General Disclaimer

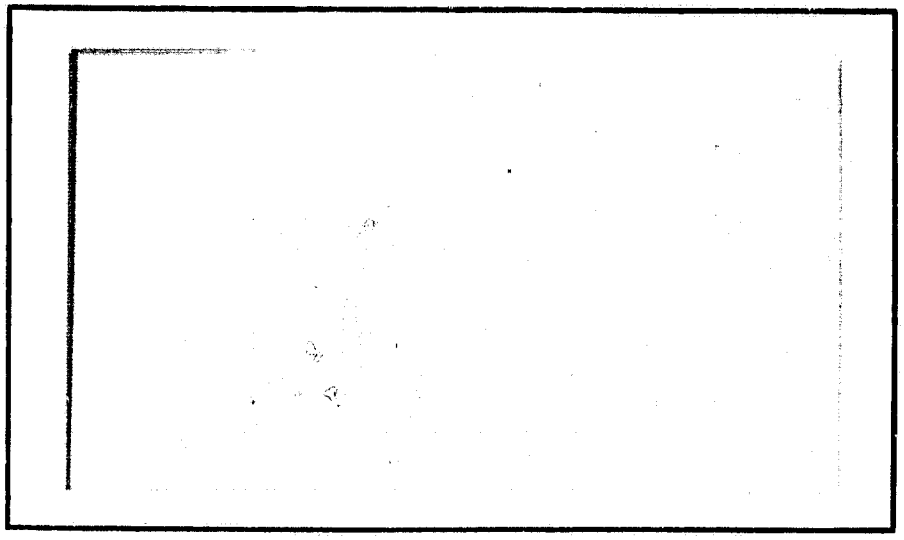
- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available much information as possible.

- This document may contain data, which exceeds the sheet parameters. It wa furnished in this condition by the organizational source and is the best copy available.

- This document may contain tone-on-tone or color graphs, charts and/or pictu which have been reproduced in black and white.

- This document is paginated as submitted by the original source.

- Portions of this document are not fully legible due to the historical nature of s of the material. However, it is the best reproduction available from the origina submission.



UNIVERSITY of PENNSYLVANIA

The Moore School of Electrical Engineering

PHILADELPHIA, PENNSYLVANIA 19104

UNIVERSITY OF PENNSYLVANIA
DEPARTMENT OF ELECTRICAL ENGINEERING AND SCIENCE

MEASUREMENT ACCURACIES IN BAND-LIMITED EXTRAPOLATION

Prepared by

H.N. Kritikos

July, 1982

Submitted to:

NASA
Goddart Space Flight Center

For Period 6/1/81 - 10/31/81

Grant No. NAG 5-92

MEASUREMENT ACCURACIES IN BAND-LIMITED EXTRAPOLATION

H.N. Kritikos

Moore School of Electrical Engineering
University of Pennsylvania
Philadelphia, PA

ABSTRACT

It is shown that in band-limited (or visible angle limited) extrapolation the larger effective aperture L' that can be realized from a finite aperture L by over sampling is a function of the accuracy of measurements. It is shown that for sampling in the interval $\frac{L}{b} < |x| < L$, $b > 1$ the signal must be known within an error e_N given by

$$e_N^2 \approx \frac{1}{4(2kL')^3} \left(\frac{e}{8b} \frac{L}{L'}\right)^{2kL'}$$

where

L is the physical aperture

L' is the extrapolated aperture

$k = 2\pi/\lambda$, λ is the wavelength

1. BACKGROUND

The extrapolation problem finds its origin in the original investigation reported by D. Stepian and H. Pollack[1,2] on the Spheroidal functions and the results on the Uncertainty Principle reported by H. Landau and H. Pollack.[2,3] Recently a resurge of interest in this area has resulted in large numbers of published papers two of which written by A. Jain[4] and R. Schafer[5] give an excellent review of the scope of the present research. K. Abend[7] also has given a review of the limitations of extrapolation methods.

One of the recurring problems occurs in any attempt to implement a useful algorithm of extrapolation is the numerical instability. Since the problems are nearly ill posed small errors are capable of generating totally erroneous results. In this communication an attempt will be made to estimate the bounds for the acceptable errors and then place a ceiling of the measurement accuracy and computational accuracy needed for the extrapolation.

2. PROBLEM STATEMENT

Consider the function $F(x)$ of finite norm known in the interval $-L/2 < x < L/2$. The Fourier transform $F(u)$ is band limited in the interval $|u| < 1_o$. One has the well known relation

$$\int_{-L/2}^{L/2} h(x - x')\, g(x')\, dx' = F(x) \qquad (1)$$

The problem of interest here is as follows:

Given the function $F(x)$ in the internal $-L < x < L$ find the source function $g(x')$ and then extrapolate into the region $|x| > L$. In its corresponding discrete form Stepian has shown that although in general a non unique answer exist the requirement that the solution has a minimum norm ensures the uniqueness of the solution.

Despite the simplicity of the formulations numerical solutions are very difficult or almost impossible to obtain. The $g(x')$ is wildly varying function as can be inferred from the following argument.

In the space of spheroidal orthogonal functions $\psi_n(x)$ one observes the following:

$$F(x) = \Sigma\alpha_n \ \psi_n(x) \quad , \quad ||F(x)|| = \Sigma\alpha_n < \infty \tag{2}$$

$$h(x-x') = \Sigma \ \psi_n(x) \ \psi(x') \quad , \quad ||\psi_n(x)|| = 1 \tag{3}$$

and

$$g(x') = \Sigma \frac{\alpha_n}{\lambda_n} \ \psi_n(x') \quad , \quad ||g(x)|| = \Sigma \frac{\alpha^2_n}{\lambda_n} \tag{4}$$

where λ_n are the eigenvalues of the spheroidal functions. The problem here is that for any functions $F(x)$ with bounded norm $||F|| < \infty$ the source function $g(x)$ may not be bounded.

The reason is that when one attempts to extrapolate beyond the degrees of freedom dictated by the bandwidth and time interval Slepian and later H. Landau[6] has shown that the asymptotic form of the eigenvalues is given by

$$\lambda_n(c) \to \frac{1}{(1 + e^b)} \tag{5}$$

when the index n is given by

$$n = \frac{m(s)\ m(T)}{2\pi}\ c + \frac{b \log c}{\pi^2} \tag{6}$$

where $m(s_o)$ is the measure of the u internal and m(T) is the measure of a finite set such that [c m(s)] = 2L. Notice that the product $\frac{[c\ m(s)]\ m(k)}{2\pi} = N_{dr}$ is known as the degrees of freedom of the interval 2L.

The important observation here is the exponential decay of the eigenvalues corresponding to indexes n larger than the index N_{dr} which is the degree of freedom of the interval. One concludes that the terms of the norm of g(x) corresponding to $n > N_{dr}$ are of the form of an exponentially increasing sequence, i.e.,

$$g_n \approx \alpha_n\ (1 + e^b)^{-1}\ ,\quad n > N_{dr} \tag{7}$$

One might insist that the product $\frac{\alpha_n}{\lambda_n}$ should converge and obtain a converging result. This requirement however eliminates interesting and useful applications. In order to counter this difficulty the problem will be looked at from a different point of view in the next section.

3. THE GREEN'S FUNCTION EXPANSION

Observing that $\frac{\sin k(x-x')}{k(x-x')}$ is the real part of the free space Green's function one for axially symmetric fields obtains the expansion

$$\frac{\sin k(x-x')}{k(x-x')} = \Sigma(2n+1)\, j_n(kx)\, j_n(kx') \begin{cases} P_n^2(0), & |x-x'| = |x| - |x'| \\ P_n(\pi)\, P_n(0), & |x-x'| = |x| + |x| \end{cases} \quad (8)$$

$P_n(0) = 1$ the integral equation becomes $P_n(\pi) = (-1)^n$

$$\sum_n G_n\, j_n(kx) = F(x) \quad , \quad |x| < L/2 \quad (9)$$

where

$$G_n = (2n+1) \int_0^{L/2} [g(x') - (-1)^n\, g(-x')]\, j_n(x')\, dx', \text{ a constant} \quad (10)$$

The problem now is that of finding some convenient numerical technique for a) deciding when to truncate the series to some point N and b) inverting the N x N matrix below

$$\sum_n{}' \; G_n\, j_n(x_m) = F(x_m) \qquad n=0,\ m=0,\ldots,N \quad (11)$$

For small values of x i.e., $x_m < n$ the Bessel functions are small numbers and the matrix becomes ill conditioned. In the next section an estimate on the required number of terms will be made.

4. THE APPROXIMATE DIMENSION OF INDEX SPACE

In order to arrive at some precise criterion of performance we consider the following problem. Consider a plane wave distribution $e^{i\bar{k}\cdot\bar{r}}$ which results from a step function distribution $\bar{I}(\bar{k})$. The distribution describes a source of angular size α thus establishing the resolution limit of the reconstructed aperture.

One obtains the one dimensional distribution in the image plane below

for $\rho = kx$ and $u = \cos\theta$

$$F(x) = \int_{-\infty}^{\infty} [e^{i\bar{k}\cdot\bar{r}}\, I(\bar{k})\, dk_x] = \int_{-\infty}^{+\infty} e^{i\rho u}\, A(u)\, du \qquad (12)$$

For a constant source of $A(u) = 1$, $|u_x| < \cos\alpha$

one obtains $A(u) = 0$, $|u| > \cos\alpha$

$$F(x) = \int_{-u_1}^{u_1} e^{ipu}\, du = .2u\, \frac{\sin u_1 \rho}{u_i \rho} \qquad (13)$$

$$u_i = \cos\alpha$$

Alternatively observing that

$$F(\rho) = 2 \int_0^{u_1} \cos u\, \rho\, du \qquad (14)$$

and $$\cos u\rho = \Sigma(2n + 1) \cos\left(\frac{n\pi}{2}\right) j_n(\rho)\, P_n(u) \qquad (15)$$

$$F(x) = \sum \alpha_n j_n(\rho) \tag{16}$$

where $$\alpha_n = 2 \cos \frac{n\pi}{2} \left[P_{n-1}(u_1) - P_n(u_1)\right] \tag{17}$$

or

$$\alpha_n = 2 \cos \frac{n\pi}{2} \frac{\sin\alpha}{n(n+1)} (2n+1) P_n'(\cos\alpha) \tag{18}$$

It is instructive to estimate the approximate dimension of the index space. The distribution directly in terms of the angle θ is

$$F(\theta) = 1 \quad , \quad \theta < \alpha$$

In the space of the orthogonal functions $\{\sqrt{n + 1/2}\ P_n(\cos\theta)\}$ one has

$$F(\theta) = \sum \alpha_n \{\sqrt{n + 1/2}\ P_n(\cos\theta)\} \tag{19}$$

$$\alpha_n = \frac{P_{n-1}(u_1) - P_{n+1}(u_1)}{\sqrt{2} \quad \sqrt{2n+1}} \tag{20}$$

In order to find the effective dimension one compares the above with the flat spectrum of the delta function which is

$$\delta(\theta) = \sum_{o}^{\infty} \sqrt{(n + 1/2)}\ \{\sqrt{n + 1/2}\ [P_n \cos\theta]\} \tag{21}$$

The spectrum is $\sqrt{(n + 1/2)}$

Using Parseyals theorem one has

$$\int_{o}^{\cos\alpha} |F(\theta)|^2 \quad d\ (\cos\theta) = \Sigma\ \alpha_n{}^2 = [1 - \cos\alpha] \tag{22}$$

Normalizing and comparing with N terms of the delta function spectrum one obtains

$$\sum_{o}^{N-1} (n + 1/2) = \frac{N}{2} + \frac{N(N-1)}{2} = \frac{(1 - \cos\alpha)}{2\alpha_o{}^2} \tag{23}$$

where α_o is the amplitude of the first term

$$\alpha_o = \frac{1 - P_1\ (\cos\alpha)}{\sqrt{2}} \tag{24}$$

and $1 - \cos\alpha$ is the quadratic content of $F(\theta)$

One obtains

$$N^2 = \frac{2}{1 - \cos\alpha} \tag{25}$$

or for large N and small α

$$N = \frac{2}{\alpha} \tag{26}$$

This result complies with our notion of resolution i.e., the accuracy of observation is inversely proportional with the number of dimensions of our observations. The following conclusion can now be reached.

1. A resolution cell of dimension 2α requires

$$N = \frac{2}{\alpha} \text{ dimensions in the aperture.}$$

A consequence of this is that if one sample at the Nyquist rate which for antennas is $\Delta x = \frac{\lambda}{2}$ corresponding to a field of view of $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. The dimension of the aperture are $L_N = 2N\frac{\lambda}{2}$. On the other hand if the aperture is less than the standard aperture $L < L_N$ one has to oversample in order to acquire the needed required dimensionality. Oversampling however imposed conditions on the accuracy of measurements and this will be examined in the next section.

5. MEASUREMENT ACCURACY

Let us assume that we have a flat delta function spectrum $\sqrt{n + 1/2}$ of dimensions N which is oversampled N times a segment given by

$$\frac{L}{2b} < |x| < \frac{L}{2} \quad , \quad b \text{ is a constant} \tag{27}$$

The bessel functions for small arguments are given by

$$j_n(\rho) \quad \frac{\rho^n}{(2n+1)!!} \quad , \quad \rho = kx \tag{28}$$

one then can obtain the smallest term which corresponds to the smallest x and largest N as follows. The amplitude of the smallest normalized term is

$$\left|\sqrt{2}\,\alpha_o\,\sqrt{N+1/2}\;\; j_n(x)\right|^2 = \left(1 - \cos\alpha\right)\sqrt{2N+1}\;\; j_n(x)\Big|^2 \tag{29}$$

where α_o is obtained from eqn. 24

More precisely from the Appendix and recognizing that

$$1 - \cos\alpha = \frac{2}{N^2}$$

$$\left|\sqrt{2}\,\alpha_o\,\sqrt{N + 1/2}\;\; J_n(x)\right|^2 \geq \frac{1}{2\,N^2\,(2N+1)}\left(\frac{xe}{2N}\right)^{2N} e^{-\frac{1}{12}}\left(1 - \frac{x^2}{2N+3}\right) \tag{30}$$

In order to carry out meaningful calculations one must contain the errors e_N^2 to levels below the smallest term which is

$$e_N^2 \leq \frac{1}{2N^2\,(2N+1)}\left(\frac{\rho e}{2N}\right)^{2N} e^{-\frac{1}{12}}\left(1 - \frac{\rho^2}{2N+3}\right) \quad \ldots\ldots \tag{31}$$

where $\rho = \frac{kL}{2b}$

For large N corresponding to small α the above equation reduces to

$$e_N^2 \sim \frac{1}{4N^3}\left(\frac{kL}{4b}\;\frac{e}{N}\right)^{2N} \quad \ldots\ldots \tag{32}$$

The above result can also cast into a different form by considering the natural degrees of freedom N_o corresponding to the long aperture of length of L. They are

$$N_o = 2kL \tag{33}$$

The error becomes

$$e_N \approx \frac{1}{4N^3}\left(\frac{e\;\;N_o}{8b\;N}\right)^{2N} \tag{34}$$

Finally in terms of an extrpolated aperture L' defined by

$$L' = \frac{N_o}{2K} \tag{35}$$

one obtains

$$e_L \approx \frac{1}{4\ (2kL')^3} \left(\frac{e}{8b}\ \frac{L}{L'}\right)^{4kL'} \tag{36}$$

6. NUMERICAL EXAMPLE

In order to illustrate the above extrapolation the following example was considered. A target in u space of the form

$$F(u) = 1 \qquad |u| < .5$$
$$F(u) = 0 \qquad |u| > .5$$

was viewed in the image plane in an aperture

$$L/2 = .810 \qquad k\frac{L}{2} = (2\pi)\ .810$$

The aperture sustains

$$N_o = 2.2 \text{ degrees of freedom} \qquad N = 1.610$$

The target however requires at least

$$N = \sqrt{\frac{2}{1 - \cos\alpha}} = 2$$

or more degrees of freedom.

Using equal sampling 11 points were observed in the image plane from $-.810 < x < +.810$. Because of the even symmetry however only 6

points were necessary. For 12 degrees of freedom one obtains that the error should be below

$$e_N^2 \leq 2.92 \times 10^{-12}$$

Figure 2 shows the results of the extrapolation that indicate that the aperture has been extended by roughly a factor of four.

If, on the other hand, only three points were taken corresponding to 6 degrees of freedom the error would be

$$e_N \leq 1.91 \times 10^{-3}$$

Figure 1 however shows that only a factor of 2 has been achieved.

7. DISCUSSION OF RESULTS

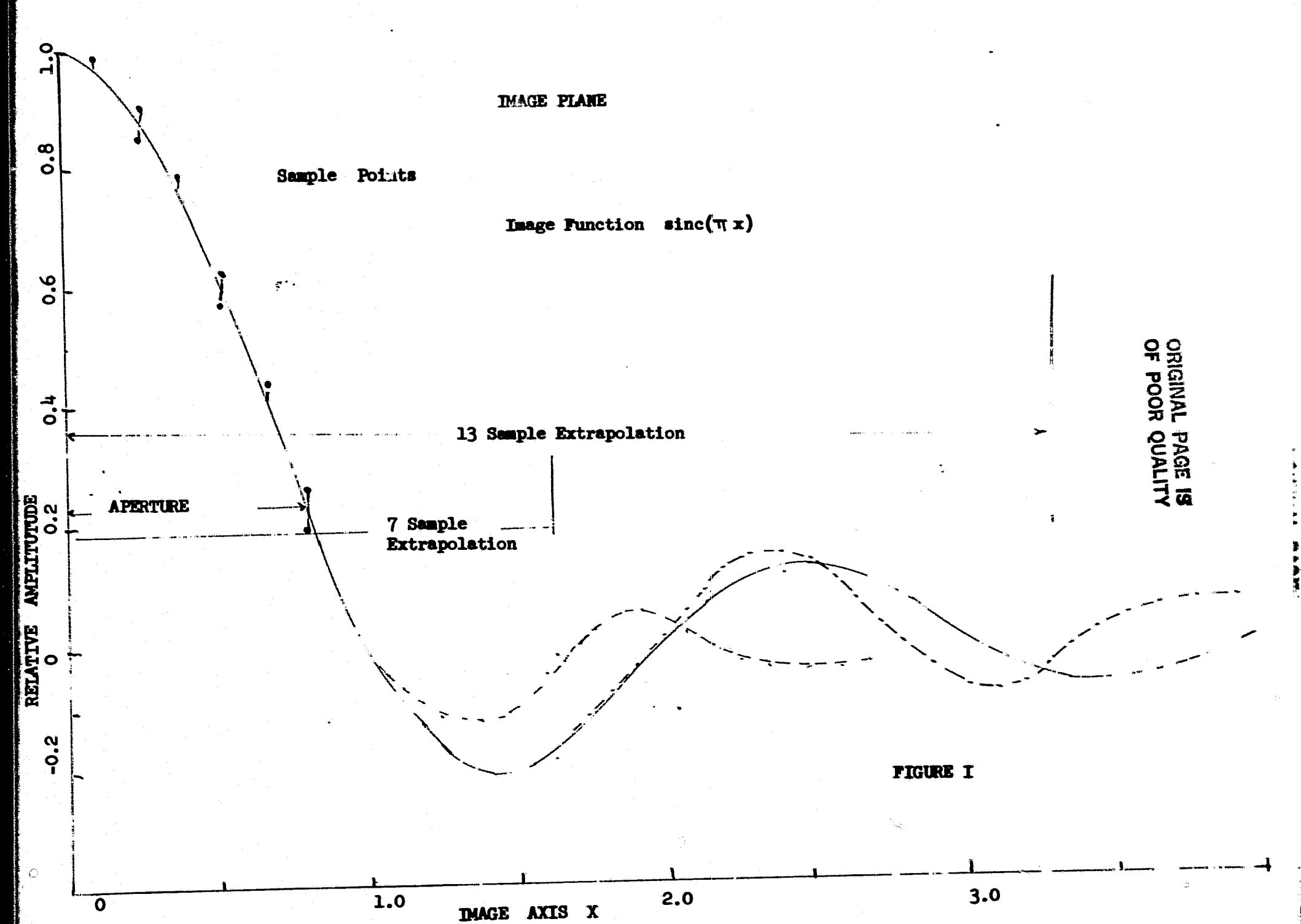
In this study one more extrapolation algorithm has been introduced to the many that already exist in the literature. Questions of uniqueness and contamination by noise were not addressed because they are adequately treated in the literature. The question which was examined here was the following: Given that we have an aperture of length L having $N_o = 2kL$ degrees of freedom and an angular resolution of $\frac{\lambda}{L}$ (λ is the wavelength) how accurate should the measurements be in order to increase the effective size to L'. In the absence of noise the results of our study show that the accuracy of measurements has to be at least as good as the term $e_{L'}$ shown below

$$e_{L'} \approx \frac{kL'}{4} \left(\frac{e}{8b} \frac{L}{L'}\right)^{2kL'} \tag{37}$$

Parameter L is an important factor in the sampling strategy. It lies in the range $1 < b < \alpha$ and it is desirable that it is as close as it is possible to L.

This indicates that the important region is the area close to the edges of the aperture. It is this area that has to be well surveyed in order to produce effective extrapolation.

The exponential behavior shown in equation (37) is similar to the one indicated by H. Landau in the asymptotic form of his eigenvalues. It seems that this is an inherent property of the extrapolation problem which makes it so difficult to implement in practice.



ORIGINAL PAGE IS OF POOR QUALITY

FIGURE I

BIBLIOGRAPHY

1. D. Slepian H.O. Pollack, Prolate Spheroidal Wave Functions Fourier Analysis and Uncertainty-I BSTJ, January 1961.

2. H.J. Landau, H.O. Pollock, "Prolate Spheroidal Wave Functions Fourier Analysis and Uncertainty-II", BSTJ, January 1961.

3. H.J. Landau, H.O. Pollock, "Prolate Spheroidal Wave Functions Fourier Analysis and Uncertainty-III", SBTJ July 1962.

4. A. Jain, S. Ranganath, "Extrapolation Algorighms for Discrete Signals with Application to Spectral Estimations IEEE Trans. on ASSP - 29 No. 4 August 1981.

5. R.W. Schaffer et. al., "Constrained Iterative Restoration Algorithms" Proc. of IEEE Vol. 69 No. 4 April 1981.

6. H.J. Landau, H. Widown, "Eigenvalue Distribution of Time and Frequency Limiting. Journal of Mathematical Analysis and Application Vol. 77 No. 2 October 1980.

7. K. Abend. Spectrum Analysis and Resolution Enhancement by Bandlimited Extropolation. Proceedings of Int. Conf. on ASSP April 1980.

8. M. Abramovitz and I. Stegun, "Handbook of Mathematical Functions Nat. Bur. Std. Appl. Math. Sr. 55 (June 1905)

APPENDIX

BOUNDS FOR BESSEL FUNCTION

The product of Bessel Functions is

$$J_p(x)\, J_q(x) = \Sigma\, (-1)^k \frac{(\frac{x}{2})^{p+q+2k}}{k!\ \Gamma(p+q+k+1)\ \Gamma(p+k+1)\ \Gamma(q+k+1)} \frac{\Gamma\ (p+q+2k+1)}{}$$

for $p = q = n + 1/2$ and $j_n = \sqrt{\frac{\pi}{2x}}\ J_{n+\frac{1}{2}}\ (x)$ one has

$$j_n^2(r) = \frac{\pi}{4} \Sigma\, (-1)^k\, (\frac{x}{2})^{2n+k}\ \frac{(2n+2k-1)!}{k!\,(2n+kn)}\ \ \frac{2^{2n+2k+2}}{[(n+k+1)!!]^2\ \pi}$$

because $\Gamma\ (n + \frac{1}{2}) = \frac{\sqrt{\pi}}{2^n}\ (2n - 1)!!$

$$j_n^2(x) = \Sigma\, (-1)\ \ \rho^{2n+2k}\ \ \frac{(2n+2k+1)!}{k!\ (2n+k+1)!\ [(2n+2k+1)!!]^2}$$

$$= \rho^{2n} \left\{ \frac{1}{[(2n+1)!!]^2} - \frac{(2n+3)\ \rho^2}{[(2n+3)!!]^2} \right\} + R$$

$$= \frac{\rho^{2n}}{(2n+1)!!}\ \ [1 - \frac{(2n+3)\ \rho^2}{(2n+3)^2}] + R$$

$$= \frac{\rho^{2n}}{[(2n+3)!!]^2} \left[1 - \frac{\rho^2}{(2n+3)}\right] + R$$

The remainder R is a positive quantity because the k+1 term is always less than the k term.

The ratio of the A_{k+1} to the A_k term is

$$\frac{A_{k+1}}{A_k} = \frac{(2n+2k+3)}{(k+1)(2n+k+2)} \quad \frac{\rho^2}{[(2n+2k+3)^2]} < 1$$

for $\rho < 1$

One obtains then

$$\frac{\rho^{2n}}{[(2n+1)!!]} \left(1 - \frac{\rho^2}{2n+3}\right) < j_n^2(x) < \frac{\rho^{2n}}{[(2n+1)!!]^2}$$

or since

$$\Gamma(n + \tfrac{1}{2}) = \frac{\sqrt{2\pi}}{2^{2n-\frac{1}{2}}} \quad \frac{\Gamma(2n)}{\Gamma(n)} = \frac{(2n-1)!!}{2^n} \quad \sqrt{\pi}$$

then

$$(2n+1)!! = \frac{(2n+2)!}{(n+1)!\ 2^{n+1}} = (2n+1)\ (2n-1)!! = (2n+1)\ \frac{(2n)!}{(n!)\ 2^n}$$

and since

$$\sqrt{2\pi n}\ \left(\frac{n}{e}\right)^n\ e^{\frac{1}{12n}} > n! > \sqrt{2\pi n}\ \left(\frac{n}{e}\right)^n$$