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Abstract

In this paper the plane elasticity problem for a nonhomogeneous medium containing a crack is considered. It is assumed that the Poisson's ratio of the medium is constant and the Young's modulus E varies exponentially with the coordinate parallel to the crack. First the half plane problem is formulated and the solution is given for arbitrary tractions along the boundary. Then the integral equation for the crack problem is derived. It is shown that the integral equation having the derivative of the crack surface displacement as the density function has a simple Cauchy type kernel. Hence, its solution and the stresses around the crack tips have the conventional square-root singularity. The solution is given for various loading conditions. The results show that the effect of the Poisson's ratio and consequently that of the thickness constraint on the stress intensity factors are rather negligible. On the other hand, the results are highly affected by the parameter β describing the material nonhomogeneity in $E(x) = E_0\exp(\beta x)$.

1. Introduction

In practical applications the material nonhomogeneity becomes an important factor to be considered particularly in two classes of problems. The first is a group of problems in geophysics in which, because of the size of the medium, the spatial variation of the material constants cannot be assumed as being negligible. The foundation and contact problems in soil mechanics and the wave propagation problems in the earth's crust may be mentioned as some of the examples. The second group of problems relates to the fracture of essentially nonhomogeneous solids. "Hydraulic fracturing" of the medium which consists of sandstone and shale, the fracture of structural materials with periodically varying material properties (as in certain laminated structures), and the fracture of variety of fuse-bonded materials

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used in electronics industry are some typical examples. The distinguishing feature of these materials is that the material constants are continuous and generally differentiable functions of the space coordinates, whereas in the standard particulate, layered, and fiber-reinforced composites the material constants are discontinuous functions. The consequence of the discontinuous behavior of the material in crack problems is that the nature of singularity of the stresses at the crack tip which is on the interface and at the point of intersection of a crack and the interface is quite different than the singularity exhibited by a crack tip which is fully imbedded in a homogeneous medium. Even though no systematic study of the problem appears to have been made, it is reasonable to expect that in nonhomogeneous materials with continuous and continuously differentiable elastic constants the nature of the stress singularity at a crack tip would be identical to that of a homogeneous solid. The existing solutions of crack and punch problems in certain specific nonhomogeneous materials seem to support this view.

In most of the existing solutions of problems relating to nonhomogeneous solids it is assumed that the material is isotropic, the Poisson's ratio is constant, and the Young's (or the shear) modulus is either an exponential or a power function of a space variable [1]. In the wedge problem described in [2] the class of functions $E(r,\theta)$ for the Young's modulus leading to a feasible solution has been investigated. Some sample studies of the Boussinesq and contact problems for a nonhomogeneous half space may be found in [3]-[8]. The corresponding "torsion" problem for a half space is described in [9] and [10]. The equivalent crack problems in a nonhomogeneous medium under torsion and under anti-plane shear loading are discussed in [11] and [12], respectively. In the studies described in [3] and [8], the shear modulus is assumed to be $\mu_0\exp(\gamma y)$ and in [12] $\mu_0\exp(x^\alpha+y^\beta)$, where $y=0$ is either the boundary of the half plane or the plane of the crack. In [4]-[7] and [9]-[11] it is assumed that $\mu = \mu_0|y|^m$, $(0<m<1)$. This latter assumption clearly has the undesirable physical feature in that at $y=0$ the shear modulus becomes zero. It is particularly difficult to attach any physical meaning to the solution of a crack problem carried out under this assumption. This difficulty has been removed in the crack problem considered in [13] where it was assumed that the shear modulus is given by $\mu=\mu_0/(1+c|y|)$, where c is a constant. It should again be noted that the solutions given in [3]-[13] are based on the assumption that the Poisson's ratio is constant.

In the studies mentioned above (with the exception of [12] which deals with the relatively simple problem of anti-plane shear) it is assumed that in the direction(s) parallel to the boundary of the half plane or the plane of the crack the shear modulus does not vary. In the case of fracture of such a nonhomogeneous medium since, generally, the plane of the crack is not a plane of symmetry, the in-plane shear component of the stress intensity factor would not be zero and hence the propagating crack would eventually align itself parallel to the direction in which the modulus varies. In the problem considered in this paper it is then assumed that the crack is located on the y=0 plane, the Young's modulus is an exponential function of x, and the Poisson's ratio is constant (Fig. 1).

2. Formulation of the Crack Problem

Consider the plane elasticity problem for a nonhomogeneous solid in which the Poisson's ratio ν is constant and the Young's modulus E is a function of x and y. Let F(x,y) be the Airy stress function. The stresses are given by

$$\sigma_{xx} = \frac{\partial^2 F}{\partial y^2}\,,\quad \sigma_{yy} = \frac{\partial^2 F}{\partial x^2}\,,\quad \sigma_{xy} = -\frac{\partial^2 F}{\partial x \partial y}\,. \tag{1}$$

Substituting from (1) through the Hooke's Law into the compatibility equation, for the plane problem we obtain

$$E^2\nabla^4 F - 2E\left(\frac{\partial E}{\partial x}\frac{\partial}{\partial x} + \frac{\partial E}{\partial y}\frac{\partial}{\partial y}\right)\nabla^2 F + 2(1+\nu)\left(2\,\frac{\partial E}{\partial x}\frac{\partial E}{\partial y} - E\,\frac{\partial^2 E}{\partial x \partial y}\right)\frac{\partial^2 F}{\partial x \partial y}$$

$$+ \left[2\left(\frac{\partial E}{\partial x}\right)^2 - 2\nu\left(\frac{\partial E}{\partial y}\right)^2 - E\,\frac{\partial^2 E}{\partial x^2} + \nu E\,\frac{\partial^2 E}{\partial y^2}\right]\frac{\partial^2 F}{\partial x^2}$$

$$+ \left[2\left(\frac{\partial E}{\partial y}\right)^2 - 2\nu\left(\frac{\partial E}{\partial x}\right)^2 - E\,\frac{\partial^2 E}{\partial y^2} + \nu E\,\frac{\partial^2 E}{\partial x^2}\right]\frac{\partial^2 F}{\partial y^2} = 0\,. \tag{2}$$

Equation (2) is for the generalized plane stress. The differential equation for plane strain is obtained by replacing E and ν by $E/(1-\nu^2)$ and $\nu/(1-\nu)$, respectively. From (2) it may easily be verified that if we let

$$E = E_0 \exp(\beta x+\gamma y) \ , \ \nu = \text{constant} \tag{3}$$

the differential equation becomes one of constant coefficients which may be written as

$$\nabla^4 F - 2\left(\beta \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial y}\right)\nabla^2 F + (\beta^2-\nu\gamma^2) \frac{\partial^2 F}{\partial x^2} + 2(1+\nu)\beta\gamma \frac{\partial^2 F}{\partial x \partial y}$$
$$+ (\gamma^2-\nu\beta^2) \frac{\partial^2 F}{\partial y^2} = 0 \ . \tag{4}$$

In problems involving the study of localized phenomena such as perturbation in stress state due to the presence of a crack or a punch a material representation such as (3) would not be very unrealistic. In most cases a reasonable approximation to the actual distribution of $E(x,y)$ can be obtained by adjusting the constants E_0, β, and γ. Referring to Fig. 1, in this problem we will further assume that E is independent of y. Thus, $y=0$ is a plane of symmetry provided we also consider only those external loads which are symmetric with respect to $y=0$. It is therefore sufficient to consider one half of the medium, $(-\infty<x<\infty, \ y>0)$ only. By letting $\gamma=0$ in (4) the differential equation of the problem becomes

$$\nabla^4 F - 2\beta\left(\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial x \partial y^2}\right) + \beta^2 \frac{\partial^2 F}{\partial x^2} - \nu\beta^2 \frac{\partial^2 F}{\partial y^2} = 0 \ . \tag{5}$$

Note that (5) reduces to the standard biharmonic equation for $\beta=0$.

Assuming the solution of (5) in the form

$$F(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y,\alpha) e^{-ix\alpha} \, d\alpha \ , \ (-\infty<x<\infty, \ y>0) \ , \tag{6}$$

we obtain

$$\frac{d^4 f}{dy^4} + (2i\beta\alpha - 2\alpha^2 - \beta^2\nu) \frac{d^2 f}{dy^2} + (\alpha^4 - 2i\beta\alpha^3 - \beta^2\alpha^2) f = 0 \ . \tag{7}$$

If we now look for a solution of (7) of the form $f = \exp(my)$ we find

$$m^4 + (2i\beta\alpha-2\alpha^2-\beta^2\nu)m^2 + (\alpha^4-2i\beta\alpha^3-\beta^2\alpha^2) = 0 \ . \tag{8}$$

The solution of (8) is found to be

$$m_1 = -m_3 = [(-\gamma_1+\gamma_2)/2]^{\frac{1}{2}} \ , \ m_2 = -m_4 = [(-\gamma_1-\gamma_2)/2]^{\frac{1}{2}} \ ,$$

$$\gamma_1 = 2i\beta\alpha-2\alpha^2-\beta^2\nu, \ \gamma_2 = (\beta^4\nu^2-4i\beta^3\nu\alpha+4\beta^2\nu\alpha^2)^{\frac{1}{2}} \ . \tag{9}$$

In (9) the roots m_j are ordered in such a way that $Re(m_1)>0$, $Re(m_2)>0$. It is assumed that the problem in the absence of the crack has been solved under the actual loading conditions, and that the crack length 2a is "small" compared to other (planar) dimensions of the solid. Through a superposition the singular part of the solution may then be reduced to that of an infinite nonhomogeneous plane with the self-equilibrating crack surface tractions as the only external loads. Thus, in the problem of interest the stresses and displacements vanish as $(x^2+y^2)\to\infty$, and the solution of (7) may be expressed as

$$f(y,\alpha) = A_1(\alpha)e^{-m_1 y} + A_2(\alpha)e^{-m_2 y} \ , \ (0<y<\infty) \ . \tag{10}$$

From (1), (6) and (9) it then follows that

$$\sigma_{xx}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_1^2 A_j m_j^2 e^{-m_j y} \ e^{-ix\alpha} \ d\alpha \ , \tag{11}$$

$$\sigma_{yy}(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha^2 \sum_1^2 A_j e^{-m_j y} \ e^{-ix\alpha} \ d\alpha \ , \tag{12}$$

$$\sigma_{xy}(x,y) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} \alpha \sum_1^2 A_j m_j e^{-m_j y} \ e^{-ix\alpha} \ d\alpha \ . \tag{13}$$

This completes the formulation of the problem for the half plane $y>0$ in which the functions A_1 and A_2 are determined from the two boundary conditions at y=0, $-\infty<x<\infty$. For example, let the half plane be subjected to tractions

$$\sigma_{yy}(x,0) = \sigma(x) \ , \ \sigma_{xy}(x,0) = \tau(x) \ , \ (-\infty<x<\infty) \tag{14}$$

on the boundary y=0 and be kept in equilibrium by a resultant force applied to the medium at infinity which is collinear with a force defined by the following x and y-components:

$$P_x = \int_{-\infty}^{\infty} \tau(x)dx \ , \ P_y = \int_{-\infty}^{\infty} \sigma(x)dx \ . \tag{15}$$

From (12) and (13) A_1 and A_2 may then be obtained as follows:

$$A_1(\alpha) = \frac{1}{m_1-m_2}\left(\frac{Q_1}{\alpha^2} + i\,\frac{Q_2}{\alpha}\right) , \tag{16}$$

$$A_2(\alpha) = -\left(1 + \frac{1}{m_1-m_2}\right)\frac{Q_1}{\alpha^2} - \frac{i}{m_1-m_2}\,\frac{Q_2}{\alpha} , \tag{17}$$

where

$$Q_1(\alpha) = \int_{-\infty}^{\infty} \sigma(x)e^{ix\alpha}\,dx \ , \ Q_2(x) = \int_{-\infty}^{\infty} \tau(x)e^{ix\alpha}\,dx \ . \tag{18}$$

3. The Integral Equation

We assume that the original cracked solid is loaded symmetrically in such a way that

$$\sigma_{xy}(x,0) = 0 \ , \ -\infty<x<\infty \ . \tag{19}$$

In the perturbation problem, in addition to (19), we then have the following mixed boundary condition

$$\sigma_{yy}(x,+0) = p(x) \ , \ (-a<x<a) \ , \tag{20}$$

$$v(x,0) = 0 \ , \ a<|x|<\infty \ , \tag{21}$$

where p(x) is a known function and v is the y-component of the displacement. From (13) and (19) it follows that

$$m_1A_1 + m_2A_2 = 0 \ . \tag{22}$$

To obtain the second equation to determine A_1 and A_2 we introduce a new unknown function g(x) by

$$g(x) = \frac{\partial}{\partial x} v\ (x,+0)\ , \tag{23}$$

From (21) and (23) it is seen that g(x) = 0 for $|x|>a$ and

$$\int_{-a}^{a} g(x)dx = 0 \quad . \tag{24}$$

By using the Hooke's law from (11) and (12) it can be shown that

$$\frac{\partial}{\partial x} v(x,y) = - \frac{1}{2\pi} \frac{1}{E(x)} \int_{-\infty}^{\infty} (\beta+i\alpha)[\frac{A_1}{m_1} (\alpha^2+\nu m_1{}^2)e^{-m_1y}$$

$$+ \frac{A_2}{m_2} (\alpha^2+\nu m_2{}^2)e^{-m_2y}]e^{-ix\alpha}\ d\alpha\ ,\ (y>0)\ . \tag{25}$$

Equation (23) and (25) would then give

$$- \frac{\beta+i\alpha}{E_0} [\frac{A_1}{m_1} (\alpha^2+\nu m_1{}^2) + \frac{A_2}{m_2} (\alpha^2+\nu m_2{}^2)] = \int_{-a}^{a} g(t)e^{(\beta+i\alpha)t}\ dt. \tag{26}$$

From (22) and (26) the functions A_1 and A_2 are now determined as follows:

$$A_1(\alpha) = \frac{E_0m_1m_2{}^2}{\alpha^2(m_1{}^2-m_2{}^2)(\beta+i\alpha)} \int_{-a}^{a} g(t)e^{(\beta+i\alpha)t}dt = - \frac{m_2}{m_1} A_2\ . \tag{27}$$

By substituting from (27) and (12) into (20) we obtain the following integral equation to determine g(x):

$$\lim_{y \to +0} \frac{1}{2\pi} \int_{-a}^{a} g(t) e^{\beta t} \, dt \int_{-\infty}^{\infty} \frac{E_o m_1 m_2}{(\beta + i\alpha)(m_1^2 - m_2^2)} (m_1 e^{-m_2 y}$$

$$- m_2 e^{-m_1 y}) e^{i(t-x)\alpha} \, d\alpha = p(x) \; , \; |x| < a \; . \tag{28}$$

To investigate and to separate a possible singular part of the kernel in (28) the asymptotic behavior of the inner integral must be examined. From (9) we observe that for $|\alpha| \to \infty$ $m_1 \to |\alpha|$ and $m_2 \to |\alpha|$. Now, by expressing the inner integral in (28) as

$$h(x,y,t) = \int_{-\infty}^{\infty} K(y,\alpha) e^{i(t-x)\alpha} \, d\alpha \tag{29}$$

and by noting that any singular part h may have must be due to the behavior of K at $|\alpha| \to \infty$, we may write h as follows:

$$h(x,y,t) = \int_{-\infty}^{\infty} [K(y,\alpha) - K_\infty(y,\alpha)] e^{i(t-x)\alpha} \, d\alpha$$

$$+ \int_{-\infty}^{\infty} K_\infty(y,\alpha) e^{i(t-x)\alpha} \, d\alpha \; . \tag{30}$$

where K_∞ is the asymptotic value of $K(y,\alpha)$ for large values of $|\alpha|$. The first integral in (30) is uniformly convergent and, therefore, when substituted into (28) the limit can be put under the integral sign. It may easily be shown that

$$K_\infty(y,\alpha) = \frac{E_o}{2i} \frac{\alpha}{|\alpha|} e^{-|\alpha| y} \; , \tag{31}$$

and the second integral in (28) may be expressed as

$$\int_{-\infty}^{\infty} \frac{E_o}{2i} \frac{\alpha}{|\alpha|} e^{-|\alpha| y} [\cos(t-x)\alpha + i \sin(t-x)\alpha] d\alpha = \frac{E_o (t-x)}{(t-x)^2 + y^2} \; . \tag{32}$$

Also, by defining

$$M(\alpha) = \frac{m_1 m_2}{(\beta+i\alpha)(m_1+m_2)} e^{i(t-x)\alpha} , \tag{33}$$

and by substituting y=0 in the first integral, (30) may now be written as follows:

$$h(x,y,t) = E_0 \int_0^\infty [M(\alpha)+M(-\alpha) - \sin(t-x)\alpha]d\alpha + \frac{E_0(t-x)}{(t-x)^2+y^2} . \tag{34}$$

Finally, if we substitute from (34) into (28) and go to limit we obtain

$$\frac{1}{\pi}\int_{-a}^{a} [\frac{e^{\beta t}}{t-x} + k(x,t)]g(t)dt = \frac{1+\kappa}{4\mu_0} p(x) , (-a<x<a) , \tag{35}$$

where the Fredholm kernel is defined by

$$k(x,t) = e^{\beta t} \int_0^\infty [M(\alpha) + M(-\alpha) - \sin(t-x)\alpha]d\alpha , \tag{36}$$

and $E_0/2$ is replaced by $4\mu_0/(1+\kappa)$ in order to cover both generalized plane stress and plane strain problems. Here μ_0 is the shear modulus at x=0, i.e., $\mu_0 = E_0/2(1+\nu)$, $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for the generalized plane stress. Note that for $\beta = 0$, $K = K_\infty$, $k(x,t) = 0$, and (35) reduces to the known integral equation of the simple (Mode I) crack problem for a homogeneous plane.

For numerical solution the interval (-a,a) is normalized by defining

$$s = t/a,\ r = x/a,\ \phi(s) = g(t),\ n(r,s) = k(x,t),\ q(r) = p(x),$$
$$-1<(r,s)<1,\ -a<(x,t)<a \ . \tag{37}$$

In terms of the normalized quantities the integral equation (35) and the single-valuedness condition (24) may be expressed as

$$\frac{1}{\pi}\int_{-1}^{1} [\frac{e^{\beta as}}{s-r} + n(r,s)]\phi(s)ds = \frac{1+\kappa}{4\mu_0} q(r) , (-1<r<1) , \tag{38}$$

$$\int_{-1}^{1} \phi(s)ds = 0 \quad . \tag{39}$$

4. Stress Intensity Factors

The index of the singular integral equation (38) is +1. Therefore its solution is of the following form:

$$e^{a\beta s} \phi(s) = \frac{G(s)}{\sqrt{1-s^2}} , \; -1<s<1 \quad , \tag{40}$$

where G(s) is a bounded function. The unknown function G may be determined from (38) and (39) to any desired degree of accuracy by using a Gaussian integration technique to solve the singular integral equation (see, for example, [14]). By observing that the left-hand side of (35) gives $\sigma_{yy}(x,0)$ for $|x|>a$ as well as for $|x|<a$, through a simple asymptotic analysis, the Mode I stress intensity factors at the crack tips defined by

$$k_1(a) = \lim_{x \to a} \sqrt{2(x-a)}\, \sigma_{yy}(x,0) \; , \tag{41}$$

$$k_1(-a) = \lim_{x \to -a} \sqrt{2(-x-a)}\, \sigma_{yy}(x,0) \; , \tag{42}$$

may be expressed in terms of G(s) as follows(*):

$$k_1(a) = -\frac{4}{1+\kappa} \mu_0 e^{\beta a} \; G(1)\sqrt{a} \quad , \tag{43}$$

$$k_1(-a) = \frac{4}{1+\kappa} \mu_0 e^{\beta a} \; G(-1)\sqrt{a} \quad . \tag{44}$$

From (23) and (37) it is seen that after obtaining G(s) the crack surface displacement may be calculated as

(*) Note that $\mu_0 \exp(\beta a) = \mu(a)$ and the expressions (43) and (44) are identical to those found for the homogeneous materials.

$$\frac{v(x)}{a} = \int_{-1}^{x/a} \frac{G(s)}{\sqrt{1-s^2}} e^{-a\beta s}\, ds. \tag{45}$$

It should again be emphasized that the structure of the integral equation (35) is essentially the same as that of a homogeneous medium, namely its kernel has a simple Cauchy singularity. Therefore, its solution and consequently the stress state around the crack tip would have the conventional square root singularity (see (40)-(42)).

5. Results and Discussion

The crack problem is solved for two types of loading. In the first it is assumed that the plane is loaded by prescribing the displacements in such a way that in the uncracked medium we have

$$\varepsilon_{yy}(x,0) = \varepsilon_0 + \varepsilon_1 (x/a)\ ,\ \varepsilon_{xy}(x,0) = 0,\ \sigma_{xx}(x,0) = 0\ . \tag{46}$$

From (46) the crack surface tractions in the perturbation problem may be expressed as follows:

$$\sigma_{yy}(x,0) = p(x) = -\varepsilon_0 E_0 e^{\beta x} - \varepsilon_1 E_0 (\tfrac{x}{a}) e^{\beta x},\ \sigma_{xy}(x,0) = 0,\ |x|<a\ . \tag{47}$$

In the second type of loading crack surface traction p(x) will simply be assumed to be a polynomial of the form

$$p(x) = -p_0 - p_1(\tfrac{x}{a}) - p_2(\tfrac{x}{a})^2 - p_3(\tfrac{x}{a})^3\ . \tag{48}$$

In the normalized integral equation (38) the material parameter β enters into the kernel through $a\beta$ only. Thus the calculated stress intensity factors are given with $a\beta$ as the variable. The results are given in Tables 1 and 2. Note that in the nonhomogeneous material problem since the kernel is also dependent on the Poisson's ratio ν, the solution must be obtained for a given value of ν, and for the cases of plane stress and plane strain separately. Tables 1 and 2 give the normalized stress intensity factors at the crack tips a and -a for plane stress and plane strain cases, respectively, by assuming that $\nu = 0.3$. The results are obtained by taking only one of the six input parameters ε_0, ε_1, p_0, p_1, p_2, and p_3 nonzero at a time. Since the problem is linear the results can be superimposed in any suitable manner. Note that for $\beta a \to 0$, that is, for a given a and $\beta \to 0$ or a given β and $a \to 0$, as expected, the stress intensity factor ratios reduce to those of a homogeneous medium, which, for an arbitrary traction p(x), are given by

$$k_1(a) = \frac{1}{\pi\sqrt{a}} \int_{-a}^{a} p(x) \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}} dx \,, \tag{49}$$

$$k_1(-a) = \frac{1}{\pi\sqrt{a}} \int_{-a}^{a} p(x) \left(\frac{a-x}{a+x}\right)^{\frac{1}{2}} dx \,. \tag{50}$$

For $\beta a = 0.5$ the effect of Poisson's ratio on the stress intensity factors in plane stress and plane strain cases is shown in Table 3. The results show that this effect is rather insignificant. Consequently, as seen from Tables 1-3 the difference between plane stress and plane strain results is also insignificant.

Since the effect of the Poisson's ratio and the thickness constraint on the stress intensity factors is negligibly small, from the results given in Tables 1 and 2 it is possible to develop simple empirical formulas for the stress intensity factors. For example, from Fig. 2 reproducing the results for a uniformly pressurized crack (columns 6 and 7, Table 1) it may be seen that the stress intensity factors vary approximately linearly with the variable βa. Hence, in this case, the following approximate formulas may be used to evaluate the stress intensity factors:

$$k_1(a) \cong p_0\sqrt{a}(1 + 0.22\beta a) \,, \tag{51}$$

$$k_1(-a) \cong p_0\sqrt{a}(1 - 0.26\beta a) \,. \tag{52}$$

A sample result showing the crack surface displacement v(x) for a uniformly pressurized crack as calculated from (45) is given in Fig. 3. The figure also shows v(x) for a homogeneous material. As expected, in the nonhomogeneous medium in the stiffer portion of the material the crack surface displacement is smaller than that of the homogeneous medium, and the reverse trend may be observed in the less stiff portion of the material.

In this as well as in the previous studies the assumption of constant Poisson's ratio has been made for analytical reasons. Even though not very conclusive, the results given in this paper show that neglecting the possible spacial variation of the Poisson's ratio is not a very restrictive assumption.

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Table 1. The normalized stress intensity factors for various loading conditions for the case of generalized plane stress (ν=0.3).

βa	$\frac{k_1(a)}{\varepsilon_0 E_0 \sqrt{a}}$	$\frac{k_1(-a)}{\varepsilon_0 E_0 \sqrt{a}}$	$\frac{k_1(a)}{\varepsilon_1 E_0 \sqrt{a}}$	$\frac{k_1(-a)}{\varepsilon_1 E_0 \sqrt{a}}$	$\frac{k_1(a)}{p_0 \sqrt{a}}$	$\frac{k_1(-a)}{p_0 \sqrt{a}}$	$\frac{k_1(a)}{p_1 \sqrt{a}}$	$\frac{k_1(-a)}{p_1 \sqrt{a}}$	$\frac{k_1(a)}{p_2 \sqrt{a}}$	$\frac{k_1(-a)}{p_2 \sqrt{a}}$	$\frac{k_1(a)}{p_3 \sqrt{a}}$	$\frac{k_1(-a)}{p_3 \sqrt{a}}$
→0	1.0	1.0	0.5	-0.5	1.0	1.0	0.5	-0.5	0.5	0.5	0.375	-0.375
0.01	1.008	0.992	0.505	-0.495	1.003	0.997	0.500	-0.500	0.501	0.499	0.375	-0.375
0.10	1.078	0.925	0.552	-0.453	1.025	0.973	0.500	-0.500	0.506	0.493	0.375	-0.375
0.25	1.202	0.820	0.640	-0.389	1.060	0.930	0.498	-0.499	0.515	0.483	0.374	-0.375
0.50	1.435	0.665	0.814	-0.302	1.113	0.861	0.495	-0.495	0.528	0.466	0.372	-0.373
0.75	1.713	0.535	1.031	-0.234	1.162	0.797	0.489	-0.489	0.540	0.450	0.369	-0.370
1.00	2.048	0.429	1.301	-0.181	1.209	0.740	0.483	-0.480	0.552	0.435	0.366	-0.366

Table 2. The normalized stress intensity factors for various loading conditions for the case of plane strain (ν=0.3).

βa	$\frac{(1-\nu^2)k_1(a)}{\varepsilon_0 E_0 \sqrt{a}}$	$\frac{(1-\nu^2)k_1(-a)}{\varepsilon_0 E_0 \sqrt{a}}$	$\frac{(1-\nu^2)k_1(a)}{\varepsilon_1 E_0 \sqrt{a}}$	$\frac{(1-\nu^2)k_1(-a)}{\varepsilon_1 E_0 \sqrt{a}}$	$\frac{k_1(a)}{p_0 \sqrt{a}}$	$\frac{k_1(-a)}{p_0 \sqrt{a}}$	$\frac{k_1(a)}{p_1 \sqrt{a}}$	$\frac{k_1(-a)}{p_1 \sqrt{a}}$	$\frac{k_1(a)}{p_2 \sqrt{a}}$	$\frac{k_1(-a)}{p_2 \sqrt{a}}$	$\frac{k_1(a)}{p_3 \sqrt{a}}$	$\frac{k_1(-a)}{p_3 \sqrt{a}}$
$\to 0$	1.0	1.0	0.5	-0.5	1.0	1.0	0.5	-0.5	0.5	0.5	0.375	-0.375
0.01	1.008	0.992	0.505	-0.495	1.003	0.997	0.500	-0.500	0.501	0.499	0.375	-0.375
0.10	1.078	0.925	0.552	-0.453	1.026	0.973	0.500	-0.500	0.506	0.493	0.375	-0.375
0.25	1.203	0.821	0.640	-0.389	1.061	0.931	0.498	-0.499	0.515	0.483	0.374	-0.375
0.50	1.439	0.667	0.814	-0.302	1.117	0.863	0.494	-0.495	0.529	0.466	0.372	-0.373
0.75	1.721	0.539	1.032	-0.234	1.170	0.801	0.489	-0.489	0.542	0.451	0.369	-0.370
1.00	2.063	0.433	1.304	-0.181	1.222	0.745	0.483	-0.481	0.555	0.436	0.366	-0.366

Table 3. The effect of Poisson's ratio on the stress intensity factors. (βa = 0.5) ($E_0^* = E_0$ for plane stress, $E_0^* = \frac{E_0}{1-\nu^2}$ for plane strain).

	ν	$\frac{k_1(a)}{\varepsilon_0 E_0^* \sqrt{a}}$	$\frac{k_1(-a)}{\varepsilon_0 E_0^* \sqrt{a}}$	$\frac{k_1(a)}{\varepsilon_1 E_0^* \sqrt{a}}$	$\frac{k_1(-a)}{\varepsilon_1 E_0^* \sqrt{a}}$	$\frac{k_1(a)}{p_0 \sqrt{a}}$	$\frac{k_1(-a)}{p_0 \sqrt{a}}$	$\frac{k_1(a)}{p_1 \sqrt{a}}$	$\frac{k_1(-a)}{p_1 \sqrt{a}}$	$\frac{k_1(a)}{p_2 \sqrt{a}}$	$\frac{k_1(-a)}{p_2 \sqrt{a}}$	$\frac{k_1(a)}{p_3 \sqrt{a}}$	$\frac{k_1(-a)}{p_3 \sqrt{a}}$
Plane Stress	0.01	1.425	0.660	0.813	-0.302	1.104	0.856	0.494	-0.494	0.525	0.464	0.372	-0.372
	0.15	1.430	0.662	0.813	-0.302	1.108	0.858	0.494	-0.495	0.526	0.465	0.372	-0.373
	0.30	1.435	0.665	0.814	-0.302	1.113	0.861	0.495	-0.493	0.528	0.466	0.372	-0.373
	0.50	1.441	0.668	0.815	-0.302	1.119	0.865	0.495	-0.496	0.529	0.467	0.372	-0.373
Plane Strain	0.01	1.425	0.660	0.813	-0.302	1.104	0.856	0.494	-0.494	0.525	0.464	0.372	-0.372
	0.15	1.431	0.663	0.813	-0.302	1.109	0.589	0.494	-0.495	0.527	0.465	0.372	-0.373
	0.30	1.439	0.667	0.814	-0.302	1.117	0.863	0.494	-0.495	0.529	0.466	0.372	-0.373
	0.50	1.457	0.677	0.817	-0.302	1.134	0.874	0.495	-0.497	0.533	0.469	0.372	-0.374

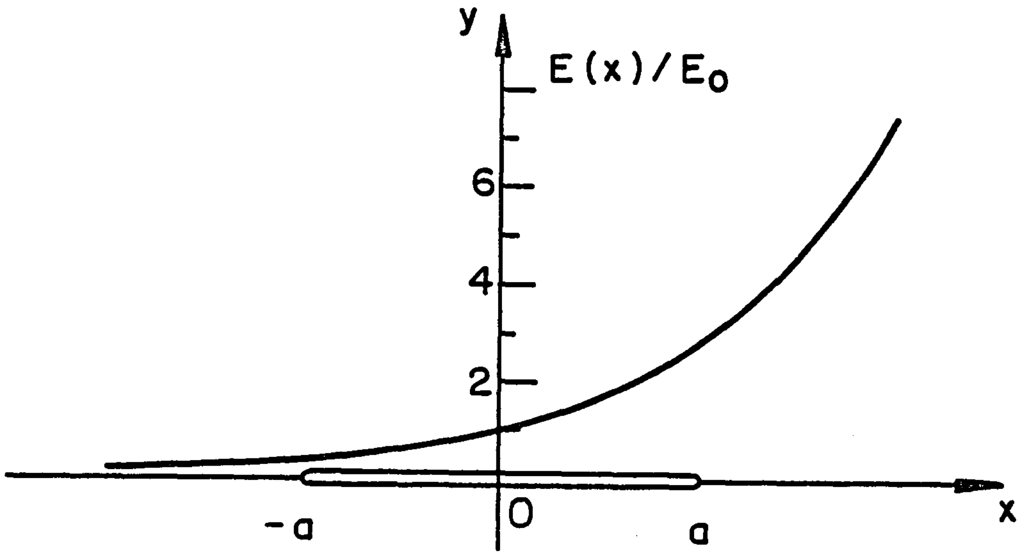

Fig. 1. The crack geometry in the nonhomogeneous medium and the variation of the Young's modulus $E = E_0 e^{\beta x}$.

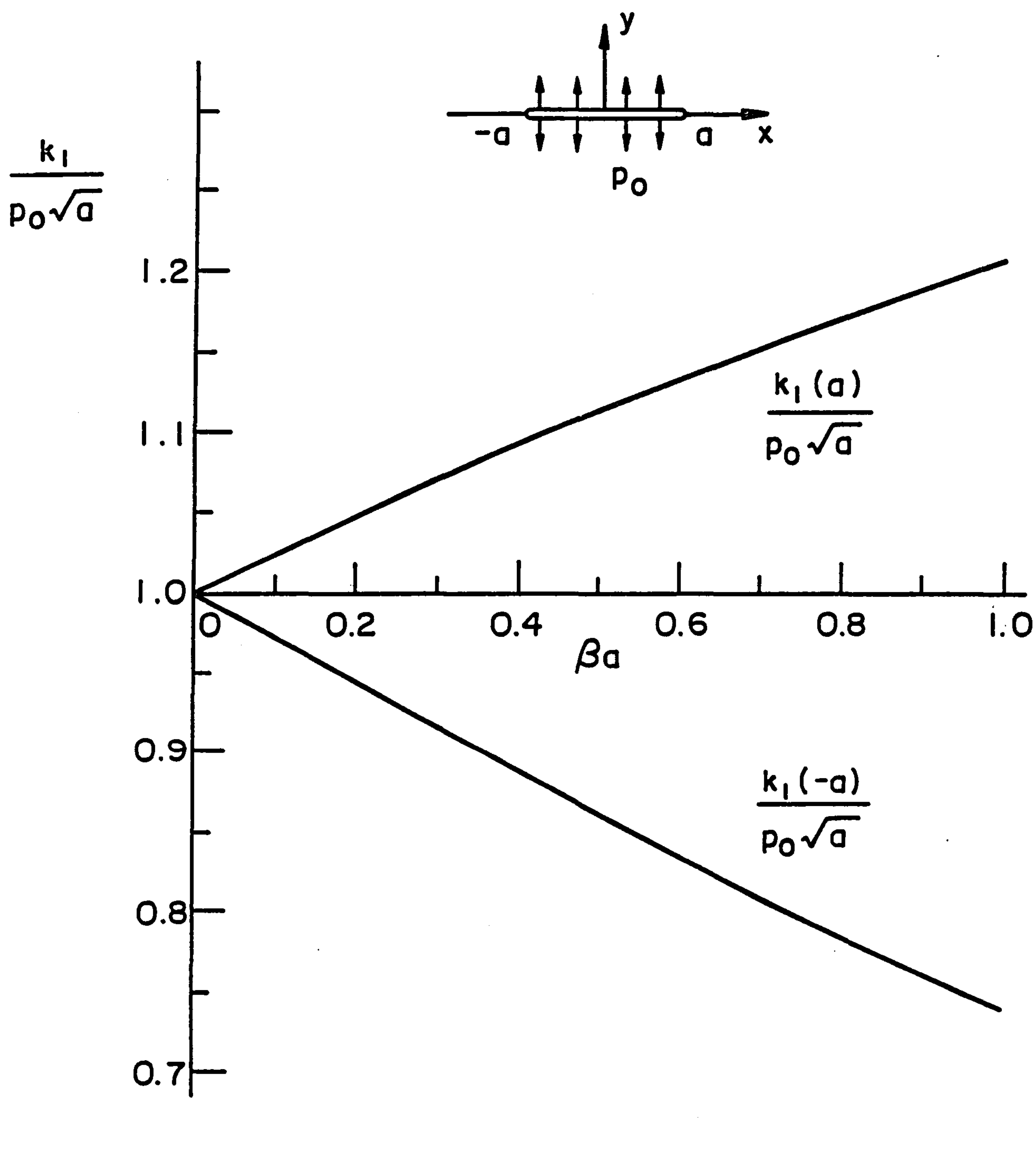

Fig. 2. Stress intensity factors in a nonhomogeneous medium having a uniformly pressurized crack ($E(x) = E_0 e^{\beta x}$, $\nu = 0.3$, plane stress conditions).

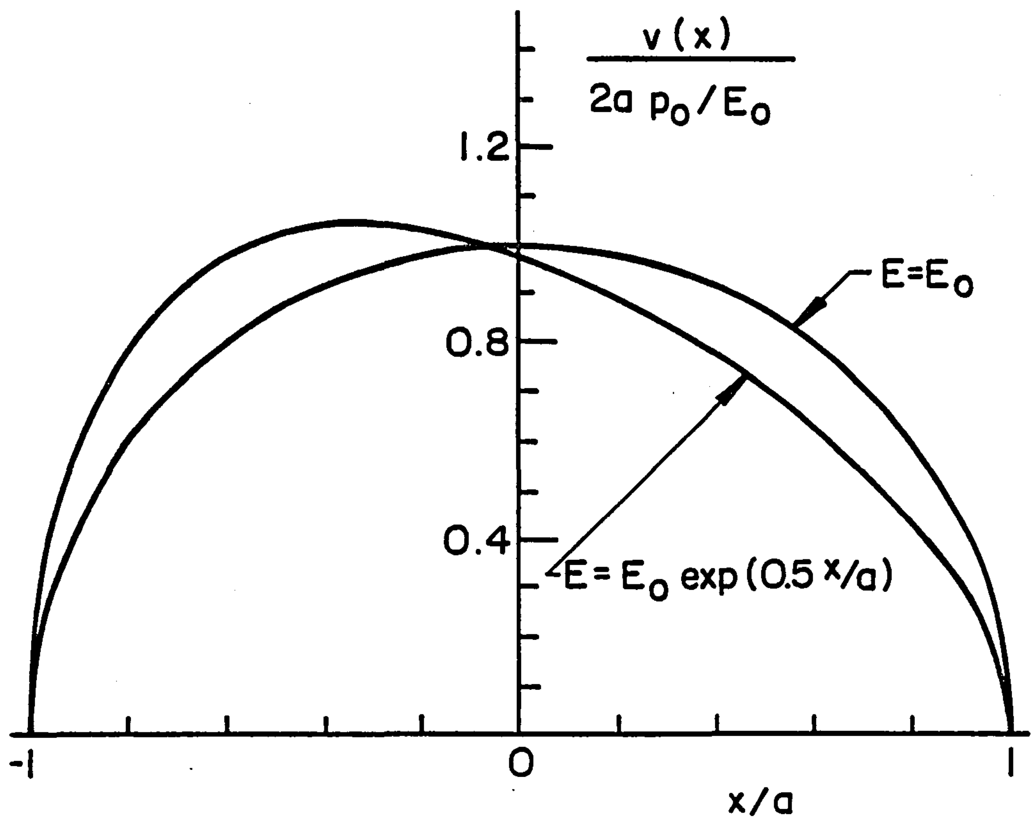

Fig. 3. Crack surface displacement v(x) in a nonhomogeneous and a homogeneous medium under uniform pressure p_0 applied to the crack surfaces. ($E(x) = E_0 \exp(0.5x/a)$ for the nonhomogeneous medium, $E(x) = E_0$ for the homogeneous medium, $\nu = 0.5$, $\beta a = 0.5$, plane stress conditions).

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16. Abstract

In this paper the plane elasticity problem for a nonhomogeneous medium containing a crack is considered. It is assumed that the Poisson's ratio of the medium is constant and the Young's modulus E varies exponentially with the coordinate parallel to the crack. First the half-plane problem is formulated and the solution is given for arbitrary tractions along the boundary. Then the integral equation for the crack problem is derived. It is shown that the integral equation having the derivative of the crack surface displacement as the density function has a simple Cauchy type kernel. Hence, its solution and the stresses around the crack tips have the conventional square-root singularity. The solution is given for various loading conditions. The results show that the effect of the Poisson's ratio and consequently that of the thickness constraint on the stress intensity factors are rather negligible. On the other hand, the results are highly affected by the parameter β describing the material nonhomogeneity in $E(x) = E_0 \exp(\beta x)$.

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