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INTEGRATING MATRIX FORMULATIONS FOR VIBRATIONS OF ROTATING BEAMS INCLUDING THE EFFECTS OF CONCENTRATED MASSES

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SUMMARY

Integrating matrices allow the efficient and accurate integration of functions whose values are given numerically on a discrete set of grid points. They also form the basis of an efficient numerical procedure for solving differential equations associated with the dynamics of rotating beams. By expressing the partial differential equations of motion in matrix notation, utilizing the integrating matrix as a spatial operator, and applying the boundary conditions, the resulting ordinary differential equations can be cast into standard eigenvalue form upon assumption of the usual time dependence. As originally developed, the technique has been limited to beams having continuous mass and stiffness properties along their lengths. This report extends integrating matrix methods to treat the differential equations governing the flap, lag, or axial vibrations of rotating beams also having concentrated masses. Inclusion of concentrated masses is shown to lead to the same kind of standard eigenvalue problem as before, but with slightly modified matrices.

INTRODUCTION

The equations of motion governing the vibrations and stability of rotating beams such as helicopter rotor blades have no closed-form solutions, and approximate methods of solution such as asymptotic techniques, Galerkin's method, or direct numerical integration must be employed. A numerical procedure based on the use of integrating matrices (refs. 1 and 2) has been employed to solve for the vibrations and stability of a wide variety of rotating beam configurations with continuous mass distributions (refs. 3 and 4). The integrating matrix provides a means for numerically integrating a function that is expressed in terms of the values of the function at a set of discrete grid points in the interval of interest. Recent work (ref. 5) has removed the previous restriction that the grid be uniform, and arbitrary increments in the independent variable are now permitted. By expressing the equations of motion of the rotating beam in matrix notation, utilizing the integrating matrix as an operator, and applying the boundary conditions, the resulting ordinary differential equations can be cast into standard eigenvalue form. Solutions can then be determined by standard methods.

As originally formulated (ref. 1) the integrating matrix method requires that the mass and stiffness distributions of the beam be at least piecewise continuous along the length of the beam. Subsequent applications of the method (refs. 2, 3 and 4) were to continuous beams. The important case in which one or more concentrated masses are located along the beam is excluded. Reference 5 allows treatment of discontinuous coefficients in the governing equations, but not the case of concentrated masses. As a number of situations of interest involve rotating beams with concentrated masses, a need exists to extend the present formulation of the integrating matrix technique.

This report describes generalizations of the integrating matrix method required to treat rotating beams with concentrated masses at arbitrary positions along their lengths. Inclusion of concentrated masses is shown to lead to a standard eigenvalue problem of the same form as before, but with slightly modified matrices. To provide a framework for the generalization and to set notation, the following section will briefly review the integrating matrix method for vibrations of rotating beams with continuous mass and stiffness distributions.

REVIEW OF THEORY FOR BEAMS WITHOUT CONCENTRATED MASSES

Let $m(x)$ $(0 \leq x \leq \ell)$ describe the continuous mass distribution of the rotating beam. Consider first the case of vertical (flap) bending. Let $w(x,t)$ be the displacement at position x and time t of the beam normal to the plane of rotation. Then, $w(x,t)$ is a solution for $0 < x < \ell$ of the partial differential equation (ref. 4):

$$[EI(x)\, w'']'' - [T(x)\, w']' + m(x)\, \ddot{w} = 0 \tag{1}$$

where $' = \dfrac{\partial}{\partial x}$, $\dot{} = \dfrac{\partial}{\partial t}$, EI is the bending stiffness, Ω is the rotation speed, and the tension T is given by:

$$T(x) = \Omega^2 \int_x^{\ell} \eta\, m(\eta)\, d\eta \tag{2}$$

Associated boundary conditions for the clamped-free beam are:

$$w(o,t) = w'(o,t) = 0 \tag{3}$$

and

$$w''(\ell,t) = w'''(\ell,t) = 0 \tag{4}$$

To obtain the so-called "fundamental derivative" w'', equation (1) may be formally integrated twice from x to ℓ using the boundary conditions in equation (4). This gives

$$EI(x)\, w''(x,t) = - \int_x^{\ell} \left\{ m(\eta)\, \Omega^2 \eta \left[w(\eta,t) - w(x,t) \right] + m(\eta)\, \ddot{w}(\eta,t)\, (\eta - x) \right\} d\eta \tag{5}$$

Equation (5) can now be converted into an integral equation for the fundamental derivative through use of the relationships

$$w'(x,t) = w'(o,t) + \int_o^x w''(s,t)\, ds \tag{6}$$

and

$$w(x,t) = w(o,t) + \int_o^x w'(s,t)\, ds \tag{7}$$

As a first step toward expressing equation (5) in matrix eigenvalue form, let $x_o, x_1, \ldots, x_N$ be $N + 1$ grid points along the beam such that

$$0 = x_o < x_1 < \ldots < x_N = \ell \tag{8}$$

Spacing between grid points must be non-zero, but need not be uniform. Also, let the $N + 1$-by-$N + 1$ integrating matrix on this grid be denoted by $[L]$ (ref. 5). Then, if $\{f\}$ is a vector giving the values of a function f(x) at the N + 1 grid points, the action of [L] on {f} can be written in the general form

$$\left\{ \int_{x_o}^{x_i} f(x)dx \right\} = [L]\{f\} \tag{9}$$

where i = 0, 1, 2, ..., N. This relation, equations (6) and (7), and the boundary conditions of equation (3) now allow w and w' to be re-expressed in terms of the fundamental derivative w'' through the matrix equations

$$\{w'\} = [L]\{w''\},$$

and

$$\{w\} = [L]^2\{w''\} \tag{10}$$

where {w}, {w'}, and {w''} are column vectors giving the values of the respective functions at the N + 1 grid points $x_o, x_1, \ldots, x_N$.

To deal with the integral in equation (5), which goes from x to ℓ rather than o to x, a slightly modified integrating matrix is required. Let [I] be the N + 1-by-N + 1 identity matrix and let $[B_1]$ be the N + 1-by-N + 1 matrix

$$[B_1] = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \tag{11}$$

Then, as

$$\int_x^{\ell} f(x)dx = \int_o^{\ell} f(x)dx - \int_o^{x} f(x)dx \tag{12}$$

the matrix

$$[J] = ([B_1] - [I])[L] \tag{13}$$

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is the required integrating matrix on the grid from x to ℓ, i.e.,

$$\left\{ \int_{x_i}^{\ell} f(x)dx \right\} = [J]\,\{f\} \tag{14}$$

where $i = 0, \ldots, N + 1$.

Assume the usual time dependence

$$w(x,t) = \overline{w}(x)\, e^{\lambda t} \tag{15}$$

so that $w' = \overline{w}'\, e^{\lambda t}$ and $\ddot{w} = \lambda^2\, \overline{w}\, e^{\lambda t}$. Equation (5) becomes

$$EI\, \overline{w}'' + \int_x^{\ell} m(\eta)\, \Omega^2 \eta\, [\overline{w}(\eta) - \overline{w}(x)]\, d\eta = \lambda^2 \int_x^{\ell} m(\eta)\, (x - \eta)\, \overline{w}(\eta)\, d\eta \tag{16}$$

Evaluating equation (16) at the N + 1 grid points, using the integrating matrices [L] and [J] as operators, and expressing the resulting equations in state variable form leads to the eigenvalue problem

$$[G]\,\{\Phi_1\} = \lambda\,[H]\,\{\Phi_1\} \tag{17}$$

where

$$\{\Phi_1\} = \begin{Bmatrix} \{\overline{w}''\} \\ \{\lambda\, \overline{w}''\} \end{Bmatrix} \tag{18}$$

The 2N + 2-by-2N + 2 matrices [G] and [H] in equation (17) have the functional forms

$$[G] = \begin{bmatrix} [G_{11}] & [o] \\ [o] & [I] \end{bmatrix} \tag{19}$$

and

$$[H] = \begin{bmatrix} [o] & [H_{12}] \\ & \\ [I] & [o] \end{bmatrix} \tag{20}$$

where the N + 1-by-N + 1 matrices $[G_{11}]$ and $[H_{12}]$ are functions of the beam properties and the integrating matrix, and [o] is the N + 1-by-N + 1 zero matrix

Eigenvalue problems for the in-plane (lag) vibrations and extensional (axial) vibrations of a rotating beam are derived in an analogous manner (ref. 4).

INCLUSION OF CONCENTRATED MASSES

This section describes generalizations of the integrating matrix method required to treat rotating beams with concentrated masses. The case of interior concentrated masses is treated first. The boundary mass situation (a tip-mass for the clamped-free beam) is treated by applying a consistent limiting procedure to the interior mass case. Flap, lag, and axial vibrations are treated separately for convenience of discussion.

Interior Masses

Consider the effect of a single concentrated mass of magnitude M located at position $x = \xi$, $o < \xi < \ell$ on the flap, lag, and axial vibrations of a rotating beam. This entails no loss of generality since the effects of multiple concentrated masses may be computed separately and then summed to obtain an aggregate effect. In what follows, the total mass distribution of the beam (distributed plus concentrated) will be denoted by $\mu(x)$. Thus, if $\delta(x)$ is the usual Dirac delta function and $m(x)$ again denotes the distributed mass

$$\mu(x) = m(x) + M\,\delta(x - \xi). \tag{21}$$

Vertical (Flap) Vibrations. The motion of the rotating beam with concentrated inboard mass is still governed by equation (1) if m is replaced by the total mass distribution μ. Further, as $o < \xi < \ell$, the boundary conditions of equations (3) and (4) still hold. Thus, the derivation of equation (5) remains valid if m is replaced by μ. The Dirac delta function has the property that

$$\int_x^\ell \delta(\eta - \xi)\ f(\eta)\ d\eta = \begin{cases} f(\xi) \text{ if } x \leq \xi < \ell \\ o \text{ if } o \leq \xi < x \end{cases} \tag{22}$$

Equation (5) now becomes

$$\begin{aligned} EI\ w''(x,t) = &-\int_x^\ell \{m(\eta)\ \Omega^2 \eta\ [w(\eta,t) - w(x,t)] \\ &+ m(\eta)\ \ddot{w}(\eta,t)\ (\eta - x)\}\ d\eta \\ &- D_\xi(x)\ \{M\ \Omega^2 \xi\, [w(\xi,t) - w(x,t)] + M\ \ddot{w}(\xi,t)(\xi - x)\} \end{aligned} \tag{23}$$

where

$$D_\xi(x) = \begin{cases} 0 \text{ if } o < \xi < x \\ 1 \text{ if } x \leq \xi < \ell \end{cases} \tag{24}$$

Assuming a solution of the form $w(x,t) = \bar{w}(x)\ e^{\lambda t}$ gives as the analogue of equation (16) the equation

$$\begin{aligned} EI\ \bar{w}'' + \int_x^\ell m(\eta)\ \Omega^2 \eta\ [\bar{w}(\eta) - \bar{w}(x)]\ d\eta + M\Omega^2 \xi\ D_\xi(x)\ [\bar{w}(\xi) - \bar{w}(x)] \\ = \lambda^2 \left\{\int_x^\ell m(\eta)(x - \eta)\ \bar{w}(\eta)\ d\eta + MD_\xi(x)(x - \xi)\ \bar{w}(\xi)\right\} \end{aligned} \tag{25}$$

The effect of the concentrated mass M at $x = \xi$ enters equation (25) only through the additive terms

$$G_1(x,\xi) = M\Omega^2 \xi\ D_\xi(x)\ [\bar{w}(\xi) - \bar{w}(x)] \tag{26}$$

and

$$\lambda^2 H_1(x,\xi) = \lambda^2 MD_\xi(x)(x - \xi)\,\overline{w}(\xi) \qquad (27)$$

with the remainder of equation (25) being exactly the same as in the previous case of distributed mass only.

To express equation (25) in matrix form, again consider the discrete grid of N + 1 points

$$0 = x_o < x_1 < \ldots < x_N = \ell$$

where ξ need not coincide with a grid point, and let [L] be the integrating matrix on this grid. Consequently, when equation (25) is evaluated at the grid points and expressed in matrix form, the resulting eigenvalue problem in state variables will have the form of equation (17) with modified matrices

$$[G] = \begin{bmatrix} [G_{11}] + [\hat{G}_{11}] & [o] \\ [o] & [I] \end{bmatrix} \qquad (28)$$

and

$$[H] = \begin{bmatrix} [o] & [H_{12}] + [\hat{H}_{12}] \\ [I] & [o] \end{bmatrix} \qquad (29)$$

where $[G_{11}]$ and $[H_{12}]$ are the same matrices as in relations (19) and (20), and $[\hat{G}_{11}]$ and $[\hat{H}_{12}]$ are N + 1-by-N + 1 matrices which give the values of $G_1(x,\xi)$ and $H_1(x,\xi)$ at the N + 1 grid points.

The principal difficulty in deriving $[\hat{G}_{11}]$ and $[\hat{H}_{12}]$ is that both $G_1(x,\xi)$ and $H_1(x,\xi)$ involve $\overline{w}(\xi)$ but ξ need not be a grid point. However, this problem may be overcome by using an interpolation formula (such as Lagrange interpolation) on the grid with coefficients that depend only on ξ and the grid points, $x_o, \ldots, x_N$. In particular, the coefficients should not depend on values of the function to be interpolated at the grid points. The interpolating expression will thus be of the form

$$\overline{w}(\xi) = a_o(\xi)\,\overline{w}(x_o) + a_1(\xi)\,\overline{w}(x_1) + \ldots + a_N(\xi)\,\overline{w}(x_N) \tag{30}$$

For example, if an Mth order Lagrange interpolating polynomial on the subset of grid points $x_\gamma, x_{\gamma+1}, \ldots, x_{\gamma+M}$ is chosen, where $x_o \leq x_\gamma < \xi < x_{\gamma+M} \leq x_N$, then in equation (30)

$$a_k(\xi) \equiv 0 \text{ for } k < \gamma \text{ or } k > \gamma + M$$

while for $\gamma \leq k \leq \gamma+M$

$$a_k(\xi) = \frac{(\xi-x_\gamma)\ldots(\xi-x_{k-1})\,(\xi-x_{k+1})\ldots(\xi-x_{\gamma+M})}{(x_k-x_\gamma)\ldots(x_k-x_{k-1})\,(x_k-x_{k+1})\ldots(x_k-x_{\gamma+M})} \tag{31}$$

By choosing M sufficiently large, e.g., $M = 7$, the resulting interpolating polynomial gives a high degree of accuracy without the need to cluster grid points around ξ. To use equation (30) in the present context, let $[\Delta_\xi]$ be the $N + 1$-by-$N + 1$ matrix with ik-th element

$$(\Delta_\xi)_{ik} = a_{k-1}(\xi) \qquad k = 1, \ldots, N + 1 \tag{32}$$

i.e., $[\Delta_\xi]$ has identical rows and each row has as elements the successive coefficients in the interpolating formula (30). Pre-multiplying $\{\overline{w}\}$ by $[\Delta_\xi]$, equation (30) now immediately shows

$$[\Delta_\xi]\,\{\overline{w}\} = \{\overline{w}(\xi)\} \tag{33}$$

Some care must be taken in defining a matrix that gives the effect at the grid points of multiplying by the discontinuous function $D_\xi(x)$ which changes from unity to zero across $x = \xi$. Suppose that ξ lies in the subinterval

$$x_{j-1} < \xi \leq x_j \qquad j = 1, \ldots, N \tag{34}$$

Then, if $[D_\xi]$ is the N + 1-by-N + 1 diagonal matrix with diagonal entries

$$(D_\xi)_{ii} = \begin{cases} 0 & i > j \\ 1 & i \leq j \end{cases} \tag{35}$$

that is,

$$[D_\xi] = \begin{bmatrix} \overbrace{1 \ddots 1}^{j} & & 0 \\ & 0 \ddots & \\ 0 & & 0 \end{bmatrix} \tag{36}$$

then $[D_\xi]$ {f} gives the vector of values of the function $D_\xi(x)\ f(x)$ at the grid points. However, it should be noted that $[D_\xi]$ {f} does not explicitly reflect the non-zero values of $D_\xi(x)\ f(x)$ on the interval $x_{j-1} \leq x < \xi$. Consequently, in specific applications, the grid should be chosen if possible so that $\xi - x_{j-1}$ is small relative to $x_j - \xi$.

Let the N + 1-by-N + 1 diagonal matrices $[T_\xi]$ and $[S_\xi]$ be defined by

$$[T_\xi] = M\Omega^2 \xi\ [I] \tag{37}$$

and

$$[S_\xi] = M\ [x - \xi] \qquad (i = 1, \ldots, N + 1) \tag{38}$$

Equations (26) and (27) now immediately show that the matrices $[\hat{G}_{11}]$ and $[\hat{H}_{12}]$ have the forms

$$[\hat{G}_{11}] = \left[{}^{\diagdown}D_{\xi\diagdown}\right]\left[{}^{\diagdown}T_{\xi\diagdown}\right]\left[[\Delta_\xi] - [I]\right][L]^2 \tag{39}$$

and

$$[\hat{H}_{12}] = \left[{}^{\diagdown}D_{\xi\diagdown}\right]\left[{}^{\diagdown}S_{\xi\diagdown}\right][\Delta_\xi][L]^2 \tag{40}$$

<u>In-plane (lag) vibrations</u>. Let $v(x,t)$ denote the deflection of the beam in the plane of rotation, where the total mass distribution is again given by $\mu(x)$ in equation (21). Seeking solutions of the form $v(x,t) = \bar{v}(x)\, e^{\lambda t}$ now gives the governing differential equation

$$\begin{aligned} EI\,\bar{v}''(x) &+ \int_x^{\ell} m(\eta)\,\Omega^2\{\eta[\bar{v}(\eta) - \bar{v}(x)] - \bar{v}(\eta)\,[\eta - x]\}\,d\eta \\ &+ D_\xi(x)\,M\Omega^2\,\{\xi[\bar{v}(\xi) - \bar{v}(x)] - \bar{v}(\xi)\,[\xi - x]\} \\ &= \lambda^2\left\{\int_x^{\ell} m(\eta)\,[x - \eta]\,\bar{v}(\eta)\,d\eta + D_\xi(x)\,M(x - \xi)\,\bar{v}(\xi)\right\} \end{aligned} \tag{41}$$

With the exception of the additive contributions

$$G_2(x,\xi) = G_1(x,\xi) - D_\xi(x)\,M\Omega^2\,\bar{v}(\xi)\,[\xi - x] \tag{42}$$

and $\lambda^2\, H_1(x,\xi)$, where $G_1(x,\xi)$ and $H_1(x,\xi)$ are defined by equations (26) and (27) with $\bar{w}$ replaced by $\bar{v}$, equation (41) is identical to the corresponding equation without concentrated masses. An argument similar to the flap case now gives that the eigenvalue problem associated with (41) for λ and $\{\Phi_2\} = \begin{Bmatrix} \{\bar{v}''\} \\ - \\ \{\lambda \bar{v}''\} \end{Bmatrix}$ is of the same form as (17), (28), and (29), with $[G_{11}]$ and $[H_{12}]$ as in the lag problem without concentrated mass, $[\hat{H}_{12}]$ again given by (40), and $[\hat{G}_{11}]$ given by the expression in (39) plus the additional term

$$+\Omega^2 \left[\diagdown D_\xi \diagdown\right] \left[\diagdown S_\xi \diagdown\right] [\Delta_\xi] \; [L]^2 \tag{43}$$

<u>Extensional (Axial) Vibrations</u>. Let u(x,t) be the axial extension of the beam and seek solutions of the form $u(x,t) = \bar{u}(x) e^{\lambda t}$. Then, the governing differential equation for the rotating beam with concentrated interior mass is

$$AE\,\bar{u}' - \int_x^\ell m(\eta)\Omega^2\,\bar{u}(\eta)\,d\eta - D_\xi(x)\,M\Omega^2\,\bar{u}(\xi)$$
$$= -\lambda^2 \left\{ \int_x^\ell m(\eta)\,\bar{u}(\eta)\,d\eta + D_\xi(x) M\,\bar{u}(\xi) \right\} \tag{44}$$

This equation differs from the case without concentrated mass only through the additive terms

$$G_3(x,\xi) = -D_\xi(x)\,M\Omega^2\,\bar{u}(\xi) \tag{45}$$

and

$$\lambda^2 H_3\,(x,\xi) = -\lambda^2 D_\xi(x) M\,\bar{u}(\xi) \tag{46}$$

Consequently, the corresponding eigenvalue problem for λ and $\{\Phi_3\} = \begin{Bmatrix} \{\bar{u}'\} \\ \lambda\{\bar{u}'\} \end{Bmatrix}$ will be of the same form as (17), (28), and (29) where $[G_{11}]$ and $[H_{11}]$ are as in the axial problem without concentrated mass, and, as $\{\bar{u}\} = [L]\{\bar{u}'\}$,

$$[\hat{H}_{12}] = -M \left[\diagdown D_\xi \diagdown\right] [\Delta_\xi] \; [L], \tag{47}$$

and

$$[\hat{G}_{12}] = \Omega^2 \; [\hat{H}_{12}]. \tag{48}$$

Boundary Masses

If the concentrated mass M is placed directly at the free end of a rotating cantilever beam, i.e. $\xi = \ell$, one of the free-end boundary conditions is replaced by the equation of motion of the concentrated mass. For example, in the flap case the boundary conditions at $x = \ell$ become

$$w'' = 0 \tag{49a}$$

and

$$(EI\ w'')' - M\{\Omega^2 \ell\ w' + \ddot{w}\} = 0 \tag{49b}$$

An appropriate limiting procedure may now be used to show that equation (23) for the interior case yields the conditions of equations (49) as ξ tends to ℓ. In this connection, let Lim denote the limit as $x \to \ell$ with $\xi > x$. This limiting procedure thus automatically ensures that $\xi \to \ell$ as $x \to \ell$. Further, as $\xi > x$, $D_\xi(x) \equiv 1$.

Applying the limiting procedure to equation (23) immediately gives

$$\text{Lim}\left(EI\ w''(x,t)\right) = EI\ w''(\ell,t) = 0$$

consistent with (49a). Differentiating (23) with respect to x with $\xi > x$ gives

$$\left(EI\ w''(x,t)\right)' = -\int_x^\ell \{m\,\Omega^2\,\eta\ w'(x,t) - m\,\ddot{w}(\eta,t)\}\,d\eta + M\,\{\Omega^2 \xi\ w'(x,t) + \ddot{w}(\xi,t)\} \tag{50}$$

Applying Lim to (50) now gives (49b). This shows that equation (23) remains valid as $\xi \to \ell$. This, in turn, implies that the integrating matrix formulation for flap vibrations with interior concentrated mass remains valid in the case of a boundary mass. Similar arguments, which will be omitted, involving equation (41), (44), and the limiting procedure show that inboard concentrated mass formulations for the lag and axial vibrations also hold in the boundary mass case.

VERIFICATION OF THE FORMULATION WITH CONCENTRATED MASSES

To verify the present formulation of the integrating matrix method including the effects of concentrated masses, a number of beam problems were analyzed. These included the flap, lag, and axial vibrations of beams with a single inboard concentrated mass, a tip mass only, and up to ten concentrated masses placed along the length of the beam. Both rotating and non-rotating situations were considered. Results for these test cases were compared to known exact solutions and solutions obtained using a finite element program. In all cases, results obtained from the integrating matrix formulation were found to be in excellent agreement with these other results. For the flap and lag vibrations, test cases also included beams with ten concentrated masses of appropriate size and no distributed masses, i.e., $m(x) \equiv 0$. Excellent agreement with other results was also obtained for these cases. It is worth noting that the integrating matrix formulation is not appropriate for the axial vibration of a beam with only concentrated masses if u' is taken as the dependent variable as is done here. Indeed, if $m(x) = 0$, equation (44) has the functional form AE $\overline{u'}$ = constant, and the eigenvalue character of the problem is lost.

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