \phi(\zeta, \eta)$	harmonic conjugate of ψ_o
Φ_n	coefficient in asymptotic expansion of stream function
$\chi(r, \theta)$	exponentially small part of stream function expansion
$\psi(x, y)$	stream function
ψ_o	potential flow stream function
$\omega(x,y), \omega(\zeta, \eta)$	magnitude of vorticity vector
$\frac{\partial \psi}{\partial n}$	normal derivative of ψ
$\vec{\nabla}$	gradient operator
∇^2	Laplacian operator

1. INTRODUCTION

Owing to the formidable nature of the Navier-Stokes equation, the history of fluid mechanics research is filled with simplifying approximations to this nonlinear problem. Included among these is a class of approximations which replaces the nonlinear inertial term $(\vec{v} \cdot \nabla)\vec{v}$ by a linear one $(\vec{v}_o \cdot \vec{\nabla})\vec{v}$ where $\vec{v}_o$ is given. Of particular interest are the Stokes and Oseen approximations where $\vec{v}_o$ is constant. These have contributed significantly to the understanding of basic fluid dynamics behavior, especially in the low Reynolds number regime where they are related asymptotically to the Navier-Stokes solution.

A linearization which has received considerably less attention is Burger's approximation (1928) in which $\vec{v}_o$ is taken to be the continuous potential flow around the body. As noted by Dryden et al. (1956), the vorticity equation in this case is identical to the temperature equation used by Boussinesq in his study of the conduction of heat from a hot body placed in an irrotational fluid. In Burgers approximation the convective velocity field $\vec{v}_o$ follows the surface of the body in its immediate neighborhood and approaches the velocity of a uniform stream at a great distance from the body. Burgers flow is asymptotically equivalent to Oseen flow far from the body, but in its immediate neighborhood Burgers flow models the exact flow more accurately. Of course Burgers approximation suffers from the defect that the convective velocity vector $\vec{v}_o$ does not tend to $\vec{0}$ as one approaches the surface of the obstacle. Furthermore at moderate values of the Reynolds number R, separation occurs on the downstream side of a bluff obstacle and the resultant velocity field $\vec{v}$ no longer resembles $\vec{v}_o$. This limits the suitability of Burgers approximation to small values of R. Nevertheless Burgers linearization provides a spatially uniform approximation to the solution of the Navier-Stokes equation and it is the purpose of this paper to investigate the extent to which it improves upon the Oseen approximation. Attention is restricted to two dimensional steady flows past circles, ellipses and flat plates.

2. MATHEMATICAL FORMULATION

The nondimensional Navier-Stokes equation has the form

$$R(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p + \nabla^2 \vec{v} \qquad (2.1)$$

where R, $\vec{v}$, p are Reynolds number, fluid velocity and pressure, respectively. The velocity must also satisfy the continuity equation

$$\vec{\nabla} \cdot \vec{v} = 0 \qquad (2.2)$$

which is guaranteed by the introduction of a stream function $\psi(x,y)$ defined by

$$\vec{v} = \text{curl } \{\psi k\} \tag{2.3}$$

The problem for the stream function corresponding to (2.1) is given by

$$[\nabla^2 + RD(\psi)]\omega = 0 \tag{2.4}$$

$$\nabla^2\psi = -\omega \tag{2.5}$$

where $\omega(x,y)$ is the magnitude of the vorticity vector and

$$D(\psi) \equiv \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} .$$

Consider a finite obstacle whose boundary is denoted by B. Let ψ_o be the stream function for the continuous potential flow around B which approaches a uniform stream at infinity. The boundary value problem for $\psi_o(x,y)$ is given by

$$\nabla^2 \psi_0 = 0 \tag{2.6}$$

$$\psi_0\Big|_B = 0, \qquad \psi_0 \sim y \text{ as } x^2 + y^2 \to \infty \tag{2.7}$$

Burgers linearization is defined in the following way:

$$[\nabla^2 + RD(\psi_0)]\omega = 0 \tag{2.8}$$

$$\nabla^2\psi = -\omega \tag{2.9}$$

$$\psi\Big|_B = \frac{\partial\psi}{\partial n}\Big|_B = 0, \qquad \psi \sim y \text{ as } x^2 + y^2 \to \infty \tag{2.10}$$

Equation (2.8) is called the vorticity equation and (2.9) is Poisson's equation. The Oseen linearization is obtained by substituting $\psi_o(x,y) = y$ in (2.8).

3. SOLUTION OF THE VORTICITY EQUATION

The exponential decay of the vorticity in Oseen flow suggests that we attempt a solution of (2.8) of the form

$$\omega(x,y) = F(x,y) \exp\,[f_o(x,y)] \tag{3.1}$$

The substitution of (3.1) into (2.8) gives a second order partial differential equation for $F(x,y)$ with non-constant coefficients involving derivatives of ψ_o and f_o. If we set the coefficients of $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ to zero, we obtain the following:

$$\frac{\partial f_o}{\partial x} = \frac{1}{2} R \frac{\partial \psi_0}{\partial y}, \qquad \frac{\partial f_o}{\partial y} = -\frac{1}{2} R \frac{\partial \psi_o}{\partial x} \tag{3.2}$$

These are the Cauchy-Riemann equations. They suggest that we choose $f_o(x,y)$ to be $\frac{1}{2} R$ times the velocity potential of the irrotational flow past B. The resulting equation for $F(x,y)$ is given by

$$\nabla^2 F - \frac{1}{4} R^2\; q(x,y)^2\; F = 0 \tag{3.3}$$

where $q(x,y) = \left[\left(\frac{\partial \psi_o}{\partial x}\right)^2 + \left(\frac{\partial \psi_o}{\partial y}\right)^2\right]^{1/2}$ is the magnitude of the potential flow velocity. In Oseen flow we have $q(x,y) \equiv 1$.

Consider a curvilinear coordinate system (ξ,η) defined by

$$x + iy = f(\xi + i\eta) \tag{3.4}$$

where f is an entire function. The metric coefficient $\alpha(\xi,\eta)$ for this transformation is defined by

$$\alpha^2 = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \tag{3.5}$$

The vorticity in (ξ, η) coordinates is expressible in the form

$$\omega(\xi, \eta) = F(\xi, \eta) \exp\left[\frac{1}{2} R\phi(\xi, \eta)\right] \tag{3.6}$$

where ϕ is the harmonic conjugate of ψ_o. From (3.3) the equation for $F(\xi, \eta)$ is given by

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} - \frac{1}{4} R^2 \rho(\xi, \eta)^2 F = 0 \tag{3.7}$$

where $\rho(\xi, \eta) = \left[\left(\frac{\partial \psi_o}{\partial \xi}\right)^2 + \left(\frac{\partial \psi_o}{\partial \eta}\right)^2\right]^{1/2} = \alpha(\xi, \eta)\, q(\xi, \eta)$. Equation (3.7) is separable provided ρ^2 is expressible in the form

$$\rho(\xi, \eta)^2 = g(\xi) + h(\eta) \tag{3.8}$$

where g and h are arbitrary functions. Under this assumption we have

$$F(\xi, \eta) = E(\xi)\, H(\eta) \tag{3.9}$$

where

$$E'' - \left[\sigma + \frac{1}{4} R^2 g(\xi)\right] E = 0 \tag{3.10}$$

$$H'' + \left[\sigma - \frac{1}{4} R^2 h(\eta)\right] H = 0 \tag{3.11}$$

and σ is the separation eigenvalue.

There are three geometries in which the vorticity equation can be solved exactly. Consider first the case of a circular cylinder of unit radius. The appropriate function in (3.4) is

$$f(z) = e^z \tag{3.12}$$

which implies that $x = e^{\xi} \cos \eta$, $y = e^{\xi} \sin \eta$. The connection with polar coordinates is $r = e^{\xi}$, $\theta = \eta$ and the unit circle is given by $\xi = 0$. The potential flow boundary value problem is given by:

$$\nabla^2 \psi_o = e^{-2\xi} \left\{ \frac{\partial^2 \psi_o}{\partial \xi^2} + \frac{\partial^2 \psi_o}{\partial \eta^2} \right\} = 0 \tag{3.13}$$

$\psi_o(0,\eta) = 0$, $\psi_o(\xi,\eta) \sim e^{\xi} \sin \eta$ as $\xi \to \infty$.

The solution is

$$\psi_o(\xi,\eta) = 2 \sinh \xi \sin \eta$$
$$\phi(\xi,\eta) = 2 \cosh \xi \cos \eta \tag{3.14}$$

from which we obtain

$$\rho(\xi,\eta)^2 = 2 \cosh 2\xi - 2 \cos 2\eta \tag{3.15}$$

Thus the separability condition (3.8) is satisfied and the corresponding separated equations are, from (3.10) and (3.11),

$$E'' - \left[\sigma + \frac{1}{2} R^2 \cosh 2\xi\right] E = 0 \tag{3.16}$$

$$H'' + \left[\sigma + \frac{1}{2} R^2 \cos 2\eta\right] H = 0 \tag{3.17}$$

These are respectively the modified and the conventional Mathieu equations. A brief discussion of their solutions is included in Appendix I.

In a streaming two dimensional flow past a symmetric body, the vorticity must be odd in η and periodic with period 2π. It must decay exponentially as $\xi \to \infty$, except possibly in the wake $\eta = 0$. The periodicity condition determines the eigenvalues σ_n in (3.17). The corresponding odd eigenfunctions are denoted by $se_n(\eta, -\frac{1}{4} R^2)$. In (3.16) the eigenfunctions which decay exponentially are denoted by $\mathrm{Gek}_n\left(\xi, -\frac{1}{4}R^2\right)$. Thus from (3.6), (3.9) and (3.14) the vorticity function has the form

$$\omega(\xi,\eta) = -\exp\left[R \cosh\xi \cos\eta\right] \sum_{n=1}^{\infty} W_n \,\mathrm{Gek}_n\left(\xi, -\frac{1}{4}R^2\right) \cdot \mathrm{se}_n\left(\eta, -\frac{1}{4}R^2\right) \qquad (3.18)$$

where the minus sign is included for convenience and the coefficients W_n are constants to be determined. The asymptotic behavior of the vorticity is, from Appendix I,

$$\omega(\xi,\eta) \sim w(\eta) \exp\left[-\frac{1}{2}\xi - \frac{1}{2}R e^{\xi} (1 - \cos\eta)\right]$$
$$= w(\theta)\, r^{-1/2} \exp\left[-\frac{1}{2}Rr (1 - \cos\theta)\right] \qquad (3.19)$$

where $w(0) = 0$, $w'(0) \neq 0$ and $w(\theta)$ has period 2π. For purposes of comparison we have solved the vorticity equation for Oseen flow past a circle in Appendix II and it is observed that the Oseen vorticity has the same asymptotic behavior.

Consider next the flow past an ellipse where the major axis of the ellipse is parallel to the flow at infinity. The appropriate coordinate system (ξ,η) is defined by

$$x + iy = a\cosh(\xi + i\eta) = a\cosh\xi\cos\eta + i\, a\sinh\xi\sin\eta \qquad (3.20)$$

The curve $\xi = \xi_o$ is an ellipse with major axis of length $2a\cosh\xi_o$ and minor axis $2a\sinh\xi_o$. In order that the unit of length be the semi-major axis, we choose

$$a = \operatorname{sech}\xi_o \qquad (3.21)$$

It is convenient to make the transformation

$$\zeta = \xi - \xi_o \qquad (3.22)$$

Thus from (3.20), the coordinate system (ζ,η) is defined by

$$x + iy = a \cosh (\zeta + \xi_o + i\eta) \tag{3.23}$$

and the ellipse is given by $\zeta = 0$. The metric coefficient for this transformation is $\alpha(\zeta,\eta)$ where

$$\alpha^2 = \left(\frac{\partial x}{\partial \zeta}\right)^2 + \left(\frac{\partial y}{\partial \zeta}\right)^2 = \frac{1}{2} a^2 \left[\cosh 2(\zeta + \xi_o) - \cos 2\eta\right] \tag{3.24}$$

The potential flow stream function $\psi_o(\zeta,\eta)$ satisfies the boundary value problem

$$\nabla^2 \psi_o = \alpha^{-2}\left\{\frac{\partial^2\psi_o}{\partial\zeta^2} + \frac{\partial^2\psi_o}{\partial\eta^2}\right\} = 0 \tag{3.25}$$

$$\psi_o(0,\eta) = 0, \quad \psi_o(\zeta,\eta) \sim \frac{1}{2} a e^{\xi_o + \zeta} \sin \eta = A e^{\zeta} \sin \eta \text{ as } \zeta \to \infty$$

where $A = \frac{1}{2} a e^{\xi_o}$. The solution is

$$\psi_o(\zeta,\eta) = 2A \sinh \zeta \sin \eta$$

$$\phi(\zeta,\eta) = 2A \cosh \zeta \cos \eta \tag{3.26}$$

Because of the similarity between (3.14) and (3.26), the solution of the vorticity equation in the elliptical case follows the method already outlined. Thus the vorticity function has the form

$$\omega(\zeta,\eta) = - \exp \left[AR \cosh \zeta \cos \eta\right] \sum_{n=1}^{\infty} W_n \, \mathrm{Gek}_n(\zeta, -\frac{1}{4} A^2R^2)$$

$$\cdot \, \mathrm{se}_n\left(\eta, -\frac{1}{4} A^2R^2\right) \tag{3.27}$$

The Reynolds number R is based on semi-major axis as length unit. The circular cylinder result in (3.18) is recovered by letting $\xi_o \to \infty$ and observing that

$$A = \frac{1}{2} e^{\xi_o} \operatorname{sech} \xi_o + 1 \tag{3.28}$$

If the ellipse is oriented so that its major axis is perpendicular to the flow at infinity, we use the coordinate system (λ, η) defined by

$$x + iy = a \sinh (\lambda + i\eta) \tag{3.29}$$

The ellipse is given by $\lambda = \lambda_o$. As before we define a modified coordinate system (μ, η) by letting $\mu = \lambda - \lambda_o$. The metric coefficient is $\alpha(\mu, \eta)$ where

$$\alpha^2 = \frac{1}{2} a^2 [\cosh 2(\mu + \lambda_o) + \cos 2\eta] \tag{3.30}$$

The potential flow stream function $\psi_o(\mu, \eta)$ satisfies a boundary value problem identical to that given in (3.25) and so the solution of the vorticity equation proceeds exactly as before. The vorticity for this case is obtained from (3.27) by making the following adjustments:

$$\zeta \to \mu$$

$$A = \frac{1}{2} e^{\lambda_o} \operatorname{sech} \lambda_o \tag{3.31}$$

4. SOLUTION OF POISSON'S EQUATION

Poisson's equation (2.9) can be solved using a Green's function approach. Equation (3.27) gives the form of the vorticity function for the three geometries considered in section 3 and so the problem reduces to the following:

$$\nabla^2\psi \equiv [\alpha(\zeta, \eta)]^{-2} \left\{ \frac{\partial^2\psi}{\partial\zeta^2} + \frac{\partial^2\psi}{\partial\eta^2} \right\} = - \omega(\zeta, \eta) \tag{4.1}$$

$$\psi(0, \eta) = 0, \quad \psi(\zeta, \eta) \sim A\, e^{\zeta} \sin \eta \text{ as } \zeta \to \infty.$$

The appropriate Green's function is defined by

$$\nabla^2 G(\zeta,\eta|\zeta',\eta') = -[\alpha(\zeta,\eta)]^{-2}\,\delta(\zeta-\zeta')\,\delta(\eta-\eta')$$
$$G(0,\eta|\zeta',\eta') = 0,\quad G(\infty,\eta|\zeta'\eta') = O(1) \tag{4.2}$$

The solution of (4.2) is given by

$$G(\zeta,\eta|\zeta',\eta') = -\frac{1}{4\pi}\,\ell n\left[\frac{\cosh(\zeta-\zeta')-\cos(\eta-\eta')}{\cosh(\zeta+\zeta')-\cos(\eta-\eta')}\right] \tag{4.3}$$

Proceeding in the usual way, we have

$$\psi(\zeta',\eta') = \iint_A [G\nabla^2\psi - \psi\nabla^2 G]\,\alpha^2\, d\zeta\, d\eta + \iint_A \omega G\alpha^2\, d\zeta\, d\eta \tag{4.4}$$

where A is the fluid region between the obstacle $\zeta = 0$ and the curve $\zeta = \Gamma$ with Γ being large.

Green's Identity transforms the first integral in (4.4) into contour integrals around the boundary curves. The integral around $\zeta = 0$ vanishes because of the boundary conditions on G and ψ. Thus we have

$$\iint_A [G\nabla^2\psi - \psi\nabla^2 G]\,\alpha^2\, d\zeta\, d\eta = \int_{-\pi}^{\pi}\left(G\frac{\partial\psi}{\partial\zeta} - \psi\frac{\partial G}{\partial\zeta}\right)_{\zeta=\Gamma} d\eta \tag{4.5}$$

Asymptotic expressions for G,ψ and their derivatives are given below:

$$\psi \sim A\, e^{\zeta}\sin\eta + O(1) \tag{4.6}$$

$$\frac{\partial\psi}{\partial\zeta} \sim A\, e^{\zeta}\sin\eta + O(e^{-\zeta}) \tag{4.7}$$

$$G \sim \frac{1}{2\pi}\zeta' + \frac{1}{\pi}\sinh\zeta'\, e^{-\zeta}\cos(\eta-\eta') + O(e^{-2\zeta}) \tag{4.8}$$

$$\frac{\partial G}{\partial\zeta} \sim -\frac{1}{\pi}\sinh\zeta'\, e^{-\zeta}\cos(\eta-\eta') + O(e^{-2\zeta}) \tag{4.9}$$

Substituting these into (4.5), we obtain the following result:

$$\lim_{\Gamma\to\infty} \int_{-\pi}^{\pi} G\left(\frac{\partial\psi}{\partial\zeta} - \psi\frac{\partial G}{\partial\zeta}\right)_{\zeta = \Gamma} d\eta = 2A \sinh\zeta' \sin\eta' = \psi_o(\zeta',\eta') \quad (4.10)$$

Thus in (4.4) we have

$$\psi(\zeta',\eta') = \psi_o(\zeta',\eta') + \int_{-\pi}^{\pi}\int_{o}^{\infty} \alpha^2(\zeta,\eta)\, \omega(\zeta,\eta)\, G(\zeta,\eta|\zeta',\eta')d\zeta d\eta \quad (4.11)$$

5. DETERMINATION OF VORTICITY COEFFICIENTS

The only boundary condition from (2.10) which remains to be satisfied is the no-slip condition. Its invocation yields unique values for the vorticity coefficients $\{W_n\}$. Thus in (4.11) we require

$$\frac{\partial\psi}{\partial\zeta'}(0,\eta') = 0 \quad (5.1)$$

which implies that

$$0 = 2A \sin\eta' + \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{o}^{\infty} \alpha^2\omega \frac{\sinh\zeta}{\cosh\zeta - \cos(\eta - \eta')} d\zeta d\eta \quad (5.2)$$

From Gradshteyn and Ryzhik (1965), we have

$$\frac{\sinh\zeta}{\cosh\zeta - \cos(\eta - \eta')} = 1 + 2\sum_{k=1}^{\infty} e^{-k\zeta}\cos k(\eta - \eta') \quad (5.3)$$

But $\alpha^2(\zeta,\eta)\,\omega(\zeta,\eta)$ is odd in η and so the even part of (5.3) makes no contribution to (5.2). After substituting (5.3) into (5.2) and equating coefficients of $\sin k\eta'$ to zero ($k = 1,2,3, \ldots$), we have

$$\int_{-\pi}^{\pi}\int_{o}^{\infty} \alpha^2\, \omega\, e^{-k\zeta} \sin k\eta\, d\zeta d\eta = -2A\pi\delta_{k1} \qquad k = 1,2,3,\ldots \quad (5.4)$$

The substitution of (3.27) into (5.4) yields the following infinite linear system for the unknowns $\{W_n\}$

$$\sum_{n=1}^{\infty} W_n C_{nk} = A\pi\delta_{k1} \qquad k = 1,2,3,\ldots \tag{5.5}$$

where

$$C_{nk} = \int_0^{\pi}\int_0^{\infty} \exp\,[-k\zeta + AR\cosh\zeta\,\cos\eta]\,\alpha(\zeta,\eta)^2 \cdot \mathrm{Gek}_n(\zeta, -\frac{1}{4}A^2R^2)\,\mathrm{se}_n(\eta, -\frac{1}{4}A^2R^2)\,\sin k\eta\,d\zeta\,d\eta \tag{5.6}$$

To obtain an approximate solution to (5.5), we truncated at $n = k = 8$. This necessitated the calculation of 64 integrals of the form (5.6) which is the chief drawback of this procedure. Nevertheless we were able to obtain good results in the range $0 < R < 5$. In calculating the coefficients C_{nk}, it is convenient to choose a modulus q for which the eigenvalues σ_n in Mathieu's equation are tabulated. The corresponding Reynolds number can be found from the relation

$$R = 2A^{-1}|q|^{1/2} \tag{5.7}$$

We examined five geometries: the circle, the ellipse with aspect ratio 3:1 oriented parallel and perpendicular to the flow and the flat plate oriented parallel and perpendicular to the flow. The first eight vorticity coefficients for a sampling of geometries and Reynolds numbers are given in Tables 1-5.

The rapid decay of the coefficients W_n does not mean that the vorticity series (3.27) converges rapidly in all regions of the flow. When $q = -1$, for example, $\mathrm{Gek}_6(0,-1) = O(10^6)$ and $\mathrm{Gek}_8(0,-1) = O(10^{10})$ and the decay of the W_n's is offset by the growth of the function values $\mathrm{Gek}_n(0,-1)$. The vorticity series may converge rather slowly therefore in the vicinity of the obstacle and indeed this is observed in the elliptical cases. The rate of convergence appears to be a maximum in the case of the circle and we therefore investigated separation phenomena for this geometry only.

Table 1. Circle.

	R = 2.0	R = 4.0
W_1	13.250	773.03
W_2	-2.2884	-617.12
W_3	0.37690×10^{-1}	11.281
W_4	-0.24210×10^{-3}	-0.71475
W_5	0.62517×10^{-6}	0.12619×10^{-1}
W_6	-0.78300×10^{-9}	-0.11800×10^{-3}
W_7	0.53897×10^{-12}	0.60380×10^{-6}
W_8	-0.21073×10^{-15}	-0.15441×10^{-8}

Table 2. Ellipse (aspect ratio 3:1) parallel to flow $\left(\xi_o = \frac{1}{2} \ln 2\right)$.

	R = 2.12	R = 4.24
W_1	5.1649	103.41
W_2	-0.41325	-46.711
W_3	0.62398×10^{-2}	1.8916
W_4	-0.46197×10^{-4}	-0.64596×10^{-1}
W_5	0.20407×10^{-6}	0.11379×10^{-2}
W_6	-0.56252×10^{-9}	-0.12250×10^{-4}
W_7	0.10014×10^{-11}	0.78668×10^{-7}
W_8	-0.10497×10^{-14}	-0.25930×10^{-9}

Table 3. Ellipse (aspect ratio 3:1) perpendicular to flow $\left(\lambda_o = \frac{1}{2} \ell n\ 2\right)$.

	R = 2.12	R = 4.24
W_1	5.8811	82.188
W_2	−0.33473	−30.625
W_3	-0.10127×10^{-2}	0.13610
W_4	0.25251×10^{-4}	0.15274×10^{-1}
W_5	0.27597×10^{-7}	-0.11094×10^{-3}
W_6	-0.30245×10^{-9}	-0.23548×10^{-5}
W_7	-0.13758×10^{-12}	0.14100×10^{-7}
W_8	0.83636×10^{-15}	0.22353×10^{-10}

Table 4. Flat plate parallel to flow ($\xi_o = 0$)

	R = 2.83	R = 4.0
W_1	6.7127	25.215
W_2	−0.58416	−5.2619
W_3	0.11229×10^{-1}	0.17616
W_4	-0.10123×10^{-3}	-0.32847×10^{-2}
W_5	0.55440×10^{-6}	0.34490×10^{-4}
W_6	-0.17737×10^{-8}	-0.22002×10^{-6}
W_7	0.35771×10^{-11}	0.82930×10^{-9}
W_8	-0.41796×10^{-14}	-0.17796×10^{-11}

Table 5. Flat plate perpendicular to flow ($\lambda_0 = 0$)

	R = 2.83	R = 4.0
W_1	9.0389	27.223
W_2	-0.40074	-3.0988
W_3	-0.73015×10^{-2}	-0.74847×10^{-1}
W_4	0.62359×10^{-4}	0.15255×10^{-2}
W_5	0.34806×10^{-6}	0.13029×10^{-4}
W_6	-0.13098×10^{-8}	-0.12114×10^{-6}
W_7	-0.25803×10^{-11}	-0.37198×10^{-9}
W_8	0.62502×10^{-14}	0.21338×10^{-11}

6. SEPARATION

Separation occurs on the downstream side of the circular cylinder provided the Reynolds number exceeds a critical value R_c defined as the value of R for which

$$\frac{\partial \omega}{\partial \eta}(0,0) = 0 \tag{6.1}$$

For Burgers flow we find $R_c = 1.12$ which is a new result. Yamada (1954) has shown that $R_c = 1.51$ for Oseen flow and Underwood (1969) has obtained $R_c = 2.88$ from a numerical solution of the full nonlinear equation.

The Burgers result should be less than the numerical value. The convective velocity field in Burgers flow is potential flow past the cylinder and this violates the no-slip condition at the cylinder's surface. The velocity field which solves the full Navier-Stokes equation satisfies this condition. Thus convection effects near the cylinder are more dominant in Burgers flow than in Navier-Stokes flow and any phenomena related to convection, such as separation, should occur at lower Reynolds numbers in Burgers flow.

The fact that the Burgers result is less than the Oseen value also can be explained. Separation begins at the rear stagnation point P of the cylinder where locally the flow appears as in Figure 1. At the onset of separation two eddies of circulating fluid form about P. (We refer to this pair of eddies as a separation vortex). The direction of motion along the axis of symmetry inside the vortex is opposed to that outside (Figure 2). In Oseen flow the convective velocity field is constant in magnitude and perpendicular to the cylinder boundary in the vicinity of P as shown in Figure 3. Oseen convection therefore deters the establishment of reverse flow at P because it directly opposes the direction of fluid motion along the axis of symmetry inside the vortex. In contrast the convective velocity field in Burgers flow vanishes at the point P and is small in magnitude near P (Figure 4). Burgers convection does not oppose the establishment of a vortex about P to the same degree that Oseen convection does and separation therefore initiates in Burgers flow at a lower Reynolds number.

As R increases beyond $R_c = 1.12$ in Burgers flow, the separation vortex grows in size. When R = 2.0 the flow appears as in Figure 5. The length of the vortex is PQ = 0.53 where OP is the unit of length, and $\angle SOP = 34.8°$. Point T, whose θ-coordinate is 83°, marks the location where the fluid pressure along the boundary is a minimum. The flow from T to S is against an adverse pressure gradient.

7. CALCULATION OF DRAG COEFFICIENTS

Several authors (Imai, 1951; Kawaguti, 1953; Dennis and Dunwoody, 1966) have shown that an obstacle's drag coefficient can be obtained from the term of O(1) in the asymptotic expansion of the stream function. This obviates the calculation of stress components which often are difficult to obtain accurately. The task therefore is to find the leading terms in the asymptotic expansion of (4.11).

We first obtain a series representation for the Green's function in (4.3). From Magnus et al. (1966), we have

$$\sum_{k=1}^{\infty} \frac{1}{k} e^{-kt} \cos kx = \frac{1}{2} t - \frac{1}{2} \ln (2 \cosh t - 2 \cos x) \qquad t > 0 \tag{7.1}$$

Manipulation of (7.1) yields the following results:

$$G(\zeta,\eta|\zeta',\eta') = \frac{1}{2\pi}\zeta + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-k\zeta'} \cdot \sinh k\zeta \cos k(\eta-\eta') \qquad \zeta < \zeta' \tag{7.2}$$

$$= \frac{1}{2\pi}\zeta' + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-k\zeta} \cdot \sinh k\zeta' \cos k(\eta-\eta') \qquad \zeta > \zeta' \tag{7.3}$$

Substituting these into (4.11) and simplifying, we obtain

$$\psi(\zeta',\eta') = \psi_o(\zeta',\eta') + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-k\zeta'} \sin k\eta' \cdot \int_{-\pi}^{\pi}\int_{o}^{\zeta'} \alpha^2\, \omega \sinh k\zeta \sin k\eta \, d\zeta d\eta + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sinh k\zeta' \sin k\eta' \cdot \int_{-\pi}^{\pi}\int_{\zeta'}^{\infty} \alpha^2\, \omega\, e^{-k\zeta} \sin k\eta \, d\zeta d\eta \tag{7.4}$$

An expression for the second integral can be obtained from (5.4):

$$\int_{-\pi}^{\pi}\int_{\zeta'}^{\infty} \alpha^2\, \omega\, e^{-k\zeta} \sin k\eta\, d\zeta d\eta = -2A\pi\delta_{k1} - \int_{-\pi}^{\pi}\int_{o}^{\zeta'} \alpha^2\, \omega\, e^{-k\zeta} \sin k\eta\, d\zeta d\eta \tag{7.5}$$

Thus a series representation for the stream function is, from (7.4),

$$\psi(\zeta',\eta') = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{-k\zeta'} \sin k\eta' \int_{o}^{\pi}\int_{o}^{\zeta'} \alpha^2\, \omega\, e^{k\zeta} \sin k\eta\, d\zeta d\eta - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}e^{k\zeta'} \sin k\eta' \int_{o}^{\pi}\int_{o}^{\zeta'} \alpha^2\, \omega\, e^{-k\zeta} \sin k\eta\, d\zeta d\eta \tag{7.6}$$

This expression can be written in the form

$$\psi(\zeta',\eta') = A\,e^{\zeta'}\sin\eta' - \frac{1}{\pi}\sin\eta'\left[\frac{\int_0^{\pi}\int_0^{\zeta'}\alpha^2\,\omega\,e^{-\zeta}\sin\eta\,d\zeta d\eta + \pi A}{e^{-\zeta'}}\right]$$

$$-\frac{1}{\pi}\sum_{k=2}^{\infty}\frac{1}{k}\sin k\eta'\left[\frac{\int_0^{\pi}\int_0^{\zeta'}\alpha^2\,\omega\,e^{-k\zeta}\sin k\eta\,d\zeta d\eta}{e^{-k\zeta'}}\right]$$

$$+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin k\eta'\left[\frac{\int_0^{\pi}\int_0^{\zeta'}\alpha^2\,\omega\,e^{k\zeta}\sin k\eta\,d\zeta d\eta}{e^{k\zeta'}}\right] \tag{7.7}$$

It can be seen from (5.4) that the first two bracketed expressions in (7.7) are indeterminate forms $\left(\frac{0}{0}\right)$ as $\zeta' \to \infty$. The third expression is of the form $\left(\frac{\infty}{\infty}\right)$ in the limit. By invoking ℓ'Hôpital's rule we obtain contributions from each expression to the $O(1)$ term in the asymptotic expansion of ψ. The result is

$$\psi(\zeta',\eta') \sim A\,e^{\zeta'}\sin\eta' + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin k\eta'$$

$$\cdot\left\{\lim_{\zeta\to\infty}\int_0^{\pi}\alpha^2\,\omega\sin k\eta\,d\eta\right\} + O\left(e^{-\zeta'}\right) \tag{7.8}$$

The limit can be computed and the series summed. The details are included in Appendix III. The expansion has the form

$$\psi(\zeta',\eta') \sim A\,e^{\zeta'}\sin\eta' - \frac{1}{2}C_D\left(\pm 1 - \frac{\eta'}{\pi}\right) + O(e^{-\zeta'}) \tag{7.9}$$

where, in the second term, the plus sign is chosen when $0 < \eta' \leq \pi$ and the minus sign when $-\pi < \eta' < 0$. This term is analytic along $\eta' = \pi$, but suffers a finite jump discontinuity along $\eta' = 0$ which coincides with the wake. Dennis and Dunwoody (1966) comment that this term must be present in order to give non-zero drag. Kawaguti (1953) shows that the constant C_D is the drag coefficient for the obstacle in the flow.

Table 6 contains drag coefficients for the circular cylinder in Ossen flow. In table 7 the Burgers flow results are given.

Table 6. Circular cylinder in Oseen flow.

R	1.0	1.51	2.0	3.0	4.0	5.0
C_D	8.08*	6.62	5.85	4.98	4.50	4.18

*From Tomotika and Aoi (1951).

Table 7. Circular cylinder in Burgers flow.

R	1.0	1.12	1.25	1.50	2.0	2.83	3.46	4.0
C_D	7.76	7.30	6.86	6.22	5.34	4.47	4.04	3.75

Figure 6 is a plot of the results contained in Tables 6 and 7 along with some of Tritton's (1959) experimental values. The graph indicates that Burgers approximation is an improvement over the Oseen approximation in modeling the flow past a circular cylinder.

In Tables 8-11, a sampling of Burgers drag coefficients is given for elliptic geometries. The results are plotted in figure 7.

Table 8. Ellipse (aspect ratio 3:1) parallel to flow.

R	2.12	3.0	4.24	5.20
C_D	3.93	3.22	2.65	2.36

Table 9. Ellipse (aspect ratio 3:1) perpendicular to flow.

R	2.12	3.0	4.24	5.20
C_D	5.35	4.56	3.91	3.58

Table 10. Flat plate parallel to flow.

R	2.83	4.0	5.66	6.93
C_D	2.42	1.94	1.56	1.34

Table 11. Flat plate perpendicular to flow.

R	2.83	4.0	5.66	6.93
C_D	4.54	3.98	3.54	3.31

8. BOUNDARY LAYER THICKNESS

The procedure outlined in section 7 can be extended to obtain higher order terms in the expansion of ψ. When expressed in polar coordinates (r,θ), this expansion has the form

$$\psi \sim r \sin\theta - \frac{1}{2} C_D \left(\pm 1 - \frac{\theta}{\pi}\right) + \sum_{n=1}^{\infty} \phi_n r^{-n} \sin n\theta + \chi(r,\theta) \tag{8.1}$$

where the ϕ_n $(n = 1,2,3, \ldots)$ are constants and $\chi(r,\theta)$ is exponentially small in r.

Except along the line $\theta = 0$, the algebraic part of the expansion is harmonic and constitutes the potential flow far from the obstacle.

Because of the tedious nature of the calculation, the expansions were computed for the circular case only. Both Oseen and Burgers expansions were calculated for purposes of comparison. The results are presented in Tables 12 and 13.

Table 12. Oseen flow.

R = 1.51	$\psi \sim r \sin\theta - 3.3118\left(\pm 1 - \frac{\theta}{\pi}\right) - 0.5995\,\frac{\sin\theta}{r} + O\left(\frac{\sin 2\theta}{r^2}\right)$		
R = 2.0		2.9247	0.5777
R = 3.0		2.4888	0.5485
R = 4.0		2.2481	0.5302
R = 5.0		2.0918	0.5171

Table 13. Burgers flow.

R = 1.12	$\psi \sim r \sin\theta - 3.6509\left(\pm 1 - \frac{\theta}{\pi}\right) - 1.1251\,\frac{\sin\theta}{r} + O\left(\frac{\sin 2\theta}{r^2}\right)$		
R = 2.0		2.6709	1.1396
R = 2.83		2.2327	1.1439
R = 3.46		2.0213	1.1486
R = 4.0		1.8763	1.1467

Since both Oseen and Burgers flows are spatially uniform approximations to the exact solution of the Navier-Stokes equation, they predict a boundary layer surrounding the obstacle. The outer edge of the boundary layer is defined to be the curve along which the algebraic part of the asymptotic expansion of ψ vanishes. The curve so defined determines the displacement body which the potential flow far from the cylinder "sees." The displacement body includes the cylinder, its separation vortex and the surrounding boundary layer.

By setting the expansions given in Tables 12 and 13 equal to zero, we obtain approximations to the displacement bodies for the various flows. A typical example is given in Figure 8. The displacement body is semi-infinite with its width at infinity numerically equal to the drag coefficient. Since the boundary of the circular cylinder is given by $r = 1$, the thickness of the boundary layer is easily calculated. Tables 14 and 15 compile these results for a variety of Reynolds numbers and locations along the cylinder boundary.

Table 14. Boundary layer thickness for Oseen flow.

$12\theta/\pi$	R = 2.0	R = 3.0	R = 4.0	R = 5.0	$\Lambda(\theta)/R^{\gamma}$
12	0.357	0.236	0.169	0.125	$0.798/R^{1.134}$
11	0.365	0.243	0.175	0.131	$0.804/R^{1.113}$
10	0.390	0.264	0.194	0.148	$0.823/R^{1.054}$
9	0.436	0.301	0.227	0.178	$0.865/R^{0.972}$
8	0.509	0.361	0.280	0.227	$0.941/R^{0.880}$
7	0.619	0.451	0.360	0.300	$1.071/R^{0.788}$
6	0.786	0.589	0.482	0.412	$1.280/R^{0.704}$
5	1.048	0.807	0.674	0.589	$1.618/R^{0.630}$
4	1.484	1.169	0.996	0.885	$2.188/R^{0.565}$
3	2.278	1.833	1.589	1.431	$3.226/R^{0.508}$
2	3.990	3.276	2.883	2.629	$5.449/R^{0.457}$
1	9.414	7.876	7.028	6.478	$12.438/R^{0.409}$

Table 15. Boundary layer thickness for Burgers flow.

$12\ \theta/\pi$	R = 2.0	R = 2.83	R = 3.46	R = 4.0	$\Lambda(\theta)/R^{\gamma}$
12	0.574	0.482	0.441	0.410	$0.800/R^{0.482}$
11	0.581	0.488	0.445	0.415	$0.811/R^{0.484}$
10	0.602	0.505	0.460	0.428	$0.843/R^{0.489}$
9	0.639	0.535	0.487	0.453	$0.900/R^{0.496}$
8	0.699	0.582	0.529	0.491	$0.991/R^{0.507}$
7	0.789	0.654	0.593	0.549	$1.130/R^{0.520}$
6	0.927	0.765	0.690	0.638	$1.342/R^{0.537}$
5	1.144	0.938	0.844	0.778	$1.679/R^{0.555}$
4	1.510	1.231	1.102	1.014	$2.244/R^{0.573}$
3	2.190	1.780	1.588	1.457	$3.285/R^{0.587}$
2	3.694	3.007	2.681	2.459	$5.541/R^{0.586}$
1	8.579	7.050	6.316	5.814	$12.639/R^{0.560}$

In both cases the thickness of the boundary layer increases as one moves around the cylinder from the forward stagnation point ($\theta = \pi$). Increasing the Reynolds number serves to compress the boundary layer. The last column in each table is a least-squares fit of the data given in each row to an expression of the form $\Lambda(\theta)/R^{\gamma}$. The function $\Lambda(\theta)$ is similar in both flows. The value of γ depends on θ in the case of Oseen flow, but appears to hover about the constant $\frac{1}{2}$ in Burgers flow in the range $\frac{\pi}{2} < \theta < \pi$. This suggests that boundary layer thickness in Burgers flow is roughly proportional to $R^{-1/2}$.

Analytic studies of the boundary layer on a semi-infinite flat plate using the full nonlinear equations show that the thickness is proportional to $R^{-1/2}$. Our work, although not conclusive, suggests a similar result for a circular cylinder using a linear model. The apparent agreement between this prediction of Burgers flow and that of nonlinear analysis verifies again its superiority over Oseen flow in describing fluid behavior near the cylinder.

APPENDIX I

The standard form of Mathieu's equation is, from McLachlan (1964),

$$y'' + (\sigma - 2q \cos 2\theta)\, y = 0 \tag{I.1}$$

where q is termed the modulus and σ is the eigenvalue. In this discussion we are only interested in solutions of (I.1) which are odd in θ. In many applications the eigenvalue is determined from the condition that $y(\theta)$ be periodic with period 2π. Thus if $q = 0$, the eigenvalue must be the square of an integer $(\sigma_n = n^2)$ and the corresponding odd eigenfunction is

$$y_n(\theta) = \sin n\theta \tag{I.2}$$

If $|q|$ is small but nonzero, the eigenvalue and eigenfunction can be expanded in series of the form

$$\sigma_n = n^2 + \sum_{k=1}^{\infty} \beta_{nk} q^k \tag{I.3}$$

$$y_n(\theta, q) = \sin n\theta + \sum_{k=1}^{\infty} S_{nk}(\theta) q^k \tag{I.4}$$

The coefficients β_{nk} and $S_{nk}(\theta)$ are determined by substituting these expansions into (I.1), equating like powers of q, and requiring that $S_{nk}(\theta)$ be odd in θ and periodic with period 2π. The expression for $y_n(\theta)$ can be rewritten in the form

$$y_n(\theta, q) = \sum_{k=1}^{\infty} D_k^{(n)}(q) \sin k\theta \tag{I.5}$$

where each $D_k^{(n)}(q)$ is a power series in q. A normalized Mathieu function is defined as follows:

ORIGINAL PAGE IS
OF POOR QUALITY

$$se_n(\theta,q) = y_n(\theta,q)/[\sum_{k=1}^{\infty} \{D_k^{(n)}(q)\}^2]^{1/2}$$

$$= \sum_{k=1}^{\infty} B_k^{(n)}(q) \sin k\theta \tag{I.6}$$

where $\sum_{k=1}^{\infty} [B_k^{(n)}(q)]^2 = 1.$

The standard form of the modified Mathieu equation is

$$Y_n''(z) - [\sigma_n - 2q \cosh 2z]\, Y_n(z) = 0 \tag{I.7}$$

When $q < 0$, the eigenfunctions which decay exponentially as $z \to \infty$ are denoted by

$$Y_n(z) = Gek_n(z,q) \tag{I.8}$$

From McLachlan (1964), we have

$$Gek_n(z,q) \sim c_n K_n(|q|^{1/2} e^z)$$

$$\sim c_n \left(\frac{1}{2}\pi\right)^{1/2} |q|^{-1/4} \exp\left[-\frac{1}{2} z - |q|^{1/2} e^z\right] \tag{I.9}$$

where $q < 0$.

ORIGINAL PAGE IS
OF POOR QUALITY

APPENDIX II

The vorticity equation for Oseen flow past a circular cylinder can be solved using the method outlined in section 3. The appropriate coordinate system (ξ,η) is defined by

$$x = e^{\xi} \cos \eta, \qquad y = e^{\xi} \sin \eta \tag{II.1}$$

and the metric coefficient is

$$\alpha(\xi,\eta) = e^{\xi} \tag{II.2}$$

The vorticity is given by

$$\omega(\xi,\eta) = F(\xi,\eta) \exp \left[\frac{1}{2} R e^{\xi} \cos \eta\right] \tag{II.3}$$

where $F(\xi,\eta)$ satisfies the equation

$$\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial \eta^2} - \frac{1}{4} R^2 e^{2\xi} F = 0 \tag{II.4}$$

Solutions to this equation are of the form

$$F_n(\xi,\eta) = E_n(\xi) \sin n\eta \tag{II.5}$$

where $E_n(\xi)$ satisfies the equation

$$E_n'' - (n^2 + \frac{1}{4} R^2 e^{2\xi}) E_n = 0 \tag{II.6}$$

From Gradshteyn and Ryzhik (1965), we have

$$E_n(\xi) = A_n I_n \left(\frac{1}{2} R e^{\xi}\right) + B_n K_n \left(\frac{1}{2} R e^{\xi}\right) \tag{II.7}$$

Since the vorticity must decay exponentially as $\xi \to \infty$, the coefficients A_n are all zero and the vorticity is given by

$$\omega(\xi,\eta) = -\exp\left[\frac{1}{2} R\, e^{\xi} \cos\eta\right] \sum_{n=1}^{\infty} B_n K_n \left(\frac{1}{2} R\, e^{\xi}\right) \sin n\eta \qquad \text{(II. 8)}$$

In terms of polar coordinates (r,θ), the asymptotic behavior of the Oseen vorticity is

$$\omega(r,\theta) \sim w(\theta)\, r^{-1/2} \exp\left[-\frac{1}{2} Rr\,(1 - \cos\theta)\right] \qquad \text{(II. 9)}$$

where $w(0) = 0$, $w'(0) \neq 0$ and $w(\theta)$ has period 2π.

APPENDIX III

The expansion of ψ in (7.8) contains the expression

$$\lim_{\zeta\to\infty} \int_0^\pi \alpha^2(\zeta,\eta)\,\omega(\zeta,\eta)\,\sin k\eta\; d\eta \tag{III.1}$$

which can be computed once the asymptotic behavior of α^2 and ω is known. From (II.2), (3.24) and (3.30), we have

$$\alpha^2(\zeta,\eta) \sim A^2\, e^{2\zeta} \tag{III.2}$$

We also know that

$$\omega(\zeta,\eta) = -\exp\,[AR\cosh\zeta\,\cos\eta]\sum_{n=1}^{\infty} W_n\,\mathrm{Gek}_n\left(\zeta,\,-\frac{1}{4}A^2R^2\right) \cdot \mathrm{se}_n\left(\eta,\,-\frac{1}{4}A^2R^2\right) \tag{III.3}$$

From (I.9) the asymptotic form of the vorticity is given by

$$\omega(\zeta,\eta) \sim -\left(\frac{\pi}{AR}\right)^{1/2} \exp\left[-\frac{1}{2}\zeta - \frac{1}{2}AR\,e^{\zeta}\,(1-\cos\eta)\right] \cdot \sum_{n=1}^{\infty} c_n W_n\,\mathrm{se}_n\left(\eta,\,-\frac{1}{4}A^2R^2\right) \tag{III.4}$$

Thus the integral in (III.1) simplifies to

$$\int_0^\pi \alpha^2\,\omega\,\sin k\eta\,d\eta \sim -\left(\frac{\pi A^3}{R}\right)^{1/2} \exp\left[-\frac{1}{2}(AR\,e^{\zeta} - 3\zeta)\right] \cdot \sum_{n=1}^{\infty} c_n W_n \int_0^\pi \exp\left[\frac{1}{2}AR\,e^{\zeta}\cos\eta\right]\sin k\eta \cdot \mathrm{se}_n\left(\eta,\,-\frac{1}{4}A^2R^2\right) d\eta \tag{III.5}$$

From McLachlan (1964), we have

$$se_n\left(\eta, -\frac{1}{4}A^2R^2\right) = \sum_{m=0}^{\infty} S_m^{(n)} \sin\,(2m + \varepsilon_n)\eta \tag{III.6}$$

where $\varepsilon_n = 1$ if n is odd,
$= 2$ if n is even.

Therefore, we have

$$se_n\left(\eta, -\frac{1}{4}A^2R^2\right) \sin k\eta = \frac{1}{2}\sum_{m=0}^{\infty} S_m^{(n)} [\cos\,(2m + \varepsilon_n - k)\eta$$
$$- \cos\,(2m + \varepsilon_n + k)\eta] \tag{III.7}$$

The substitution of (III.7) into (III.5) yields integrals of the form

$$\int_0^{\pi} \exp\,[u \cos\theta] \cos n\theta\, d\theta = \pi\, I_n(u) \tag{III.8}$$

where I_n is the modified Bessel function of the first kind. After evaluating the integral in (III.5) and replacing the Bessel functions by their asymptotic forms, we have

$$\int_0^{\pi} \alpha^2\, \omega \sin k\eta d\eta \sim -\frac{2\pi}{R^2} k \sum_{n=1}^{\infty} c_n W_n \sum_{m=0}^{\infty} (2m + \varepsilon_n)\, S_m^{(n)} + O(e^{-\zeta}) \tag{III.9}$$

Taking the limit in (III.1) and substituting back into (7.8), we obtain

$$\psi(\zeta',\eta') \sim A\, e^{\zeta'} \sin \eta' - \frac{1}{\pi} C_D \sum_{k=1}^{\infty} \frac{1}{k} \sin k\eta' + O(e^{-\zeta'}) \tag{III.10}$$

where

$$C_D = \frac{4\pi}{R^2} \sum_{n=1}^{\infty} c_n W_n \sum_{m=0}^{\infty} (2m + \varepsilon_n)\, S_m^{(n)} \tag{III.11}$$

But

$$\sum_{k=1}^{\infty} \frac{1}{k} \sin k\eta' = \frac{1}{2} (\pm \pi - \eta') \tag{III.12}$$

where the plus sign is chosen when $0 < \eta' < \pi$ and the minus sign when $-\pi < \eta' < 0$. Thus the asymptotic form of the stream function for Burgers flow past an elliptical cylinder is given by (7.9).

This work was supported by the National Aeronautics and Space Administration under contract no. NAG1-197.

REFERENCES

Burgers, J.M. 1928. Proc. Acad. Sci. Amsterdam 31, 433-453.

Dennis, S.C.R. and Dunwoody, J. 1966. J. Fluid Mech. 24, 577-595.

Dryden, H.L., Murnaghan, F.D. and Bateman, H. 1956. Hydrodynamics, Dover Publications, Inc., 265-266.

Gradshteyn, I.S. and Ryzhik, I.M. 1965. Table of Integrals, Series and Products, Academic Press, pp. 42, 971.

Imai, I. 1951. Proc. Roy. Soc. A., 208, 487-516.

Kawaguti, M. 1953. J. Phys. Soc. Japan, 8, 747.

Magnus, W., Oberhettinger, F. and Soni, R.P. 1966. Formulas and Theorems for the Special Functions of Mathematical Physics, Springer-Verlag New York Inc., p. 469.

McLachlan, N.W. 1964. Theory and Application of Mathieu Functions, Dover Publications, Inc., pp. 10, 22, 369.

Skinner, L.A. 1975. Quart. J. Mech. Appl. Math. 28, 333-340.

Tomotika, S. and Aoi, T. 1951. Quart. J. Mech. Appl. Math. 4, 401-406.

Tritton, D.J. 1959. J. Fluid Mech. 6, 547-567.

Underwood, R.L. 1969. J. Fluid Mech. 37, 95-114.

Van Dyke, M. 1975. Perturbation Methods in Fluid Mechanics (Annotated Edition), The Parabolic Press, p. 236.

Yamada, H. 1954. Rep. Res. Inst. Appl. Mech. Kyushu Univ. 3, 11-23.

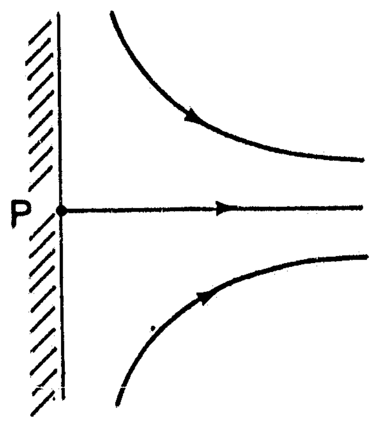

Figure 1. Streamlines at the rear stagnation point P of the cylinder prior to separation.

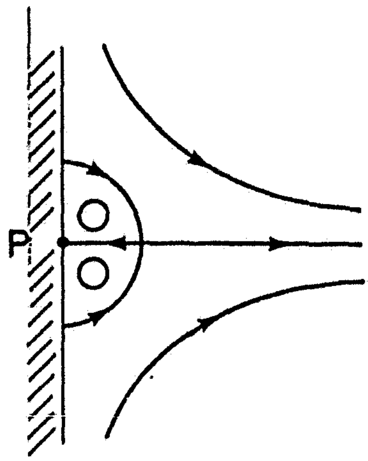

Figure 2. Streamlines after separation.

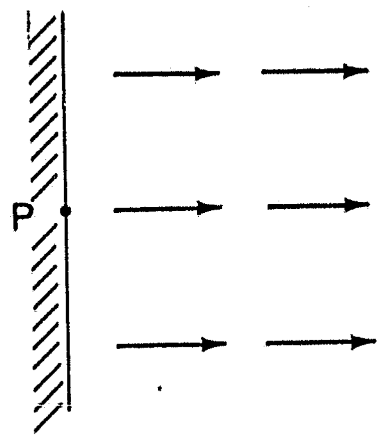

Figure 3. Convective velocity field at the rear stagnation point of the cylinder in Oseen flow.

ORIGINAL PAGE IS
OF POOR QUALITY

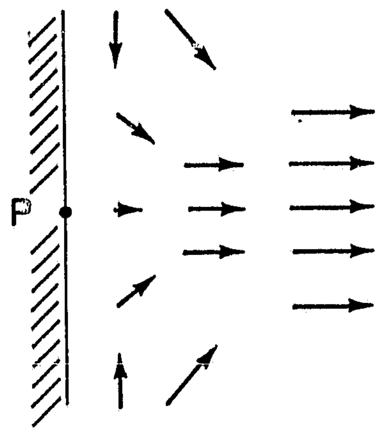

Figure 4. Convective velocity field in Burgers flow.

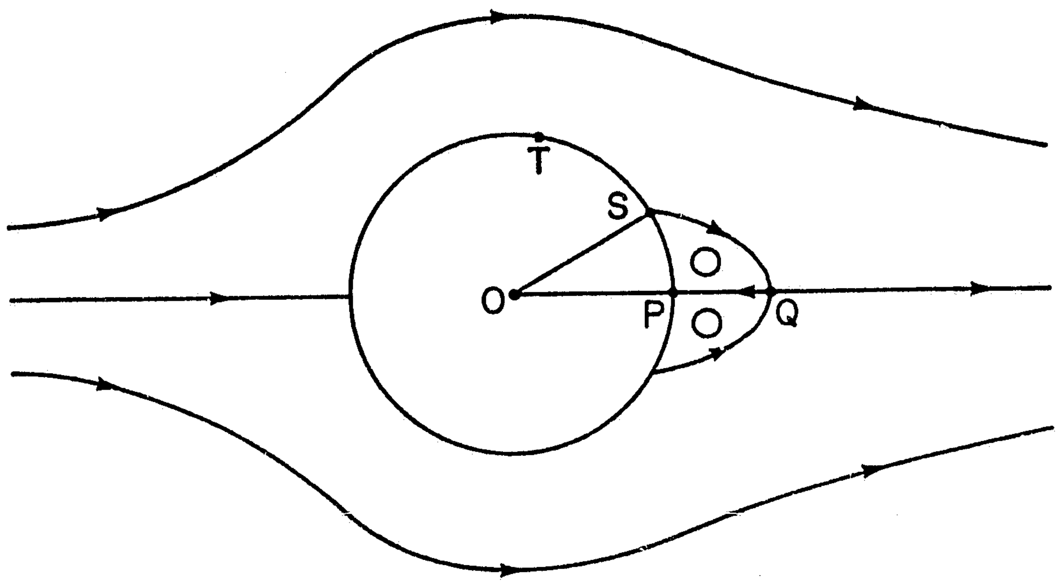

Figure 5. Burgers flow past a circular cylinder at R = 2.0.

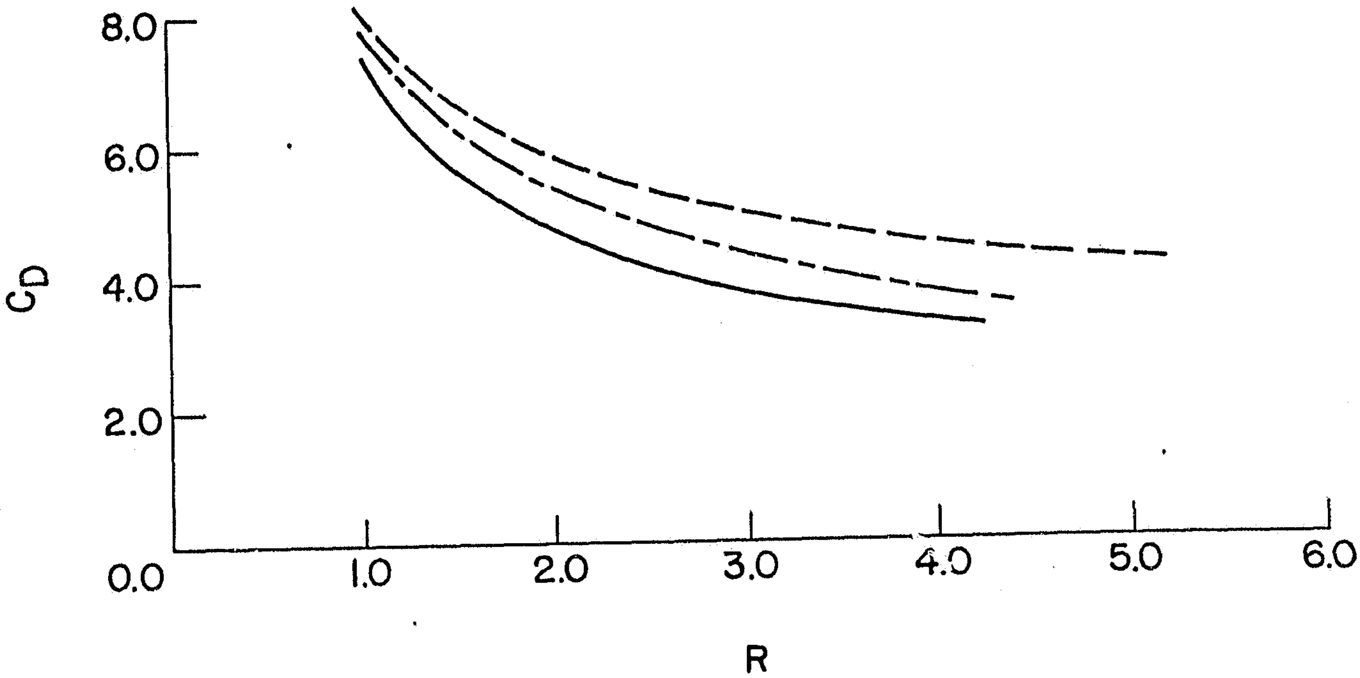

Figure 6. Drag coefficient vs. Reynolds number for flow past a circular cylinder. - - - - - -, Oseen flow; —— - - —— - ——, Burgers flow; ——— experiment (Tritton).

ORIGINAL PAGE IS
OF POOR QUALITY

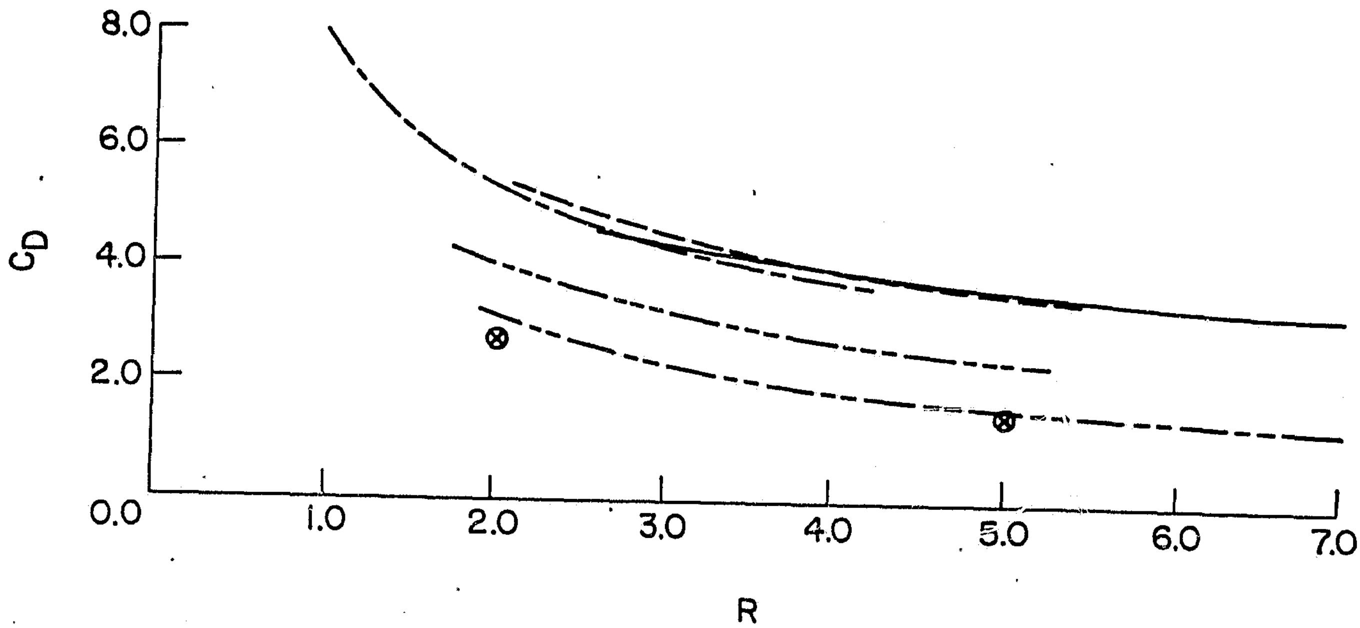

Figure 7. Drag coefficient vs. Reynolds number for Burgers flow past a variety of geometries. ————, flat plate perpendicular to flow; - - - -, ellipse (aspect ratio 3:1) perpendicular to flow; —— - —— -, circle; —— --- —— ---, ellipse (aspect ratio 3:1) parallel to flow; —— -- —— -- ——, flat plate parallel to flow; ⊗, numerical solution (Dennis and Dunwoody) of the full nonlinear equation for flow past a flat plate.

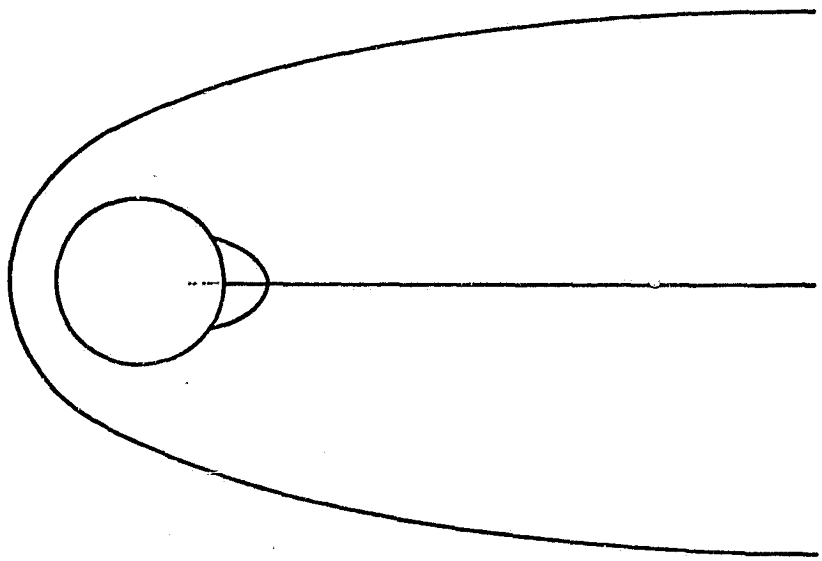

Figure 8. Displacement body for Burgers flow past a circular cylinder at $R = 2.0$.