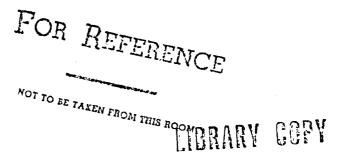NASA Technical Memorandum 84246

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SOLUTION OF A SECOND-ORDER LINEAR SYSTEM

BY MATCHED ASYMPTOTIC EXPANSIONS

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SUMMARY

For the purpose of giving a clear exposition of the method, matched asymptotic expansions (MAE) are used to obtain a first-order approximation to the solution of a singularly perturbed second-order system. A special case is considered in which the uniform asymptotic solution obtained by MAE is shown to converge to the exact solution. Ways in which the method can be used to solve higher-order linear systems, including those which are not singularly perturbed, are also discussed.

INTRODUCTION

One of the simplest singular-perturbation problems is that of finding an asymptotic solution to a constant-coefficient linear system, with one slow and one fast variable subject to initial conditions. Such a system is given by

$$\frac{dx}{dt} = Ax + By \quad ; \quad x(\varepsilon,0) = \alpha$$

$$\varepsilon \frac{dy}{dt} = Cx + Dy \quad ; \quad y(\varepsilon,0) = \beta \tag{1}$$

where x and y are called the slow and fast variables, respectively, and ε is a "small" parameter. In this paper we will obtain an asymptotic solution to this problem to first order in ε on $0 \leq t \leq T \leq \infty$ by the method of matched asymptotic expansions (MAE) under the following two assumptions:

$$E = AD - BC \neq 0 \tag{2}$$

$$D < 0 \tag{3}$$

The first of these assumptions assures the existence of a solution and the second assures boundary-layer stability, the significance of which will become apparent later.

The purpose of solving such a simple example is to make as transparent as possible the use of the MAE method. This method was developed in connection with certain fluid mechanics problems and is discussed in depth in references 1 and 2. Reference 3 adapts and applies MAE to a wide range of problems involving systems of ordinary differential equations.

There are four steps in the MAE method: (1) formulation of the outer problem and its asymptotic solution to the desired order; (2) formulation of the inner (boundary-layer) problem and its asymptotic solution to the desired order; (3) matching of the outer and inner solutions to obtain unknown constants of integration and the common parts; and (4) formulation of a composite solution that is uniformly valid in all dependent variables for $0 \leq t \leq T \leq \infty$.

Although the MAE method is somewhat more cumbersome than other singular-perturbation techniques that have been developed in recent years, it has two advantages over the more recent methods. The first is that it explicitly splits the problem into two (or more) separate ones. The solutions of these problems may be useful in their own right, in addition to being the constituents of composite solutions. Further, the inner problem(s) satisfies the boundary conditions as opposed to most other methods. The second advantage lies in the great flexibility that is afforded in the form of the assumed stretching transformations and inner and outer expansions, as well as in the manner in which these expansions are combined to give the uniform composite solution. This flexibility is often desirable and occasionally necessary.

The solution to equation (1) will be obtained to first-order, since determination of the zero-order solution is essentially trivial. The zero-order uniform solution consists of the solution to the reduced problem [Eq. (1), with $\varepsilon = 0$ and the initial condition on y discarded] added to the solution of the zero-order inner problem [eq. (1), with t transformed to $\tau = t/\varepsilon$ and $\varepsilon = 0$]. Steps three and four of the MAE method are then not needed. Further, the first-order solution gives much better numerical results than the zero-order one, unless ε is "very small." This point is dealt with in the final section (*Discussion and Generalizations*).

By definition, $f(\varepsilon)$ is an nth-order asymptotic approximation to $F(\varepsilon)$ if

$$\lim_{\varepsilon \to 0} \frac{|F(\varepsilon) - f(\varepsilon)|}{\varepsilon^{n+1}} < \infty$$

OUTER SOLUTION

The outer system associated with equation (1) describes the solution away from the initial point. It is simply equation (1) without the initial conditions:

$$\frac{dx^o}{dt} = Ax^o + By^o$$

$$\varepsilon \frac{dy}{dt} = Cx^o + Dy^o$$

(4)

To solve this to first-order we assume that power-series expansions are valid and set

$$x^o(\varepsilon,t) = x^o_o(t) + x^o_1(t)\varepsilon$$

$$y^o(\varepsilon,t) = y^o_o(t) + y^o_1(t)\varepsilon$$

(5)

in equations (4) and retain only first-order terms to get

$$\frac{dx^o_o}{dt} + \frac{dx^o_1}{dt}\varepsilon = Ax^o_o + Ax^o_1\varepsilon + By^o_o + By^o_1\varepsilon$$

$$\frac{dy^o}{dt}\varepsilon = Cx^o_o + Cx^o_1\varepsilon + Dy^o_o + Dy^o_1\varepsilon$$

(6)

Equating the coefficients of ε^o gives the zero-order outer problem

$$\frac{dx^o_o}{dt} = Ax^o_o + By^o_o$$

$$0 = Cx^o_o + Dy^o_o$$

(7)

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The solutions to equations (7) under assumptions (2) and (3), are

$$x_o^o = C_o^o \, e^{\frac{E}{D} t} \quad ; \quad y_o^o = -C_o^o \, \frac{C}{D} \, e^{\frac{E}{D} t} \tag{8}$$

where C_o^o is a constant of integration that is as yet undetermined. Since equations (7) are a first-order system, both of the initial conditions cannot be satisfied (unless, of course, $\beta = -\alpha \frac{C}{D}$). Thus, at best, equations (8) are zero-order approximations only on $0 < t \leq T \leq \infty$, and no initial conditions have been imposed on the outer problem.

Similarly, the first-order problem is obtained from the coefficients of ε in equations (6) as

$$\frac{dx_1^o}{dt} = Ax_1^o + By_1^o$$

$$\frac{dy_o^o}{dt} = Cx_1^o + Dy_1^o \tag{9}$$

Since $y_o^o(t)$ is a known function from the solution of the previous order, this is a first-order problem. Using (8),

$$\frac{dx_1^o}{dt} = \frac{E}{D} x_1^o - C_o^o \, \frac{CBE}{D^3} \, e^{\frac{E}{D} t}$$

$$y_1^o = -C_o^o \, \frac{CE}{D^3} \, e^{\frac{E}{D} t} - \frac{C}{D} x_1^o \tag{10}$$

which has solution

$$x_1^o = C_1^o \, e^{\frac{E}{D} t} - C_o^o \, \frac{CBE}{D^3} \, t \, e^{\frac{E}{D} t}$$

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$$y_1^o = -C_o^o \frac{CE}{D^3} e^{\frac{E}{D}t} - C_1^o \frac{C}{D} e^{\frac{E}{D}t} + C_o^o \frac{C^2 BE}{D^4} t \; e^{\frac{E}{D}t} \qquad (11)$$

where C_1^o is another as yet unknown constant of integration.

INNER SOLUTION

Because the purpose of the inner system is to model the solution near the initial point, its solution is required to satisfy both initial conditions. Postulating that the inner motion occurs on a time scale of order ε, we stretch the independent variable by

$$\tau = \frac{t}{\varepsilon} \qquad (12)$$

Substituting equation (12) in (1) gives the inner (boundary-layer) equations

$$\frac{dx^i}{d\tau} = \varepsilon(Ax^i + By^i) \quad ; \quad x^i(\varepsilon,0) = \alpha$$

$$\frac{dy^i}{d\tau} = Cx^i + Dy^i \qquad ; \quad y^i(\varepsilon,0) = \beta$$

$$\qquad (13)$$

To solve this to first-order, assume a power-series expansion and set

$$x^i(\varepsilon,\tau) = x_o^i(\tau) + x_1^i(\tau)\varepsilon$$

$$y^i(\varepsilon,\tau) = y_o^i(\tau) + y_1^i(\tau)\varepsilon$$

$$\qquad (14)$$

in (13) to get

$$\frac{dx_o^i}{d\tau} + \frac{dx_1^i}{d\tau} \varepsilon = Ax_o^i\varepsilon + By_o^i\varepsilon \qquad\qquad ; \quad x_o^i(0) + x_1^i(0)\varepsilon = \alpha$$

$$\frac{dy_o^i}{d\tau} + \frac{dy_1^i}{d\tau} \varepsilon = Cx_D^i + Cx_1^i\varepsilon + Dy_o^i + Dy_1^i\varepsilon \quad ; \quad y_o^i(0) + y_1^i(0)\varepsilon = \beta$$

$$\qquad (15)$$

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The zero-order problem from (15) is

$$\frac{dx_o^i}{d\tau} = 0 \qquad ; \quad x_o^i(0) = \alpha$$

$$\frac{dy_o^i}{d\tau} = Cx_o^i + Dy_o^i \quad ; \quad y_o^i(0) = \beta \tag{16}$$

with the solution

$$x_o^i = \alpha$$

$$y_o^i = \left(\beta + \frac{C\alpha}{D}\right)e^{D\tau} - \frac{C\alpha}{D} \tag{17}$$

Note that x_o^i is a constant. In effect, the boundary-layer motion is so rapid that to zero-order the slow variable has not had time to begin its motion.

The first-order problem is

$$\frac{dx_1^i}{d\tau} = Ax_o^i + By_o^i \quad ; \quad x_1^i(0) = 0$$

$$\frac{dy_1^i}{d\tau} = Cx_1^i + Dy_1^i \quad ; \quad y_1^i(0) = 0 \tag{18}$$

which has the solution

$$x_1^i = \frac{E}{D}\,\alpha\tau + \frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right)\left(e^{D\tau} - 1\right)$$

$$y_1^i = \frac{EC\alpha}{D^3}\,e^{D\tau} - \frac{BC}{D^2}\left(\beta + \frac{C\alpha}{D}\right)e^{D\tau} - \frac{EC\alpha}{D^2}\,\tau - \frac{EC\alpha}{D^3} + \frac{BC}{D}\left(\beta + \frac{C\alpha}{D}\right)\tau\, e^{D\tau} + \frac{BC}{D^2}\left(\beta + \frac{C\alpha}{D}\right) \tag{19}$$

Note that to first-order there is a variation in the slow variable x in the boundary layer.

Matching is the key step in the MAE method. It identifies any faulty assumptions regarding the form of the asymptotic expansions and stretching transformations and serves to determine the unknown constants C_o^o and C_1^o. Matching essentially requires that the behavior of the outer solution as $t \to 0$ is the same as that of the inner solution as $\tau \to \infty$, that is, the outer solution extended into the inner region must agree with the inner solution extended into the outer region. This "limit matching principle" may be stated as

$$\lim_{\substack{t \to 0 \\ \tau \to \infty \\ \varepsilon \to 0}} [x^o(\varepsilon, t) - x^i(\varepsilon, \tau)] = 0 \tag{20a}$$

$$\lim_{\substack{t \to 0 \\ \tau \to \infty \\ \varepsilon \to 0}} [y^o(\varepsilon, t) - y^i(\varepsilon, \tau)] = 0 \tag{20b}$$

These equations are to be regarded as shorthand for the requirement that the outer and inner solutions must agree in an "overlap region" between the inner and outer regions.

The limiting behaviors of x^o and y^o are obtained by expanding equations (8) and (10) about $t = 0$; for small t,

$$x_o^o \approx C_o^o \left(1 + \frac{E}{D} t + \ldots \right)$$

$$y_o^o \approx -C_o^o \frac{C}{D} \left(1 + \frac{E}{D} t + \ldots \right)$$

$$x_1^o \approx C_1^o \left(1 + \frac{E}{D} t + \ldots \right) - C_o^o \frac{CBE}{D^3} \left(t + \ldots \right) \tag{21}$$

$$y_1^o \approx -C_o^o \frac{CE}{D^3} \left(1 + \frac{E}{D} t + \ldots \right) - C_1^o \frac{C}{D} \left(1 + \frac{E}{D} t + \ldots \right) + C_o^o \frac{C^2 BE}{D^4} \left(t + \ldots \right)$$

For large τ, (17) and (19) are approximately

$$x_o^i \approx \alpha$$

$$y_o^i \approx -\frac{C}{D}\alpha$$

$$x_1^i \approx \frac{E}{D}\alpha\tau - \frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right) \qquad (22)$$

$$y_1^i \approx -\frac{EC\alpha}{D^2}\tau - \frac{EC\alpha}{D^3} + \frac{BC}{D^2}\left(\beta + \frac{C\alpha}{D}\right)$$

By assumption (3), terms involving $e^{D\tau}$ do not appear in (22).

Using equations (5), (14), (21), and (22) in equation (20a), we now match x to first-order:

$$\lim_{\substack{t\to 0 \\ \tau\to\infty \\ \varepsilon\to 0}} \left\{ c_o^o + c_o^o \frac{E}{D}t + \dots + \varepsilon\left(c_1^o + c_1^o \frac{E}{D}t + \dots - c_o^o \frac{CBE}{D^3}t - \dots\right)\right.$$
$$\left. - \alpha - \varepsilon\left[\frac{E}{D}\alpha\tau - \frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right)\right]\right\} = 0 \qquad (23)$$

Noting that ε goes to zero faster than t, and using equation (12), equation (23) can be true only if

$$c_o^o - \alpha = 0 \quad \text{(coefficients of } t^o \varepsilon^o) \qquad (24a)$$

$$c_o^o \frac{E}{D} - \frac{E}{D}\alpha = 0 \quad \text{(coefficients of } t^1 \varepsilon^o) \qquad (24b) \qquad (24)$$

$$c_1^o + \frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right) = 0 \quad \text{(coefficients of } t^o \varepsilon^1) \qquad (24c)$$

Equations (24a) and (24c) give

$$c_o^o = \alpha$$

$$c_1^o = -\frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right) \qquad (25)$$

and (24b) is then satisfied trivially. Terms in equation (23) that involve higher powers of $t^m \epsilon^n$ than $t^0 \epsilon^1$ are not matched at this order of approximation.

It is now clear why assumption (3) is necessary. If $D > 0$, then y_o^i grows exponentially and cannot be matched by any algebraic term. It is in fact a general requirement of singular-perturbation analysis that the zero-order boundary-layer equations, equations (16) in our case, be asymptotically stable. In the MAE method, this requirement arises naturally from the matching relation.

The matching relation for y to first-order is

$$\lim_{\substack{t \to 0 \\ \tau \to \infty \\ \epsilon \to 0}} \left\{ \left(- C_o^o \frac{C}{D} - C_o^o \frac{CE}{D^2} t - \dots \right) - \epsilon \left(C_o^o \frac{CE}{D^3} + C_o^o \frac{CE^2}{D^4} t + \dots + C_1^o \frac{C}{D} + C_1^o \frac{CE}{D^2} t + \dots \right. \right.$$

$$\left. \left. - C_o^o \frac{C^2 BE}{D^4} t - \dots \right) + \frac{C\alpha}{D} - \epsilon \left[- \frac{EC\alpha}{D^2} \tau - \frac{EC\alpha}{D^3} + \frac{BC}{D^2} \left(\beta + \frac{C\alpha}{D} \right) \right] \right\} = 0 \qquad (26)$$

This implies

$$- C_o^o \frac{C}{D} + \frac{C\alpha}{D} = 0$$

$$- C_o^o \frac{CE}{D^2} + \frac{EC\alpha}{D^2} = 0 \qquad (27)$$

$$- C_o^o \frac{CE}{D^3} - C_1^o \frac{C}{D} + \frac{EC\alpha}{D^3} - \frac{BC}{D^2} \left(\beta + \frac{C\alpha}{D} \right) = 0$$

which agrees with (25). In general, matching the slow variables is sufficient to determine all unknown constants of integration.

The fact that we have been able to match both x and y validates our selection of the linear stretching transformation (12) and power-series expansions (5) and (14). If matching had not been possible with these assumptions, more general transformations and expansions would be required.

From equations (25), (8), and (10) the outer solution is

$$x^o_o = \alpha \; e^{\frac{E}{D} t} \quad ; \quad y^o_o = -\alpha \frac{C}{D} e^{\frac{E}{D} t}$$

$$x^o_1 = -\frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right) e^{\frac{E}{D} t} - \alpha \frac{CBE}{D^3} t \; e^{\frac{E}{D} t} \tag{28}$$

$$y^o_1 = -\alpha \frac{CE}{D^3} e^{\frac{E}{D} t} + \frac{BC}{D^2}\left(\beta + \frac{C\alpha}{D}\right) e^{\frac{E}{D} t} + \alpha \frac{C^2 BE}{D^4} t \; e^{\frac{E}{D} t}$$

FORMATION OF COMPOSITE SOLUTION

We now have a representation for the solution of (1) near $t = 0$, as given by
equations (14), with (17) and (19), and a representation away from $t = 0$, as given
by (5), with (28). However, it is generally more useful to have one solution that
is valid everywhere, and this is the purpose of composite solutions. The most common
composite solution (in principle the number of solutions is infinite) is the additive
one formed by simply adding the inner and outer solutions. The result must be
adjusted by subtracting out the "common part," that is, the portion of the solutions
that explicitly cancels in the matching relation; otherwise, this portion would be
added in twice. In particular, the initial conditions would not be met. Thus, the
additive composite has the form

$$x^a(\varepsilon,t) = x^o(\varepsilon,t) + x^i\left(\varepsilon, \frac{t}{\varepsilon}\right) - CP_x(\varepsilon,t)$$

$$y^a(\varepsilon,t) = y^o(\varepsilon,t) + y^i\left(\varepsilon, \frac{t}{\varepsilon}\right) - CP_y(\varepsilon,t) \tag{29}$$

To first-order,

$$x^a_1 = x^o_o + x^o_1 \varepsilon + x^i_o + x^i_1 \varepsilon - CP_{x_1}$$

$$y^a_1 = y^o_o + y^o_1 \varepsilon + y^i_o + y^i_1 \varepsilon - CP_{y_1} \tag{30}$$

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From equations (23) and (26), or equivalently (24) and (27), the common parts to first-order for the problem at hand are

$$CP_{x_1} = \alpha + \frac{E}{D}\alpha t - \frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right)\epsilon$$

$$CP_{y_1} = -\frac{C}{D}\alpha - \frac{EC}{D^2}\alpha t - \frac{EC}{D^3}\alpha\epsilon + \frac{BC}{D^2}\left(\beta + \frac{C\alpha}{D}\right)\epsilon$$

(31)

Note that although the matching of y is not required for determining constants, it is required for determining common parts. According to (30), the first-order additive composite solution is

$$x_1^a = \alpha\, e^{\frac{E}{D}t} - \epsilon\left[\frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right) + \alpha\frac{CBE}{D^3}t\right]e^{\frac{E}{D}t} + \epsilon\frac{B}{D}\left(\beta + \frac{C\alpha}{D}\right)e^{\frac{D}{\epsilon}t}$$

(32)

$$y_1^a = -\alpha\frac{C}{D}e^{\frac{E}{D}t} + \epsilon\left[-\alpha\frac{CE}{D^3} + \frac{CB}{D^2}\left(\beta + \frac{C\alpha}{D}\right) + \alpha\frac{C^2 BE}{D^4}t\right]e^{\frac{E}{D}t}$$

$$+ \left(\beta + \frac{C\alpha}{D}\right)e^{\frac{D}{\epsilon}t} + \epsilon\left[\frac{EC\alpha}{D^3} - \frac{BC}{D^2}\left(\beta + \frac{C\alpha}{D}\right) + \frac{BC}{D}\left(\beta + \frac{C\alpha}{D}\right)\frac{t}{\epsilon}\right]e^{\frac{D}{\epsilon}t}$$

(33)

This solution is a first-order, uniform asymptotic approximation to the problem (1), that is, if $x(\epsilon,t)$ and $y(\epsilon,t)$ are the exact solutions, then

$$\lim_{\epsilon\to 0}\frac{\left|x(\epsilon,t) - x_1^a(\epsilon,t)\right|}{\epsilon^2} < \infty$$

$$\lim_{\epsilon\to 0}\frac{\left|y(\epsilon,t) - y_1^a(\epsilon,t)\right|}{\epsilon^2} < \infty$$

on the interval $0 \le t \le T < \infty$.

A SPECIAL CASE

In this section, a special case is considered for the purpose of easily determining higher-order approximations and comparing the approximate solution with the exact one. Specifically, consider (1) with $C = 0$:

$$\frac{dx}{dt} = Ax + By \quad ; \quad x(\varepsilon,0) = \alpha$$

$$\frac{dy}{dt} = Dy \qquad ; \quad y(\varepsilon,0) = \beta$$

(34)

In this system, the second equation is uncoupled from the first and the exact solution is easily obtained as

$$x^e = \alpha \, e^{At} - \frac{B\beta}{(D/\varepsilon - A)} \left(e^{At} - e^{\frac{D}{\varepsilon} t} \right)$$

$$y^e = \beta \, e^{\frac{D}{\varepsilon} t}$$

(35)

Solving the outer and inner problems associated with (34) gives

$$x_j^o = C_j^o \, e^{At} \quad ; \quad j = 0,1,2, \ldots$$

$$y_j^o = 0 \qquad ; \quad j = 0,1,2, \ldots$$

$$x_1^i = A\alpha\tau + \frac{B\beta}{D} \left(e^{D\tau} - 1 \right)$$

$$x_2^i = \frac{1}{2} A^2 \alpha \tau^2 - \frac{AB\beta}{D} \tau + \frac{AB\beta}{D^2} \left(e^{D\tau} - 1 \right)$$

(36)

$$\vdots$$

$$y_o^i = \beta \, e^{D\tau}$$

$$y_j^i = 0 \qquad ; \quad j = 1,2, \ldots$$

Forming the additive composite for y gives

$$y_j^a = \beta \, e^{\frac{D}{\varepsilon} t} \quad ; \quad j = 0,1,2, \ldots \tag{37}$$

so that

$$y_j^a = y^e \quad ; \quad j = 0,1,2, \ldots \tag{38}$$

This will happen whenever the fast equation does not contain slow variables and the initial condition on y does not depend on ε .

Forming the additive composite for x to second-order,

$$x_2^a = \alpha \, e^{At} - \varepsilon \, \frac{B\beta}{D} \left(e^{At} - e^{\frac{D}{\varepsilon} t} \right) - \varepsilon^2 \, \frac{AB\beta}{D^2} \left(e^{At} - e^{\frac{D}{\varepsilon} t} \right) \tag{39}$$

From this we conjecture that

$$x_j^a = \alpha \, e^{At} - \frac{B\beta}{A} \left(e^{At} - e^{\frac{D}{\varepsilon} t} \right) \left[\sum_{k=1}^{j} \left(\frac{\varepsilon A}{D} \right)^k \right] \tag{40}$$

This clearly shows the need for $\left| \varepsilon (A/D) \right| < 1$. By the binomial expansion:

$$\frac{1}{(D/\varepsilon - A)} = \frac{1}{A} \sum_{k=1}^{\infty} \left(\frac{\varepsilon A}{D} \right)^k \tag{41}$$

From (35) and (41),

$$x^e = \alpha \, e^{At} - \frac{B\beta}{A} \left(e^{At} - e^{\frac{D}{\varepsilon} t} \right) \sum_{k=1}^{\infty} \left(\frac{\varepsilon A}{D} \right)^k \tag{42}$$

Comparing equations (40) and (42) we see that

$$\lim_{j \to \infty} x_j^a = x^e \tag{43}$$

so that the uniform asymptotic expansion resulting from the method of MAE converges to the exact solution.

DISCUSSION AND GENERALIZATIONS

We now return to the problem stated in the *Introduction*. The fact that the solution (32) obtained by MAE is a first-order, uniform, asymptotic approximation to the exact solution gives no assurance either that the asymptotic expansion is convergent or that it is numerically close to the exact solution. The first question, that of convergence, is relatively unimportant. (This is somewhat surprising in view of the heavy emphasis placed on convergence in elementary mathematics.) The second question, that of numerical accuracy, is obviously of great importance in practical applications and will be taken up now.

The solution (32) consists of two types of terms, those with factor $\exp[(E/D)t]$ and those with factor $\exp[(D/\varepsilon)t]$. The first type arises from the outer solution and the second from the inner. The more rapidly the inner terms decay relative to the outer terms the better the approximation (32) will be. Thus, it is not the absolute magnitude of ε that is important for numerical accuracy but rather the size of $|D/\varepsilon|$ relative to $|E/D|$. The larger $|D/\varepsilon|$ is relative to $|E/D|$, the better will be the approximation. We write this accuracy requirement as

$$\left| \frac{E}{D} \right| \ll \left| \frac{D}{\varepsilon} \right| \tag{44}$$

Now suppose $\varepsilon = 1$, that is, the system is given by

$$\frac{dx}{dt} = Ax + By \quad ; \quad x(\varepsilon, 0) = \alpha \tag{45a}$$

$$\frac{dy}{dt} = Cx + Dy \quad ; \quad y(\varepsilon, 0) = \beta \tag{45b}$$

Then (44) becomes, using (2),

$$|AD - BC| \ll D^2 \qquad (46)$$

This will be true if

$$|D| \geqslant \text{Max}(|A|, |B|, |C|) \qquad (47)$$

Thus we can solve the nonsingularly perturbed problem (45) by MAE, provided (47) holds. The solution to first-order is (32) with $\varepsilon = 1$. The condition (47) may also be deduced by studying the eigenvalues of the coefficient matrix of (45). These eigenvalues are

$$\lambda_1, \lambda_2 = \frac{1}{2} \left\{ A + D \pm [(A - D)^2 + 4BC]^{\frac{1}{2}} \right\} \qquad (48)$$

If (47) is satisfied, one of these eigenvalues will be approximately equal to D, a large negative number, and the other will be relatively small in magnitude, exactly the situation we require.

Solution of (45) by MAE may be viewed another way. Let

$$\varepsilon = \frac{\text{Max}(|A|, |B|, |C|)}{|D|} \qquad (49)$$

Multiply (45b) by ε to get the system

$$\frac{dx}{dt} = Ax + By \quad ; \quad x(\varepsilon, 0) = \alpha$$

$$\varepsilon \frac{dy}{dt} = C'x + D'y \quad ; \quad y(\varepsilon, 0) = \beta \qquad (50)$$

and let

$$E' = AD' - BC' \qquad (51)$$

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where

$$C' = \varepsilon C \quad ; \quad D' = \varepsilon D \tag{52}$$

We now have a problem to which the MAE method may be applied, and the first-order solution is given by (32), with C, D, and E replaced by C', D', and E'. Transforming back into unprimed quantities then gives (32), with $\varepsilon = 1$. Therefore, the same answer is obtained by simply artificially inserting ε in front of dy/dt in (45b), applying MAE, and then setting $\varepsilon = 1$. This technique greatly broadens the applicability of singular-perturbation methods.

Since there is no distinction between x and y in equations (45), we conclude that for a valid and useful solution of (45) by MAE (or any other singular-perturbation method) it is sufficient that either

$$|D| \gg \text{Max}(|A|, |B|, |C|), \text{ and } D < 0 \tag{53}$$

or

$$|A| \gg \text{Max}(|D|, |B|, |C|), \text{ and } A < 0 \tag{54}$$

It is obvious that these conclusions may be easily extended to higher-order systems. Consider

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}(\varepsilon, t)\underline{x} + \underline{\underline{B}}(\varepsilon, t)\underline{y} \quad ; \quad \underline{x}(\varepsilon, 0) = \underline{\alpha}$$

$$\tag{55}$$

$$\frac{d\underline{y}}{dt} = \underline{\underline{C}}(\varepsilon, t)\underline{x} + \underline{\underline{D}}(\varepsilon, t)\underline{y} \quad ; \quad \underline{y}(\varepsilon, 0) = \underline{\beta}$$

where $\underline{\underline{A}}(\varepsilon, t)$, $\underline{\underline{B}}(\varepsilon, t)$, $\underline{\underline{C}}(\varepsilon, t)$, and $\underline{\underline{D}}(\varepsilon, t)$ are of class C^{∞} in t and have asymptotic power-series expansions in ε. A valid and useful solution of this problem can be obtained by MAE, provided the solution consists of combinations of fast decaying and slow modes. This will occur when the eigenvalues of $\underline{\underline{D}}$ (or equivalently, of $\underline{\underline{A}}$) have real parts that are relatively large (in absolute value) negative numbers. Therefore, we can give the following general procedure for solving

linear systems by MAE: (1) put the system in a form such that eigenvalues of D have the required properties by rearranging the equations or transforming the variables, or both; (2) insert ϵ in front of dy/dt to create a singularly perturbed system; (3) apply the four steps of MAE; and (4) set $\epsilon = 1$.

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16. Abstract

The method of matched asymptotic expansions (MAE) is used to obtain a first-order approximation to the solution of a singularly perturbed second-order system for the purpose of giving a clear exposition of the method. A special case is considered in which the uniform asymptotic solution obtained by MAE is shown to converge to the exact solution. A discussion is given of how the method can be used to solve higher-order linear systems, including those which are not singularly perturbed.

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